SISSA
Entrance
Examination

Elementary
Particle Theory
Sector

Trieste, 22 July 2003

Four problems are given. Each of them has three questions, the first being introductory. Try to do as much as you can within three hours. You are not expected to solve all the problems.

Problem 1. Quantum Field Theory

Consider the quantum field theory describing two interacting real scalar fields,Φ with mass M and ϕ with mass m, with $M > m$, with the action:

$$S = \int d^4x \{ \frac{1}{2}(\partial^\mu \Phi \partial_\mu \Phi - M^2 \Phi^2) + \frac{1}{2}(\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2) - \frac{g}{2} \Phi \phi^2 \} \tag{1}$$

1) Write the expression for the probability amplitude describing the decay of the Φ particle, at the lowest order in the coupling constant g.

2) Find the decay probability per unit time, and therefore its inverse that is the lifetime, of the Φ particle, at the lowest order in g and in the limit $M \gg m$. (It is allowed to ignore numerical factors, powers of π, etc. You can do the standard computation or use a shortcut argument).

3) Consider the elastic scattering of two ϕ particles, say

$$\phi(p_1) + \phi(p_2) \to \phi(p_3) + \phi(p_4) \,, \tag{2}$$

where p_1, p_2, p_3 and p_4 are the four-momenta of the corresponding particles. Draw the Feynmann diagrams contributing at the lowest order in g; find the value of the corresponding scattering amplitude in the resonance limit $(p_1+p_2)^2 \to M^2$ and in the limit $g \to 0$ (make use of the result of question 2).

Problem 2. Majorana Spin 1/2 Fields and Particles

Consider a *free massive spin 1/2* field $\chi_\alpha(x)$, $\alpha = 1, 2, 3, 4$, which satisfies the Dirac equation and the additional *Majorana* condition:

$$C(\bar{\chi}(x))^T = \xi \ \chi(x), \quad |\xi|^2 = 1, \tag{1}$$

where C is the 4×4 charge-conjugation matrix, $\bar{\chi}(x) = (\chi(x))^\dagger \gamma_0$ is the Dirac conjugated field, γ_0 being the 4th Dirac gamma matrix, and ξ is a phase factor. The matrix C satisfies

$$C^{-1} \ \gamma_\mu \ C = -(\gamma_\mu)^T, \tag{2}$$

where γ_μ are the Dirac gamma matrices. In the Bjorken-Drell representation, for instance, one has: $(\gamma_{1,2,3})^\dagger = -\gamma_{1,2,3}$, $(\gamma_0)^\dagger = \gamma_0$. It follows from eq. (2) that $C^T = -C$. One can always choose C to be unitary: $C^\dagger = C^{-1}$.

1) What, according to you, are the physical consequences of the Majorana condition? Show also that the Majorana condition is not invariant with respect to global U(1) phase transformations. Give a physical interpretation of this fact.

2) Prove that the vector, tensor and pseudo-tensor currents of a free spin 1/2 (second-quantized) Majorana field are equal to 0:

$$: \ \bar{\chi}(x)) \ \gamma_\mu \ \chi(x) \ : \quad = 0, \tag{3}$$

$$: \ \bar{\chi}(x)) \ \sigma_{\mu\nu} \ \chi(x) \ : \quad = 0, \qquad \sigma_{\mu\nu} \equiv \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \ , \tag{4}$$

$$: \ \bar{\chi}(x)) \ \sigma_{\mu\nu}\gamma_5 \ \chi(x) \ : \quad = 0 \quad (\gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3). \tag{5}$$

Indicate possible physical consequences of these results.

N.B.: The double-dot sign : ... : in equations (3) - (5) means "normal ordering of the operators". Under this sign $\bar{\chi}(x)$ and $\chi(x)$ anti-commute.

3) Under the CP-symmetry operation the field $\chi(x)$ transforms as follows:

$$U_{CP}\ \chi_\alpha(x)\ U_{CP}^\dagger = \rho_{CP}\,(\gamma_0)_{\alpha\beta}\chi_\beta(x'), \tag{6}$$

where U_{CP} is the unitary operator of the CP-transformation, $x' = (x_0, -\mathbf{x})$, and ρ_{CP} is the CP-parity of the field $\chi_\alpha(x)$. Prove that

$$\rho_{CP} \ = \ \pm\, i. \tag{7}$$

Note that the operator U_{CP} does not act on the Lorentz index of the field $\chi_\alpha(x)$, e.g., that $U_{CP}\ (\gamma_0)_{\alpha\beta}\chi_\beta(x)\ U_{CP}^\dagger = (\gamma_0)_{\alpha\beta}U_{CP}\ \chi_\beta(x)\ U_{CP}^\dagger$, etc.

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Problem 3. Classical Field Theory

Consider a classical real scalar field $\phi\,(x^\mu)$, with lagrangian

$$L = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V\,(\phi) \tag{1}$$

(let the metric be $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$), which undergoes a phase transition that shifts V as follows:

$$V\,(\phi) = \frac{1}{2}m^2\,\left(\phi\,(x^\mu)\right)^2 \quad \Rightarrow \quad V\,(\phi) = \frac{\lambda}{4}\left[\left(\phi\,(x^\mu)\right)^2 - \sigma^2\right]^2 \,, \tag{2}$$

where m, λ and σ are real constants, and $\lambda > 0$.

1) Sketch the two potentials and find the corresponding shift of the ground-state values of the field ϕ, say, e.g.,

$$\phi_0 \quad \Rightarrow \quad \phi_0^+ \;\text{ and }\; \phi_0^- \,. \tag{3}$$

2) Suppose that, after the transition is completed and the system has come to a stationary equilibrium state, instead of taking everywhere the same configuration, the system has evolved in such way that in some region of space the ground state is ϕ_0^+ while in others it is ϕ_0^-; for simplicity assume ϕ to be just a function of the spatial coordinate z, i.e. $\phi(t,x,y,z) \equiv \phi(z)$, with $\phi(z=-\infty) = \phi_0^-$ and $\phi(z=+\infty) = \phi_0^+$. Verify that the classical solution with the appropriate boundary conditions has the form:

$$\phi(z) = A\,\tanh\left(\frac{z}{B}\right) \tag{4}$$

and find the parameters A and B.

3) Find the energy density ρ and the pressure $\vec{p}$ for the system, using their relation to the stress energy tensor

$$T^{\mu}_{\nu} = \mathrm{diag}(\rho, -p_x, -p_y, -p_z); \tag{5}$$

write down the Poisson's equation for the resulting Newtonian gravitation potential Φ applying the Newtonian limit $R_{00} \to \Delta\Phi$ in Einstein's equation:

$$R_{00} = 8\pi G_N \left(T_{00} - \frac{1}{2} T^{\alpha}_{\alpha}\, g_{00} \right) \tag{6}$$

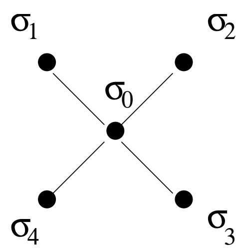
(T^{α}_{α} is the trace of T^{μ}_{ν}). Qualitatively, how would a test mass behave in the resulting Newtonian gravitation potential Φ?

Problem 4. Statistical Mechanics

Consider a system of five spins as shown in the figure below, where only the central spin (denoted as σ_0) interacts with the other four spins (denoted as $\sigma_1, \dots \sigma_4$), with coupling constant J; thus the Hamiltonian of the system is:

$$H = -J\,\sigma_0 \sum_{i=1}^{4} \sigma_i\,. \tag{1}$$

Each spin has values $\sigma = \pm 1$.



1) Write the expression for the partition function for the system.

2) Compute the partition function and the internal energy.

3) Derive the magnetization in the presence of an external magnetic field **h**. Has the system spontaneous magnetization?

Hint:

$$\sum_{\sigma_1=\pm} \dots \sum_{\sigma_N=\pm} \sigma_1 \cdot \dots \cdot \sigma_N = \sum_{\sigma_1=\pm} \sigma_1 \cdot \dots \cdot \sum_{\sigma_N=\pm} \sigma_N$$