Problem D. Majorana Fields and Discrete Symmetries

Let $\chi(x)$ be a free spin-1/2 Majorana field in 4 space-time dimensions, i.e., a 4-component (second quantized) spin-1/2 field which satisfies the Dirac equation and the Majorana condition

$$C(\bar{\chi}(x))^T = \xi\ \chi(x), \qquad (i)$$

where C is the charge conjugation matrix,

$$C^T = -C, \quad C^{-1}\gamma_\mu C = -\gamma_\mu^T, \qquad (ii)$$

(γ_μ being the Dirac matrices), and ξ is a phase factor, $|\xi|^2 = 1$. Under the $CP-$conjugation operation the field $\chi(x)$ transforms as follows:

$$U_{CP}\chi(x)U_{CP}^\dagger = \eta_{CP}\ \gamma_0\ \chi(x'), \qquad (iii)$$

where $|\eta_{CP}|^2 = 1$, and $x' = (x_0, -x_1, -x_2, -x_3)$. The factor η_{CP} is the $CP-$parity of the Majorana field $\chi(x)$.

1. Prove that $\eta_{CP} = \pm i$.

2. Prove that

$$:\bar{\chi}(x)\gamma_\mu\chi(x): \ = 0,$$

where the sign : means normal ordering. What is the physical meaning of this result?