SISSA
Entrance
Examination

Elementary
Particles
Sector

Trieste, October 1998

THE problems are grouped in two classes. Problems in **class A** are in field theory. Problems in **class B** deal with quantum mechanics, general relativity and statistical mechanics. The candidate should try and **solve completely at least one problem out of each class**. In some problems, parts that are more difficult and are considered optional are marked by an asterisk *. The candidate should finish within 3 hours.

A.1. Fermion Masses, Mixings and Couplings to Scalars and Vectors

Consider the following Lagrangian of two fermionic fields ψ_A and ψ_B:

$$\begin{aligned} L_1 &= i\bar{\psi}_A(x)\gamma^\mu\partial_\mu\psi_A(x) + i\bar{\psi}_B(x)\gamma^\mu\partial_\mu\psi_B(x) \\ &\quad - m\left[\bar{\psi}_A(x)\psi_B(x) + \bar{\psi}_B(x)\psi_A(x)\right] \\ &\quad - M\,\bar{\psi}_B(x)\psi_B(x) \end{aligned} \tag{1}$$

where the mass parameters m and M are real.

1. Find the mass eigenstates ψ_1 and ψ_2, their masses and the mixing angle as a function of such masses.

> Comment: Remember that if you obtain a negative mass eigenvalue for one of the fermions you should not worry since the minus sign can be reabsorbed in a phenomenologically viable way.

Consider now the interactions of $\psi_A(x)$ and $\psi_B(x)$ with a real scalar field $\phi(x)$:

$$\begin{aligned} L_2 &= L_1 + a\left[\bar{\psi}_A(x)\psi_B(x)\;\phi(x) + \bar{\psi}_B(x)\psi_A(x)\;\phi(x)\right] \\ &\quad + b\,\bar{\psi}_B(x)\psi_B(x)\phi(x) + \frac{1}{2}\phi(x)\left[\nabla^2 + \mu^2\right]\phi(x)\,, \end{aligned} \tag{2}$$

with a and b real parameters of the same order of magnitude.

Assume that its mass μ is much larger than M and m. Take also $M \gg m$.

2. Compute the ratios of the following partial decay rates:

$$\frac{\Gamma(\phi \to \bar{\psi}_1\psi_1)}{\Gamma(\phi \to \bar{\psi}_2\psi_2)} \quad \text{and} \quad \frac{\Gamma(\phi \to \bar{\psi}_1\psi_2)}{\Gamma(\phi \to \bar{\psi}_2\psi_2)}. \tag{3}$$

3. Which relation of the parameters a and b with m and M enforces $\Gamma(\phi \to \bar{\psi}_1\psi_2) = \Gamma(\phi \to \bar{\psi}_2\psi_1) = 0$?

LET us add to L_1 the interactions of ψ_A and ψ_B with the vector fields V_μ and V'_μ:

$$\begin{aligned} L_3 &= L_1 + g_1 \left[\bar{\psi}_A(x)\gamma^\mu V_\mu(x)\psi_A(x) + \bar{\psi}_B(x)\gamma^\mu V_\mu(x)\psi_B(x)\right] \\ &\quad + g_2 \left[\bar{\psi}_A(x)\gamma^\mu V'_\mu(x)\psi_B(x) + \text{c.c.}\right] \end{aligned} \tag{4}$$

with g_1 and g_2 real coupling constants. Consider the limit $M \gg m$ and V and V' very much heavier than all fermions.

4. Compute the ratios of the following rates:

$$\frac{\Gamma(V_\mu \to \bar{\psi}_1\psi_2)}{\Gamma(V'_\mu \to \bar{\psi}_2\psi_2)} \quad \text{and} \quad \frac{\Gamma(V'_\mu \to \bar{\psi}_1\psi_2)}{\Gamma(V_\mu \to \bar{\psi}_2\psi_2)}. \tag{5}$$

CONSIDER the up- and down-quarks of the first two generations (i.e., u, c, d and s). Take their mass matrices to be ($\psi_{L,R} = (1 \pm \gamma_5)/2$):

$$L_m = v\, h_{ij}\bar{u}_{L_i}\, u_{R_i} + v\, k_{ij}\bar{d}_{L_i}\, d_{R_i} + \text{c.c.}\,, \tag{6}$$

with v the vacuum expectation value of the scalar Higgs field providing quark masses and

$$h_{ij} = \delta_{ij}\,, \quad k_{12} = k_{21} = x\,, \quad k_{22} = y\,, \quad k_{11} = 0. \tag{7}$$

5.* Compute the Cabibbo angle in terms of the physical quark masses (remember that this angle represents the relative rotation of the up- and down- quark mass matrices to bring them into a diagonal form).

6.* Compute the ratio:

$$\frac{\Gamma(W_\mu \to \bar{u}d)}{\Gamma(W_\mu \to \bar{u}s)} \tag{8}$$

where W denotes the charged vector boson of the electroweak standard model.

7.* Prove that the neutral vector bosons γ (photon) and Z^0 of the electroweak standard model cannot have the couplings:

$$(\gamma, Z^0) - d^0 - s^0 \tag{9}$$

where d^0 and s^0 denote the mass eigentstates of the down-quark mass matrix.

HOWEVER, we experimentally observe transitions between a quark d^0 and a quark s^0 (for instance, in the oscillation $K^0 - \bar{K}^0$).

8.* How would you account for these processes in view of what you have seen in this exercise?

A.2. Free Massive QED in Three Dimensions

Consider the field theory model in $d = 3$ space–time dimensions defined by the following Lagrangian:

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{\lambda}{2} \epsilon^{\mu\nu\rho} A_\mu \, F_{\nu\rho} - A_\mu J^\mu ,$$

where A_μ is an Abelian vector, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ its field strength, and J^μ an external classical source. The external current J^μ is conserved, *i.e.* $\partial_\mu J^\mu = 0$, and g and λ are coupling constants.

1. Discuss in which sense the model is gauge–invariant.

2. Find the mass spectrum of the model. Is there a mass gap?

Hint: There are many ways to compute the mass spectrum, the easiest of which is to consider the equation satisfied by the field strengths.

3. Compute the spin of each particle in the spectrum.

4. Discuss the properties of the various particles under P, T and C.

5. Consider the topological "magnetic" charge $Q = \oint_\infty A_\mu \, dx^\mu$ (*i.e.* the Wilson loop taken along the circle at infinity in space). Show that there is a conserved current V^μ associated to this conserved charge.

6. Consider the sector of the Hilbert space in which the 'magnetic' charge Q has a definite value. Let $q = \int d^2x \, J^0$ be the usual (external) electric charge. Find the allowed values of q in each sector of definite Q.

Hint: The Hilbert space is defined by the constraint given by Gauss' law.

7.* Find the static field configuration induced by a static external point charge. What is its spin?

8.* Consider two "localized" states with charge pairs (q, Q) and (q', Q') which are not in the list of (6). Using the constraint (Gauss's law) discuss in which sense there is a "confining potential" between them.

9.* Give the propagator of A_μ in a gauge of your choice. Say a few words on the interpretation of the various poles, and their relation with the previous issues.

B.1. Relativistic Scattering

Comment: This problem is made up of seven steps to be solved in the given order. They are of somewhat increasing difficulty. Steps 1 to 3 only require knowledge of elementary quantum mechanics. Steps from 4 to 7 make use of a rudimentary form of quantum field theory. If, after having solved all steps up to number 7, you still have "$\Delta t\,\Delta E > 0$", you can try step 8.

Consider a particle of mass m and charge e moving under the influence of a scalar potential

$$\phi(x) = \begin{cases} \phi_0 & x \geq 0 \\ 0 & x < 0\,. \end{cases} \tag{1}$$

1. Write the time-independent relativistic equation (Klein-Gordon equation).

Hint: Recall that in the relativistic case $(p_\mu p^\mu + m^2)\psi(x) = 0$, where p is the four-vector of the particle's momentum, and that the external electro-magnetic field is introduced (minimal coupling) by replacing $p_\mu \to p_\mu - eA_\mu$, where A_μ is the electro-magnetic potential. The index μ runs from 0 to 3—e.g., $p = (E, \vec{p})$, where E is the particle's energy—and the metric signature is $+ - - -$.

2. Find the solution $\psi(x)$ of the equation of point 1 above in terms of plane waves of momentum p and q, respectively for $x < 0$ and $x \geq 0$.

Hint: The solution of this relativistic equation is the same as the corresponding non-relativistic Schrodinger equation once the appropriate values for the momenta p and q are taken. Write the normalization of the plane waves in terms of the reflection amplitude $r(\xi)$ and trasmission amplitude $t(\xi)$, where $\xi = q/p$.

3. Write the coefficients of reflection R and of trasmission T and briefly discuss the three cases:

- $E > e\,\phi_0$
- $E < e\,\phi_0$ and $e\,\phi_0 - E < m$
- $E < e\,\phi_0$ and $e\,\phi_0 - E > m$,

where E is again the energy of the particle. What is taking place in the third case?

Hint: The third case above is called the Klein's paradox in the textbooks.

In order to better understand the Klein's paradox, we can think of the scattering as accompanied by a process of particle-antiparticle pair creation from the vacuum in the presence of the strong potential. However, this is still a rather vague idea. The three following points try to make this idea more precise.

4. Write the solutions of the Klein-Gordon equation which contain:

- only an asymptotically outgoing particle for $x > 0$, call it $\psi_1(x)$
- only an asymptotically incoming particle for $x > 0$, call it $\psi_2(x)$
- only an asymptotically outgoing particle for $x < 0$, call it $\psi_3(x)$
- only an asymptotically incoming particle for $x < 0$, call it $\psi_4(x)$.

Hint: Introduce $\varphi_p^{in} = e^{ipx}/\sqrt{p}$ and $\tilde{\varphi}_p^{in} = e^{iqx}/\sqrt{|q|}$ and similarly for the outgoing states. Write the $\psi_i(x)$ in terms of these four normalized single-particle states.

You have thus defined two complete sets of states—one representing asymptotically only an outgoing particle (or antiparticle) and those representing only an incoming particle (or antiparticle). These two sets are not independent and are related by an unitary trasformation, the S matrix:

$$|\psi(\text{in})\rangle = S\,|\psi(\text{out})\rangle\,. \tag{2}$$

5. Consider an arbitrary field operator in terms of the two complete sets of states you have defined at point 4

$$\psi(x) = \sum_k \hat{b}_k^{in}\varphi_k^{in}(x) + \hat{d}_k^{in\dagger}\varphi_k^{in}(x) \tag{3}$$

and

$$\psi(x) = \sum_k \hat{b}_k^{out}\varphi_k^{out}(x) + \hat{d}_k^{out\dagger}\varphi_k^{out}(x) \tag{4}$$

where the creation operators are defined by

$$\hat{b}_k^{in\dagger}|0(\text{in})\rangle = |k(\text{in})\rangle \qquad \hat{d}_k^{in\dagger}|0(\text{in})\rangle = |\tilde{k}(\text{in})\rangle\,,$$

the annhilation operators by

$$\hat{b}_k^{in}|0(\text{in})\rangle = 0 \qquad \hat{d}_k^{in}|0(\text{in})\rangle = 0\,,$$

and similarly for the outgoing states. Find the relations among the various out and in creation and annhilation operators (Bogolioubov's relations).

Hint: Write $\psi_1(x)$ in terms of $\psi_4(x)$ and $\psi_2(x)$ and so on.

6. By means of the previous results, compute the vacuum to vacuum amplitude

$$\langle 0(\text{out})|0(\text{in})\rangle \equiv e^W\,. \tag{5}$$

Hint: Compute first the amplitude for the creation of 1 pair

$$A_{1\,pair} = \langle 0(\text{out})|\hat{d}^{out}\,\hat{b}^{out}|0(\text{in})\rangle \tag{6}$$

and than that for n pairs. Impose the unitarity condition that

$$\sum_{k=0}^{\infty} |A_{k\,pair}|^2 = 1\,. \tag{7}$$

Remember that the n-particle states are normalized as $|n\rangle = (\hat{b}^\dagger)^n|0\rangle/\sqrt{n!}$ and that $[\hat{d}^{out}, \hat{d}^{out\dagger}] = 1$.

7. Write the amplitude for a particle to be reflected with no pair being created

$$A^{ref}_{0\,pair} = \langle 0(\text{out})|\hat{b}^{out}\,\hat{b}^{in\dagger}|0(\text{in})\rangle \tag{8}$$

and that for reflection with n pairs being created $A^{ref}_{n\,pair}$. Verify that the total probability for scattering accompanied by the production of no, one, two etc. pairs is one (unitarity).

8.* Retrace your steps from 1 to 7 in the case of a particle of spin 1/2 (Dirac equation).

Hint: Anticommuting implies that only a single pair can be created since more than one would violate the Pauli exclusion principle.

[Reference: B. R. Holstein, Am. J. Phys. **66** (1998) 507.]

B.2. Light Deflection by the Sun

Comment: This is a problem in general relativity which only requires an elementary knowledge of the theory.

Consider a light ray (photon) in the gravitation field of the sun.

1. Show that the equation of its trajectory in the spherical space coordinates (r, θ, ϕ) can be written as

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = 3u^2 \,,$$

where $u \equiv GM_\odot/r$. G is Newton constant and $M_\odot$ the mass of the sun.

Hint: Write the condition $g^{\mu\nu}p_\mu p_\nu = 0$ for the momentum p of the massless photon and bear in mind that the momenta conjugated to time, p_0, and ϕ, p_ϕ are constants of motion (call them E and L, respectively) and that the motion takes place in the plane $\theta = \pi/2$. Be careful with the up and down indices and recall that space-time around the sun is described in general relativity by the metric element

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM_\odot}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM_\odot}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \,,$$

which defines the metric tensor $g_{\mu\nu}$ via the relation $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$.

2. Introduce the impact parameter b as the closest approach distance of the light ray (photon) to the sun. Compute for $GM_\odot/b \ll 1$ and to the lowest order in $GM_\odot/b$, the deflection angle $\Delta\varphi$ of the light rays (photons) in the gravitation field of the sun.

Hint: Write the zeroth-order solution (straight line) as

$$u_0 = \frac{2GM_\odot}{b} \sin\phi$$

and expand $u = u_0 + u_1 + \cdots$. Compute by means of u_1 the two angles at $r = \infty$ and obtain the deflection angle $\Delta\varphi$ as their difference.

B.3. Three-Dimensional Harmonic Oscillator

Given a tridimensional harmonic oscillator, whose Hamiltonian is written as

$$\mathcal{H} = \frac{1}{2}\omega\left[\mathbf{r}^2 + \mathbf{p}^2\right] = \omega\left[\mathbf{a}^\dagger \cdot \mathbf{a} + 3/2\right]$$

1. Verify that the degeneration of the energy levels is higher than that which is implied by the $SU(3)$ rotational invariance.

2. Verify that the other conserved observables allow for the construction of three further $SU(2)$ algebras.

> Hint: Remember that a two-dimensional harmonic oscillator admits an $SU(2)$ symmetry.

3. Write down some examples of perturbations that only partially removes the degeneration of the levels.

4. Explain why, although the various $SU(2)$'s allow for the construction of a complete algebra of $SU(3)$, it is not possible to find among the eigenstates of $\mathcal{H}$ bases for all the representations of $SU(3)$.

B.4. Statistical Mechanics of a Spin Chain

Consider a spin-chain, that is a one-dimensional lattice with a spin S_n placed at each site $x = n$ (with $n \in \mathbf{Z}$). Each spin S_n takes two possible values $+1$ and -1. The energy (Hamiltonian) of a configuration of spins is given by a nearest neighborhood interaction plus a magnetic term

$$H = J \sum_n (S_{n+1} - S_n)^2 + h \sum_n S_n.$$

1. Consider a finite chain of N spins and assume periodic boundary conditions $S_{N+1} \equiv S_1$. Compute the corresponding partition function Z_N for all values of the temperature T and magnetic field h.

2. Compute the free energy $\mathcal{F}$ in the thermodynamical limit $N \to \infty$ (infinite chain).

3. Assume $J > 0$. For the infinite chain, find for which temperatures T there is a spontaneous magnetization at zero magnetic field $h = 0$.

> Comment: By spontaneous magnetization we mean a spontaneous breaking of the $\mathbf{Z}_2$ symmetry of the $h = 0$ model signaled by a non–vanishing statistical expectaction value $\langle S_n \rangle|_{h=0}$.

4. Let $J < 0$. Describe the thermal state of the infinite chain in the limit $h \to 0$, $T \to 0$.

5.* In the infinite chain, compute the connected correlation function

$$\langle S_n \ S_m \rangle^{\text{conn}} = \langle S_n \ S_m \rangle - \langle S_n \rangle \langle S_m \rangle.$$

6.* Compute the correlation length ξ.