July 2, 2013

SISSA Entrance Examination

PhD in Theoretical Particle Physics

SOLVE two out of the four exercises given below. You are allowed to choose among these four combinations: 1-3, 1-4, 2-3 or 2-4.

Problem 1

Consider a charged point particle P_1, with electric charge q_1, subject to the action of a magnetic field with potential $\vec{A}(x)$ via the so-called minimal coupling, given by the Hamiltonian

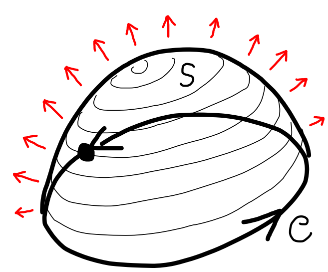
$$H = \frac{(\vec{p} - q_1 \vec{A})^2}{2m}$$

1. Show that the Schrödinger equation is invariant under the $U(1)$ gauge transformation defined by (work in natural units, $\hbar = c = 1$)

   $$\vec{A}(\vec{x}) \to \vec{A}'(\vec{x}) = \vec{A}(\vec{x}) + \vec{\nabla}\alpha(x)\,, \qquad \psi(\vec{x}, t) \to \psi'(\vec{x}, t) = e^{i\,q_1 \alpha(x)}\psi(\vec{x}, t)$$

   where ψ is the wave-function of the particle. Choose $\alpha(\vec{x})$ such that $\vec{A}' = 0$ and use this to prove that the particle wave function $\psi(x, t)$ depends on the magnetic potential only through a phase.

2. Consider the particle P_1 moving on a closed path $\mathcal{C}$, in presence of an external magnetic field. Using the result of point 1, prove that while moving along $\mathcal{C}$ the wave-function picks-up a phase proportional to the flux of the magnetic field $\vec{B}$ across a surface S enclosed by $\mathcal{C}$.



3. Consider the previous setting in the specific case in which the magnetic field is generated by a point-particle P_2 which carries a *magnetic* charge g_2 (this is a so-called magnetic monopole: a particle which sources a magnetic field). Use the previous result, and impose that the phase factor depends only on the contour $\mathcal{C}$ and not on the particular choice

of the enclosed surface S. Derive from this requirement the famous *Dirac quantization condition*:

$$q_1 g_2 = 2\pi n \quad , \qquad n \in Z$$

Hint: Consider two different enclosed surfaces S and S' such that $S - S'$ contains the monopole P_2 in the interior.

4. For particles having both electric and magnetic charges, (q_1, g_1) and (q_2, g_2) respectively, the Dirac quantization condition generalizes to

$$\begin{pmatrix} q_1 & g_1 \end{pmatrix} \cdot M \cdot \begin{pmatrix} q_2 \\ g_2 \end{pmatrix} = 2\pi n$$

with M a 2×2 matrix. Find the explicit form of M by using the invariance of the electro-magnetic interactions under

$$\begin{pmatrix} q & g \end{pmatrix} \rightarrow O \begin{pmatrix} q & g \end{pmatrix} \quad , \quad \begin{pmatrix} \vec{E} & \vec{B} \end{pmatrix} \rightarrow O \begin{pmatrix} \vec{E} & \vec{B} \end{pmatrix}$$

with $O \in SO(2)$.

PROBLEM 2

A spin 1/2 particle in a magnetic field along the direction

$$\hat{n} = (sin\,\theta\,cos\,\phi, sin\,\theta\,sin\,\phi, cos\,\theta)$$

has Hamiltonian

$$H = \hbar\,\omega\,\hat{n}\cdot\vec{\sigma},$$

neglecting the orbital part.

1. Find the eigenstates $|\pm, \hat{n} >$ of H in terms of the usual eigenstates $|\pm >$ of σ_z.

2. Evaluate the single-particle canonical partition function $Z = tre^{-\beta H}$, with $\beta = 1/kT$. What is its limit for $T \to \infty$?

3. Consider the operator (density matrix)

$$\rho = \frac{1}{4\pi}\int sin\theta d\theta d\phi |+, \hat{n} >< \hat{n}, +|.$$

   Compute the expectation values $< \vec{\Sigma} >= tr\rho\vec{\Sigma}$, where $\vec{\Sigma} = \hbar\vec{\sigma}/2$ is the spin operator.

4. Is there any (pure) state $|\psi >$ such that $< \psi|\vec{\Sigma}|\psi >$ reproduces the results of 3.?

Hint: Notice that $\hat{n}\cdot\vec{\sigma}$ can be obtained from σ_z by a combination of two rotations with appropriate angles around the y-axis and z-axis.

Problem 3

Consider the following Lagrangian

$$\mathcal{L} = \bar{Q}_L i\partial\!\!\!/ Q_L + \bar{q}_R i\partial\!\!\!/ q_R + |\partial_\mu H|^2 + m^2 H^\dagger H - \frac{\lambda}{2}(H^\dagger H)^2 + (y\bar{Q}_L H q_R + h.c.)\,,$$

where H is a complex doublet scalar field, Q_L is a left-handed chiral fermion doublet and Q_R is a right-handed chiral fermion singlet. The above Lagrangian can be seen as a toy version of the Standard Model where all gauge interactions have been switched off and one fermion pair only is kept.

1. What are the global symmetries of this system? Consider separately the cases $y = 0$ and $y \neq 0$.

2. Find the spectrum of the system when $m^2 > 0$ and $\lambda > 0$. Is the scalar spectrum of the theory that expected from the Goldstone theorem? Again, consider separately the cases $y = 0$ and $y \neq 0$.

3. Draw the tree-level diagram that leads to the decay of a fermion in the theory and evaluate the associated Feynman rules.

4. Calculate the tree-level decay rate of the unstable fermion. Consider an unpolarized fermion in its rest frame.

Problem 4

Consider the following approximation for the pion lagrangian density:

$$\mathcal{L} = \frac{f^2}{4}\,\mathrm{Tr}[(\partial_\mu \Sigma)^\dagger (\partial^\mu \Sigma)], \quad \Sigma = e^{i\pi_a\sigma_a/f},$$

where π_a are the pion fields ($\sqrt{2}\pi^\pm = \pi_1 \mp i\pi_2$, $\pi^0 = \pi_3$), $f \approx 93\,\mathrm{MeV}$ is the pion decay constant, σ_a are the Pauli matrices. Pion masses, electromagnetic interactions, and higher derivative interactions have been neglected.

1. Expand the above lagrangian in powers of the pion field and write the four-pion interaction in terms of $\pi = \pi_a\sigma_a/2$.
2. Use the above expression to obtain the interaction term containing four charged pions (two π^+ and two π^-).
3. Write the tree level amplitude for the process

   $$\pi^+(p_1)\pi^-(p_2) \to \pi^+(p_3)\pi^-(p_4),$$

   where $p_1 \ldots p_4$ are the pion 4-momenta.
4. Using energy-momentum conservation and the massless approximation for the pions, write the above amplitude in terms of $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ (Mandelstam's variables).