Problem A. Field theory and spontaneous symmetry breaking.

The lagrangian of a field theory in 4 dimensions is

$$\mathcal{L} = \sum_{i,j=1}^{3} \Big(\frac{1}{2} K_{i,j} \partial_\mu \phi^i \partial^\mu \phi^j - M_{i,j} \phi^i \phi^j\Big) - V(\phi^i) \tag{1}$$

where $K_{i,j}$ and $M_{i,j}$ are real symmetric constant matrices; $K_{i,j}$ is non–singular positive definite matrix.

1. What is the largest internal symmetry of the kinetic term? Call the corresponding group G. What is the form of $M_{i,j}$ such that the mass term is invariant too under G?
2. Suppose now that $V(\phi^i)$ is also invariant under G. Write the conserved currents of this symmetry and compute the canonical commutators of the corresponding charges.
3. Can there be in $V(\phi^i)$ (which is invariant under G) a term of the third order in ϕ^i? Write the fourth order term in $V(\phi^i)$. In the presence of the latter write a possible scenario of spontaneous breakdown of the symmetry G.

Hint: for any positive definite symmetric matrix K one can find a real matrix E such that $E^t E = K$, where E^t is the transpose of E.