SISSA
Entrance
Examination

Elementary
Particles
Sector

Trieste, 12 October 2000

There are 6 problems. Problems 1, 3 and 6 deal with quantum mechanics; problems 2 and 5 with the phenomenology of particle physics and problem 4 with a simple example of field theory. The candidate should try and **solve as much as possible of two problems**. The candidate should finish within 3 hours.

1. Charged Particle in a Constant Magnetic Field

Consider the motion of a particle of charge e in a constant magnetic field in the direction z, denoted by $\mathbf{B} = B\,\hat{\mathbf{e}}_z$. Choose the following vector potential

$$\mathbf{A} = (-By, 0, 0) \quad \text{and} \quad V = 0\,. \tag{1}$$

1. Write the time-independent Schrödinger equation for the problem.

2. Show that the following separation of variables

$$\psi(x, y, z) = \exp i\left[(p_x x + p_z z)/\hbar\right] \varphi(y)\,, \tag{2}$$

used after substituting $y = y' - p_x c/(eB)$, leads to the equation of a harmonic oscillator.

3. What are the energy eigenvalues? The energy states thus defined are known as Landau levels.

Consider now the same problem for a relativistic fermion.

4. What is the time-dependent equation in this case?

5. By writing the 4-component wave function as

$$\psi = \exp i\left[Et/\hbar\right] \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \tag{3}$$

where φ and χ are 2-component spinors, show that the Dirac equation leads again to a harmonic oscillator plus a part depending on the spin.

> Hint: Recall that $(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = \vec{A} \cdot \vec{B} + i\,\vec{\sigma} \cdot (\vec{A} \times \vec{B})$ and then try again the separation of variables $\varphi(x, y, z) = \exp i\left[(p_x x + p_z z)/\hbar\right] u(y)$.

6. Show that in the relativistic case the eigenvalues are

$$E_{n,k}^2 = c^4 m^2 + p_z^2 c^2 + \hbar \frac{eB}{c}\,(2n + 1 - k) \tag{4}$$

where n labels the level of the oscillator and $k = \pm 1$ depends on the spin eigenfunction.

Hint: The 2-component spin wave function are of the form

$$u = \begin{pmatrix} u_1 \\ 0 \end{pmatrix} \quad \text{for} \quad k = 1 \quad \text{and} \quad u = \begin{pmatrix} 0 \\ u_{-1} \end{pmatrix} \quad \text{for} \quad k = -1\,.$$

2. Discrete Symmetries in Strange Decays

Consider the hadronic decays of the neutral strange mesons K^0 into two or three pions: $K^0 \to \pi\pi$. Knowing that kaons and pions are pseudo-scalar particles:

1. Show that the two-particle states $|\pi^+\pi^-\rangle$ and $|\pi^0\pi^0\rangle$ are CP **even** eigenstates, where C represents the charge conjugation operation, which acts as follows

$$\begin{array}{rcl} C|K^0, \mathbf{r}_K\rangle & = & |\bar{K}^0, \mathbf{r}_{\bar{K}}\rangle \\ C|\pi^+, \mathbf{r}_+\rangle & = & |\pi^-, \mathbf{r}_-\rangle \\ C|\pi^0, \mathbf{r}_0\rangle & = & |\pi^0, \mathbf{r}_0\rangle\,, \end{array} \tag{1}$$

and P is the parity transformation,

$$P|\pi, \mathbf{r}_\pi\rangle = -|\pi, -\mathbf{r}_\pi\rangle \quad . \tag{2}$$

Hint: consider the decaying kaon in its rest frame and use translational invariance to write $|\pi_1\pi_2, \mathbf{r_1}, \mathbf{r_2}\rangle \propto |\pi_1\pi_2, \mathbf{r_1} - \mathbf{r_2}\rangle$.

2. Show that the CP eigenvalues of the three pion state $|\pi^+\pi^-\pi^0\rangle$ depend on the orbital angular momentum ℓ of π^0 with respect to the center of mass of the $\pi^+\pi^-$ system:

$$CP|\pi^+\pi^-\pi^0\rangle = (-)^{\ell+1}\, |\pi^+\pi^-\pi^0\rangle\,, \tag{3}$$

while the state $|\pi^0\pi^0\pi^0\rangle$ is CP **odd**:

$$CP|\pi^0\pi^0\pi^0\rangle = -|\pi^0\pi^0\pi^0\rangle\,. \tag{4}$$

The physical eigenstates of the $K^0 - \bar{K}^0$ system are denoted by K_L (long-lived) and K_S (short-lived). What can you infer on the CP symmetry of weak interactions and the $K_{L,S}$ properties from the experimental evidence that:

3. Both $K_L \to \pi^+\pi^-$ and $K_L \to \pi^0\pi^0\pi^0$ are seen, and

4. The decay rate for $K_L \to \pi^+\pi^-$ is much smaller than the rate for $K_S \to \pi^+\pi^-$?

CONSIDERING the results of the discussion above and given that $m_K \simeq 500$ MeV and $m_\pi \simeq 140$ MeV,

5. Can you give a qualitative argument for the large difference in lifetimes of K_L and K_S (experimentally $\tau_L/\tau_S \approx 600$) ?

Assume that the 2 and 3 pion modes are responsible for a large fraction of the $K_{L,S}$ decay widths.

3. Spin-1/2 Particle in a Magnetic Field

A spin 1/2 particle with magnetic moment $\vec{\mu} = g\vec{S} = g\hbar\vec{\sigma}/2$ is initially in the state $S_z = \hbar/2$. A magnetic field of absolute value B is switched on in the x-direction for $0 \leq t \leq t_1$.

1. Disregarding the space dependence, what is the spin eigenfunction of the particle during this interval? How much time does it take for the spin to flip to $S_z = -\hbar/2$?

Call t_0 the time found at point 1 and set $t_1 = 3\,t_0/2$. Now suppose that for $t > t_1$ the magnetic field gets doubled.

2. What is the eigenfunction for $t > t_1$?

3. What is the probability of finding the particle in the eigenstate $S_x = \hbar/2$ at $t = 2\,t_1$?

Notation: The Pauli matrices needed in the problem are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

4. Scalar Field Theory

LET ϕ_i, $i = 1, 2$, be two real scalar fields. The Lagrangian

$$L_0 = 2\partial_\mu \phi_1 \partial^\mu \phi_2 \tag{1}$$

describes two independent modes.

1. Explain why one of them is unphysical. Write the Lagrangian in a form that makes this explicit.

2. What is the symmetry group G of L_0, besides the rigid translations of ϕ_i.

CONSIDER now the interacting Lagrangian

$$L = L_0 - \frac{1}{2}\sum_{i=0}^{2} m_i^2 \phi_1^i \phi_2^{i-2} - \frac{1}{4!}\sum_{i=0}^{4} \lambda_i \phi_1^i \phi_2^{4-i} \tag{2}$$

3. Find the values of m_i, λ_i for which the unphysical mode decouples from the physical one.

4. Find the values of m_i, λ_i for which the symmetry G is preserved.

5. In the latter case, indicate graphically the Feynman rules and the contributions to the 2-, 3- and 4-point functions up to one loop.

Notation: The metric has signature $(1, -1, -1, -1)$.

5. Propagation of High-Energy Cosmic Rays

1. Find the energy threshold of the reaction

$$\gamma + p \to \pi + N, \tag{1}$$

where N is the nucleon and the initial proton is at rest.

The cross-section of the reaction (1) is about 5×10^{-28} cm^2 at the energy of photon $E \approx 1.2 - 1.3$ GeV.

2. Consider propagation of the cosmic rays (take for definiteness, protons in the Universe filled in by the cosmic microwave (electromagnetic) background radiation with temperature $2.7K$.) Find the energy E_c of cosmic rays at which the reaction (1) becomes important.

3. Calculate the distance from which cosmic rays with energies $E > E_c$ can reach the Earth?

The density of the cosmic microwave background radiation is ≈ 400 cm^{-3}; $1\text{eV} = 1.16 \times 10^4 K$.

6. Nonrelativistic Scattering

Consider a non relativistic scattering process for a spinless particle with mass m from a potential V. In the Dirac formalism (set $\hbar = 1$) the two kets $|\vec{p}_1\rangle$ and $|\vec{p}_2\rangle$ represent states of defined momentum of the particle before and after the collision.

The observable properties of the scattering are provided by the so called *scattering amplitude* $f(\vec{p}_1, \vec{p}_2)$ whose squared modulus gives the differential cross section of the process. In the so called *Born approximation* one has:

$$f(\vec{p}_1, \vec{p}_2) = -4\,\pi^2 m\,\langle \vec{p}_2\,|\mathrm{V}|\,\vec{p}_1\rangle \tag{1}$$

where V is the operator associated to the potential energy of the problem.

Call $\vec{q} = \vec{p}_2 - \vec{p}_1$ the momentum transfer of the process, q its modulus, p the modulus of $\vec{p}_1$ and θ the scattering angle between the final and the initial momentum, such that

$$q = 2\,p\,\sin\frac{\theta}{2}\,. \tag{2}$$

1. What relationship must exist between the two vectors $\vec{p}_1, \vec{p}_2$ in order that the collision is an *elastic* one?

2. Write the expression of the scattering amplitude for a central potential V=V(r), where r is the (three dimensional) distance between the particle and the scattering centre (suggestion: introduce a complete set of projectors on the position $\vec{r}$ variable).

3. Show that in the limit $\theta \to 0$ (forward scattering) the value of the scattering amplitude is finite if the potential vanishes at infinite r more quickly than the inverse third power of r. Show that in this case the value of the amplitude does not depend on the particle energy.

4. In the previous limit, the scattering amplitude is real in Born approximation. What fundamental property of the scattering process does this approximation violate? In particular, what theorem is violated?