SISSA
Entrance
Examination

Elementary
Particle Theory
Sector

FOUR PROBLEMS are given. Each of them has three questions of increasing difficulty. Do as much as you can within four hours. You are not required to solve all the problems.

Trieste, 18 July 2005

Problem 1. Classical Field Theory

Consider a system of two complex scalar fields ϕ_i, $i = 1, 2$ in four space-time dimensions.

1. Write the most general lagrangian density $\mathcal{L}_1$ for the two fields invariant under the U(1) transformation

$$\phi_i(x) \to e^{i\alpha}\phi_i(x), \quad i = 1, 2$$

   for any value of α.

   [Here and in the following, consider only lagrangians that are Poincarè (i.e. Lorentz and space-time translation) invariant, renormalizable (i.e. such that the dimension in energy of the couplings is non-negative), and local (i.e. in the form $\mathcal{L}_1(x) = \mathcal{L}_1(\phi_i(x), \partial\phi_i(x))$].

2. Write the most general lagrangian $\mathcal{L}_2$ that is also invariant under the transformation

$$\phi_i(x) \to \sum_j U_{ij}\phi_j(x), \quad i, j = 1, 2$$

   for any 2×2 unitary matrix U ($U(2)$ group). For which range of values of its parameters is $\mathcal{L}_2$ physically viable? Justify the answer.

3. Is $\mathcal{L}_2$ invariant under a group of transformations that is larger than $U(2)$? If so, which one? Finally, determine the most general transformation under which both $\mathcal{L}_2$ and the specific value of the fields

$$\begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

   are invariant.

   [Consider only linear and global, i.e. space-time independent, transformations].

Problem 2. Statistical Mechanics

Consider a system in 3 space dimensions having energy levels $E_n = n$ with n positive integer $n = 1, 2, \cdots, \infty$. Each level is degenerate: the states having energy E_n are eigenstates of the angular momentum $\hat{J}^2$ with eigenvalues $j(j+1)$ and j takes integer values from 1 to n: $j = 1, \ldots, n$.

The system is in a heat bath at temperature T (set the Boltzmann constant $= 1$).

1) Compute the entropy of the system at $T = 0$.

2) Find the asymptotic form of the partition function of the system in the limit $T \to \infty$ (write just the leading term).

[Hint: set $\beta = 1/T$ and note that $ae^{a\beta} = \frac{d}{d\beta} e^{a\beta}$]

3) Find the asymptotic form of the entropy of the system in the limit $T \to \infty$ (write just the leading term).

[Hint: remember that, given the probability distribution P, the entropy S is the average of $-log(P)$, that is $S = - < log(P) >$]

Problem 3. $\beta-$Decay Hamiltonian

Consider the $\beta-$decay Hamiltonian (density):

$$\mathcal{H}_I^\beta(x) = \sum_{i=S,P,V,A,T} \bar{p}(x)O^i n(x)\ \bar{e}(x)O_i(G_i - G_i'\gamma_5)\nu(x) + h.c. \tag{1}$$

Here $p(x)$, $n(x)$, $e(x)$ and $\nu(x)$ are the proton, neutron, electron and neutrino (Dirac) fields, respectively, $O^S = I$ (scalar), $O^P = \gamma_5$ (pseudo-scalar), $O^V = \gamma^\alpha$ (vector), $O^A = \gamma^\alpha\gamma_5$ (axial-vector), $O^T = \sigma^{\alpha\beta}$ (tensor), where γ_α are the Dirac gamma matrices and $\sigma_{\alpha\beta} \equiv \frac{1}{2}(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha)$.

[In the Bjorken-Drell representation, for instance, one has:
$(\gamma_{1,2,3})^\dagger = -\gamma_{1,2,3}$, $(\gamma_0)^\dagger = \gamma_0$]

The complex constants G_i and G_i' characterize respectively the scalar and pseudo-scalar parts of the Hamiltonian.

1. Under the P-symmetry operation, $x \to x'$, where $x = (x_0, \mathbf{x})$, $x' = (x_0, -\mathbf{x})$, the fields $p(x)$, $n(x)$, $e(x)$ and $\nu(x)$ transform as follows:

$$U_P\, \psi_{ja}(x)\, U_P^\dagger = (\gamma_0)_{ab}\, \psi_{jb}(x'),\ j = 1,2,3,4,\ \psi_1(x) \equiv p(x), \psi_2(x) \equiv n(x),\ etc., \tag{2}$$

where U_P is the unitary operator of the P-transformation and a, b are Lorentz indexes.

Show that the requirement of invariance of the Hamiltonian (1) under the $P-$transformation (2), that is the requirement $U_P\ \mathcal{H}_I^\beta(x)\ U_P^\dagger = \mathcal{H}_I^\beta(x')$, implies $G_i' = 0$, $i = S, P, V, A, T$.

[Note that, e.g., $U_P\ \gamma_\alpha\ U_P^\dagger = \gamma_\alpha$]

(*cont.*)

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2. Consider the left-handed (LH) and right-handed (RH) neutrino field components $\nu_L(x)$ and $\nu_R(x)$:

$$\nu_L(x) \equiv \frac{1}{2}\,(1-\gamma_5)\,\nu(x), \quad \nu_R(x) \equiv \frac{1}{2}\,(1+\gamma_5)\,\nu(x)\ . \tag{3}$$

Suppose that instead of the neutrino field $\nu(x)$, only its left-handed component $\nu_L(x)$ enters into the expression of the Hamiltonian (1) (two-component neutrino theory).

What form the $\beta-$decay Hamiltonian (1) takes in this case? Write the corresponding expression.

How does the field $\nu_L(x)$ transform under the $P-$transformation?

Can the new Hamiltonian be made invariant under the $P-$symmetry transformation by suitably choosing G_i and G'_i?

Can a theory which contains only the LH neutrino field $\nu_L(x)$ (and no RH field $\nu_R(x)$) be invariant under the $P-$symmetry transformation ?

3. Assume that the $\beta-$decay Hamiltonian (1) is formed by the LH components not only of the neutrino field, but also of all other fermion fields. Following this hypothesis make the change

$$p(x) \to p_L(x),\ n(x) \to n_L(x),\ e(x) \to e_L(x),\ \nu(x) \to \nu_L(x), \tag{4}$$

in the Hamiltonian in eq. (1). Write the expression for the new Hamiltonian obtained after the change (4). Comment the result.

Problem 4. Quantum Mechanics

Let us consider a non relativistic particle in one space dimension with Lagrangian

$$\mathcal{L}(q,\dot{q}) = \frac{1}{2}[e(q)]^2(\dot{q})^2$$

where $e(q)$ is a positive function on the real line and, as usual, $\dot{q} = \frac{dq}{dt}$.

1. Compute the Hamiltonian of the system and the action in the Hamiltonian formalism.

2. Determine a function $f(q,p)$ such that the canonical transformation

$$q' = f(q,p) \quad \text{and} \quad p' = \frac{p}{e(q)}$$

satisfies the relation

$$p'dq' = pdq.$$

Write the Hamiltonian in the new phase space variables (q', p').

3. Let's finally consider the system from a quantum mechanical point of view. Show that for the position eigenstates the relation

$$|q >= \sqrt{e(q)}|q' >$$

holds.

Compute

$$< q_f|e^{-it\hat{\mathcal{H}}}|q_i >,$$

where $\hat{\mathcal{H}}$ is the operator Hamiltonian, that is the transition probability amplitude from an initial position eigenstate at q_i at initial time $t_i = 0$ to a final position eigenstate at q_f at final time $t_f = t$. Keep $\hbar = 1$ for simplicity.

[Hint:

$$\int \frac{dp'}{2\pi} \mathrm{e}^{-i\frac{(p')^2}{2}t + iq'p'} = \frac{\mathrm{e}^{i\frac{(q')^2}{2t}}}{\sqrt{2\pi i t}}]$$