SISSA
Entrance
Examination

Elementary
Particle Theory
Sector

Trieste, July 15, 2004

FOUR PROBLEMS are given. The candidates are requested to solve two of them.

Problem 1.

The one-dimenional Schroedinger equation for a particle of mass m and charge e in the presence of a scalar potential $V(x)$ is

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + eV(x) \right] \psi(x,t) \tag{1}$$

Suppose that $V(x) = -\frac{1}{x}$, $x > 0$ and $V = \infty$ for $x \leq 0$.

1. Show that $\psi_0(x,t) = B\,(x - \beta x^2)\, e^{-\beta x} e^{-i\frac{E_1}{\hbar}t}$ is an eigenfunction of (1) if β takes a precise value. Determine β and E_1 and the normalization constant B.

2. Compute the average position of the particle $< x >$ and its indeterminacy $\Delta x = \sqrt{< x^2 > - < x >^2}$.

3. Compute the average momentum. Can you give an *a priori* explanation of the result?

Hint: for questions 2 and 3, express the result in terms of β.

Suppose now that $V(x) = -\frac{1}{|x|}$, $-\infty < x < \infty$.

4. Show that $\psi_0(x,t) = A\,|x|\, e^{-\alpha|x|} e^{-i\frac{E_0}{\hbar}t}$, with appropriate α and E_0, satisfies (1). Is it acceptable as en eigenfunction of (1)?

Problem 2.

Consider a point particle moving in the upper half plane

$$H \equiv \{z \in \mathbb{C}| \quad \mathrm{Im}(z) > 0\} \tag{1}$$

with action functional

$$S[z; t_1, t_2] = \int_{t_1}^{t_2} [g(z, \bar{z})\dot{z}\dot{\bar{z}}]^{1/2}\, dt \tag{2}$$

for any path $z(t)$ and end points $z_i = z(t_i)$, $i = 1, 2.$ and where $g(z, \bar{z})$ is a positive real function on H and, as usual, $\dot{z} = \frac{d}{dt}z$.

1. Prove that H is closed under the action of the group of real linear fractional transformations $\mathbf{G}$, that is the group of maps $\gamma(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$.

2. Determine (up to a multiplicative constant) the function $g(z, \bar{z})$ in (2) such that the action is symmetric under $\mathbf{G}$.

Hint: Consider first invariance under the subgroup $\gamma_b(z) = z + b$. Then study invariance under the whole group.

3. Calculate the equations of motion for the action resulting at point 2 for trajectories with fixed end points.

4. Verify that particles moving monotonically along circular arcs centered on the boundary of H are solutions of the equation of motion resulting at point 3.

Problem 3.

Problem 3. Consider the neutrino Yukawa type interaction Lagrangian:

$$\mathcal{L}_{\mathrm{Y}}(x) = \sum_{j=1}^{3} \sum_{l=e,\mu,\tau} \lambda_{jl}\, \overline{N_j}(x)\, \Phi^{\dagger}(x)\, \psi_{lL}(x) + h.c. \tag{1}$$

Here λ_{jl} are, in general, complex constants, $N_j(x)$ are the fields of three heavy Majorana neutrinos with masses $M_j > 0$, $\psi_{lL}(x)$, $l = e, \mu, \tau$, is the Standard Model (SM) lepton doublet field, $\psi^{\mathrm{T}}_{lL}(x) = (\nu^{\mathrm{T}}_{lL}(x)\ l^{\mathrm{T}}_{L}(x))$, $\nu_{lL}(x)$ and $l_L(x)$ being the left-handed (LH) flavour neutrino and charged lepton fields, and $\Phi^{\dagger}(x)$ is the SM Higgs doublet field, $\Phi^{\dagger}(x) = ((\Phi^{(0)}(x))^{\dagger}\ (\Phi^{(-)}(x))^{\dagger})$. The fields $N_j(x)$ satisfy the Majorana condition:

$$C(\overline{N}_j(x))^T = N_j(x),\ j = 1, 2, 3, \tag{2}$$

where C is the charged conjugation matrix: $C^{-1}\gamma_{\alpha}C = -\gamma^{\mathrm{T}}_{\alpha}$, γ_{α} being the Dirac gamma matrices ($\alpha = 0, 1, 2, 3$), $C^{\mathrm{T}} = -C$, $C^{\dagger} = C^{-1}$.

1. Derive the constraints which the requirement of CP-invariance of $\mathcal{L}_{\mathrm{Y}}(x)$ imposes on the neutrino Yukawa coupling constants λ_{jl}, knowing that $N_j(x)$, $\nu_{lL}(x)$, $l_L(x)$, $\Phi^{(0)}(x)$ and $\Phi^{(-)}(x)$ transform as follows under the CP-symmetry operation:

$$\begin{aligned} U_{\mathrm{CP}}\, N_j(x)\, U^{\dagger}_{\mathrm{CP}} &= \eta^{NCP}_{j}\, \gamma_0\, N_j(x')\,, \quad \eta^{NCP}_{j} = i\rho^{N}_{j} = \pm i\,, && (3)\\ U_{\mathrm{CP}}\, f_{lL}(x)\, U^{\dagger}_{\mathrm{CP}} &= i\, \gamma_0\, C(\overline{f_{lL}}(x'))^T\,,\ f_{lL} = \nu_{lL}, l_L\,,\ l = e, \mu, \tau\,, && (4)\\ U_{\mathrm{CP}}\, \phi(x)\, U^{\dagger}_{\mathrm{CP}} &= \phi^{\dagger}(x')\,, \quad \phi = \Phi^{(0)}, \Phi^{(-)}\,, && (5) \end{aligned}$$

where U_{CP} is the unitary CP-transformation operator, $x = (x_0, \mathbf{x})$, $x' = (x_0, -\mathbf{x})$ and $\eta^{NCP}_{j} = i\rho^{N}_{j} = \pm i$ is the CP-parity of the heavy Majorana neutrino N_j. Comment the result obtained.

2. Assume that the flavour neutrino fields

$$\nu_{l\mathrm{L}}(x) = \sum_{k=1}^{3} U_{lk}\, \chi_{k\mathrm{L}}(x), \quad l = e, \mu, \tau, \tag{6}$$

where $\chi_{k\mathrm{L}}(x)$ is the LH component of the field $\chi_k(x)$ of a light Majorana neutrino having a mass m_k and U is the 3×3 unitary neutrino mixing matrix. Derive the CP-invariance constraints on the elements U_{lk} of the matrix U using eqs. (4), (6) and

$$\begin{aligned} C(\overline{\chi_{kL}}(x))^T &= \chi_{kR}(x)\,,\ k = 1, 2, 3, \,, && (7)\\ U_{\mathrm{CP}}\, \chi_{kL}(x)\, U^{\dagger}_{\mathrm{CP}} &= \eta^{\nu CP}_{k}\, \gamma_0\, \chi_{kR}(x')\,,\ \eta^{\nu CP}_{k} = i\rho^{\nu}_{k} = \pm i\,, && (8) \end{aligned}$$

where $\chi_{kR}(x)$ is the right-handed (RH)component of the field $\chi_k(x)$. Comment the result.

[N.B. Use the Bjorken-Drell representation for the γ-matrices: $(\gamma_{1,2,3})^\dagger = -\gamma_{1,2,3}$, $(\gamma_0)^\dagger = \gamma_0$, $(\gamma_0)^2 = \mathbf{1}$ - the unit 4×4 matrix. Note also that, e.g., $U_{CP}\ \gamma_\alpha\ U^\dagger_{CP} = \gamma_\alpha$.]

Problem 4.

Consider two sets of real classical ($\hbar = 0$) scalar fields in four dimensions

$$\phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \end{pmatrix} \quad \text{and} \quad \rho(x) = \begin{pmatrix} \rho_1(x) \\ \rho_2(x) \\ \rho_3(x) \end{pmatrix},$$

with lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_i \partial^\mu \phi_i + \frac{1}{2}\partial_\mu \rho_i \partial^\mu \rho_i - a(\phi^2 - m^2)^2 - b\,(\rho^2 - \mu^2)^2 - c\,(\phi\,\rho)^2,$$

where $a, b, m, \mu > 0$, $c \geq 0$, $\phi^2 \equiv \sum_i \phi_i^2$, $\rho^2 \equiv \sum_i \rho_i^2$, $\phi\,\rho \equiv \sum_i \phi_i \rho_i$.

Assume first that $c = 0$:

1a. Identify the group of the global, linear symmetry transformations on the fields.

1b. Determine the value of the fields minimizing the potential up to a symmetry transformation (consider only field configurations constant in space-time). What is the group of symmetries leaving the vacuum invariant?

1c. Consider fluctuations around the minimum field configuration. How many real fields turn out to be massive? What are their masses?

Suppose next that $c \neq 0$ is switched on:

2. Answer the above questions in the case $c \neq 0$.