Problem C. Statistical Mechanics

Let us consider a quantum system of volume V described by the Hamiltonian H. Let $e_m(V)$ $(m = 0, 1, \ldots, \infty)$ be the energy spectrum. The system is put in contact with a heat reservoir at temperature T.

1. Prove that the expectation value of H, denoted by $E(T, V)$, is a monotonic function of the temperature T.

2. Assume that the energy levels are homogeneous functions of the volume V according to the formula

$$e_m(\lambda V) \;=\; \lambda^x \, e_m(V) \;\; .$$

Show that $E(T, V)$ satisfies a differential equation of the form

$$\left(a\, T \frac{\partial}{\partial T} + b\, V \frac{\partial}{\partial V} + c \right) E(T, V) \;=\; 0 \;\; .$$

Determine a, b and c.