Problem B. Classical Statistical Mechanics of a q-state Model

Let us consider an open one-dimensional lattice of N sites, with a fluctuating variable σ_i defined on each site i. The possible values assumed by σ_i are $\sigma_i = 1, 2, \ldots, q$ and the Hamiltonian of the model is given by

$$H(\sigma_1, \ldots, \sigma_N) = J \sum_{i=1}^{N-1} \delta(\sigma_i, \sigma_{i+1}) \ ,$$

where J is a real constant and

$$\delta(a,b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

1. Under which transformations is the Hamiltonian invariant? Describe the ground state for $J > 0$ and for $J < 0$ and determine its degeneracy in the two cases.

2. Let us put the model in contact with a heat reservoir at temperature β^{-1}. Compute the partition function $Z_N(\beta J, q)$ defined by

$$Z_N(\beta J, q) = \sum_{\sigma_1=1}^{q} \cdots \sum_{\sigma_N=1}^{q} e^{-\beta H(\sigma_1, \ldots, \sigma_N)} \ .$$

   (*Hint.* Obtain a recursive equation $Z_N \to Z_{N-1}$ by using the mathematical identity $e^{x\delta(a,b)} = 1 + (e^x - 1)\delta(a,b)$).

3. Since $Z_N(\beta J, q)$ above computed is an analytic function of the variable q, the model results to be also defined for continous values of q and in particular in the "unphysical" region $q < 1$. Determine the occurrence of phase transitions for $J > 0$ and $J < 0$ and the critical temperature $\beta_c^{-1}(q)$ as a function of q.