July 19, 2006

SISSA Entrance Examination

Elementary Particle Theory Sector

SOLVE two out of the four problems below

Problem 1

The most general form of the matrix element of the electromagnetic current $j^\mu(x)$ between two electron states ψ_1, ψ_2 characterized by momenta p_1, p_2 and by spinor wave functions u_1, u_2 respectively is

$$\langle\psi_1|j^\mu(x)|\psi_2\rangle = \\ e^{-iqx}\bar{u}_1\left[iq^\mu(f_1+ig_1\gamma_5)+\gamma^\mu(f_2+g_2\gamma_5)+i\sigma^{\mu\nu}q_\nu(f_3+ig_3\gamma_5)\right]u_2,$$

where $q = p_2 - p_1$, $f_i = f_i(q^2)$, $g_i = g_i(q^2)$, $\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]$.

1. Show that the conservation of the electromagnetic current implies $q^2 f_1 = 0$ and $q^2 g_1 + 2mg_2 = 0$, where m is the electron mass.
2. Derive the constraints on the functions f_i and g_i following from the hermiticity of the electromagnetic current.
   [Reminder: $u^\dagger = \bar{u}\beta$, with $\beta\gamma_\mu^\dagger\beta^\dagger = \gamma_\mu$, $\beta\gamma_5^\dagger\beta^\dagger = -\gamma_5$, $\beta^\dagger\beta = \mathbf{1}$]
3. Consider a parity transformation on the electron states such that $p \to p_P = (p_0, -p_1, -p_2, -p_3)$ and $u \to u_P = e^{i\theta}\gamma_0 u$, where θ is an arbitrary phase. Assuming invariance under parity, so that the electromagneric current correspondingly transforms as, $j^\mu(x) \to j_\mu(x_P)$, show that the functions g_i, $i = 1, 2, 3$ must vanish.

Problem 2.

Consider slow neutrons with spin initially in some given direction.

1. What are the spin measurements in the x and y directions expected to yield if the neutron spin is in the z direction?

2. At time $t = 0$ a constant uniform magnetic field of absolute value H is switched on along the positive y direction.

   2a. Write the Schrödinger equation for these neutrons.

   2b. Disregarding the kinetic energy of the neutrons, what is their eigenfunction at time t, if their spin is initially aligned with the positive y axis?.

   2c. When are the neutrons expected to have the spin aligned to the positive x axis?

3. Answer question 2a, in the case in which $H = H_0 t$.

Hint: write the neutron magnetic moment as $\vec{\mu}= \frac{g\hbar}{2} \vec{\sigma}$, with

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Problem 3.

Consider the following classical Lagrangian describing the interactions between two complex scalar field doublets $\phi = (\phi_1\,, \phi_2)^T$ and $\chi = (\chi_1\,, \chi_2)^T$:

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) + (\partial_\mu \chi)^\dagger (\partial^\mu \chi) - \lambda |\phi_1^* \chi_1 + \phi_2^* \chi_2|^2 - \lambda' |\phi_1 \chi_2 - \phi_2 \chi_1|^2\,, \quad (1)$$

where $\lambda, \lambda' > 0$.

1. Identify the global continuous internal symmetries of $\mathcal{L}$.
2. Find the corresponding conserved currents expressed in terms of the fields ϕ and χ.
3. Add to $\mathcal{L}$ the following term:

   $$\mathcal{L}_\epsilon = \epsilon (\phi_1^* \phi_2 - \phi_2^* \phi_1)\,, \quad (2)$$

   where ϵ is a purely imaginary parameter. Identify the global internal symmetries and conserved currents of $\mathcal{L} + \mathcal{L}_\epsilon$.
4. Which is the maximal global continuous internal symmetry of $\mathcal{L}$ when $\lambda = \lambda' = 0$?

Hint: you may use $\sigma_a^* = -\sigma_2 \sigma_a \sigma_2$, where σ_a are the standard Pauli matrices ($a = 1, 2, 3$).

Problem 4. Majorana Neutrinos and Neutrinoless Double Beta Decay

Consider the effective (V-A) β-decay Hamiltonian in the case of three-neutrino mixing:

$$\mathcal{H}_{\mathcal{I}}^{\beta} = \frac{G_F}{\sqrt{2}}\, \bar{p}(x)\gamma^{\mu}(1-\gamma_5)n(x)\ \bar{e}(x)\gamma_{\mu}(1-\gamma_5)\sum_{k=1}^{3} U_{ek}\chi_k(x) + \text{h.c.}, \qquad (1)$$

where G_F is the Fermi constant, $p(x)$, $n(x)$ and $e(x)$ are the proton, neutron and electron fields, respectively, U_{ek} are the elements of the first row of the 3×3 unitary neutrino mixing matrix U and $\chi_k(x)$, $k=1,2,3$, is the (4-component) field of Majorana neutrino with mass m_k satisfying the Majorana condition,

$$C\ (\bar{\chi}_k(x))^{\mathrm{T}} = \chi_k(x),\ C^{-1}\gamma_{\mu}C = -(\gamma_{\mu})^{T},\ C^{-1}\gamma_5 C = (\gamma_5)^{T}\ ,\ C^{T} = -C\ , \quad (2)$$

C being the charge conjugation matrix and $\bar{\chi}_k(x) \equiv (\chi_k(x))^{\dagger}\gamma_0$.

1. Using the Hamiltonian (1) show that if the neutrino fields $\chi_k(x)$ satisfy the Majorana condition (2), the process of neutrinoless double beta ($(\beta\beta)_{0\nu}$-) decay $\mathbf{n}+\mathbf{n}\rightarrow \mathbf{p}+\mathbf{p}+\mathbf{e}^{-}+\mathbf{e}^{-}$ is allowed and can proceed in second order of perturbation theory in the Fermi constant G_F. Draw the Feynman diagram of the process.

2. Consider the case when the two initial state neutrons and the two final state protons are at rest. Show that if the neutrino masses m_k, $k=1,2,3$, are sufficiently small, i.e., if m_k^2 are negligible with respect to $(m_n - m_p)^2 - m_e^2$ in the case specified above, m_n, m_p and m_e being the neutron, proton and electron masses, respectively, the dependence of the $(\beta\beta)_{0\nu}$-decay amplitude $A(\beta\beta)_{0\nu}$ on m_k and U_{ek} factorizes in *the effective Majorana mass* $<\!m\!>$:

$$A(\beta\beta)_{0\nu} = <\!m\!>\, M, \qquad |<\!m\!>| = \left|m_1 U_{\mathrm{e}1}^2 + m_2 U_{\mathrm{e}2}^2 + m_3 U_{\mathrm{e}3}^2\right|\ , \quad (3)$$

   where M is matrix element of the process. Only a schematic expression for M can be given.

3. Derive approximate *leading order* analytic expressions for $|<m>|$ for the following three types of neutrino mass spectrum:
   a) $m_1 \ll m_2 \ll m_3$ (normal hierarchical or NH), with $m_1 = 0$;
   b) $m_3 \ll m_1 \ll m_2$ (inverted hierarchical or IH), with $m_3 = 0$;
   c) $m = m_1 \cong m_2 \cong m_3$, $m^2 \cong m_k^2 \gg |\Delta m_{21}^2|, |\Delta m_{31}^2|$, $k = 1, 2, 3$, where $\Delta m_{kj}^2 \equiv m_k^2 - m_j^2$ (quasi-degenerate or QD).
   In solving this problem use i) the standard parametrisation of U_{ek}, $U_{e1}^2 = \cos^2\theta_{12}(1 - \sin^2\theta_{13})$, $U_{e2}^2 = e^{i\alpha} \sin^2\theta_{12}(1 - \sin^2\theta_{13})$, $U_{e3}^2 = e^{i\beta} \sin^2\theta_{13}$, where θ_{12} and θ_{13} are neutrino mixing angles and α and β are physical phases (Majorana CP-violating phases), and ii) the fact that, as it follows from the existing experimental data, $\sin^2\theta_{13} \ll \cos 2\theta_{12}, \sin^2\theta_{12}$, and $\Delta m_{21}^2 \ll |\Delta m_{31}^2| \cong |\Delta m_{32}^2|$. Express the results in terms of the mixing angles, Δm_{kj}^2 and phases.

4. For each of the three types of spectra, NH, IH, and QD, determine the interval of values $|<m>|$ can take using the existing data on θ_{12}, θ_{13}, Δm_{21}^2 and Δm_{31}^2, $\sin^2\theta_{12} = 0.30$, $\sin^2\theta_{13} < 0.04$, $\Delta m_{21}^2 = 8.1 \times 10^{-5}$ eV^2, $|\Delta m_{31}^2| = 2.5 \times 10^{-3}$ eV^2, and the fact that the phases α and β are unconstrained and can have any value in the interval $0 \leq \alpha, \beta \leq 2\pi$. In the case of the QD spectrum use 0.1 eV $\leq m \leq$ 0.6 eV. Comment the results. What is the role played by the Majorana phases α and β?

Hint: use the Majorana condition and take into account the fact that the Majorana neutrino field $\chi_k(x)$ has a standard propagator

$$< 0|T(\chi_{ka}(x)\bar{\chi}_{jb}(y))|0 >= \delta_{kj}\, S_{ab}^{Fk}(x-y) = \delta_{kj} \int \frac{d^4q}{(2\pi)^4} \frac{e^{i(x-y)q}\ (q^\mu\gamma_\mu + m_k\mathbf{1})_{ab}}{q^2 - m_k^2}\,, \tag{4}$$

where $\mathbf{1}$ is the unit 4×4 matrix.