Problem D. Second Quantization of the Dirac Equation in the Majorana Representation

Consider the massless Dirac equation in the Majorana representation for the γ matrices:

$$\gamma_0 = \begin{pmatrix} 0 & -i1 \\ i1 & 0 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix} \qquad \gamma_3 = \begin{pmatrix} 0 & i1 \\ i1 & 0 \end{pmatrix}$$

where each entry represents a 2×2 matrix and σ_k are the usual Pauli matrices.

1. Find the x_1, x_2 independent (i.e. with momentum in the third direction) plane wave solutions of the Dirac equation in the Majorana representation.

2. Prove that the Dirac equation is compatible with the constraint

$$\psi = \psi^\star$$

3. Recalling the second quantised representation of a real ($\Phi = \Phi^\dagger$) massless scalar field with momentum in the third direction:

$$\Phi(t, x_3) = \int \frac{dp}{2\mid p\mid}[a_p\, e^{-i(|p|t - px_3)} + a_p^\dagger\, e^{i(|p|t - px_3)}] \ ,$$

write, using the solutions found at 1), a similar representation for the real Majorana field ($\psi = \psi^\dagger$). How many one-particle states with a given momentum are there?