Problem A. Two–dimensional harmonic oscillator.

A quantum two–dimensional harmonic oscillator may be described in terms of two creation and two annihilation operators $a_i, a_i^\dagger$, $(i = 1, 2)$, with commutation relations

$$[a_i, a_j] = 0 = [a_i^\dagger, a_j^\dagger], \qquad [a_i, a_j^\dagger] = \delta_{i,j}$$

The Hamiltonian is $H = 2J + 1$, where $J = \frac{1}{2}\left(a_1^\dagger a_1 + a_2^\dagger a_2\right)$.

1) Consider the operators

$$\begin{aligned} J_1 &= \frac{1}{2}\left(a_2^\dagger a_1 + a_1^\dagger a_2\right) \\ J_2 &= \frac{i}{2}\left(a_2^\dagger a_1 - a_1^\dagger a_2\right) \\ J_3 &= \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right) \end{aligned}$$

Compute $[H, J_i]$ and $[J_i, J_j]$ and interpret the latter.

2) Find the expression of $\mathbf{J}^2 = J_1^2 + J_2^2 + J_3^2$ in terms of J.

3) Find the eigenvalues of $\mathbf{J}^2, J$ and J_3 corresponding to the states $(a_1^\dagger)^k (a_2^\dagger)^l |0>$, with k and l non–negative real numbers. Consider the subset of such states with fixed eigenvalue j of J: what is the condition on k, l and j for such subset to form a finite complete basis of eigenstates of J and J_3?
(*Hint: use* $J_\pm = J_1 \pm iJ_2$).

4) What is the degeneracy of the eigenvalues of H? Devise possible generalizations of the present model to remove such degeneracy.