Problem C. Field Theory Description of a Statistical Model

The aim of this exercise is to study the continuum version of a lattice statistical model of microscopic lattice length ϵ near its critical temperature T_c. Let $\varphi(x)$ and

$$\begin{aligned} \mathcal{A} &= \int d^D x \left[\frac{1}{2} (\partial_\mu \varphi)^2 + V(\varphi) \right] \\ V(\varphi) &= a\varphi^2 + b\varphi^4 + g\varphi^6 \end{aligned} \tag{1}$$

be respectively the continuum limit of the order–parameter of the model and its Euclidean action in D–dimensions. In eq. (1) a and b are real parameters whereas $g > 0$ and in our units, the action $\mathcal{A}$ is a dimensionless quantity. Assume the validity of the tree level (alias mean field) approximation to answer the following questions.

1. Determine the curve $b(a)$ where three distinct vacua coexist.

2. Consider now the case $b = 0$ and $a = a_0 t$, where t is the temperature difference $t = T - T_c$. The correlation length ξ of the system is defined in terms of the 2–point function in momentum space as $\langle \varphi(k) \varphi(-k) \rangle \sim \frac{1}{k^2 + \xi^{-2}}$

   Show that

   $$\xi \sim \xi_0^{\pm} \mid t \mid^{-1/2}$$

   and compute the ratio of the amplitudes $\xi_0^{\pm}$ above and below T_c.

3. Determine by dimensional analysis how the coupling constant g depends on the lattice length ϵ. Use this information to find the dimensions D for which the tree level approximation is expected to be exact in the continuum limit of the model.