Problem C. Extimate of Ground State Energy in Quantum Mechanics

Exact results in Physics are quite rare but a quite good estimate of physical quantities can be often obtained by means of some approximation schemes. The aim of this exercise is to apply two of these approximations to a simple example of a quantum mechanical system. In the following h is the usual Planck constant and $\hbar = \frac{h}{2\pi}$.

Consider the motion of a quantum particle of mass m on a real axis $-\infty < x < +\infty$ with Hamiltonian given by

$$H = \frac{p^2}{2m} + k \mid x \mid \quad , \quad k > 0$$

1. Discuss the symmetry of the Hamiltonian, the nature of the spectrum (if continuous or discrete) and its degeneracy. Using dimensional analysis, construct a quantity $\mathcal{E}$ in terms of $\hbar$, m and k which has the dimension of an energy and which can be used as the scale for all the energy eigenvalues.

2. Using the uncertainty principle $\Delta p \, \Delta x \geq \hbar$, give an estimate of the ground state energy E_0 of the system and the size of the corresponding wave function.

3. Using the semi-classical quantization condition

$$\oint p(x)\, dx = 2\pi \left(n + \frac{1}{2}\right) \hbar$$

find the (approximate) energy levels E_n of the system. In particular, calculate E_0 and compare with the result of point (3).