Conformal field theories

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February 6, 2020

Chapter 1

Introduction

There are systems in nature that are characterized by *scale invariance*, that is they appear to have the same features if we blow up or down distances between points (or elements) of the system. This is the case for statistical systems at critical points. In such systems singular thermodynamical quantities are related to the correlation length by a set of critical exponents (universality principle). At critical points the correlation length becomes infinite and the systems become scale invariant, since there is no scale left to measure distances.

The situation just outlined occurs for systems such as the vapor-liquid one at the critical point, characterized by a critical pressure p_c and temperature T_c. Below T_c density is not well defined, while it is well defined at and above T_c. A similar picture holds for a ferromagnetic material with magnetization m. In nature there are many other systems with analogous characteristics. Density and magnetization are example of *order parameters*, temperature T and external magnetic field h are instead examples of *external fields*. To be more specific, in ferromagnetic materials the source of magnetization is the spin of the electrons in incomplete atomic shells, each electron carrying one unit of magnetic moment. Such spins can be imagined to be attached to lattice points and to interact with the neighboring ones in such a way that the state of lowest energy corresponds to all the spins being parallel. At $T = 0$ all the spins are aligned. When the temperature increases the thermal motion destroys this order, but not completely if the temperature is low enough; there remains patches where the spin are all aligned, with the result that a finite magnetization survives. As T reaches T_c and goes beyond it, order is completely destroyed and magnetization vanishes (disordered or paramagnetic phase).

The physics near a critical point is experimentally characterized by critical exponents. For ferromagnetic materials, when $h = 0$, we have

$$m \sim (T_c - T)^\beta \tag{1.1}$$

with $\beta \approx 0.33$, typically. At $T = T_c$ the magnetization is not an analytic function of h

$$m \sim h^{\frac{1}{\delta}} \tag{1.2}$$

and, typically, $\delta \approx 4$. The magnetic susceptibility χ is also singular

$$\chi \equiv \left(\frac{\partial m}{\partial h}\right)_T \sim |T - T_c|^{-\gamma}, \qquad T \approx T_c \tag{1.3}$$

(with $\gamma \approx 1.33$) as well as the specific heat C

$$C \sim |T - T_c|^{-\alpha} \tag{1.4}$$

($\alpha \approx 0.10$). There are two more critical exponents. We can think of a ferromagnet as a continuous distribution of spin, represented (in the uniaxial case) by a field $\sigma(x)$. Let us denote by $G(k)$ the Fourier transform of the two-point correlator of $\sigma(x)$:

$$G(k) = \int d^d x \langle \sigma(x)\sigma(0)\rangle e^{-ikx}$$

This quantity can be measured by means of neutron scattering. It is observed that, at the critical point,

$$G(k) \sim k^{-2+\eta} \tag{1.5}$$

with $\eta \approx 0.067$. Finally, $G(k)$ peaks at $k = 0$, so let us define the width ξ^{-1} by

$$\xi^{-2} = -\frac{1}{2} G^{-1}(0) \left(\frac{\partial^2 G(k)}{\partial k^2}\right)_{k=0} \tag{1.6}$$

The correlation length ξ diverges at $T = T_c$:

$$\xi \sim |T - T_c|^{-\nu} \qquad \text{at} \quad h = 0 \tag{1.7}$$

and $\nu \approx 0.7$.

The remarkable thing is that these critical exponents $\beta, \delta, \alpha, \gamma, \eta$ and ν are universal, i.e. the physical relations (1.1,1.2, ...) do not depend on the particular material. Moreover there are phenomenological relations among them

$$\alpha + 2\beta + \gamma = 2, \qquad \beta\delta = \beta + \gamma, \qquad \alpha = 2 - d\nu, \qquad \delta = \frac{d + 2 - \eta}{d - 2 + \eta} \tag{1.8}$$

so that only two of them are independent.

The universality and simplicity of the critical phenomena just illustrated prompt us to suppose that such systems can be described by scale invariant Euclidean field theories (time is not involved). Such theories contain fields $A_i(x)$ that, under a coordinate dilatation

$$x^a \longrightarrow \lambda x^a, \qquad \lambda > 0, \qquad a = 1, \dots d,$$

transform as

$$A_i(x) \longrightarrow \lambda^{\Delta_i} A_i(\lambda x)$$

The number Δ_i is called the *scaling weight* of $A_i(x)$. The example of the ferromagnetism, discussed above, suggests that what is at work at the critical point is not simply scale invariance, but a larger symmetry, *confomal invariance*. For the correlation length is interpreted as the average size of the polarized spin patches; the fact that at the critical point this becomes infinite, means that we have patches of any size. Therefore the physical picture does not change, not only when we rescale the system rigidly, but also when we rescale it with a varying scale from point to point. But these are precisely conformal transformations (which locally change size without changing shape).

It is obvious that conformal invariance is more restrictive than simply scale invariance with obvious consequences on the predictivity of the theory. It is generally the case that scale invariant theories are endowed with a larger symmetry, *the conformal symmetry*. Actually, to date no unitary scale invariant field theory has been obtained that is not also conformal invariant. When $d > 2$ the conformal group is finite dimensional, when $d = 2$ it is infinite dimensional. It is not surprising that this infinite symmetry will play an important role in the description of $2d$ conformal fields theories. Two-dimensional conformal field theories are important not only because they can describe two-dimensional statistical systems at the critical point, but also because specific 2d conformal field theories are the backbone of (super)string theory.

Beside the successes of CFT in $2d$, conformal symmetry and conformal field theories are at the forefront also in higher dimensions. The truth is that conformal symmetry is intrinsic to quantum field theory via the renormalization group (RG) and the Wilsonian philosophy. Suppose we start from a microscopic system represented by a distribution of spins. We are interested in seeing how the system evolves when our resolution decreases (i.e. the typical energy, or scale, decreases). To this end we can replace the system with another one constructed as follows: we subdivide the system into small patches and replace the spins in each patch by their average over the patch. If we next rescale the distances and the resulting spins, we obtain a system of the same type as before. We can repeat the process over and over and accordingly decrease our resolution. In many cases this process will converge to a new theory. Although such a process is more effectively carried out in momentum space, the space description may help intuition of what the renormalization group approach is. The RG in quantum field theory is mathematically described by the Callan-Symanzik equation, valid for any correlator among (composite) fields $A_i(x)$:

$$\left(\frac{\partial}{\partial \log \Lambda} + \beta^I \frac{\partial}{\partial g^I}\right) \langle A_{i_1}(x_1) \dots A_{i_n}(x_n)\rangle$$
$$= \gamma_{i_1}^{j_1} \langle A_{j_1}(x_1) \dots A_{i_n}(x_n)\rangle + \dots + \gamma_{i_n}^{j_n} \langle A_{i_1}(x_1) \dots A_{j_n}(x_n)\rangle \tag{1.9}$$

where Λ is the scale of energy, g^I are the couplings in the theory, $\beta^I = \frac{\partial g^I}{\partial \log \Lambda}$ the beta-functions and γ_i^j the matrix of anomalous dimensions. This equation says that when the energy scale changes the correlators remain formally the same provided the couplings and the anomalous dimensions of the fields suitably change. The β^Is are the crucial objects of our concern. Suppose for simplicity there is only one coupling. When the corresponding β-function vanishes it means that the coupling does not evolve with the scale of energy. Therefore a fixed point of the beta function means that the theory has evolved to scale invariance. Since we have seen that scale invariance means conformal invariance, fixed points of the RG provide plenty of examples of (not necessarily Lagrangian) conformal field theories. The fixed points can be UV or IR and we have no guarantee that a theory flowing from the UV to the IR along an RG trajectory will end up in a fixed point. However this turns out to be the case in many physically interesting examples.

Finally one needs not stress the importance of the role CFT in the AdS/CFT duality or generalizations thereof, where CFT on the boundary and gravity in the bulk are different descriptions of the same physics.

1.1 Preliminaries

1.1.1 $d \geq 3$

Conformal transformations are coordinate transformations that 'change the size without changing the shape', that is change the distances between points without changing the angles. The conforml group in $d \geq 3$ dimensions is finite dimensional. It encompasses Poincaré transformations, dilatations and special conformal transformations (sct's). The latter are

$$x^\mu \to x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2} \tag{1.10}$$

where b^μ are constant parameters. This of course can be viewed as a particular general coordinate transformation. For infinitesimal b^μ

$$\frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2} \approx x^\mu - b^\mu x^2 + 2b\cdot x\, x^\mu \tag{1.11}$$

and the general coordinate transformation takes the form: $x^\mu \to x^\mu + \xi^\mu$ where $\xi^\mu = -b^\mu x^2 + 2b\cdot x\, x^\mu$.

Looking at how the square line element $ds^2 = dx^\mu \eta_{\mu\nu} dx^\nu$ transforms,

$$ds^2 \longrightarrow (1 + 4b\cdot x) ds^2, \tag{1.12}$$

(1.10) can be understood as a special local rescaling of the metric. Conformal transformations can in fact be defined as those coordinate transformations that correspond to a (local) rescaling of the metric.

The Lie algebra generators of the conformal group (CG) are

$$\begin{aligned}
&P_\mu = i\partial_\mu \\
&D = ix^\mu\,\partial_\mu \\
&L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \\
&K_\mu = i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)
\end{aligned}$$

They generate translations, dilatations, Lorentz and SCT's, respectively. The Lie brackets among them are

$$\begin{aligned}
&[L_{\mu\nu}, L_{\lambda\rho}] = -i\left(\eta_{\mu\lambda}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\lambda} - \eta_{\nu\lambda}L_{\mu\rho} + \eta_{\nu\rho}L_{\mu\lambda}\right) \\
&[P^\mu, P^\nu] = 0 \\
&[L_{\mu\nu}, P_\lambda] = -i\left(\eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu\right) \\
&[D, P^\mu] = -iP^\mu \\
&[D, K^\mu] = iK^\mu \\
&[P^\mu, K^\nu] = 2i\eta^{\mu\nu}D - 2iL^{\mu\nu} \\
&[K^\mu, K^\nu] = 0 \\
&[L^{\mu\nu}, D] = 0 \\
&[L^{\mu\nu}, K^\lambda] = -i\eta^{\lambda\nu}K^\mu + i\eta^{\lambda\mu}K^\nu
\end{aligned} \tag{1.13}$$

They form the Conformal Algebra (CA), which is the maximally extended bosonic algebra of the Poincaré algebra. So far we have not specified what background metric we understand in the previous formulas. It can be either Minkowskian or Euclidean. In the first case the conformal algebra turns out to be isomorphic to $so(d,2)$ in the latter case to $so(d+1,1)$, the Lie algebra of the orthogonal group $SO(d,2)$ and $SO(d+1,1)$, respectively.

1.1.2 $d = 2$

In two dimensions the conformal group is infinite dimensional, for the diffeomorphisms that correspond to local rescaling of the metric are arbitrary functions (see 1.1.3). Here we introduce the notation when the background metric is Euclidean. By defining $z = \frac{1}{\sqrt{2}}(x^1 + ix^2)$ we will adopt the complex variable notation. Infinitesimal conformal transformations are defined by

$$z \to z + \epsilon(z), \qquad \bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z}) \tag{1.14}$$

where $\epsilon(z)(\bar{\epsilon}(\bar{z}))$ are arbitrary infinitesimal holomorphic (antiholomorpphic) functions in the entire complex plane with the possible exclusion of the origin and the point at infinity. The group of infinitesimal conformal transformations is thus a direct product of its holomorphic and antiholomorphic subgroups: $\mathcal{C} \times \bar{\mathcal{C}}$. This induces the characteristic chirality splitting of conformal field theories in $d = 2$, with the consequence that the holomorphic and antiholomorphic sectors of these theories can be studied separately: for instance the results for the full theory correlators are obtained by putting together the results of the two sectors and requiring reality.

Let us see a few properties of the holomorphic conformal transformations. From now on we will limit our attention to the $\epsilon(z)$ parameters that admit a Laurent expansion

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} \epsilon_{-n} z^{n+1} \tag{1.15}$$

The generator corresponding to the n-th mode of this expansion is the vector field

$$\ell_n = -z^{n+1}\frac{d}{dz} \tag{1.16}$$

as can be seen by noting that, for any holomorphic function $f(z)$,

$$\delta_\epsilon f(z) \approx f(z+\epsilon) - f(z) \approx \pounds_{\sum_n \epsilon_{-n}\ell_n} f(z)$$

where $\pounds_X$ denotes the Lie derivative with respect to the vector field X. The Lie bracket between the ℓ_n's define the *Witt algebra* of the holomorphic vector fields on the unit circle

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m} \tag{1.17}$$

A finite subalgebra of both algebras is spanned by $\ell_{-1}, \ell_0, \ell_1$ (or L_{-1}, L_0, L_1), which generate $s\ell(2,\mathbb{C})$, the Lie algebra of the group $SL(2,\mathbb{C})$. The latter is the group of fractional transformations of the complex plane with unit determinant:

$$z \to \frac{az+b}{cz+d}, \qquad ad - bc = 1 \tag{1.18}$$

If we set $a = 1+\alpha$, $b = \beta$, $c = \gamma$, $d = 1+\delta$, with $\alpha, \beta, \gamma, \delta$ infinitesimal parameters, the corresponding fractional transformation becomes, to first order,

$$z \to z + \beta + (\alpha - \delta)z - \gamma z^2$$

This is to be compared with the transformation $z \to \epsilon_1 + \epsilon_0 z + \epsilon_{-1} z^2$ generated by $\ell_1, \ell_0, \ell_{-1}$. One can easily recognize the parameters of the 2d conformal group: the translation parameters are given by the two components of ϵ_1; $\epsilon_0 = \rho e^{i\theta}$ gives the dilataion and rotation parameters; finally the sct parameters b_1 and b_2 correspond to $-\Re(\epsilon_{-1})$ and to $\Im(\epsilon_{-1})$, respectively.

Radial quantization

When the spacetime metric is Euclidean the quantization of a field theory is better carried out with the radial ordering rather the usual time ordering. This means that the spacetime is foliated by spheres centered at the origin (instead of being foliated by space-like hyperplanes as in the ordinary Lorentzian quantization). States are propagated not by the Hamiltonian but by D. A more detailed introduction to radial quantization will be given in Part II. For the time being the notion of radial ordering of operators inserted in correlators will be enough.

OPE

In general in quantum field theory, given two fields $\varphi_1(x), \varphi_2(y)$ at nearby points ($x \approx y$) their (radial ordered) product can be expanded in the complete set of fields and their derivatives of the theory, located at some intermediate point (we will mostly use the second point y):

$$\mathcal{R}\left(\varphi_1(x)\varphi_2(y)\right) \approx \sum_n C_n(x-y)\varphi_n(y). \tag{1.19}$$

The coefficients are in general distributions with some kind of singularity at $x = y$. The symbol $\approx$ here means that the series on the RHS is asymptotic, that is the RHS the better approximates the LHS the closer is x to y. In CFT a remarkable difference is that the series on the RHS of (1.19) is convergent and $\approx$ is replaced by $=$.

1.1.3 Appendix A: Conformal transformations in $d = 2$

In a Euclidean 2d space diffeomorphisms are transformations $x^a \to x^a + \xi^a(x)$, with $a = 1, 2$, where ξ^a are infinitesimal generic functions. The metric transforms according to

$$\delta_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a \tag{1.20}$$

where $\xi_a = g_{ab}\xi^b$. Under a Weyl transformation it transforms according to

$$\delta_\omega g_{ab} = 2\omega\, g_{ab} \tag{1.21}$$

where $\omega(x)$ is a generic positive function. A conformal transformation is a diffeomorphism which coincides with a Weyl transformation. So if we start from a flat metric η_{ab} it is specified by

$$\partial_a \xi_b + \partial_b \xi_a = 2\omega \eta_{ab} \tag{1.22}$$

from which it follows that

$$\partial_1 \xi_2 + \partial_2 \xi_1 = 0, \qquad \partial_1 \xi_1 - \partial_2 \xi_2 = 0 \tag{1.23}$$

where $\partial_i = \partial_{x^i}$. Let us introduce the complex coordinates $z = \frac{1}{\sqrt{2}}(x^1 + ix^2), \bar{z} = \frac{1}{\sqrt{2}}(x^1 - ix^2)$ and the combinations $\epsilon(z,\bar{z}) = \frac{1}{\sqrt{2}}(\xi_1 + i\xi_2)$ and $\bar{\epsilon}(z,\bar{z}) = \frac{1}{\sqrt{2}}(\xi_1 - i\xi_2)$. One can verify that

$$\partial_{\bar{z}}\epsilon(z,\bar{z}) = \frac{1}{2}(\partial_1 + i\partial_2)(\xi_1 + i\xi_2) = 0, \qquad \partial_z \bar{\epsilon}(z,\bar{z}) = 0 \tag{1.24}$$

due to (1.23). So ϵ is a function only of z, and $\bar{\epsilon}$ only of $\bar{z}$. Thus in a 2d Euclidean framework conformal transformations take the form (1.14).

We need at times the compact notation z^α, where $z^1 = z, z^2 = \bar{z}$. In this notation we have

$$z^\alpha \to z^\alpha + \epsilon^\alpha, \qquad \epsilon^1(z) \equiv \epsilon^z(z) = \epsilon(z), \qquad \epsilon^2(\bar{z}) \equiv \epsilon^{\bar{z}}(\bar{z}) = \bar{\epsilon}(\bar{z}). \tag{1.25}$$

Since with these new variables the flat metric is the unit anti-diagonal matrix, we have

$$\epsilon^z(z) = \epsilon_{\bar{z}}(z), \qquad \epsilon^{\bar{z}}(\bar{z}) = \epsilon_z(\bar{z}) \tag{1.26}$$

Part I

$d = 2$ conformal field theories

In this part we introduce and analyze Euclidean conformal field theories in two dimensions. The presentation follows closely the Belavin-Polyakov-Zamolodchikov construction. It consists in extracting the conformal Ward identities (WI) on field amplitudes from the path integral formulation of an abstract conformally invariant field theory and split them in their holomorphic and anitiholomorphic parts. Such split WI's can be written in different equivalent ways, a prominent one being the operator product expansion (OPE). A theory is defined by the central charge of the underlying Virasoro algebra and the weights of the primary fields, and a fundamental role is played by the energy-momentum tensor (a descendant of the identity operator). The primary fields and their descendants, applied to the vacuum at the origin of the complex plane, generate states which turn out to be lowest weight representations (LWR) of the Virasoro algebra. In this way the classification problem of CFT's is translated into the mathematical problem of classification of the unitary (or non-unitary) irreducible (or reducible) representations of the Virasoro algebra. This problem is completely solved. In particular for central charge $c < 1$ there is a discrete series of singular reducible LWR's (both unitary and non-unitary), where singular means endowed with singular vectors. The presence of the latter allows us to write down partial differential equations for split (holomorphic and anti-holomorphic) correlators. More practically, instead of solving partial differential equations, one can resort to the Coulomb gas technique, which permits to write down the split correlators as explicit contour integrals. Once these are known the physical correlators can be determined by studying the monodromy properties of the split ones.

In order to overcome the barrier of $c < 1$ solvable models one has to increase the symmetry of the theory, for instance by adding supersymmetry. This part of the lectures ends with the example of superconformal $N = 1$ and $N = 2$ field theories.

Chapter 2

The BPZ construction

Two–dimensional statistical systems at a critical point call for an interpretation in terms of two–dimensional Euclidean conformal field theories. The construction of such theories was fostered by the work of Belavin, Polyakov and Zamolodchikov. They were able to give a general formulation, using a path-integral approach, but abstracting from the specific form of the Lagrangian. Their basic achievements were to extract from the path integral the Ward identities corresponding to the conformal symmetry, to clarify the role of the energy-momentum tensor and of the *primary fields* (i.e. fields tensorial with respect to the conformal group, which are identified with representations of the Virasoro algebra) and to classify all possible theories.

Let us assume the theory has a local action $S[\phi, g]$ in terms of elementary fields, collectively denoted ϕ, and of a background metric g. We wish to compute the correlators of n fields $A_{i_1}, \ldots, A_{i_n}$, which are in general supposed to be composite. The amplitudes are defined by

$$\langle \mathcal{A} \rangle \equiv \langle A_{i_1}(x_1) \ldots A_{i_n}(x_n) \rangle = \frac{\int \mathcal{D}\phi A_{i_1}(x_1) \ldots A_{i_n}(x_n)\, e^{-S[\phi,g]}}{\int \mathcal{D}\phi\, e^{-S[\phi,g]}}$$

Let us consider the consequence of an infinitesimal deformation $\phi(x) \to \phi(x) + \delta_\phi \phi(x)$ in the path integral on the numerator of the RHS. To first order we have

$$\int \mathcal{D}\phi \left(\sum_{k=1}^{n} A_{i_1}(x_1) \ldots \delta_\phi A_{i_k}(x_k) \ldots A_{i_n}(x_n) \right) e^{-S[\phi,g]}$$
$$= \int \mathcal{D}\phi\, A_{i_1}(x_1) \ldots A_{i_n}(x_n)\, \delta_\phi S(x)\, e^{-S[\phi,g]}$$

or, more syntetically,

$$\delta_\phi \langle A_{i_1}(x_1) \ldots A_{i_n}(x_n) \rangle = \langle A_{i_1}(x_1) \ldots A_{i_n}(x_n) \delta_\phi S(x) \rangle \tag{2.1}$$

Now, let us suppose that $\delta_\phi S + \delta_g S = 0$. Then we have

$$\delta_\phi S = -\delta_g S = -\frac{1}{2}\int \sqrt{g}\,\delta g_{ab}\,T^{ab}$$

which has to be inserted in (2.1). If the transformations are diffeomorphisms, so that $\delta g_{ab} = \nabla_a \epsilon_b + \nabla_b \epsilon_a$, $a, b = 1, 2$, then

$$\delta_\epsilon \langle A_{i_1}(x_1)\dots A_{i_n}(x_n)\rangle = \int d^2x\sqrt{g}\,\epsilon_a \nabla_b \langle T^{ab}(x) A_{i_1}(x_1)\dots A_{i_n}(x_n)\rangle \tag{2.2}$$

If the transformations are Weyl rescalings of the metric $\delta g_{ab} = 2\sigma g_{ab}$ we have

$$\delta_\sigma \langle A_{i_1}(x_1)\dots A_{i_n}(x_n)\rangle = -\int d^2x\sqrt{g}\,\sigma(x)\langle T_a{}^a(x) A_{i_1}(x_1)\dots A_{i_n}(x_n)\rangle \tag{2.3}$$

The insertion of the energy-momentum tensor in the RHS of (2.2,2.3) generates singularities when x coincides with $x_1, \dots, x_n$. But since ϵ and σ are arbitrary functions of x, the two equations mean that outside the points $x_1, \dots x_n$ we have

$$\nabla_b \langle T^{ab}(x) A_{i_1}(x_1)\dots A_{i_n}(x_n)\rangle = 0, \qquad \langle T_a{}^a(x) A_{i_1}(x_1)\dots A_{i_n}(x_n)\rangle = 0$$

These equations simplify further if the metric g is the flat one. Passing to the complex notation[1], and denoting by $\mathcal{A}$ the product $A_{i_1}(x_1)\dots A_{i_n}(x_n)$, the previous equations write

$$\partial_z \langle T^{zz}\mathcal{A}\rangle = 0, \qquad \partial_{\bar z}\langle T^{\bar z\bar z}\mathcal{A}\rangle = 0, \qquad \langle T^{z\bar z}\mathcal{A}\rangle = 0 \tag{2.4}$$

From now on we will set

$$T = T_{zz} = T^{\bar z\bar z} = \frac{1}{2}\left(T_{11} - 2iT_{12} - T_{22}\right), \quad \bar T = T_{\bar z\bar z} = T^{zz} = \frac{1}{2}\left(T_{11} + 2iT_{12} - T_{22}\right)$$
$$T_{z\bar z} = T_{\bar z z} = T_{11} + T_{22} = T_a{}^a$$

Eq.(2.3) means that $\langle T\,\mathcal{A}\rangle$ and $\langle \bar T\,\mathcal{A}\rangle$ are holomorphic and anti-holomorphic, respectively, except at the points $x_1, \dots, x_n$. We interpret this by formulating the theory in terms of field operators, the em tensor being conserved and traceless, in other words $T = T(z)$ and $\bar T = \bar T(\bar z)$ and $T_{z\bar z} = 0$[2]. The singularities in the correlators are produced by inserting these operators at coincident points.

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[1] In complex notation α, β runs over $z, \bar z$ and the flat metric $\eta_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

[2] This can be verified to be true a posteriori for flat metric. It may not be true for non-flat metric due to anomalies.

Now let us apply the previous Ward identities to the case of conformal transformations. We will start by considering a transformation such that $\epsilon = \epsilon(z,\bar{z})$ is nonvanishing, while $\bar{\epsilon}(z,\bar{z}) = 0$.

$$\begin{aligned}
\langle \delta_\epsilon \mathcal{A} \rangle &= \int dz d\bar{z}\, \epsilon_\alpha\, \partial_\beta\, \langle T^{\alpha\beta} \mathcal{A} \rangle \\
&= \int dz d\bar{z}\, \epsilon(z,\bar{z})\, \left(\partial_{\bar{z}} \langle T^{\bar{z}\bar{z}}\, \mathcal{A} \rangle + \partial_z \langle T^{z\bar{z}} \mathcal{A} \rangle \right) \\
&= \int dz d\bar{z}\, \epsilon(z,\bar{z})\, \bar{\partial} \langle T\, \mathcal{A} \rangle \qquad (2.5)
\end{aligned}$$

where we have introduced (and will often use henceforth) the notation $\partial \equiv \partial_z, \quad \bar{\partial} = \partial_{\bar{z}}$.

Let us elaborate on (2.5). We expect singularities at the points $x_1, \ldots, x_n$, and also at the origin and at infinity due to the singularity of $\epsilon(z)$. For simplicity, we suppose that $x_1, \ldots, x_n$ are all finite and different from the origin. We will choose a simple closed contour C that surrounds all of them and excludes the origin. Since $\epsilon(z,\bar{z})$ is arbitrary, we can choose it so that it coincides with a holomorphic transformation $\epsilon(z)$ inside C and vanishes outside it[3]. Using Stokes' theorem we can integrate over $\bar{z}$ and we find

$$\langle \delta_\epsilon \mathcal{A} \rangle = \frac{1}{2\pi i} \oint_C dz\, \epsilon(z) \langle T\, \mathcal{A} \rangle \qquad (2.6)$$

where the integral is taken in the anticlockwise direction along C. It follows that T is the generator of holomorphic conformal transformations.

We can rewrite (2.6) in suggestive operator form. Since the singularities of the correlator are only at the points $x_1, \ldots, x_n$, we can deform the contour C and reduce it to a union of small separate contours C_i around each point x_i. Let $A(z,\bar{z})$ be one of the operators inserted in the correlator and let C_z be the small contour about it. Then (2.6) can be split operatorially into n relations of the type

$$\delta_\epsilon A(z,\bar{z}) = \frac{1}{2\pi i} \oint_{C_z} dw\, \epsilon(w)\, \mathcal{R}\left(T(w) A(z,\bar{z})\right) \qquad (2.7)$$

where $\mathcal{R}$ is the radial ordering (which replaces the time ordering in Euclidean field theories):

$$\mathcal{R}\left(A(z)B(w)\right) = \begin{cases} A(z)B(w), & |z| > |w| \\ B(w)A(z), & |z| < |w| \end{cases}$$

[3] We actually need smoothing out $\epsilon(z,\bar{z})$ in a small neighborhood of C, because $\epsilon(z,\bar{z})$ has to be sufficiently regular.

Eq.(2.7), in turn, can be further simplified by choosing two circular contours, C_i and C_e around the origin such that C_i excludes C_z and is clockwise oriented and C_e includes C_z and is anticlockwise oriented. Since the only singularity between the two circles is where z coincides with w, we can deform C_z so that it coincides with the union of C_i and C_e. Thus

$$\begin{aligned}\delta_\epsilon A(z,\bar{z}) &= \frac{1}{2\pi i}\oint_{C_z} dw\, \epsilon(w)\, \mathcal{R}\left(T(w)A(z,\bar{z})\right)\\ &= \frac{1}{2\pi i}\oint_{C_e} dw\, \epsilon(w)\, T(w)A(z,\bar{z}) + A(z,\bar{z})\frac{1}{2\pi i}\oint_{C_i} dw\, \epsilon(w)T(w)\end{aligned}$$

Now there is no singularity left and we can deform the two contours so that they coincide. Due to their opposite orientation we finally get

$$\delta_\epsilon A(z,\bar{z}) = \frac{1}{2\pi i}\Big[\oint_{C_0} dw\, \epsilon(w)T(w), A(z,\bar{z})\Big] \equiv \Big[T_\epsilon, A(z,\bar{z})\Big] \tag{2.8}$$

where C_0 is a contour surrounding the origin.

For an antiholomorphic conformal transformation $\bar{\epsilon}(\bar{z})$ we have similarly

$$\langle \delta_{\bar{\epsilon}} \mathcal{A}\rangle = \frac{1}{2\pi i}\oint_C d\bar{z}\, \bar{\epsilon}(\bar{z})\langle \bar{T}\, \mathcal{A}\rangle \tag{2.9}$$

and

$$\delta_{\bar{\epsilon}} A(z,\bar{z}) = \frac{1}{2\pi i}\Big[\oint_{C_0} dw\, \bar{\epsilon}(\bar{w})\bar{T}(\bar{w}), A(z,\bar{z})\Big] \equiv \Big[\bar{T}_{\bar{\epsilon}}, A(z,\bar{z})\Big] \tag{2.10}$$

2.1 Properties of the energy-momentum tensor

The energy momentum tensor is itself a composite field of the theory. So we can apply the previous formulas to it as well. We thus find the following equation

$$\delta_\epsilon T(z) = \Big[T_\epsilon, T(z)\Big] = \epsilon(z)T'(z) + 2\epsilon'(z)T(z) + \frac{c}{12}\epsilon'''(z) \tag{2.11}$$

where a prime denotes derivation with respect to z. The rightmost equality follows from the transformation property of $T(z)$. The energy momentum tensor in $d = 2$ is almost a quadratic differential, but not quite; it behaves in fact as a projective connection, i.e., under a holomorphic transformation $z \to w(z)$, it transform as follows

$$T(z)(dz)^2 = \Big((w')^2\tilde{T}(w) + \frac{c}{12}\{w, z\}\Big)\,(dz)^2 \tag{2.12}$$

where

$$\{w, z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2$$

is called the *Schwarzian derivative*. The transformation (2.11) is the infinitesimal version of (2.12) when $w(z) \approx z + \epsilon(z)$.

Similar considerations and formulas hold for $\bar{T}(\bar{z})$.

Let us Laurent-expand $T(z)$ and $\bar{T}(\bar{z})$.

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \qquad \bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{L}_n \bar{z}^{-n-2} \tag{2.13}$$

Using an analogous expansion for ϵ and $\bar{\epsilon}$ (viz. $\epsilon(z) = \sum_{n\in\mathbb{Z}} \epsilon_{-n} z^{n+1}$) and identifying the modes on the left and the right sides one finds that (2.11) corresponds to the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{n+m} + \frac{c}{12}(m^3 - m)\delta_{n+m,0} \tag{2.14}$$

An analogous commutator is obtained for $\bar{L}_n$. In addition $[L_n, \bar{L}_m] = 0$.

We see that c has to be identified with the central charge of the Virasoro algebra.

2.1.1 Operator product expansions

Consider a field theory with a complete basis of fields $A_i(x)$. It can be proven that the product of any two such fields $A(x)B(y)$ at nearby points $x \approx y$, can be expressed in terms of a series of basis fields with coefficients given by (in general) singular functions (i.e. distributions) of $x - y$. This is referred to as operator product expansion (OPE). Such expansion has to be understood as inserted in correlators. In a generic field theory the OPE series are generically asymptotic, while in a conformal field theory they are convergent. In the language of one complex variable adopted in this chapter we have, for instance,

$$\mathcal{R}(A(z)B(w)) = \sum_i C_i(z-w)A_i(w)$$

where the functions $C_i(z-w)$ are meromorphic with possible poles at $z = w$ (later on also cuts will appear). In the sequel we will mostly understand the radial ordering symbol $\mathcal{R}$ in OPE's.

The first example we present is the OPE of $T(z)$ with itself. It takes the form

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \text{Reg} \tag{2.15}$$

where Reg stands for regular terms.

This OPE is equivalent (up to the regular terms) to the transformation law (2.11). For, let us consider (2.7) applied to the operator $T(z)$,

$$\delta_\epsilon T(w) = \frac{1}{2\pi i}\oint_{C_w} dz\,\epsilon(z)\,\mathcal{R}\left(T(z)T(w)\right) \tag{2.16}$$

and let us replace (2.15) inside it. By the Cauchy theorem we see that the regular part of the OPE does not contribute and the simple pole can be easily integrated. As for the other poles we use the residue theorem. If an analytic function $f(z)$ has a pole of order n at $z = z_0$, the residue at that pole is

$$\mathrm{Res} f(z_0) = \lim_{z\to z_0} \frac{1}{(n-1)!}\left(\frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n f(z)\right]\right)$$

The end result is easily seen to be (2.11), which in turn contains the same information as (2.14).

Example: the Gaussian model

The Gaussian model is the model of a free scalar field φ. The full propagator is

$$\langle \varphi(z,\bar z)\varphi(w,\bar w)\rangle = \ln|z-w|^2$$

We can isolate the holomorphic part

$$\langle \varphi(z)\varphi(w)\rangle = \ln(z-w) \tag{2.17}$$

The holomorphic energy momentum tensor is

$$T(z) = \frac{1}{2} :\partial_z\varphi\partial_z\varphi: \tag{2.18}$$

Using the Wick theorem and expanding in Taylor series when necessary, we obtain

$$\begin{aligned} T(z)T(w) &= \frac{1}{2}:\partial_z\varphi(z)\partial_z\varphi(z):\frac{1}{2}:\partial_w\varphi(w)\partial_w\varphi(w): \\ &= :\partial_z\varphi(z)\partial_w\varphi(w):\partial_z\partial_w\ln(z-w) + \frac{1}{2}\left(\partial_z\partial_w\ln(z-w)\right)^2 \\ &\quad + \frac{1}{4}:\partial_z\varphi(z)\partial_z\varphi(z)\partial_w\varphi(w)\partial_w\varphi(w): \\ &= \frac{1}{2(z-w)^4} + 2\frac{T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \mathrm{Reg} \end{aligned}$$

and a similar OPE for $\overline{T}(\bar z)$, which confirms the general form of the OPE for energy-momentum tensors. It also tells us that the central charge of the model is 2 (1 for the holomorphic and 1 for the antiholomorphic part).

Example: massless Majorana fermion

A massless Majorana field has two components ψ and $\bar{\psi}$. The Lagrangian is

$$L = \psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi} \tag{2.19}$$

and the propagators are

$$\langle\psi(z)\psi(w)\rangle = \frac{1}{z-w}, \qquad \langle\bar{\psi}(\bar{z})\bar{\psi}(\bar{w})\rangle = \frac{1}{\bar{z}-\bar{w}} \tag{2.20}$$

The energy-momentum tensor components write

$$T(z) = -\frac{1}{2} : \psi(z)\partial\psi(z) :, \qquad \bar{T}(\bar{z}) = -\frac{1}{2} : \bar{\psi}(\bar{z})\bar{\partial}\bar{\psi}(\bar{z}) : \tag{2.21}$$

Using OPE and Wick theorem we can compute

$$\langle T(z)T(w)\rangle = \frac{1}{4}\frac{1}{(z-w)^4} + 2\frac{T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{Reg}$$

and an analogous expression for the antiholomorphic part. This means that the central charge of the holomorphic part is $\frac{1}{2}$. The central charge of the full model is 1.

2.2 Primary fields

Basic ingredients in conformal field theories are the *primary fields*. Primary fields are covariant under the action of the conformal group. A primary field is defined by the following transformation rule under a conformal change of coordinates $z \to w(z)$ and $\bar{z} \to \bar{w}(\bar{z})$:

$$\tilde{\phi}(z,\bar{z})(dz)^h(d\bar{z})^{\bar{h}} = \phi(w,\bar{w})(dw)^h(d\bar{w})^{\bar{h}} \tag{2.22}$$

The number $\Delta = h+\bar{h}$ is the *scaling weight*. This is the same concept introduced at the beginning of this chapter as it represents the response of the field with respect to a rigid coordinate rescaling. The number $s = h - \bar{h}$ is the spin, as it represents the response of the field under a coordinate rotation $z \to e^{i\theta}z$.

Let us concentrate from now on on the holomorphic part of ϕ. Under an infinitesimal conformal trasformation $z \to w(z) \approx z + \epsilon(z)$ we get

$$\begin{aligned}\tilde{\phi}(z) = \phi(w)\left(\frac{\partial w}{\partial z}\right)^h &\approx \phi(z+\epsilon(z))(1+\epsilon'(z))^h \\ &= \phi(z) + \epsilon(z)\partial\phi(z) + h\,\epsilon'(z)\,\phi(z)\end{aligned}$$

i.e., setting $\delta_\epsilon\phi(z) = \tilde{\phi}(z) - \phi(z)$,

$$\delta_\epsilon\phi(z) = \epsilon(z)\partial\phi(z) + h\,\epsilon'(z)\,\phi(z) \tag{2.23}$$

But, since this transformation is generated by T, we have also

$$\delta_\epsilon\phi(z) = \left[\frac{1}{2\pi i}\oint dw\,\epsilon(w)T(w), \phi(z)\right] = \epsilon(z)\partial\phi(z) + h\,\epsilon'(z)\,\phi(z) \tag{2.24}$$

By considering for instance a transformation $\epsilon(z) = \epsilon_{-n}z^{n+1}$ we can derive from this

$$[L_n, \phi(z)] = z^{n+1}\partial\phi(z) + h(n+1)z^n\phi(z) \tag{2.25}$$

Alternatively one can use OPE to express the same property

$$T(w)\phi(z) = \frac{h}{(w-z)^2}\phi(z) + \frac{1}{w-z}\partial\phi(z) + \mathrm{Reg} \tag{2.26}$$

Examples of primary fields

It is easy to prove that in the Gaussian model we have

$$T(z)\,\partial\varphi(w) = \frac{1}{(z-w)^2}\partial\varphi(w) + \frac{1}{z-w}\partial^2\varphi(w) + \dots$$

Hence $\partial\varphi$ is a primary field of weight 1.

In the free Majorana fermion model we have

$$T(z)\psi(w) = \frac{1}{2}\frac{1}{(z-w)^2}\psi(w) + \frac{1}{z-w}\partial\psi(w) + \dots$$

Therefore ψ is a primary field of weight $\frac{1}{2}$.

2.2.1 Secondary fields

Any primary field ϕ of weight h defines an infinite family of fields, called *conformal family*. Consider the OPE

$$T(w)\phi(z) = \sum_{k=0}^{\infty}(w-z)^{k-2}\phi^{(-k)}(z)$$

where $\phi^{(0)}(z) = h\phi(z)$ and $\phi^{(-1)}(z) = \partial\phi(z)$. The remaining ones are defined as follows:

$$\phi^{(-k)}(z) = L_{-k}(z)\phi(z), \qquad \text{where} \qquad L_{-k}(z) = \frac{1}{2\pi i}\oint_{C_z} dw\frac{T(w)}{(w-z)^{k-1}}$$

We have in particular $L_{-k}(0) = L_{-k}$. The fields $\phi^{(-k)}(z)$ are the *secondary* or *descendants* fields of the family. The process of creating descendants can continue *ad infinitum* by applying $L_{-k}(z)$ to the descendants, and so on.

One important secondary field is the em tensor itself. In fact the identity operator I is a primary field of 0 weight. Therefore it generates a conformal family. The second secondary field of this family

$$I^{(-2)}(z) = \frac{1}{2\pi i}\oint dw\, \frac{T(w)}{w-z} = T(z)$$

is precisely the em tensor.

The transformation property of the secondary field $\phi^{(k)}$ is given by

$$\begin{aligned}\delta_\epsilon \phi^{(-k)}(z) &= \epsilon(z)\partial\phi^{(-k)}(z) + (h+k)\epsilon'(z)\phi^{(-k)}(z) \\ &+ \sum_{l=1}^{k-1} \frac{k+l}{(l+1)!}\left(\frac{d^{l+1}}{dz^{l+1}}\epsilon(z)\right)\phi^{(l-k)}(z) \\ &+ \frac{c}{12}\frac{1}{(k-2)!}\left(\frac{d^{k+1}}{dz^{k+1}}\epsilon(z)\right)\phi(z)\end{aligned} \tag{2.27}$$

2.2.2 State-operator correspondence

A given 2d conformal field theory is defined by its energy-momentum tensor and its primary fields. The energy-momentum tensor is equivalent to the datum of the L_n's. Let us introduce the vacuum vector $|0\rangle$. In any theory it is defined by

$$L_n|0\rangle = 0, \qquad n \geq -1 \tag{2.28}$$

This is known as the $SL(2,\mathbb{C})$ invariant vacuum since it is left unchanged by the $SL(2,\mathbb{C})$ generators L_{-1}, L_0, L_1. By acting with the fields of the theory on the vacuum we can generate the states of the theory. Any primary field is required to be regular at the origin of the complex plane[4]. So, given a primary field $\phi(z)$ we can define the state

$$|\phi\rangle = \phi(0)|0\rangle \tag{2.29}$$

Due to (2.25) this implies

$$\begin{aligned}L_n|\phi\rangle &= L_n\phi(0)|0\rangle = [L_n, \phi(0)]|0\rangle = 0, \qquad n>0 \\ L_0|\phi\rangle &= [L_0, \phi(0)]|0\rangle = h\phi(0)|0\rangle = h|\phi\rangle\end{aligned} \tag{2.30}$$

[4] This does not mean regular as a field operator, but that, after applying the field to the vacuum, there is no singularity left at the origin

This means that a primary field defines a lowest weight representation of the Virasoro algebra (see below)

The secondary fields define other states of the same representation, for

$$\phi^{(-k)}(0)|0\rangle = L_{-k}|\phi\rangle$$

and more states are obtained by applying to the vacuum the descendants of the descendants, and so on. Therefore giving a primary field is tantamount to specifying a lowest weight representation of the Virasoro algebra (see below).

This means that the process $field \to state$ can be inverted. From a lowest weight state we can go back to the corresponding field[5]. This is called the $state - operator$ correspondence.

The dual vacuum $\langle 0|$ gives rise to the dual $state - operator$ correspondence as follows:

$$\langle\phi| \sim \langle 0|\phi(\infty)$$

In defining the dual state $\langle\phi|$ one has to be careful and take into account the transformation properties of the field ϕ under conformal transformations:

$$\phi(z) = \left(\frac{dw}{dz}\right)^h \tilde{\phi}(w)$$

The point $z = \infty$ is outside the patch where the coordinate z is a good coordinate. A good coordinate at $z = \infty$ is instead $w = -\frac{1}{z}$. Therefore the relevant relation is

$$\tilde{\phi}(w = 0) = \lim_{z\to\infty} z^{2h}\phi(z)$$

So

$$\langle\phi| = \langle 0|\tilde{\phi}(0) = \lim_{z\to\infty} z^{2h}\langle 0|\phi(z) \tag{2.31}$$

2.3 Lowest weight representations of the Virasoro algebra

The concept of primary field and relevant conformal family of secondary fields is strictly connected, and actually identified, with the idea of lowest weight representation of the Virasoro algebra (Vir). The Virasoro algebra is defined, in a more

[5]We may argue as follows. Knowing the primary state and its descendants allows us to compute all the correlators involving primary and secondary fields. On the other hand the reconstruction theorem of axiomatic field theory tells us that knowing all the correlators we can reconstruct the fields of the theory. The Coulomb gas (see below) is an example of explicit field operator construction from LWR data.

formal way than above, by the generators L_n and $\hat{c}$. The latter commutes with all the L_n's and its eigenvalue is the central charge. Lowest weight modules are defined in analogy with ordinary Lie algebras by singling out a maximal commuting subalgebra. This is chosen to be formed by the generators $\hat{c}$ and L_0. Different representations are defined by their eigenvalues. More formally, a lowest weight representation (LWR) $V_{h,c}$ is constructed over a lowest weight vector $|\phi_h\rangle$ such that

$$\begin{aligned} \hat{c}|\phi_h\rangle &= c|\phi_h\rangle \\ L_0|\phi_h\rangle &= h|\phi_h\rangle \\ L_n|\phi_h\rangle &= 0, \qquad n > 0 \end{aligned} \tag{2.32}$$

$V_{h,c}$ is spanned by the vectors

$$L^{r_n}_{-k_n} L^{r_{n-1}}_{-k_{n-1}} \dots L^{r_1}_{-k_1} |\phi_h\rangle, \qquad 0 < k_1 < k_2 ... < k_n \tag{2.33}$$

where $r_1, \dots, r_n$ are positive integers. The L_0 eigenvalue of the generic state (2.33) is $h + \sum_{i=1}^n k_i r_i$. If all the states (2.33) are independent, $V_{h,c}$ is called a *Verma module.*

Remark

LWR's with $h = \bar{h} > 0$ are sometime called 'positive energy representations'. This comes from interpreting $L_0 + \bar{L}_0$ as Hamiltonian. Let $z \in \mathbb{C}$ be the complex coordinate and let us introduce coordinates σ, τ in the cylinder $\mathbb{S}^1 \times \mathbb{R}$ by the map $z = e^{\tau + i\sigma}$. Then

$$\ell_0 + \bar{\ell}_0 = \frac{\partial}{\partial \tau}, \qquad \ell_0 - \bar{\ell}_0 = i \frac{\partial}{\partial \sigma}$$

Therefore the representative $L_0 + \bar{L}_0$ of $\ell_0 + \bar{\ell}_0$ plays the role of Hamiltonian. In this context $|\phi_h\rangle \otimes |\bar{\phi}_h\rangle$ is the state of lowest energy as all the other states of $V_{h,c} \otimes \bar{V}_{h,c}$, see eq.(2.33), have a higher energy eigenvalue.

The two important questions concerning a LWR $V_{h,c}$ are:

- when is $V_{h,c}$ irreducible?
- when is $V_{h,c}$ unitary?

Starting from the position

$$\langle \phi_h | \phi_h \rangle = 1 \tag{2.34}$$

and the 'reality condition' (see below)

$$L_k^\dagger = L_{-k} \tag{2.35}$$

and using the defining commutator (2.14), it is possible to define an inner product in $V_{h,c}$. Unitarity is understood with respect to this inner product.

A comment on the 'reality' condition

In CFT there are two types of conjugations, the *bpz* and the *hermitean* conjugation. The first is the conjugation under the transition map $z \to -\frac{1}{z}$. Since under this map $T(z)$ is mapped to $\frac{1}{z^4}T\left(-\frac{1}{z}\right)$, it follows that L_n is mapped to $(-1)^n L_{-n}$. In proving this one should notice that the Schwarzian derivative $\{w, z\}$ vanishes when $w(z) = -\frac{1}{z}$.

So we define

$$bpz(L_n) = (-1)^n L_{-n} \tag{2.36}$$

On the other hand we can define another conjugation, by calling real a quantity that coincides with its complex conjugate on the unit circle of the z plane. The classical e.m. tensor must satisfy this condition

$$\left(T(z)(dz)^2\right)^* = T(z)(dz)^2, \qquad z = e^{i\theta} \tag{2.37}$$

This implies $L_n^* = L_{-n}$. This classical condition induces the hermitean conjugation (2.35) in the quantum theory. Therefore

$$L_n^\dagger = (-1)^n bpz(L_n). \tag{2.38}$$

It is worth remarking that the hermitean conjugation is connected not to the transition function $z \to -\frac{1}{z}$, but to another map, the inversion map: $x^a \to x^a/x^2$, see section 4.0.1, which in complex notation becomes $z \to \frac{1}{\bar{z}}$. This maps the origin to ∞ and viceversa, and leaves the unit circle $|z| = 1$ unchanged. This fixed locus of the inversion map is what is needed to impose *reflection positivity*, the property of Euclidean field theory amplitudes that implies unitarity in the Wick-rotated (Minkowskian) field theories. In ordinary Euclidean theories the reflection positivity is defined with respect to a suitable hyperplane in spacetime. The present case is better understood on the cylinder via the map $z = e^{\tau+i\sigma}$, where the unit circle corresponds to the section $\tau = 0$. Such a section splits the infinite cylinder into two symmetrical halves. To implement reflection positivity the states and operators localized at $\tau = -\infty$ ($z = 0$), must be hermitean conjugates to the states and operators at $\tau = \infty$. This the origin of (2.35).

2.3.1 Unitary LWR's

For a representation $V_{h,c}$ to be unitary it is necessary that the just defined inner product be nonnegative. In particular, for any $n > 0$, we must have

$$\begin{aligned} ||L_{-n}|\phi_h\rangle||^2 &= \langle\phi_h|L_n L_{-n}|\phi_h\rangle = \langle\phi_h|[L_n, L_{-n}]|\phi_h\rangle \\ &= \left(2nh + \frac{c}{12}(n^3 - n)\right)\langle\phi_h|\phi_h\rangle = 2nh + \frac{c}{12}(n^3 - n) \geq 0 \end{aligned} \tag{2.39}$$

This implies

$$c \geq 0, \qquad h \geq 0 \tag{2.40}$$

To obtain more stringent unitarity conditions we first notice that any state of $V_{h,c}$ can be labeled by the *level* N, which for the generic basis state (2.33) is the number defined by

$$N = \sum_{i=1}^{n} k_i r_i \tag{2.41}$$

(recall that the eigenvalue of L_0 is $N + h$). States belonging to different levels are orthogonal. Let us examine the generic state at each level:

- N=0: the only state is $|\phi_h\rangle$ and $\langle\phi_h|\phi_h\rangle = 1$.
- N=1: the only state is $L_{-1}|\phi_h\rangle$ and $||L_{-1}|\phi_h\rangle||^2 = 2h$.
- N=2: the generic state is a superposition

  $$|\psi\rangle = a\, L_{-2}|\phi_h\rangle + b\, L_{-1}^2|\phi_h\rangle \equiv a|\psi_1\rangle + b|\psi_2\rangle$$

  According to the theory of quadratic forms the state $|\psi\rangle$ has positive, negative or 0 norm according to the value of the determinant of the matrix

  $$M_2(h,c) = \begin{pmatrix} \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_2\rangle \\ \langle\psi_2|\psi_1\rangle & \langle\psi_2|\psi_2\rangle \end{pmatrix} = \begin{pmatrix} 4h + \frac{1}{2}c & 6h \\ 6h & 8h^2 + 4h \end{pmatrix}$$

  A necessary condition for absence of negative norm states is

  $$\det M_2(h,c) = 2h(c + 2(c-5)h + 16h^2) \geq 0 \tag{2.42}$$

  Since $h \geq 0$, this condition excludes a region between $c = 0$ and $c = 1$ enclosed by a parabola $c + 2(c-5)h + 16h^2 = 0$ passing through the origin of the (h,c) plane and the point $c = 0, h = \frac{5}{8}$. On the parabola we have the solutions $h = h_{1,2} \equiv h_+, h = h_{2,1} = h_-$, where

  $$h = h_\pm = \frac{1}{16}\left[5 - c \pm \sqrt{(c-1)(c-25)}\right] \tag{2.43}$$

For N generic calculations are more involved. The number $p(N)$ of states at level N is given by the formula

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{N=0}^{\infty} p(N)\, q^N \tag{2.44}$$

where q is a free variable. In general one has to analyze a determinant of a $p(N) \times p(N)$ matrix, $M_N(c,h)$. The analysis was carried out by V.Kac and by B.L.Feigin and D.B.Fuchs, [3, 2]. Here are the results. The Kac determinant is

$$\det M_N(c,h) \sim \prod_{k=1}^{N} \prod_{nm=k} [h - h_{n,m}(c)]^{p(N-k)} \tag{2.45}$$

where

$$\begin{aligned} h_{n,m}(c) &= \frac{1}{48}\Big[(13-c)(n^2+m^2) + (n^2-m^2)\sqrt{(c-1)(c-25)} \\ &\qquad -24nm + 2c - 2\Big] \end{aligned} \tag{2.46}$$

with $1 \leq nm \leq N$.

Theorem. *The representation $V_{h,c}$ of Vir is unitary and irreducible for $c \geq 1, h \geq 0$.*

All the states of these representations have *positive* norm. There are no zero norm states (singular vectors).

There are moreover other unitary representations for discrete values of $c < 1$:

$$c = 1 - \frac{6}{p(p+1)}, \qquad p = 3, 4, 5, \ldots \tag{2.47}$$

Replacing this in (2.46) one obtains the discrete weights

$$h = h_{n,m}(p) = \frac{[(p+1)m - pn]^2 - 1}{4p(p+1)}, \qquad n = 1, \ldots, p-1, \quad m = 1, \ldots, n \tag{2.48}$$

The case $p = 2$ corresponds to the trivial representation $c = 0, h = 0$. Formulas (2.47) and (2.48) define central charge and operators weights of the minimal models of CFT.

2.3.2 Reducible and irreducible LWR's

We have mentioned above that $V_{h,c}$ with $c > 1, h > 0$ is an irreducible unitary module of Vir. We did not specify what irreducibility means in this context. Let us expand on this point.

At level $N = 2$ consider the combination

$$|\chi\rangle = \left(L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2\right) |\phi_h\rangle \tag{2.49}$$

It satisfies

$$\begin{aligned}
L_0|\chi\rangle &= (h+2)|\chi\rangle \\
L_1|\chi\rangle &= \left(3 - \frac{3}{4h+2}(4h+2)\right) L_{-1}|\phi_h\rangle = 0 \\
L_2|\chi\rangle &= \frac{16h^2 + 2(c-5)h + c}{4h+2}|\phi_h\rangle = 0, \qquad h = h_{1,2}, h_{2,1} \\
L_n|\chi\rangle &= 0, \qquad n > 2
\end{aligned}$$

This means that χ is a LWV and

$$\langle\chi|\chi\rangle = 0 \tag{2.50}$$

So for $h = h_{1,2}, h_{2,1}$, $|\chi\rangle$ is a singular vector orthogonal to all the states of $V_{h,c}$.

Starting from $|\chi\rangle$ we can construct for $h = h_{1,2}, h_{2,1}$ the module $V_{h+2,c}$: $V_{h+2,c} = \{|\chi\rangle, L_{-1}|\chi\rangle, \ldots\}$. $V_{h+2,c}$ is orthogonal to all the states of $V_{h,c}$. So $V_{h+2,c}$ is a subrepresentation of $V_{h,c}$. The latter is reducible and we can obtain a new module by the quotient

$$\frac{V_{h,c}}{V_{h+2,c}}$$

According to Feigin and Fuchs we can generalize this construction: for any level k, with k integer, we can construct a singular vector $|\chi\rangle$ such that

$$\begin{aligned}
L_0|\chi\rangle &= (h+k)|\chi\rangle \\
L_n|\chi\rangle &= 0, \qquad n > 0
\end{aligned} \tag{2.51}$$

Moreover $|\chi\rangle$ is orthogonal to all the vectors in $V_{h,c}$. In particular $\langle\chi|\chi\rangle = 0$.

Feigin and Fuchs' conclusions are summarized in the following

Theorem.

- *The module $V_{h,c}$ is irreducible if and only if it does not contain any singular vectors.*
- *A singular vector of level k generates in $V_{h,c}$ a submodule isomorphic to $V_{h+k,c}$.*
- *The factor module $M_{h,c}$ by the maximal proper submodule is irreducible and every irreducible LW module is of the form $M_{h,c}$.*

As far as the unitary series (2.47,2.48) is concerned, let us denote by $V_{p,n,m}$ the module $V_{h,c}$ identified by the integers p, n and m. We can conclude that all $V_{p,n,m}$ are reducible. The irreducible module $M(h,c)$ is given by the quotient

$$M(h,c) = \frac{V_{p,n,m}}{V_{p,2p-n,m} \oplus V_{p,n+p,p+1-m}} \tag{2.52}$$

Finally, the modules belonging to the unitary series, are not the only reducible modules: there is a series of degenerate modules classified by two relatively prime positive integers p, q, whose central charge and weights are given by

$$\begin{aligned} c &= 1 - \frac{6(p-q)^2}{pq} \\ h_{n,m} &= \frac{(mp-nq)^2-(p-q)^2}{4pq}, \qquad 1 \leq m \leq q-1, \quad 1 \leq n \leq p-1 \end{aligned} \tag{2.53}$$

For $q = p+1$ and $m \leq n$ these formulas give back the unitary series. When $q \neq p+1$ these representations are not unitary.

2.3.3 LWR's of Vir and CFT

As we have seen above, a primary field ϕ in CFT contains the information of the conformal family generated by it. Applying ϕ and all its secondary fields, evaluated at the origin, to the vacuum generates (a basis of) a LWR of the Virasoro algebra. On the other hand the data of LWR's of Vir allow us to reconstruct a corresponding primary field. Therefore primary fields of CFT and LWR's of the Virasoro algebra contain an isomorphic set of data. Thus the classification of LWR's of Vir provides an essential tool for the classification of CFT's. In particular the list of unitary LW modules with definite central charge provides a list of unitary CFT's. We will focus from now on on the minimal unitary series, which underlies the conformal unitary minimal models.

In a field theory the final goal is of course the calculation of its correlation functions. In 2d CFT we have the advantage of an infinite dimensional symmetry algebra, which puts strong conditions on the correlators. These conditions are encoded in the underlying LWR's corresponding to the primary operators of the theory.

2.4 Correlation functions in 2d CFT

To start with let us establish a few general properties of the correlators in a given CFT. Let $\phi_1, \ldots, \phi_N$ be the primary fields of the theory and $h_1, \ldots, h_N$ their weights. From (2.7) we have

$$\delta_\epsilon \langle \phi_1(z_1) \ldots \phi_n(z_n) \rangle = \frac{1}{2\pi i} \oint_C dz\, \epsilon(z) \langle T(z) \phi_1(z_1) \ldots \phi_n(z_n) \rangle \tag{2.54}$$

where C is a closed contour surrounding $z_1, \ldots, z_n$, but not the origin. On the other hand, using OPE inside the correlator beside (2.54) we have

$$\begin{aligned} & \delta_\epsilon \langle \phi_1(z_1) \ldots \phi_n(z_n) \rangle \\ = \; & \frac{1}{2\pi i} \oint_C dz\, \epsilon(z) \sum_{i=1}^{n} \left(\frac{h_i}{(z-z_i)^2} + \frac{1}{z-z_i} \partial_{z_i} \right) \langle \phi_1(z_1) \ldots \phi_n(z_n) \rangle \end{aligned} \tag{2.55}$$

since the regular terms of the OPE do not contribute. Comparing (2.54) with (2.55), since $\epsilon(z)$ is a generic function, we get

$$\begin{aligned} & \langle T(z) \phi_1(z_1) \ldots \phi_n(z_n) \rangle \\ & = \sum_{i=1}^{n} \left(\frac{h_i}{(z-z_i)^2} + \frac{1}{z-z_i} \partial_{z_i} \right) \langle \phi_1(z_1) \ldots \phi_n(z_n) \rangle \end{aligned} \tag{2.56}$$

so, once we know a correlator for a set of primary fields it is easy to get the same correlator with the insertion of T. With a slight generalization we can also derive correlators with several T insertions.

Eq.(2.56) is an example of correlator involving a secondary field. For a generic secondary field $\phi^{(-k)}$, with $k \geq 2$, using (2.56), we have

$$\begin{aligned} & \langle \phi^{(-k)}(z)\, \phi_1(z_1) \ldots \phi_n(z_n) \rangle \\ = \; & \frac{1}{2\pi i} \oint_{C_z} dw\, \frac{1}{(w-z)^{k-1}} \langle T(w)\, \phi(z)\, \phi_1(z_1) \ldots \phi_n(z_n) \rangle \qquad (2.57) \\ = \; & \sum_{i=1}^{n} (-1)^k \left(\frac{(k-1)h_i}{(z-z_i)^k} + \frac{1}{(z-z_i)^{k-1}} \partial_{z_i} \right) \langle \phi(z)\, \phi_1(z_1) \ldots \phi_n(z_n) \rangle \end{aligned}$$

Therefore, once we know the correlators for primary fields we can compute the correlators with any kind of insertions of secondary fields. Therefore, what remains to be computed are the correlators of primary fields.

2.4.1 Correlators of primary fields

For correlators in CFT there are general results valid in any dimension. As a consequence of conformal invariance we have, for primary fields ϕ_i of scaling weight Δ_i, (below $x_{ij} = x_i - x_j$)

- a two point function $\langle \phi_1(x_1) \phi_2(x_2) \rangle$ vanishes unless ϕ_1 coincides with ϕ_2, and

$$\langle \phi_i(x_1) \phi_i(x_2) \rangle \sim \frac{1}{|x_{12}|^{2\Delta_i}}$$

- three point functions:

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = C_{123}\,|x_{12}|^{\Delta_3-\Delta_1-\Delta_2}|x_{13}|^{\Delta_2-\Delta_1-\Delta_3}|x_{23}|^{\Delta_1-\Delta_2-\Delta_3}$$

  where C_{123} is a suitable constant.

- four point functions

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \prod_{i,k}|x_{ik}|^{\frac{1}{3}\Delta-\Delta_i-\Delta_k}f(u,v),$$
$$u = \frac{x_{12}x_{34}}{x_{13}x_{24}}, \qquad v = \frac{x_{14}x_{23}}{x_{13}x_{24}}$$

  where $\Delta = \sum_{i=1}^4 \Delta_i$ and f is an arbitrary function of ξ and η.

In 2d one can say much more than this. In many cases all the correlators of primary fields are known exactly. In minimal models integrability is guaranteed by the existence of singular vectors in the corresponding LWR's. For a degenerate field (that is a field that generates a degenerate LWR) at level 2 we have the singular state (see above)

$$|\chi\rangle = \left(L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2\right)|\phi\rangle$$

Let us define the singular field

$$\chi(z) = \left(L_{-2}(z) - \frac{3}{2(2h+1)}L_{-1}^2(z)\right)\phi(z) \tag{2.58}$$

Then modding out the submodule built over $|\chi\rangle$ corresponds to setting $\chi = 0$ in correlators like $\langle\chi(z)\phi_1(z_1)\dots\phi_n(z_n)\rangle$. So we have

$$\begin{aligned} 0 &= \langle\chi(z)\phi_1(z_1)\dots\phi_n(z_n)\rangle \qquad (2.59)\\ &= \langle L_{-2}(z)\phi(z)\phi_1(z_1)\dots\phi_n(z_n)\rangle - \frac{3}{2(2h+1)}\partial_z^2\langle\phi(z)\phi_1(z_1)\dots\phi_n(z_n)\rangle \\ &= \left(\sum_{i=1}^n\left(\frac{h_i}{(z-z_i)^2} + \frac{1}{(z-z_i)}\partial_{z_i}\right) - \frac{3}{2(2h+1)}\partial_z^2\right)\langle\phi(z)\,\phi_1(z_1)\dots\phi_n(z_n)\rangle \end{aligned}$$

We obtain a partial differential equation for the correlator. Solving it will allow us to determine the correlator.

$SL(2,\mathbb{C})$ Ward identities

An often useful condition on correlators comes from the $SL(2,\mathbb{C})$ of the vacuum. This involves the generators L_{-1}, L_0, L_1, which annihilate both $\langle 0|$ and $|0\rangle$ vacua. We can therefore write for any correlator $\langle \phi_1(z_1)\ldots\phi_n(z_n)\rangle$ the Ward identity

$$\langle [L_i, \phi_1(z_1)\ldots\phi_n(z_n)]\rangle = 0, \qquad i=-1,0,1 \tag{2.60}$$

This becomes very effective if the fields ϕ_i are primary or, at least, *quasi-primary*, which means that their commutation with the L_i generators take the form (2.25), for $n=-1,0,1$. In such a case we can write (2.60) as

$$\begin{aligned}
&\sum_{k=1}^{n} \partial_{z_k} \langle \phi_1(z_1)\ldots\phi_n(z_n)\rangle = 0 \\
&\sum_{k=1}^{n} (z_k\partial_{z_k} + h_k)\langle \phi_1(z_1)\ldots\phi_n(z_n)\rangle = 0 \\
&\sum_{k=1}^{n} \left(z_k^2\partial_{z_k} + 2z_k h_k\right)\langle \phi_1(z_1)\ldots\phi_n(z_n)\rangle = 0
\end{aligned} \tag{2.61}$$

An example: the critical Ising model

The Ising model is a model for a uniaxial ferromagnet. In 2d at the critical point it can be represented by a conformal invariant local Euclidean field theory. This is simply the theory of a free Majorana fermion with components ψ and $\bar{\psi}$ (see above). The critical Ising model admits different descriptions, one in terms of the primary fields $I, \psi, \bar{\psi}, \epsilon =: \bar{\psi}\psi :$ of weights $(0,0), (\frac{1}{2},0), (0,\frac{1}{2}), (\frac{1}{2},\frac{1}{2})$ respectively, another in terms of the order parameter $\sigma(z,\bar{z})$ of weight $(\frac{1}{16},\frac{1}{16})$ with primaries I, σ, ϵ, and a third one in terms of the disorder parameter $\mu(z,\bar{z})$ of weight $(\frac{1}{16},\frac{1}{16})$ with primaries I, μ, ϵ. The central charge of the model is $c=\frac{1}{2}$, corresponding to $p=3$ in the minimal series (2.47) and the weights $h=0,\frac{1}{2},\frac{1}{16}$ are those allowed by formula (2.48).

The fields σ and μ are nonlocal with respect to ψ

$$\begin{aligned}
\psi(w)\sigma(z,\bar{z}) &= \frac{1}{\sqrt{w-z}}(\mu(z,\bar{z}) + O(w-z)) \\
\psi(w)\mu(z.\bar{z}) &= \frac{1}{\sqrt{w-z}}(\sigma(z,\bar{z}) + O(w-z))
\end{aligned}$$

The singular vector at level 2 for the primary operator σ with weight $\frac{1}{16}$ is

$$|\chi\rangle = (L_{-2} - \frac{4}{3}L_{-1}^2)|\sigma\rangle \tag{2.62}$$

(we consider here only the holomorphic part, there is a similar singular vector for the antiholomorphic part). The differential equation (2.59) for the σ four points function contains a quadratic derivative with respect to z and three simple derivatives with respect to z_1, z_2, z_3. The latter can be eliminated in favor of a simple derivative ∂_z using (2.61), so that (2.59) becomes an ordinary differential equation

$$\left\{\frac{4}{3}\frac{d^2}{dz^2} + \sum_{i=1}^{3}\left[\frac{1}{z-z_i}\frac{d}{dz} - \frac{1}{16}\frac{1}{(z-z_i)^2}\right]\right.$$
$$\left. + \frac{1}{8}\sum_{i<j}\frac{1}{(z-z_i)(z-z_j)}\right\}\langle\sigma(z)\sigma(z_1)\sigma(z_2)\sigma(z_3)\rangle = 0 \qquad (2.63)$$

Next we exploit the $SL(2,\mathbb{C})$ invariance of the vacuum to fix the three insertion points z_1, z_2, z_3 at $\infty, 1$ and 0, so that the above equation becomes

$$\left\{\frac{4}{3}\frac{d^2}{dz^2} + \left(\frac{1}{z} + \frac{1}{z-1}\right)\frac{d}{dz} - \frac{1}{16}\left(\frac{1}{z^2} + \frac{1}{(z-1)^2}\right)\right.$$
$$\left. + \frac{1}{8}\frac{1}{z(z-1)}\right\}\langle\sigma(\infty)\sigma(1)\sigma(z)\sigma(0)\rangle = 0 \qquad (2.64)$$

Setting

$$\langle\sigma(\infty)\sigma(1)\sigma(z)\sigma(0)\rangle = z^{-\frac{1}{8}}(z-1)^{-\frac{1}{8}} f(z) \qquad (2.65)$$

one gets

$$\left\{z(1-z)\frac{d^2}{dz^2} + (\frac{1}{2} - z)\frac{d}{dz} + \frac{1}{16}\right\} f(z) = 0$$

This is a standard hypergeometric differential equation, which admits two independent solutions

$$f(z) = \begin{cases} {}_2F_1(\frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, z) & = \sqrt{\frac{1+\sqrt{1-z}}{2}} \equiv \frac{W_1(z)}{\sqrt{2}} \\ \sqrt{z}\, {}_2F_1(\frac{3}{4}, \frac{1}{4}, \frac{3}{2}, z) & = \sqrt{\frac{1-\sqrt{1-z}}{2}} \equiv \frac{W_2(z)}{\sqrt{2}} \end{cases} \qquad (2.66)$$

Using these two solution it is possible to construct a univalent correlator

$$\begin{aligned} G(z,\bar{z}) &= \langle\sigma(\infty)\sigma(1,1)\sigma(z,\bar{z})\sigma(0,0)\rangle \\ &= |z|^{-\frac{1}{4}}|1-z|^{-\frac{1}{4}}\,|W_1\bar{W}_1 + W_2\bar{W}_2| \end{aligned} \qquad (2.67)$$

To complete this short presentation of the results of the critical Ising model let us compute the critical exponents. Using the identification

$$\Delta_\sigma = h_\sigma + \bar{h}_\sigma = \frac{d-2+\eta}{2}$$

we obtain $\eta = \frac{1}{4}$. Using the other identification

$$\nu = \frac{1}{d-(h_\epsilon + \bar{h}_\epsilon)}$$

we get $\nu = 1$. Finally using the relations (1.8) we obtain: $\alpha = 0, \beta = \frac{1}{8}$, $\gamma = \frac{7}{4}$ and $\delta = 15$.

2.5 Coulomb gas

It is clear from the previous example that, since the primary operators of the unitary minimal series are all degenerate, for all primary field correlators we can exploit differential equations such as (2.59). In principle, by solving these equations we can determine all the correlators. In practice this is very complicated. Fortunately there exists a way to (implicitly) integrate these differential equations. This is the so-called Coulomb gas method, [4], which utilizes a modification of the Gaussian model introduced above. The Gaussian model for a scalar field has central charge 1. Thus it is of very limited use, in particular it cannot describe the minimal series. We can radically change the situation if we add a background charge. This can be achieved by modifying the holomorphic energy-momentum tensor as follows

$$T(z) = -\frac{1}{4}(\partial_z\varphi)^2(z) + i\,\alpha_0\,\partial_z^2\varphi(z) \tag{2.68}$$

while the propagator is unchanged

$$\langle\varphi(z)\varphi(w)\rangle = -2\ln(z-w) \tag{2.69}$$

(we have slightly changed the definition of the scalar field $\varphi \to -\frac{i}{\sqrt{2}}\varphi$). It is easy to show that with respect to the Gaussian model the central charge is shifted from $c = 1$ to

$$c = 1 - 24\,\alpha_0^2 \tag{2.70}$$

Moreover, defining the vertex operator

$$V_\alpha(z) =: e^{i\alpha\varphi(z)} : \tag{2.71}$$

it is easy to prove the OPE

$$T(z)V_\alpha(w) = \frac{h_\alpha}{(z-w)^2}V_\alpha(w) + \frac{1}{z-w}V'_\alpha(w) + Reg \tag{2.72}$$

where

$$h_\alpha = \alpha^2 - 2\alpha_0\alpha \tag{2.73}$$

Thus V_α is a primary field of weight h_α. One can also prove that

$$: e^{i\alpha\varphi(z)} :: e^{i\beta\varphi(w)} := (z-w)^{2\alpha\beta} : e^{i(\alpha\varphi(z)+\beta\varphi(w))} : \tag{2.74}$$

In order to compute correlators one has to take into account that there is a charge $-2\alpha_0$ at infinity. This can be represented by a vertex operator $V_{-2\alpha_0}(z)$ inserted at infinity. Taking into account (2.31) we write, for instance, the two point function as follows[6]

$$\langle\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)\rangle\rangle = \lim_{z\to\infty} z^{8\alpha_0^2}\langle 0|V_{-2\alpha_0}(z)V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)|0\rangle \tag{2.75}$$

Using (2.74) this turns out to make sense and equal $(z_1 - z_2)^{2\alpha_1\alpha_2}$ only when $\alpha_1 + \alpha_2 - 2\alpha_0 = 0$. In fact inserting the operator $\frac{1}{2\pi i}\oint dw\,\partial\varphi(w)$ [7] in a generic correlator and using the OPE, one finds

$$\begin{aligned} &\langle 0|V_{-2\alpha_0}(z)V_{\alpha_1}(z_1)\dots V_{\alpha_n}(z_n)\frac{1}{2\pi i}\oint_C dz\,\partial\varphi(z)|0\rangle \qquad (2.76)\\ &= \frac{1}{2\pi i}\oint_C dw\left(\sum_{i=1}^n \frac{-2i\alpha_i}{w-z_i} + \frac{4i\alpha_0}{w-z}\right)\langle 0|V_{-2\alpha_0}(z)V_{\alpha_1}(z_1)\dots V_{\alpha_n}(z_n)|0\rangle = 0 \end{aligned}$$

because the contour C is around the origin and does not contain any of the point $z, z_1, \dots, z_n$. On the other hand, by reversing the picture we can understand this contour, with opposite orientation, as surrounding all these points. The result is

$$(2\alpha_0 - \sum_{i=1}^n \alpha_i)\lim_{z\to\infty} z^{8\alpha_0^2}\langle 0|V_{-2\alpha_0}(z)V_{\alpha_1}(z_1)\dots V_{\alpha_n}(z_n)|0\rangle = 0 \tag{2.77}$$

So, the correlator can be non-vanishing if and only if the total charge is conserved:

$$2\alpha_0 = \sum_{i=1}^n \alpha_i \tag{2.78}$$

[6] See Ch.3 for the relation of the Coulomb gas approach with the general method of bosonization.

[7] This charge is well defined even though $\partial\varphi$ is not strictly speaking a primary field of weight 1. This is due to the fact that under a conformal transformation $\delta_\epsilon\varphi = \partial(\epsilon\varphi + 2i\alpha_0\epsilon')$

In general

$$\langle\langle V_{\alpha_1}(z_1)\ldots V_{\alpha_n}(z_n)\rangle\rangle = \begin{cases} \prod_{i<j}(z_i - z_j)^{2\alpha_i\alpha_j}, & \sum_{i=1}^n \alpha_i = 2\alpha_0 \\ 0 & \sum_{i=1}^n \alpha_i \neq 2\alpha_0 \end{cases} \tag{2.79}$$

Notice that $h_\alpha = h_{2\alpha_0-\alpha}$. Therefore, for instance, we can write the four points function of an operator of weight h_α in various ways. However

$$\begin{array}{ccc} \langle\langle V_\alpha(z_1)\, V_\alpha(z_2)\, V_\alpha(z_3)\, V_\alpha(z_4)\rangle\rangle \neq 0, & \Rightarrow & \alpha = \frac{\alpha_0}{2} \\ \langle\langle V_\alpha(z_1)\, V_\alpha(z_2)\, V_\alpha(z_3)\, V_{2\alpha_0-\alpha}(z_4)\rangle\rangle \neq 0, & \Rightarrow & \alpha = 0 \\ \langle\langle V_\alpha(z_1)\, V_\alpha(z_2)\, V_{2\alpha_0-\alpha}(z_3)\, V_{2\alpha_0-\alpha}(z_4)\rangle\rangle \neq 0, & \Rightarrow & \alpha_0 = 0 \end{array}$$

all reduce to very particular or uninteresting cases. The way out is to introduce screening operators.

2.5.1 Screening operators

Screening operators S_α are defined by

$$S_\alpha = \frac{1}{2\pi i}\oint_C dz V_\alpha(z) \tag{2.80}$$

where C is a contour to be specified. This makes sense only if the conformal weight of the integrand is 1, i.e. if $h_\alpha = 1$. This has two solutions

$$\alpha = \alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 1} \tag{2.81}$$

So, for a given α_0, there are two screening operators

$$S_+ = S_{\alpha_+}, \qquad S_- = S_{\alpha_-}$$

These screening operators have vanishing weight but non-vanishing charge. Inserting a certain number of such operators does not change the total weight but changes the balance of charges. For instance, we can represent the four points functions of an operator of weight h_α as follows

$$\langle\langle V_\alpha(z_1)\, V_\alpha(z_2)\, V_\alpha(z_3)\, V_{2\alpha_0-\alpha}\,(z_4)\, S_+^{\tilde m}\, S_-^{\tilde n}\rangle\rangle \tag{2.82}$$

For it to be nonvanishing we must have

$$2\alpha + \tilde m\, \alpha_+ + \tilde n\, \alpha_- = 0 \tag{2.83}$$

This implies that the charge must be quantized

$$\alpha = \alpha_{n,m} = -\frac{1}{2}(\tilde m\, \alpha_+ + \tilde n\, \alpha_-) \equiv \frac{1}{2}((1-m)\, \alpha_+ + (1-n)\, \alpha_-) \tag{2.84}$$

The corresponding weight is

$$h_{n,m} = \alpha_{n,m}^2 - 2\,\alpha_0\,\alpha_{n,m} = \frac{1}{4}\left[(n\,\alpha_- + m\,\alpha_+)^2 - (\alpha_+ + \alpha_-)^2\right] \tag{2.85}$$

This reproduces the *Kac formula* for degenerate representations (up to the exchange $(n \leftrightarrow m)$:

$$\begin{aligned} h_{n,m}(c) &= \frac{1}{48}\Big[(13-c)(n^2+m^2) + (n^2-m^2)\sqrt{(c-1)(c-25)} \\ &\qquad -24nm + 2c - 2\Big] \end{aligned} \tag{2.86}$$

since $c = 1 - 24\alpha_0^2$, $\alpha_+\alpha_- = -1$ and $\alpha_+ + \alpha_- = 2\alpha_0$.

In other words (2.82) incorporates the information contained in a given singular vector, therefore it is a solution to the corresponding differential equation for correlators.

Remark. If in (2.85) we set

$$\frac{\alpha_-}{\alpha_+} = -\frac{p}{q} \tag{2.87}$$

where p, q are positive integers, we get

$$\begin{aligned} h_{n,m} &= \frac{1}{4}\left[(mp - nq)^2 - (p-q)^2\right] \\ c &= 1 - \frac{6(p-q)^2}{pq} \end{aligned}$$

i.e. we reproduce the formulas for the minimal series.

The monodromy problem

Let us consider the four points function for a minimal model primary field $\phi_{n,m}$:

$$\begin{aligned} &W(z_1, z_2, z_3, z_4) = \Big\langle \phi_{n,m}(z_1)\phi_{n,m}(z_2)\phi_{n,m}(z_3)\phi_{n,m}(z_4)\Big\rangle \\ &= \frac{1}{2\pi i}\oint_{C_1} du_1 \ldots \frac{1}{2\pi i}\oint_{C_{n-1}} du_{n-1}\frac{1}{2\pi i}\oint_{C_1'} dv_1 \ldots \frac{1}{2\pi i}\oint_{C_{m-1}'} dv_{m-1} \\ &\langle\!\langle V_{\alpha_{n,m}}(z_1)\, V_{\alpha_{n,m}}(z_2)\, V_{\alpha_{n,m}}(z_3)\, V_{2\alpha_0 - \alpha_{n,m}}(z_4) \\ &\qquad \cdot V_{\alpha_-}(u_1) \ldots V_{\alpha_-}(u_{n-1}) V_{\alpha_+}(v_1) \ldots\, V_{\alpha_+}(v_{m-1})\rangle\!\rangle \end{aligned} \tag{2.88}$$

where

$$2\alpha_{n,m} + (n-1)\alpha_- + (m-1)\alpha_+ = 0 \tag{2.89}$$

W depends of course on the contours $C_1, \ldots, C_{n-1}$ and $C'_1, \ldots, C'_{m-1}$, for which there are several possible choices: they may include one or more points $z_1, \ldots, z_4$. The problem that ensues is known as the *monodromy problem*.

Let us denote by an index $i = 1, \ldots, N$ the just mentioned distinct possibilities $W_i(z)$ (after setting as usual $z_1 = \infty, z_2 = 1, z_3 = z, z_4 = 0$). We will have as many antiholomorphic possibilities $\bar{W}_i(\bar{z})$. When going with z around the other points along a contour (labeled by l) we will have

$$W_i(z) \longrightarrow \sum_j (g_l)_{ij} W_j(z) \tag{2.90}$$

The matrices g_l define a representation of the monodromy group. A similar transformation hold for the antiholomorphic part. To recover locality of the four points correlation function one must find a combination

$$G(z, \bar{z}) = \sum_{ij} I_{ij} W_i(z) \bar{W}_j(\bar{z}) \tag{2.91}$$

invariant under the monodromy group.

2.6 Operator algebras and fusion algebras

For any two operators $\phi_n(z, \bar{z}), \phi_m(z, \bar{z})$ of weight $h_n, \bar{h}_m$ and $h_m, \bar{h}_m$, respectively we have the OPE

$$\begin{aligned}\phi_n(z, \bar{z})\phi_m(0,0) \;\; = \;\; & \sum_p \sum_{\{k\}} \sum_{\{\bar{k}\}} C^p_{n,m} \beta^{p;\{k\}}_{n,m}, \bar{\beta}^{p;\{\bar{k}\}}_{n,m} \; z^{h_p - h_n - h_m + \sum_i k_i} \\ & \cdot \bar{z}^{\bar{h}_p - \bar{h}_n - \bar{h}_m + \sum_i \bar{k}_i} \; \phi_p^{\{k\},\{\bar{k}\}}(0,0)\end{aligned}$$

where $C^p_{n,m}$ are constants that play the role of Clebsh-Gordan coefficients in the decomposition of the tensor product of two representations. β and $\bar{\beta}$ are constants related to the secondary fields. The summations are in general infinite. Requiring associativity

$$\Big(\phi_n(z, \bar{z})\phi_m(w, \bar{w})\Big)\phi_p(0,0) = \phi_n(z, \bar{z})\Big(\phi_m(w, \bar{w})\phi_p(0,0)\Big) \tag{2.92}$$

we have enough equations to determine the constants $C^p_{n,m}$. However this method is too cumbersome in general.

We can write a compact form of the operator algebra that involves conformal families, rather than fields. Denoting by $[\phi_n]$ the family with primary ϕ_n, the fusion algebra takes the form

$$[\phi_n] \times [\phi_m] \sim \sum_k C^k_{n,m} \, [\phi_k] \tag{2.93}$$

For families of the minimal series the summation is finite. For fields of weight (2.86) we have, for instance,

$$
\begin{aligned}
[\phi_{(1,2)}] \times [\phi_{(n,m)}] &= [\phi_{n,m+1}] + [\phi_{n,m-1}] \\
[\phi_{(2,1)}] \times [\phi_{(n,m)}] &= [\phi_{n+1,m}] + [\phi_{n-1,m}]
\end{aligned}
\tag{2.94}
$$

Example: the critical Ising model. As we have seen the critical Ising model has three different descriptions, represented by three set of primaries: $\{1, \psi, \bar{\psi}, \epsilon\}$,$\{1, \sigma, \epsilon\}$ and $\{1, \mu, \epsilon\}$.

The fusion rules are

$$
\begin{array}{ccc}
[\psi] \times [\psi] \sim [1], & [\psi] \times [\bar{\psi}] \sim [\epsilon], & [\epsilon] \times [\epsilon] \sim [1] \\
[\epsilon] \times [\psi] \sim [\bar{\psi}], & [\epsilon] \times [\bar{\psi}] \sim [\psi], & [\psi] \times [\mu] \sim [\sigma] \\
[\psi] \times [\sigma] \sim [\mu], & [\epsilon] \times [\sigma] \sim [\sigma], & [\sigma] \times [\sigma] \sim [1] + [\epsilon]
\end{array}
$$

Intermezzo. So far we have taken into consideration only CFT's characterized uniquely by being invariant with respect to the Virasoro algebra alone. As we have seen, this invariance implies such strong restrictions that it is possible, in the case of minimal models, to determine all the correlators. The absence of singular vectors for theories with $c \geq 1$, on the other hand, renders it impossible to apply the previously outlined methods to them. It is possible however to define predictive theories with $c \geq 1$, provided we enlarge their symmetry. In other words we will consider 2d theories with symmetry Lie algebras that include the Virasoro algebra as a particular subalgebra. Since they all split into a holomorphic and antiholomorphic subalgebras they are generically referred to as *chiral algebras*. They include in paricular Kac-Moody algebras and superconformal algebras. In the sequel we will consider theories invariant under $N = 1$ and $N = 2$ superconformal algebras.

2.7 $N = 1$ superconformal field theories

Let us consider first the generalization of a Gaussian field theory to the $N = 1$ supersymmetric case. The model consists of the usual scalar field φ and its superpartner, a Majorana spinor field, with two components $\chi, \bar{\chi}$. The action is

$$
S = \int d^2z (\varphi_z \varphi_{\bar{z}} - \chi\, \chi_{\bar{z}} - \bar{\chi}\, \bar{\chi}_z) \tag{2.95}
$$

This is invariant with respect to the supersymmetric transformations

$$\begin{aligned} &\delta_\kappa \varphi = \kappa\,\chi, \qquad \delta_\kappa \chi = \kappa\,\partial_z \varphi, \qquad \delta_\kappa \bar{\chi} = 0 \\ &\delta_{\bar\kappa} \varphi = \bar\kappa\,\bar\chi, \qquad \delta_{\bar\kappa} \chi = 0, \qquad \delta_{\bar\kappa} \bar\chi = \bar\kappa\,\partial_{\bar z} \varphi \end{aligned} \tag{2.96}$$

where κ and $\bar\kappa$ are anticommuting parameters (constant spinor components).

The holomorphic current and charge corresponding to the supersymmetry transformations are

$$J(z) = \chi(z)\partial_z\varphi(z), \qquad Q = \frac{1}{2\pi i}\oint dz\, J(z) \tag{2.97}$$

with $\delta_\kappa = \kappa[Q, \ \cdot\ ]$, while the holomorphic energy-momentum tensor is

$$T(z) = \frac{1}{2}\partial\varphi\,\partial\varphi - \frac{1}{2}\chi\partial\chi \tag{2.98}$$

Of course we have similar antiholomorphic expressions for $\bar J, \bar Q$ and $\bar T$.

Using the propagators for the bosonic Gaussian and the Majorana field model one can compute the OPE's of T and J and see that the central charge is $\frac{3}{2}$. Moreover

$$T(z)J(w) = \frac{3}{2}\frac{1}{(z-w)^2}J(w) + \frac{1}{z-w}J'(w) + \text{Reg} \tag{2.99}$$

$$J(z)J(w) = \frac{1}{(z-w)^3} + \frac{2}{z-w}T(w) + \text{Reg} \tag{2.100}$$

The first OPE implies that J has weight $\frac{3}{2}$.

The above relations for the superGaussian model can be generalized to a generic $N = 1$ superconformal model. Any such model will be defined by an em tensor $T(z)$ and a supercurrent $J(z)$ (with their antiholomorphic counterpart) satisfying the OPE's

$$T(z)T(w) = \frac{3}{4}\frac{\hat c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T(w)'}{z-w} + \text{Reg} \tag{2.101}$$

$$T(z)J(w) = \frac{3}{2}\frac{1}{(z-w)^2}J(w) + \frac{1}{z-w}J'(w) + \text{Reg} \tag{2.102}$$

$$J(z)J(w) = \frac{\hat c}{(z-w)^3} + \frac{2}{z-w}T(w) + \text{Reg} \tag{2.103}$$

It is customary to denote by $\hat c$ the supersymmetric central charge (the sum of the bosonic + $\frac{1}{2}$ fermionic degrees of freedom), so that the relation with the usual central charge is $c = \frac{3}{2}\hat c$.

Superconformal transformations are spanned by two arbitrary (holomorphic) functions, an ordinary bosonic $\epsilon(z)$ and an anticommuting one $\eta(z)$. The variations $\delta_\epsilon T$ is the same as in the bosonic case, $\delta_\epsilon J$ is the same as in the bosonic case for a field of weight $\frac{3}{2}$. The variations δ_η are as follows

$$\delta_\eta J(w) = \frac{1}{2\pi i}\oint_{C_w} dz\,\eta(z)J(z)J(w) = 2\eta(w)\,T(w) + \frac{\hat{c}}{2}\,\eta''(w) \tag{2.104}$$

$$\delta_\eta T(w) = \frac{1}{2\pi i}\oint_{C_w} dz\,\eta(z)J(z)T(w) = \frac{3}{2}\eta'(w)\,J(w) + \frac{1}{2}\,\eta(w)J'(w) \tag{2.105}$$

Using the analog of (2.8) we can extract from these equations the commutators of $J_\eta = \frac{1}{2\pi i}\oint_{C_0}\eta(w)\,J(w)$ with J and T. The latter are met more frequently in terms of modes. In this case, since fermions are involved, by analogy with the superGaussian model where J is linear in the Majorana field, we will have two sectors of the theory:

- the NS sector with $J(e^{2\pi i}z) = J(z)$;
- the R sector with $J(e^{2\pi i}z) = -J(z)$.

Accordingly, the Laurent expansion of J takes the form

$$J(z) = \sum_{\alpha\in\mathbb{Z}+\frac{1}{2}} J_\alpha\, z^{-\alpha-\frac{3}{2}}, \qquad \text{NS} \tag{2.106}$$

$$J(z) = \sum_{\alpha\in\mathbb{Z}} J_\alpha\, z^{-\alpha-\frac{3}{2}}, \qquad \text{R} \tag{2.107}$$

while for T it takes the usual form

$$T(z) = \sum_{n\in\mathbb{Z}} L_n\, z^{-n-2} \tag{2.108}$$

Likewise we can expand

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} \epsilon_{-n}\, z^{n+1}, \qquad \eta(z) = \sum_{\alpha} \eta_{-\alpha} z^{\alpha+\frac{1}{2}} \tag{2.109}$$

where $\alpha \in \mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$, according to the sector. Using (2.101,2.102,2.103) or, more directly, (2.104,2.105) we arrive at the algebras

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{\hat{c}}{8}(n^3 - n)\delta_{n+m,0} \tag{2.110}$$

$$[L_n, J_\alpha] = (\frac{1}{2}n - \alpha)J_{n+\alpha} \tag{2.111}$$

$$[J_\alpha, J_\beta] = 2L_{\alpha+\beta} + \frac{\hat{c}}{2}(\alpha^2 - \frac{1}{4})\delta_{\alpha+\beta,0} \tag{2.112}$$

2.7.1 Primary fields

A primary multiplet is specified by a couple of a bosonic field ϕ and a fermionic one λ, which are covariant with respect to the superconformal Lie algebra, i.e. they transform like

$$
\begin{aligned}
&\delta_\epsilon \phi(z) = \epsilon(z)\partial\phi(z) + h\,\epsilon'(z)\,\phi(z) \\
&\delta_\epsilon \lambda(z) = \epsilon(z)\partial\lambda(z) + (h+\frac{1}{2})\epsilon'(z)\,\lambda(z) \\
&\delta_\eta \phi(z) = \eta(z)\,\lambda(z), \qquad \delta_\eta \lambda(z) = \eta(z)\,\phi'(z) + 2h\,\eta'(z)\,\phi(z)
\end{aligned}
\tag{2.113}
$$

The first two lines are obvious, the third corresponds to the OPE:

$$
J(z)\phi(w) = \frac{1}{z-w}\lambda(w) + \ldots \tag{2.114}
$$

$$
J(z)\lambda(w) = \frac{2h}{(z-w)^2}\phi(w) + \frac{1}{z-w}\phi'(w) + \ldots \tag{2.115}
$$

or

$$
[J_\alpha, \phi(w)] = w^{\alpha+\frac{1}{2}}\lambda(w) \tag{2.116}
$$

$$
[J_\alpha, \lambda(w)]_+ = w^{\alpha+\frac{1}{2}}\phi'(w) + 2\,h\,(\alpha+\frac{1}{2})w^{\alpha-\frac{1}{2}}\phi(w). \tag{2.117}
$$

Now let us set $L_n^\dagger = L_{-n}$, $J_\alpha^\dagger = J_{-\alpha}$ and define the $SL(2|2,\mathbb{C})$ invariant vacuum $|0\rangle$ for the NS sector by:

$$
\begin{aligned}
L_n|0\rangle = 0 = \langle 0|L_{-n}, \qquad & n \geq -1 \\
J_\alpha|0\rangle = 0 = \langle 0|J_{-\alpha}, \qquad & \alpha \geq -\frac{1}{2}
\end{aligned}
$$

It is easy to verify that, setting

$$
|\phi\rangle = \phi(0)|0\rangle, \qquad |\lambda\rangle = \lambda(0)|0\rangle, \tag{2.118}
$$

we get

$$
L_0|\phi\rangle = h|\phi\rangle, \qquad L_0|\lambda\rangle = (h+\frac{1}{2})|\lambda\rangle \tag{2.119}
$$

and

$$
\begin{aligned}
&J_{\frac{1}{2}}|\phi\rangle = 0, \qquad J_{-\frac{1}{2}}|\phi\rangle = |\lambda\rangle \\
&J_{\frac{1}{2}}|\lambda\rangle = 2h|\phi\rangle, \qquad J_{-\frac{1}{2}}|\lambda\rangle = L_{-1}|\phi\rangle,
\end{aligned}
\tag{2.120}
$$

while

$$
\begin{aligned}
L_n|\phi\rangle = 0 = L_n|\lambda\rangle, \qquad & n > 0 \\
J_\alpha|\phi\rangle = 0 = J_\alpha|\lambda\rangle, \qquad & \alpha > \frac{1}{2}
\end{aligned}
\tag{2.121}
$$

Eqs.(2.119,2.120,2.121) are the definition of a $N = 1$ superconformal LWV.

2.7.2 Singular vectors

By definition a *singular* or *null* vector satisfies

$$L_n|\chi\rangle = 0, \quad n > 0; \qquad J_\alpha|\chi\rangle = 0, \quad \alpha \geq \frac{1}{2} \tag{2.122}$$

Starting with a primary of weight h the first singular vector is met at level $\frac{3}{2}$, for:

$$|\chi\rangle = \big(L_{-1}J_{-\frac{1}{2}} - (h+\frac{1}{2})J_{-\frac{3}{2}}\big)|\phi\rangle \tag{2.123}$$

It does satisfy (2.122) provided

$$h = \frac{3-\hat{c} \pm \sqrt{(3-\hat{c})^2 - 4\hat{c}}}{4} \tag{2.124}$$

The corresponding singular field is

$$\chi(z) = \left(L_{-1}(z)J_{-\frac{1}{2}}(z) - (h+\frac{1}{2})J_{-\frac{3}{2}}(z)\right)\phi(z) \tag{2.125}$$

where $L_{-1}(z)$ was defined above and

$$J_{-\alpha}(z) = \frac{1}{2\pi i}\oint_{C_z} dw\, \frac{J(w)}{(w-z)^{\alpha-\frac{1}{2}}}$$

At this point one can repeat the analysis of LWR's made for the bosonic case and also introduce a super-Coulomb gas representation. The parameters are

$$\hat{c} = 1 - 16\alpha_0^2, \qquad \alpha_\pm = \frac{1}{4}\left(\sqrt{1-\hat{c}} \pm \sqrt{9-\hat{c}}\right) \tag{2.126}$$

Singular fields are met at

$$h_{n,m} = \frac{1}{4}\left(n\alpha_+ + m\alpha_-\right)^2 - \frac{1}{16}(1-\hat{c}) \tag{2.127}$$

Parametrizing

$$\hat{c} = 1 - \frac{8}{p(p+2)} \tag{2.128}$$

we get

$$h_{n,m} = \frac{1}{8p(p+2)}\left[\big(n(p+2) - mp\big)^2 - 4\right] \tag{2.129}$$

The corresponding LWR's are *unitary* provided

$$p = 3, 4, 5, \ldots \quad 1 \leq n \leq p-1, \quad 1 \leq m \leq p+1, \tag{2.130}$$

where

- $n - m$ even, in the NS sector,
- $n - m$ odd, in the R sector.

The case $p = 3$ corresponds to $\hat{c} = \frac{7}{15}$, i.e. $c = \frac{7}{10}$. Primary weights are $h = 0, \frac{1}{10}, 1$. This corresponds to the second minimal unitary model of the bosonic series ($p = 4$) and describes the field content of the tricritical Ising model.

2.8 $N = 2$ superconformal field theories

In the $N = 2$ case we have two supercurrents $G^+(z), G^-(z)$ instead of one, as in the $N = 1$ case. In addition the supersymmetry algebra requires the presence of a $U(1)$ current $J(z)$. Setting

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \quad G^\pm(z) = \sum_{\alpha\in\tilde{\mathbb{Z}}} G^\pm_\alpha z^{\alpha-\frac{3}{2}}, \quad J(z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1} \tag{2.131}$$

where

$$\tilde{\mathbb{Z}} = \begin{cases} \mathbb{Z} & \text{R} \\ \mathbb{Z} + \frac{1}{2} & \text{NS} \end{cases}$$

we obtain the following algebra

$$\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m} + \frac{\tilde{c}}{4}\,(n^3 - n)\,\delta_{n+m,0} \\
&[G^+_\alpha, G^-_\beta]_+ = 2L_{\alpha+\beta} + (\alpha-\beta)\,J_{\alpha+\beta} + \frac{\tilde{c}}{4}\left(4\alpha^2 - 1\right)\delta_{\alpha+\beta,0} \\
&[L_n, G^\pm_\alpha] = \left(\frac{1}{2} - \alpha\right) G^\pm_{n+\alpha}, \qquad [J_n, G^\pm_\alpha] = \pm G^\pm_{n+\alpha} \\
&[L_n, J_m] = -m\,J_{n+m}, \qquad [J_n, J_m] = \tilde{c}\,n\,\delta_{n+m,0}
\end{aligned} \tag{2.132}$$

The reason for the form of the central charge in the first commutator is that $\tilde{c}$ counts the number of supermultiplets (2 bosons + 2 fermions), so the relation with the ordinary bosonic central charge is $c = 3\,\tilde{c}$.

The vacuum is defined by

$$L_n|0\rangle = 0, \quad n \geq -1, \qquad G^\pm_\alpha|0\rangle = 0, \quad \alpha \geq -\frac{1}{2}, \qquad J_n|0\rangle = 0, \quad n \geq 0$$

A LWV (primary state) is defined by a triple of states $|\phi\rangle, |\lambda^\pm\rangle$ as follows

$$\begin{aligned}
&L_n|\phi\rangle = 0, \quad n > 0, \qquad L_0|\phi\rangle = h\,|\phi\rangle \\
&G^\pm_\alpha|\phi\rangle = 0, \quad \alpha > 0, \qquad G^\pm_{-\frac{1}{2}}|\phi\rangle = |\lambda_\pm\rangle \\
&L_0|\lambda_\pm\rangle = (h + \frac{1}{2})|\lambda_\pm\rangle
\end{aligned} \tag{2.133}$$

Since $[L_0, J_0] = 0$, $|\phi\rangle$ can be chosen to be an eigenvector of J_0:

$$J_0|\phi\rangle = q|\phi\rangle \tag{2.134}$$

The analysis of LWR's, singular fields and Coulomb gas can be repeated as in the previous cases. As a result here are the unitary series for the NS sector ($\alpha, \beta \in \mathbb{Z} + \frac{1}{2}$):

- $\tilde{c} = 1 - \frac{2}{p+2} \longrightarrow c = \frac{3p}{p+2}, \qquad p \geq 1$
- $h_{\alpha,\beta} = \frac{\alpha\beta - \frac{1}{4}}{p+2}, \qquad 0 < \alpha, \beta, \alpha + \beta \leq p + 1$
- $q = \frac{\alpha - \beta}{p+2}$.

The weight and $U(1)$ charge are often written in the form

- $h = \frac{l(l+2) - m^2}{4(p+2)}, \qquad l = 0, 1, 2, \ldots, p$
- $q = \frac{m}{p+2}, \qquad m = -l, -l + 2, \ldots, l - 2, l$

Here $m, l \in \mathbb{Z}$.
The unitary series for the R sector ($\alpha, \beta \in \mathbb{Z}$) is

- $\tilde{c} = 1 - \frac{2}{p+2} \longrightarrow c = \frac{3p}{p+2}, \qquad p \geq 1$
- $h_{\alpha,\beta} = \frac{\tilde{c}}{8} + \frac{\alpha\beta}{p+2}$
- $q = \frac{n-m}{p+2}, \qquad 0 \leq \alpha - 1, \beta, \alpha + \beta \leq p + 1.$

There is also a twisted unitary minimal series.

2.8.1 Chiral ring in $N = 2$ SCFT's

The definition (2.133) corresponds to the following OPE's

$$\begin{aligned} T(z)\Phi(w) &= \frac{h}{(z-w)^2}\phi(w) + \frac{1}{z-w}\phi'(w) + ... \\ J(z)\phi(w) &= \frac{q}{z-w}\phi(w) + ... \\ G^{\pm}(z)\phi &= \frac{1}{z-w}\lambda^{\pm}(w) + ... \end{aligned} \tag{2.135}$$

The field $\phi(z)$ is *chiral* if

$$G^{+}_{-\frac{1}{2}}\phi(z) = 0 \tag{2.136}$$

this corresponds to

$$G^+_{-\frac{1}{2}}|\phi\rangle = 0, \qquad G^+(z)\phi(w) = \mathrm{Reg} \tag{2.137}$$

Now lets us remark that $((G^\pm_\alpha)^\dagger = G^\mp_{-\alpha})$

$$\begin{aligned} 0 &\leq \|G^\pm_{\frac{1}{2}}|\phi\rangle\|^2 + \|G^\mp_{-\frac{1}{2}}|\phi\rangle\|^2 = \langle\phi|[G^\pm_{\frac{1}{2}}, G^\mp_{-\frac{1}{2}}]_+|\phi\rangle \\ &= \langle\phi|2L_0 \pm J_0|\phi\rangle = 2h + q \end{aligned}$$

that is

$$2h + q \geq 0, \qquad \text{or} \qquad h \geq \frac{1}{2}|q| \tag{2.138}$$

The *chiral ring* $\mathcal{R}$ is formed by the set of fields for which

$$h = \frac{1}{2}q > 0 \tag{2.139}$$

For, if this is true, we have

$$\|G^-_{\frac{1}{2}}|\phi\rangle\|^2 + \|G^+_{-\frac{1}{2}}|\phi\rangle\|^2 = \langle\phi|[G^-_{\frac{1}{2}}, G^+_{-\frac{1}{2}}]_+|\phi\rangle = 2h - q = 0 \tag{2.140}$$

i.e.

$$G^-_{\frac{1}{2}}|\phi\rangle = 0 = G^+_{-\frac{1}{2}}|\phi\rangle \tag{2.141}$$

On the other hand, if $|\phi\rangle$ is a chiral primary state for which (2.138) is true we have

$$0 = \langle\phi|[G^-_{\frac{1}{2}}, G^+_{-\frac{1}{2}}]_+|\phi\rangle = 2h - q \tag{2.142}$$

Thus (2.139) is satisfied and a chiral ring can be characterized either by (2.136) or (2.139).

Similarly we have an antichiral ring $\bar{\mathcal{R}}$ characterized by

$$h = -\frac{1}{2}q \quad \Longleftrightarrow \quad G^-_{-\frac{1}{2}}|\phi\rangle = 0 \tag{2.143}$$

The reason why the chiral ring is called ring is the following one. It can be shown that if ϕ_i, ϕ_j are two chiral fields, then, either

$$\lim_{z\to w} \phi_i(z)\phi_j(w) = 0 \tag{2.144}$$

or

$$\lim_{z\to w} \phi_i(z)\phi_j(w) = \phi_k(w) \tag{2.145}$$

where ϕ_k is a chiral field. Thus, from (2.144,2.145), we can define on $\mathcal{R}$ a product for which we have

$$\begin{aligned} &\text{either} \qquad \phi_i \cdot \phi_j = 0 \\ &\text{or} \qquad \phi_i \cdot \phi_j = \phi_k \end{aligned}$$

that is a product that closes on $\mathcal{R}$.

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Chapter 3

Bosonization in 2d

The chiral splitting characteristic of conformal field theories represents an enormous simplification in the calculation of conformal correlators. One of the most prominent tools allowed by chiral splitting is *bosonization*, i.e. the possibility to represent primary fields by means of elementary bosonic fields. This technique is not exclusive of 2d, but it becomes particularly powerful in the framework of the holomorphic-antiholomorphic splitting of conformal field theories. The term bosonization is somewhat misleading, because it may seem to apply only to fermionic field. In fact we can bosonize not only fermion fields, but also, if not principally, bosonic fields, representing them by means of *vertex operators*.

3.1 Modification of the Gaussian Model

The starting point of bosonization is the Gaussian model introduced in Part I, see eq.(2.18). While the propagator remains the same (2.17), we modify the holomorphic energy momentum tensor as follows[1]

$$T(z) = \frac{1}{2} : \partial_z\varphi\partial_z\varphi(z) : -\beta_0\partial_z^2\varphi(z) = \frac{1}{2} : j(z)^2 : -\frac{Q}{2}\partial_z j(z) \tag{3.2}$$

where $j(z) = \partial_z\varphi(z)$. The number β_0 is a real number related to Q, by the relation $\beta_0 = -\frac{Q}{2}$, and Q is interpreted as a background charge. Calculating the OPE of T with itself one can compute the central charge of this model

$$c_{\beta_0} = 1 - 12\beta_0^2 \tag{3.3}$$

[1]The energy-momentum tensor (3.2) can be derived from the action

$$S_\varphi = \frac{1}{2}\int d^2z\,\sqrt{h}\left(h^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - \beta_0 R\varphi\right) \tag{3.1}$$

where R is the Ricci curvature of $h_{\alpha\beta}$.

One next introduces the vertex operators

$$V_\beta(z) =: e^{-\beta\varphi(z)} : \tag{3.4}$$

β is interpreted as the charge of the vertex operator. In fact let us define the canonical charge operator $q = -\frac{1}{2\pi i}\oint dz j(z)$, then the OPE

$$j(z)V_\beta(w) = \frac{\beta}{z-w}V_\beta(w) + \mathrm{Reg}$$

implies

$$[q, V_\beta(z)] = \beta V_\beta(z) \tag{3.5}$$

Computing the OPE of V_β with T one finds that it is a primary field of weight

$$h_\beta = \frac{\beta^2}{2} - \beta_0\beta \tag{3.6}$$

It is evident from (3.3) and (3.5) that this formalism provides a simple way to represent the primary operators of weight h_β in a theory with central charge c_{β_0}. But before going to the applications let us explain why we interpret Q as a background charge.

3.1.1 The background charge

To analyze this point it is opportune to expand φ in oscillators

$$\varphi(z) = ix_0 + (j_0 - \beta_0)\ln z - \sum_{n\neq 0}\frac{j_n}{n}z^{-n} \tag{3.7}$$

which corresponds to

$$j(z) = \sum_n (j_n - \beta_0\delta_{n,0})z^{-n-1} \tag{3.8}$$

Next we expand T in Virasoro modes, $T(z) = \sum_n L_n z^{-n-2}$. We get

$$L_n = \frac{1}{2}\sum_k j_{n-k}j_k + \beta_0 n j_n - \frac{1}{2}\beta_0^2\delta_{n,0} \tag{3.9}$$

We quantize the theory by imposing the Dirac brackets

$$[j_n, j_m] = n\delta_{n+m,0}, \qquad [x_0, j_0] = i \tag{3.10}$$

After normal ordering (3.9) one can compute

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}(1-12\beta_0^2)(n^3-n)\delta_{n+m,0} \tag{3.11}$$

Similarly, defining the normal ordered vertex

$$V_\beta(z) = e^{\beta\sum_{n<0} j_n \frac{z^{-n}}{n}} e^{-i\beta x_0} e^{\beta(-j_0+\beta_0)\ln z} e^{\beta\sum_{n>0} j_n \frac{z^{-n}}{n}} \tag{3.12}$$

one can verify

$$[L_n, V_\beta(z)] = z^{n+1}\partial_z V_\beta(z) + h_\beta\,(n+1)z^n V_\beta(z) \tag{3.13}$$

as appropriate for a primary field.

The next step is to analyze the vacuum of this theory, which is herein denoted by $|\tilde{0}\rangle$. As usual we have $j_n|\tilde{0}\rangle = 0$ for $n > 0$. Then we set $j_0|\tilde{0}\rangle = \lambda|\tilde{0}\rangle$ and try to identify λ. To this end we require $|\tilde{0}\rangle$ to be SL_2 invariant: $L_n|\tilde{0}\rangle = 0$ for $n \geq -1$. In particular this leads to

$$\begin{aligned} L_0|\tilde{0}\rangle &= \frac{1}{2}\left(j_0^2 - \beta_0^2\right)|\tilde{0}\rangle = 0 \\ L_{-1}|\tilde{0}\rangle &= (j_0 - \beta_0)\,j_{-1}|\tilde{0}\rangle = 0 \end{aligned}$$

i.e. $\lambda = \beta_0$. We cannot assume j_0 to be selfadjoint, but $j_n^\dagger = j_{-n}$ for $n \neq 0$ and $L_n^\dagger(\beta_0, j_0) = L_{-n}(-\beta_0, j_0^\dagger)$. Imposing that also the dual vacuum be SL_2 invariant, i.e. $\langle 0|L_n = 0$, for $n \leq 1$, we get

$$\langle\tilde{0}|j_0 = -\beta_0\langle\tilde{0}| \tag{3.14}$$

This implies that $\langle\tilde{0}|\tilde{0}\rangle = 0$, but $\langle\tilde{0}|e^{-2i\beta_0 x_0}|\tilde{0}\rangle \neq 0$. We assume

$$\langle\tilde{0}|e^{-2i\beta_0 x_0}|\tilde{0}\rangle = 1 \tag{3.15}$$

Let us remark that

$$[q, e^{-2i\beta_0 x_0}] = 2\beta_0 e^{-2i\beta_0 x_0} = -Qe^{-2i\beta_0 x_0}. \tag{3.16}$$

This means that the vacuum carries a charge $Q = 2\beta_0$. In order to obtain a non vanishing vev, we have to soak it with insertions having the opposite charge.

3.1.2 Vertex operator correlators

The previous result is clearly visible if we compute the vev of any product of vertex operators $\langle 0|V_{\beta_1}(z_1)\ldots V_{\beta_n}(z_n)|\tilde{0}\rangle$. Inserting j_0 and repeatedly using

$$[j_0, V_\beta(z)] = -\beta V_\beta(z) \tag{3.17}$$

we get

$$
\begin{aligned}
&-\beta_0\langle\tilde{0}|V_{\beta_1}(z_1)\dots V_{\beta_n}(z_n)|\tilde{0}\rangle = \langle 0|j_0\, V_{\beta_1}(z_1)\dots V_{\beta_n}(z_n)|\tilde{0}\rangle \\
&= -\sum_{i=1}^{n}\beta_i\langle\tilde{0}|V_{\beta_1}(z_1)\dots V_{\beta_n}(z_n)|\tilde{0}\rangle + \langle\tilde{0}|V_{\beta_1}(z_1)\dots V_{\beta_n}(z_n)j_0|\tilde{0}\rangle
\end{aligned}
$$

I.e. $\langle\tilde{0}|V_{\beta_1}(z_1)\dots V_{\beta_n}(z_n)|\tilde{0}\rangle \neq 0$, if and only if $\sum_{i=1}^n \beta_i = 2\beta_0$. This vev can be explicitly evaluated by using, for instance, the normal ordered expression (3.12) and commuting the creation operators to the left. One easily obtains

$$
\begin{aligned}
\langle\tilde{0}|V_{\beta_1}(z_1)\dots V_{\beta_n}(z_n)|\tilde{0}\rangle &= \prod_{i<j}(z_i - z_j)^{\beta_i\beta_j}\langle\tilde{0}|e^{-i(\beta_1+\dots+\beta_n)x_0}|\tilde{0}\rangle \\
&= \prod_{i<j}(z_i - z_j)^{\beta_i\beta_j}\,\delta\left(\sum_{i=1}^{n}\beta_i - 2\beta_0\right)
\end{aligned}
\tag{3.18}
$$

The background charge $Q = -2\beta_0$ can be conveniently located at infinity with the following trick. Using again (3.12) one remarks that

$$
\lim_{|z|\to\infty} z^{2h_\beta}\langle 0|V_\beta(z) = \langle\tilde{0}| \tag{3.19}
$$

where $\langle 0|$ is the ordinary vacuum. Then one defines

$$
\langle\!\langle V_{\beta_1}(z_1)\dots V_{\beta_n}(z_n)\rangle\!\rangle = \lim_{|z|\to\infty} z^{2h_\beta}\langle 0|V_\beta(z)V_{\beta_1}(z_1)\dots V_{\beta_n}(z_n)|0\rangle \tag{3.20}
$$

with $\beta = -2\beta_0$. With this definition we have

$$
\langle\!\langle V_{\beta_1}(z_1)\dots V_{\beta_n}(z_n)\rangle\!\rangle = \prod_{i<j}(z_i - z_j)^{\beta_i\beta_j}\,\delta\left(\sum_{i=1}^{n}\beta_i - 2\beta_0\right) \tag{3.21}
$$

We can also set

$$
\lim_{|z|\to\infty} z^{2h_\beta}\langle\tilde{0}|V_\beta(z) = \langle 0| \tag{3.22}
$$

with $\beta = 2\beta_0$ and define

$$
\langle V_{\beta_1}(z_1)\dots V_{\beta_n}(z_n)\rangle \equiv \langle 0|V_{\beta_1}(z_1)\dots V_{\beta_n}(z_n)|0\rangle = \prod_{i<j}(z_i - z_j)^{\beta_i\beta_j}\,\delta\left(\sum_{i=1}^{n}\beta_i\right) \tag{3.23}
$$

3.1.3 Bosonization of various fields

Bosonization of Majorana fermions

Superstring theory is formulated in terms of 2d Majorana fermions, see (2.19). The latter are often usefully bosonized. To do so one assembles the Majorana fields in couples to form complex spinors. For instance ψ_1 and ψ_2 form

$$\psi(z) = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2), \qquad \bar{\psi}(z) = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2) \tag{3.24}$$

The OPE of the latter are as follows

$$\psi(z)\bar{\psi}(0) \sim \frac{1}{z}, \qquad \psi(z)\psi(0) \sim O(z), \qquad \bar{\psi}(z)\bar{\psi}(0) \sim O(z) \tag{3.25}$$

The central charge of $\psi(z)$ is 1, and the weight $\frac{1}{2}$. Therefore in order to bosonize it we introduce the Gaussian model above based on the bosonic field φ, with $\beta_0 = 0$ and $\psi = e^{\varphi(z)}$, i.e. $\beta_\psi = 1$. The customary notation is somewhat different, one uses $H(z) = -i\varphi(z)$ and

$$\psi(z) = e^{iH(z)}, \qquad \bar{\psi}(z) = e^{-iH(z)} \tag{3.26}$$

In this way one has $c = 1$ and $h_\psi = h_{\bar{\psi}} = \frac{1}{2}$. Using the propagator (2.17) one can easily prove the OPE (3.25).

One can also reproduce the anticommutation properties of ψ and $\bar{\psi}$. Using (3.12), the product of two vertex operators $V_\alpha(z)$ and $V_\beta(w)$ can be rearranged as

$$V_\alpha(z)V_\beta(w) = (z-w)^{\alpha\beta} \,:e^{-\alpha\varphi(z)}e^{-\beta\varphi(w)}: \tag{3.27}$$

where $|z| > |w|$. We can next consider the inverted product $V_\beta(w)V_\alpha(z)$ and compare the two RHS's by analytically continuation. We get

$$V_\alpha(z)V_\beta(w) = (-1)^{\alpha\beta}V_\beta(w)V_\alpha(z) \tag{3.28}$$

Therefore if $\alpha\beta = \pm 1$ the two vertex operators anticommute. This is precisely the case of $e^{iH(z)}$ and $e^{-iH(z)}$ above.

Bosonization of b, c ghosts

The b, c ghosts of bosonic string theory are a free field system with central charge $c = -26$ and weight 2 and -1, respectively. In this case a background charge $Q = -3$ is definitely needed, so that $c = 1 - 12\left(\frac{3}{2}\right)^2 = -26$. Let us call the bosonizing scalar field χ. Then if we set

$$c(z) =: e^{\chi(z)} :, \qquad b(z) =: e^{-\chi(z)} : \tag{3.29}$$

one can verify that $h_c = -1$ and $h_b = 2$. Moreover they anticommute for the same reason as before. Using (2.17) one can easily prove that

$$\langle c(z)b(w)\rangle = \frac{1}{z-w} \tag{3.30}$$

where $\langle \quad \rangle$ is the notation (3.23).

Bosonization of β, γ ghosts

In superstring theory the commuting ghosts $\beta(z)$ and $\gamma(z)$ make their appearance. They have central charge 11 and weight $\frac{3}{2}$ and $-\frac{1}{2}$, respectively. The first remark is that the central charge is greater than 1, therefore the formula (3.3) is not fit in this case. We will proceed by making the following changes:

$$Q \to iQ, \qquad \beta_0 \to i\kappa_0, \qquad \varphi \to i\phi, \qquad \beta \to i\kappa \tag{3.31}$$

So $\langle \phi(z)\phi(w)\rangle = -\ln(z-w)$, and

$$T(z) = -\frac{1}{2} : \partial_z\phi\partial_z\phi(z) : +\kappa_0\partial_z^2\phi(z) = -\frac{1}{2}\left(: j(z)^2 : -Q\partial_z j(z)\right) \tag{3.32}$$

where $j(z) = -\partial_z\phi(z)$. The number κ_0 is a real number with $\kappa_0 = -\frac{Q}{2}$. The central charge of this model is

$$c_{\kappa_0} = 1 + 12\kappa_0^2 \tag{3.33}$$

The vertex operators are

$$V_\kappa(z) =: e^{\kappa\phi(z)} : \tag{3.34}$$

whose weight is $h_\kappa = -\frac{1}{2}\kappa^2 + \kappa_0\kappa$.

Contrary to the previous case, however, it is not possible to satisfy the requirements with one single bosonizing field and rational weights. The way out is to split the commuting β, γ system into two anticommuting subsystems. One is the ξ, η anticommuting system, where the weights are 0 and 1, respectively, and the central charge is -2. This system is the analog of the b, c system, with different weights. It has the same propagator $\langle \eta(z)\xi(w)\rangle = \frac{1}{z-w}$, and can be itself bosonized (although we will not do it). The second anticommuting system is obtained via bosonization. We set $Q = 2, \kappa_0 = -1$, so that its central charge is 13. Then we use $: e^{\phi} :$ and $: e^{-\phi} :$, whose weights are $-\frac{3}{2}, \frac{1}{2}$ respectively. So the central charge of the overall system is $13 - 2 = 11$ and

$$\beta(z) =: e^{-\phi(z)}\partial\xi(z) :, \qquad \gamma(z) =: e^{\phi(z)}\eta(z) : \tag{3.35}$$

have weight $\frac{3}{2}, -\frac{1}{2}$, respectively. They are commuting operators and their propagator is

$$\langle \gamma(z)\beta(w) \rangle = \frac{1}{z-w} \tag{3.36}$$

with the notation (3.23), as expected.

The Coulomb gas

The Coulomb gas method of section 2.5 is a particular example of bosonization. To make contact with section 2.5 one must make the following changes:

$$\begin{aligned}
&\varphi \to -\frac{i}{\sqrt{2}}\varphi && (3.37)\\
&\beta_0 \to \sqrt{2}\alpha_0 &&\\
&\beta \to \sqrt{2}\alpha && (3.38)\\
&|\tilde{0}\rangle \to |0\rangle && (3.39)\\
&\langle\tilde{0}| \to \langle 0|e^{-2i\alpha_0 x_0} &&
\end{aligned}$$

Bibliography

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Part II

Conformal field theory in d dimensions

The second part of the lecture notes is devoted to conformal field theories in d dimensions, with particular attention and explicit applications to $d = 3$ and $d = 4$. The difference between the case $d = 2$ and the case $d \geq 3$ is mostly due to the difference of the conformal groups. In $2d$ it is infinite-dimensional, while for $d \geq 3$ the dimension of the conformal group is finite. It is in fact isomorphic to the group $SO(d, 2)$ (or $SO(d+1, 1)$) when the metric is Minkowskian (Euclidean). It is nevertheless possible to repeat to a large extent the two-dimensional approach: radial quantization, OPE, state-operator correspondence, lowest weight representations of the conformal algebra, all such tools can be generalized to higher dimensions. An additional powerful tool is introduced in $d \geq 3$, the *null-cone method*. It is outrun by more powerful tools in 2d, but it becomes precious in higher dimensions. It is based on the remark that the conformal d-dimensional group is isomorphic to the pseudo-orthogonal group in d+2 dimensions. Since the SCT transformation is a complicated transformation that does not allow for any simple intuitive way to construct covariants, while the pseudo-orthogonal (Lorentz) covariants are easy to construct, it is natural to start with very simple and compact expressions in d+2 dimensions and try to project them to d dimensions while preserving their relevant covariant properties. This is done by restricting the d+2 dimensional expressions to a suitable section of the light-cone in d+2 (from which the name). It is possible to derive in this way several remarkable results for correlators.

In general the high expectations for solutions of CFT's in any dimension arise from the *bootstrap* idea, also called *OPE associativity*. This is based on OPE and the crossing symmetric properties of amplitudes. In particular we can expand a four-point amplitude in a superposition of two-point amplitudes by using OPE in any two distinct couples of fields. Eventually the different expressions we obtain in this way must be the same because of crossing symmetry. This puts restrictions on the expansion coefficients, which may help to determine them.

Chapter 4

Conformal algebra representations

Conformal transformations and conformal algebra in d dimensions have been introduced in section 1.1. The representations of the CA algebra we are interested in are those in terms of local fields. Fields that transform according to an irreducible representation of the CA are called *primary*. In particular a primary field has a definite weight Δ. For a generic scalar or fermion primary field $\mathcal{O}(x)$ of weight Δ (we understand the tensor or spinor labels) we have

$$\begin{aligned}
&[P_\mu, \mathcal{O}(x)] = -i\partial_\mu O(x) \\
&[L_{\mu\nu}, \mathcal{O}(x)] = -i(x_\mu\partial_\nu - x_\nu\partial_\mu)\mathcal{O}(x) + \Sigma_{\mu\nu}\mathcal{O}(x) \\
&[D, \mathcal{O}(x)] = -i(\Delta + x^\mu\partial_\mu)\mathcal{O}(x) \\
&[K_\mu, \mathcal{O}(x)] = -i(2\Delta x_\mu + 2x_\mu x^\lambda\partial_\lambda - x^2\partial_\mu - 2ix^\lambda\Sigma_{\lambda\mu})\mathcal{O}(x)
\end{aligned} \tag{4.1}$$

where $\Sigma_{\mu\nu}$ are the spinor part of the Lorentz generators.

For a current and the e.m. tensor, in particular, the explicit form of the commutators is

$$\begin{aligned}
&[D, J_\mu(x)] = -i(d - 1 + x^\lambda\partial_\lambda)J_\mu(x) \\
&[K_\lambda, J_\mu] = -i\left(2(d-1)x_\lambda + 2x_\lambda\, x\cdot\partial - x^2\partial_\lambda\right) J_\mu \\
&\qquad -2i\left(x^\alpha J_\alpha\eta_{\lambda\mu} - x_\mu J_\lambda\right)
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&[D, T_{\mu\nu}(x)] = -i(d + x^\lambda\partial_\lambda)T_{\mu\nu}(x) \\
&[K_\lambda, T_{\mu\nu}] = -i\left(2dx_\lambda + 2x_\lambda\, x\cdot\partial - x^2\partial_\lambda\right) T_{\mu\nu} \\
&\qquad -2i\left(x^\alpha T_{\alpha\nu}\eta_{\lambda\mu} + x^\alpha T_{\mu\alpha}\eta_{\lambda\nu} - x_\mu T_{\lambda\nu} - x_\nu T_{\mu\lambda}\right)
\end{aligned} \tag{4.3}$$

For later use let us remark that if we disregard the spin part, the last equation in (4.1) for infinitesimal b_μ, can be viewed as a local rescaling, for it is equivalent to

$$\tilde{\mathcal{O}}(x^\mu) \approx (1 + 4b\cdot x)^{\frac{\Delta}{2}}\mathcal{O}(x^\mu - b^\mu x^2 + 2b\cdot xx^\mu) \tag{4.4}$$

Note. The commutators (4.1,4.2,4.3) can be obtained, much like in 2d, as follows. For any tensorial primary field $\Phi_{\mu_1\ldots\mu_n}$ of weight Δ, one can write the invariance relation

$$\tilde{\Phi}_{\mu_1\ldots\mu_n}(x)dx^{\mu_1}\ldots dx^{\mu_n}(ds)^{\Delta-n} = \Phi_{\mu_1\ldots\mu_n}(x')dx'^{\mu_1}\ldots dx'^{\mu_n}(ds')^{\Delta-n} \tag{4.5}$$

for any transformation $x^\mu \to x'^\mu$. If the transformation is a sct generated by K_μ, for instance, we can then write

$$e^{ibK}\Phi_{\mu_1\ldots\mu_n}(x)e^{-ibK} = \Phi_{\nu_1\ldots\nu_n}(x')\,\frac{\partial x'^{\nu_1}}{\partial x^{\mu_1}}\cdots\frac{\partial x'^{\nu_n}}{\partial x^{\mu_n}}\left(\frac{ds'}{ds}\right)^{\Delta-n}, \tag{4.6}$$

from which the commutators follow using (1.11,1.12). Similarly one can proceed for the other transformations.

4.0.1 The inversion

In these notes, in general, when speaking of conformal invariance, we will mean invariance under infinitesimal conformal transformations and we will not consider global aspects related to conformal group and spacetime. However it is convenient to introduce a transformation that does not belong to such a category, the inversion $\mathcal{I}$:

$$\mathcal{I}(x^\mu) = \frac{x^\mu}{x^2}, \qquad x^2 = x^\mu\eta_{\mu\nu}x^\nu \tag{4.7}$$

which in particular maps the origin to infinity. The reason is that a special conformal transformation can be viewed as an inversion followed by a translation followed by another inversion:

$$x^\mu \quad \to \quad \frac{x^\mu}{x^2} \quad \to \quad \frac{x^\mu}{x^2}+b^\mu \quad \to \quad \frac{x^\mu+b^\mu x^2}{1+2b\cdot x+b^2x^2} \tag{4.8}$$

The operation of inversion does not belong to the conformal group, it is an outer automorphism. In fact under the same sequence

$$\text{inversion} \times \text{operation} \times \text{inversion}$$

we obtain the following correspondences:

- translation $\longrightarrow$ sct
- sct $\longrightarrow$ translation
- dilatation $\longrightarrow$ inverse dilatation
- Lorentz transformation $\longrightarrow$ same Lorentz transformation

Adding the inversion to the conformal transformations also form a group. When speaking about the conformal group and conformal invaraince/covariance we will always understand this extended group (including inversion). In particular, primary fields are understood to be covariant also with respect to inversion.

4.1 Properties of the e.m. tensor

In a local field theory Poincaré invariance implies the existence of a symmetric and conserved e.m. tensor

$$T_{\mu\nu} - T_{\nu\mu} = 0, \qquad \partial^\mu T_{\mu\nu} = 0 \tag{4.9}$$

This implies in particular that the Lorentz current $\mathcal{M}_\lambda^{\mu\nu} = x^{[\mu} T_\lambda^{\nu]}$ is conserved.

Scale invariance requires the existence of a current J^μ (the *virial* current) such that

$$T_\mu^\mu - \partial_\mu J^\mu = 0 \tag{4.10}$$

This implies that the current $D_\mu = x^\lambda T_{\lambda\mu} - J_\mu$ is conserved.

Invariance under special conformal transformations requires

$$T_\mu^\mu = 0 \tag{4.11}$$

so that the current $K_\mu^{(\rho)} = [\rho_\nu x^2 - 2x_\nu(\rho{\cdot}x)]T_\mu^\nu$ is conserved for any constant vector ρ_μ.

For a conformal field theory these three properties (4.9,4.10,4.11) are true classically. At a quantum level they appear in a weaker form, as constraints for the correlators where, for instance, T_μ^μ is inserted.

In a theory with a Lagrangian density $\mathcal{L}$ an e.m. tensor is obtained via the functional formula

$$T^{\mu\nu} = \sum_i \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_i} \partial^\nu \varphi^i - \eta^{\mu\nu} \mathcal{L} \tag{4.12}$$

where φ^i denote the generic fields in the theory. This e.m. tensor is, in general, not unique. There is some freedom to redefine it without breaking the conservation law $\partial_\mu T^{\mu\nu} = 0$. For instance it may happen that $\partial_\mu T^{\mu\nu} = 0$ and $T_\mu^\mu = \partial_\mu \partial_\nu L^{\mu\nu}$. Then one can define

$$\begin{aligned} \Theta^{\mu\nu} &= T^{\mu\nu} + \frac{1}{d-2}\left(\partial^\mu \partial_\lambda L^{\lambda\nu} + \partial^\nu \partial_\lambda L^{\lambda\mu} - \eta^{\mu\nu} \partial_\lambda \partial_\rho L^{\lambda\rho} - \Box L^{\mu\nu}\right) \\ &\quad + \frac{1}{(d-2)(d-1)} \left(\eta^{\mu\nu}\Box - \partial^\mu \partial^\nu\right) L_\lambda{}^\lambda \end{aligned} \tag{4.13}$$

which satisfies both $\partial_\mu \Theta^{\mu\nu} = 0$ and $\Theta_\mu^\mu = 0$.

The most practical way to derive the e.m. tensor of a theory defined by a local action S is to couple it to an external metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ in a covariant way and define the e.m. tensor as

$$T^{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S[g]}{\delta g_{\mu\nu}}\Big|_{g_{\mu\nu}=\eta_{\mu\nu}}, \qquad T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S[g]}{\delta g^{\mu\nu}}\Big|_{g^{\mu\nu}=\eta^{\mu\nu}} \tag{4.14}$$

where $S[\eta] = S$. For instance, in the case of a Yang-Mills theory

$$\begin{aligned} S &= \frac{1}{4g_{YM}} \int d^d x \, \mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}) \\ &\quad \to S[g] = \frac{1}{4g_{YM}} \int d^d x \sqrt{g} g^{\mu\mu'} g^{\nu\nu'} \mathrm{Tr}\,(F_{\mu\nu}F_{\mu'\nu'}) \end{aligned} \tag{4.15}$$

and in the case of a free massless fermion theory

$$S = \int d^d x \, i\bar{\psi}\gamma^\mu \partial_\mu \psi \to S[g] = \int d^d x \, \sqrt{g} \, i\bar{\psi}\gamma^\mu (\partial_\mu + \frac{1}{2}\omega_\mu)\psi \tag{4.16}$$

where $\gamma^\mu = e^\mu_a \gamma^a$ and ω_μ is the spin connection:

$$\omega_\mu = \omega_\mu^{ab} \Sigma_{ab}$$

where $\Sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$ are the Lorentz generators.

In this type of theories Poincaré invariance extends to diffeomorphism invariance, i.e. invariance under $x^\mu \to x^\mu + \xi^\mu(x)$, and conformal invariance to Weyl invariance, i.e. invariance under local metric rescalings. In the first case $\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$, with $\xi_\mu = g_{\mu\nu}\xi^\nu$ and

$$\begin{aligned} 0 = \delta_\xi S[g] &= \int d^d x \, \frac{\delta S[g]}{\delta g_{\mu\nu}(x)} \delta_\xi g_{\mu\nu}(x) = -\int d^d x \, \left(\frac{\sqrt{g}}{2} T^{\mu\nu} \left(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu\right)\right) \\ &= \int d^d x \sqrt{g} \xi_\nu \nabla_\mu T^{\mu\nu} \to \nabla_\mu T^{\mu\nu} = 0 \end{aligned} \tag{4.17}$$

after integration by parts, because ξ^μ is arbitrary. In flat spacetime the RHS becomes $\partial_\mu T^{\mu\nu} = 0$. In the second passage in (4.17) we have disregarded terms proportional to the equation of motion. Therefore the conclusion is valid on shell.

In the case of Weyl invariance, i.e. invariance under metric rescalings $g_{\mu\nu} \to 2\omega g_{\mu\nu}$, we have analogously

$$0 = \delta_\omega S[g] = -\int d^d x \, \frac{\delta S[g]}{\delta g_{\mu\nu}(x)} \delta_\omega g_{\mu\nu}(x) = -\int d^d x \, \omega T^\mu_\mu, \quad \to \quad T^\mu_\mu = 0 \tag{4.18}$$

4.1.1 A few examples

Here are some examples of classically conformal invariant field theories.

Scalar field theory. Let us consider the scalar field ϕ and the action

$$S[g] = \frac{1}{2} \int d^d x \, \sqrt{g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \tag{4.19}$$

Proceeding as above we find $T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{\eta_{\mu\nu}}{2}\partial_\lambda\phi\partial^\lambda\phi$. So, on shell,

$$T^\mu_\mu = \frac{2-d}{4}\Box\phi^2 = \partial^\mu\partial^\nu L_{\mu\nu}, \qquad L_{\mu\nu} = \frac{2-d}{4}\eta_{\mu\nu}\phi^2 \tag{4.20}$$

Now we can use formula (4.13) to define a traceless e.m. tensor or, better, remark that (4.19) can be made conformal invariant by adding a term involving the Ricci scalar R:

$$S'[g] = \frac{1}{2}\int d^dx\,\sqrt{g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{d-2}{4(d-1)}R\phi^2\right) \tag{4.21}$$

Now the model is Weyl invariant under the metric rescaling $\delta_\omega R = -2\omega R - 2(d-1)\Box\omega$ and $\delta_\omega\phi = -\frac{d-2}{2}\omega\phi$; consequently $T'^\mu_\mu = 0$.

Free massless Dirac fermion. This is the theory introduced above, (4.16). It is conformal invariant under the Weyl transformation. The flat e.m. tensor is

$$T_{\mu\nu} = \frac{i}{4}\left(\bar\psi\gamma_\mu\overleftrightarrow{\partial}_\nu\psi + \bar\psi\gamma_\nu\overleftrightarrow{\partial}_\mu\psi\right) \tag{4.22}$$

It is traceless on shell in any dimension.

Maxwell theory in 4d. The Maxwell action in 4d

$$S = \frac{1}{4}\int d^4x\,\sqrt{g}\,g^{\mu\mu'}g^{\nu\nu'}F_{\mu\nu}F_{\mu'\nu'} \tag{4.23}$$

is Weyl invariant under $\delta_\omega A_\mu = 0$. Its e.m. tensor

$$T_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho - \frac{\eta_{\mu\nu}}{4}F_{\lambda\rho}F^{\lambda\rho} \tag{4.24}$$

is clearly traceless.

4.1.2 Correlators in free field theories

In CFT the relevant objects to be calculated are the correlators with various field insertions. In free field theories this can be done by means of the Wick theorem. To this end we need the propagators. For instance, in 4d, in configuration space representation, we have:

- for a free scalar field ϕ: $\langle 0|\mathcal{T}\phi(x)\phi(y)|0\rangle = \frac{1}{(x-y)^2}$
- for a free Dirac fermion ψ: $\langle 0|\mathcal{T}\psi(x)\bar\psi(y)|0\rangle = \frac{\not{x}-\not{y}}{(x-y)^4} = \gamma_\mu\frac{(x^\mu-y^\mu)}{(x-y)^4}$

since $\Box\frac{1}{x^2} = -4\pi^2\delta^{(4)}(x)$, etc.

Now let us consider for instance the normal ordered composite operators $\Phi(x) =: \phi^2 : (x)$ and $J_\mu(x) =: \bar{\psi}\gamma_\mu\psi : (x)$, with conformal weight 2 and 3, respectively. Using the Wick theorem we can straightforwardly compute

$$\langle 0|\mathcal{T}\Phi(x)\Phi(y)|0\rangle = \frac{2}{(x-y)^4}, \tag{4.25}$$

$$\langle 0|\mathcal{T}J_\mu(x)J_\nu(y)|0\rangle = 4\frac{2(x_\mu - y_\mu)(x_\nu - y_\nu) - \eta_{\mu\nu}(x-y)^2}{(x-y)^8} \tag{4.26}$$

In the same way one can compute correlators with any number of insertions of Φ and J_μ. Of course they become more and more complicated as the number of insertions increases. They are automatically conformal covariant, although proving it is not quite straightforward. Of course massless free field theories are but a particular example of conformal field theory. It is clear that we have to go beyond free theories. The interest of CFT's is that, using conformal symmetry, correlators can be computed independently of a perturbative approach and even of the existence of a Lagrangian formulation.

It is necessary to introduce formalism and procedures independent of the existence of a local action for a given theory, and depending only on the conformal properties of the relevant insertion operators. We need to confront with conformal invariance from a general point of view.

4.1.3 Scale invariance vs conformal invariance

One of the important general questions raised recently in the framework of CFT is whether scale invariant theories are also conformal invariant. If we enlarge the Poincaré group by adjoining (only) scale transformations we obtain a new group. If a field φ transforms as

$$\varphi(x) \to \lambda^\Delta \varphi(\lambda x), \qquad \lambda > 0 \tag{4.27}$$

it is called *quasi-primary*. A field like φ may transform or not as an irreducible representation of the full CA. In the former case it is a primary field. But for the time being let us consider two quasi-primary scalar fields φ_1 and φ_2 with scaling weights Δ_1 and Δ_2, respectively. Due to Poincaré invariance the two-point correlator can only take the following form

$$\langle \varphi_1(x)\varphi_2(y)\rangle = \frac{c_{12}}{|x-y|^{\Delta_1+\Delta_2}} \tag{4.28}$$

where $|x| = \sqrt{x^2}$. As we shall see, a full conformal invariant 2-point function is nonvanishing only if $\Delta_1 = \Delta_2$, so that for a primary field $O(x)$ of weight Δ, the conformal 2-point correlator is

$$\langle O(x)O(y)\rangle = \frac{c}{(x-y)^{2\Delta}} \tag{4.29}$$

where c_{12}, c are constants.

The three-point function of three quasi-primary fields φ_1, φ_2 and φ_3 may have the form

$$\langle \varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\rangle = \sum_{\delta_1,\delta_2} \frac{c_{123}^{\delta_1,\delta_2}}{|x_1-x_2|^{\delta_1}|x_2-x_3|^{\delta_2}|x_1-x_3|^{\Delta_{123}-\delta_1-\delta_2}} \tag{4.30}$$

where $\Delta_{123} = \sum_{i=1}^{3}\Delta_i$. Poincaré and scale covariance are satisfied for any value of δ_1, δ_2 and of the constants $c_{123}^{\delta_1,\delta_2}$ (the latter depend on the model). In the case of full conformal invariance for the 3-point function of three scalar primary fields O_i, $i = 1, 2, 3$, we have instead (see below)

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle = \frac{c_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_1}} \tag{4.31}$$

The only model-dependent quantity is the constant c_{123}.

It is evident that in a scale invariant theory requiring also covariance under special conformal transformations has dramatic consequences on correlators. On the other hand a fixed point of the RG can be assumed to represent a scale invariant theory, which, however, a priori is not guaranteed to be conformal invariant. Therefore it is of utmost importance to answer the following question: is a scale invariant theory also conformal invariant? In general the answer is negative, it is possible to construct scale invariant theories which are not conformal invariant. However the answer is positive (at least in $d \leq 4$) if we require unitarity, causality and discrete spectrum. This means:

- unitarity
- causality: $[O_i(x), O_j(y)] = 0$, $(x-y)^2 < 0$, for any $O_i(x), O_j(y)$;
- discrete spectrum of D.

Although these conditions seem to be quite natural, there are nevertheless examples of nonunitary theories describing meaningful physics in condensed matter theory.

From now on we will deal only with conformal invariant theories.

4.2 Radial quantization

Ordinary quantization implies a foliation of space-time in space-like slices and a privileged role for time. One prepares the relevant system at $t_{in} = -\infty$ in an asymptotic state Ψ_{in}, belonging to a Hilbert space $\mathcal{H}_{in}$, where any interaction with other states can be ignored. One then evolves this state to $t_{out} = \infty$, when it will be transformed into a state in $\mathcal{H}_{out}$. The propagation is mediated by a unitary operator $U = e^{iP^0(t_{out}-t_{in})}$. The issue is to compute the amplitudes

$$\langle \Psi_{out}|U(+\infty, -\infty)|\Psi_{in}\rangle \tag{4.32}$$

for $\langle \Psi_{out}| \in \mathcal{H}_{out}$.

This approach is the favorite one in ordinary quantum field theory, where the aim is the calculation of S-matrix elements. But it is not the only one. As we have seen in 2d another scheme is provided by radial quantization. The latter makes sense when the metric is Euclidean: the spacetime is foliated by spheres centered at the origin, and the privileged role is played by the radius r. The system is supposed to evolve radially from the origin to infinity. The evolution is operated by the unitary operator

$$U = e^{iD \log r} \tag{4.33}$$

Just as in 2d we associate the '*in*' vacuum $|0\rangle$ to the origin and the '*out*' vacuum $\langle 0|$ to the sphere at infinity. Any weight Δ primary operator $\Phi(x)$ applied to the vacuum at the origin generates a state

$$|\Phi\rangle = \Phi(0)|0\rangle \tag{4.34}$$

To see the properties of these states let us return to the primary field definition (4.1) and restrict it to the origin

$$\begin{aligned}
&[P_\mu, \mathcal{O}(0)] = -i\partial_\mu \mathcal{O}(0)\\
&[L_{\mu\nu}, \mathcal{O}(0)] = \Sigma_{\mu\nu}\mathcal{O}(0)\\
&[D, \mathcal{O}(0)] = -i\Delta \mathcal{O}(0)\\
&[K_\mu, \mathcal{O}(0)] = 0
\end{aligned} \tag{4.35}$$

Since all the generators annihilate the vacuum it follows in particular

$$K_\mu|\Phi\rangle = 0, \qquad D|\Phi\rangle = -i\Delta|\Phi\rangle \tag{4.36}$$

Now in the algebra (1.13) consider in particular the commutators

$$[D, P_\mu] = -iP_\mu \tag{4.37}$$

$$[D, K_\mu] = iK_\mu \tag{4.38}$$

$$\tag{4.39}$$

From (4.37) it is easy to see that $D(P_\mu|\Phi\rangle) = -i(\Delta+1)(P_\mu|\Phi\rangle)$, i.e. P_μ raises the eigenvalue of D by one unit. Analogously (4.38) says that K_μ is a lowering operator. By repeatedly acting with P_μ on $|\Phi\rangle$ we can create all the states of the representation defined by (4.35), i.e. by the primary field Φ.

4.2.1 Hermiticity and reflection positivity

To complete the presentation of radial quantization we need the definition of scalar product and hermitean conjugation. We will proceed in analogy with 2d. There we used various geometries. The ordinary 2d flat Euclidean space with coordinates x^1, x^2 and the complex plane (or Riemann sphere) with local coordinate $z = x^1 + ix^2$. In string theory we also consider the mapping from the infinite cylinder with coordinates σ, t, $0 \le \sigma \le 2\pi$, to the complex z plane,

$$z = e^{t+i\sigma}, \qquad \text{so} \quad \text{that} \quad x^1 = e^t \cos\sigma, \qquad x^2 = e^t \sin\sigma$$

This implies

$$ds^2 = dz d\bar{z} = e^{2t}\left(dt^2 + d\sigma^2\right) \tag{4.40}$$

We can use similar geometries in $d \ge 3$, that is we can connect cylindrical coordinates to flat Euclidean and spherical ones via a metric rescaling

$$\begin{aligned} ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu = dr^2 + r^2 d\Omega_{d-1} \\ &= r^2\left(\frac{dr^2}{r^2} + d\Omega_{d-1}\right) = e^{2t_E}\left(dt_E^2 + d\Omega_{d-1}\right) \end{aligned} \tag{4.41}$$

where $r = e^{t_E}$ (t_E is to be identified with the Wick-rotated time) and $d\Omega_n$ is the measure on the unit sphere in n dimensions. More explicitly, we can introduce a unit vector $\vec{n}$ ($\vec{n}\cdot\vec{n} = 1$) on the unit sphere and write $d\Omega_n = d\vec{n}\cdot d\vec{n}$. Then $x^\mu = e^{t_E} n^\mu$ and $x^2 = e^{2t_E}$.

Under a change of coordinates a primary scalar field Φ of weight Δ will transform as

$$\Phi_{flat}(x) = \Phi_{sph}(r,\vec{n}) = e^{-\Delta t_E}\Phi_{cyl}(t_E,\vec{n}) \tag{4.42}$$

Now, in order to find the prescription for hermitean conjugation we start from a theory defined on a flat Minkowski spacetime. Let $\mathcal{O}$ be a hermitean field operator. We have

$$\mathcal{O}(x)^\dagger = \mathcal{O}(x), \qquad \mathcal{O}(x) = e^{iPx}\mathcal{O}(0)e^{-iPx}$$

If we Wick-rotate x^0, i.e. $x^0 \to it_E$ and wish to preserve the hermiticity of P^0_E, we must have

$$\mathcal{O}(x)^\dagger = \left(e^{-P^0_E t_E - i\vec{P}\cdot\vec{x}}\mathcal{O}(0)e^{P^0_E t_E + i\vec{P}\cdot\vec{x}}\right)^\dagger = e^{P^0_E t_E - i\vec{P}\cdot\vec{x}}\mathcal{O}(0)e^{-P^0_E t_E + i\vec{P}\cdot\vec{x}}$$

In other words

$$\mathcal{O}^\dagger(t_E, \vec{x}) = \mathcal{O}(-t_E, \vec{x}) \tag{4.43}$$

I.e. the operation of hermitean conjugation of a local field must be accompanied by the Euclidean time reversal. There is a reflection symmetry with respect to the hyperplane $t_E = 0$. This implies two consequences:

- since we have associated the vacuum $|0\rangle$ to the point $r = 0$ ($t_E = -\infty$), we must associate the dual vacuum $\langle 0|$ to the point $r = \infty$ ($t_E = \infty$), with $\langle 0|0\rangle = 1$;
- the rule for hermitean conjugation is

$$(\mathcal{O}_1(t_{E1}, \vec{x}_1)\ldots\mathcal{O}_n(t_{En}, \vec{x}_n)|0\rangle)^\dagger = \langle 0|\mathcal{O}_n(-t_{En}, \vec{x}_n)\ldots\mathcal{O}_1(-t_{E1}, \vec{x}_1) \tag{4.44}$$

This simultaneously guarantees the existence of a scalar product and of the hermitean conjugation in the Wick rotated theory. If one starts directly from a Euclidean theory, these are the rules (*reflection positivity*) that have to be satisfied.

It is also clear from the above that, since $t_E = \ln r$, the evolution operator in the Euclidean, i.e. P^0_E, has to be identified with the dilatation operator D. We will see below that the correct identification is $P^0_E = iD$.

For many purposes it is more convenient to work in spherical coordinates. In flat or spherical coordinates reflection is represented by inversion. So the dual vacuum $\langle 0|$ is associated to the sphere at infinity. A primary field Φ is always regular at the origin and at infinity, when applied to the respective vacua. Thus, in order to define the state dual to $|\Phi\rangle$, since we use a single coordinate patch, like in 2d we have to take into account the transformation properties of the field. Under inversion $\mathcal{I}$ the square line element in spherical coordinates changes to

$$ds^2 \to ds'^2 = \frac{1}{r^4}(dr^2 + r^2 d\Omega_{d-1}) \tag{4.45}$$

Thus a primary field Φ of weight Δ transforms as

$$\Phi(x) = r^{-2\Delta}(\mathcal{I}\circ\Phi)\,(\mathcal{I}(x)) \tag{4.46}$$

As in 2d we will identify the dual state created by Φ as

$$\langle\Phi| = \lim_{r\to\infty} r^{2\Delta}\langle 0|\Phi\,(r) \tag{4.47}$$

This property refines the primary field definition: primary fields must be covariant with respect to inversion, like in (4.46).

We can therefore summarize the rules for conjugation as follows

$$\langle 0| = (|0\rangle)^\dagger, \qquad \langle \Phi| = (|\Phi\rangle)^\dagger, \qquad (\Phi(x))^\dagger = (\mathcal{I} \circ \Phi)(\mathcal{I}(x)) \tag{4.48}$$

Moreover, since we have seen that inversion maps translations to sct's, we have

$$(P_\mu)^\dagger = K_\mu, \tag{4.49}$$

and since inversion maps a dilatation to its inverse we have

$$D^\dagger = -D, \tag{4.50}$$

while the Lorentz generators are hermitean.

4.2.2 The state-operator correspondence

Next let us consider the same primary field Φ applied to the vacuum not at the origin but at the generic point x. We have

$$\Phi(x)|0\rangle = e^{iPx}\phi(0)e^{-iPx}|0\rangle = e^{iPx}\phi(0)|0\rangle = \sum_{n=0}^{\infty} \frac{1}{n!}(iPx)^n|\Phi\rangle. \tag{4.51}$$

We see that in this case all the states of the representation are 'excited'.

Remark. This is analogous to the LWR's of the Virasoro algebra in 2d, the operators D, P_μ, K_μ playing the role of L_0, L_n with $n < 0$ and L_n with $n > 0$, respectively.

From (4.51) we see that, much like in 2d, the notion of a primary field allows us to know all the states of the corresponding representation of the CA. Is the converse also true? That is, does knowing all the states of a representation allow us to reconstruct the field? The answer is yes, if we accept the idea that knowing all the amplitudes of a theory allows us to reconstruct the fields themselves. This is what can be rigorously proven in axiomatic field theory (*reconstruction theorem*), see [7].

Let us assume this for any QFT. Let us start from a primary state $|\Delta\rangle$, which is the lowest weight state of a representation of the CA. We want to interpret it as obtained from the vacuum by the action of an operator $\Phi(0)$. The problem is to reconstruct the operator $\Phi(0)$. Knowing all the correlators that involve $|\Delta\rangle$,

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n)|\Delta\rangle = \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n)\Phi(0)|0\rangle,$$

determines $\Phi(0)$. Finally $\Phi(x) = e^{iPx}\Phi(0)e^{-iPx}$.

4.2.3 Some consequences of radial quantization

Let us consider the 2-point correlator of two quasi-primary fields Φ_1, Φ_2. As we saw above, (4.28), on the basis of scale invariance alone it is given by

$$\langle \Phi_1(x_1)\Phi_2(x_2)\rangle = \frac{c_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{4.52}$$

We can choose to set $x_2 = 0, x_1 \to \infty$, using (4.47). We see that the result is 0 or infinite unless $\Delta_1 = \Delta_2$, in which case it is finite[1]. Therefore, in order for (4.52) to be conformal we must have $\Delta_1 = \Delta_2$.

As another application let us consider

$$\langle \Phi | K_\mu P_\nu | \Phi \rangle \tag{4.53}$$

where $|\Phi\rangle$ is a spinless lowest weight state of weight Δ. Now use $[K_\mu, P_\nu] = -2i\,(\eta_{\mu\nu} D + L_{\mu\nu})$, $K_\mu|\Phi\rangle = 0$ and $L_{\mu\nu}|\Phi\rangle = 0$. Then

$$\langle \Phi | K_\mu P_\nu | \Phi \rangle = -2\Delta\eta_{\mu\nu} = 2\Delta\delta_{\mu\nu} \tag{4.54}$$

because the Wick-rotated metric has negative signature. Since the matrix element (4.53) must be positive in a unitary theory, due to (4.49), it follows that unitarity requires $\Delta > 0$. This condition is in fact necessary but not sufficient. A more complete argument (a generalization of the previous one) shows that unitarity requires

$$\Delta \geq \frac{d}{2} - 1 \tag{4.55}$$

for scalar primary fields, and

$$\Delta \geq s + d - 2, \qquad s = 1, 2, \ldots \tag{4.56}$$

for operators with integer spin s, and

$$\Delta \geq \frac{d-1}{2}, \tag{4.57}$$

for $s = \frac{1}{2}$.

[1]The correlator is expected to be regular, except possibly at coincident points.

Chapter 5

Ward identities and null cone method

The purpose of CFT is to compute conformal correlators. We have seen some examples above. Poincaré and scale covariance is rather easy to comply with. The most difficult part is to implement covariance with respect to special conformal transformations (sct's). To this end correlators must satisfy the appropriate Ward identities (WI's). This chapter is devoted to deriving them.

5.1 Deriving WI's

In this section we show how to derive the relevant Ward identities for the e.m. tensor. The derivation of the others is similar.

The special conformal transformation (SCT) of the em tensor $T_{\mu\nu}$ in coordinate representation is given by

$$\begin{aligned} & i[K_\lambda, T_{\mu\nu}] \\ & = \left(2dx_\lambda + 2x_\lambda\, x\cdot\partial - x^2\partial_\lambda\right) T_{\mu\nu} + 2\left(x^\alpha T_{\alpha\nu}\eta_{\lambda\mu} + x^\alpha T_{\mu\alpha}\eta_{\lambda\nu} - x_\mu T_{\lambda\nu} - x_\nu T_{\mu\lambda}\right) \end{aligned} \tag{5.1}$$

Let us couple $T_{\mu\nu}$ to an external source $h_{\mu\nu}$. The generating function of connected Green functions (in Minkowski spacetime) is

$$\begin{aligned} W[h_{\mu\nu}] &= W[0] \\ &\quad + \sum_{n=1}^{\infty} \frac{(-i)^{n+1}}{2^n n!} \int \prod_{i=1}^{n} dx_i\; h^{\mu_i\nu_i}(x_i)\, \langle 0|\mathcal{T}\{T_{\mu_1\nu_1}(x_1)\ldots T_{\mu_n\nu_n}(x_n)\}|0\rangle_c, \end{aligned} \tag{5.2}$$

In order for W to be invariant under sct's the external source $h_{\mu\nu}$ must transform as $\delta_b h_{\mu\nu} = -i[b^\lambda K_\lambda(x), h_{\mu\nu}(x)] \equiv -i[b\cdot K(x), h_{\mu\nu}(x)]$, where

$$\begin{aligned} & i[K_\lambda(x), h_{\mu\nu}(x)] \\ & = \left(2x_\lambda\, x\cdot\partial - x^2\partial_\lambda\right) h_{\mu\nu} - 2\left(x^\alpha h_{\alpha\nu}\eta_{\lambda\mu} + x^\alpha h_{\mu\alpha}\eta_{\lambda\nu} - x_\mu h_{\lambda\nu} - x_\nu h_{\mu\lambda}\right) \end{aligned} \tag{5.3}$$

Invariance of $W[h]$ leads to

$$\begin{aligned}0 = \delta_b W = \int d^d x \frac{\delta W}{\delta h^{\mu\nu}} \delta h^{\mu\nu} &= -\frac{i}{2} \int d^d x \, [b{\cdot}K(x), h^{\mu\nu}(x)] \langle\!\langle T_{\mu\nu}(x \rangle\!\rangle \\ &= \frac{i}{2} \int d^d x \, h^{\mu\nu}(x) [b{\cdot}K(x), \langle\!\langle T_{\mu\nu}(x) \rangle\!\rangle] = 0 \end{aligned} \tag{5.4}$$

where

$$\begin{aligned}\langle\!\langle T_{\mu\nu}(x) \rangle\!\rangle &= 2\frac{\delta W[h]}{\delta h^{\mu\nu}(x)} = \sum_{n=1}^{\infty} \frac{(-i)^{n+2}}{2^n n!} \int dx_1 \ldots \int dx_n h^{\mu_1\nu_1}(x_1) \ldots h^{\mu_n\nu_n}(x_n) \\ &\quad \times \langle 0 | \mathcal{T} T_{\mu\nu}(x) T_{\mu_1\nu_1}(x_1) \ldots T_{\mu_n\nu_n}(x_n) | 0 \rangle_c \end{aligned} \tag{5.5}$$

Differentiating (5.4) with respect to $h_{\mu\nu}(x)$ and setting $h_{\mu\nu} = 0$ we get $0 = 0$, because $\langle 0 | T_{\mu\nu}(x) | 0 \rangle = \langle 0 | T_{\mu\nu}(0) | 0 \rangle$. Differentiating twice (5.4) and integrating by parts we get

$$(b{\cdot}K(x) + b{\cdot}K(y)) \langle 0 | \mathcal{T} T_{\mu\nu}(x) T_{\lambda\rho}(y) \rangle | 0 \rangle = 0 \tag{5.6}$$

Differentiating three times (5.4)

$$(b{\cdot}K(x) + b{\cdot}K(y) + b{\cdot}K(z)) \langle 0 | \mathcal{T} T_{\mu\nu}(x) T_{\lambda\rho}(y) T_{\alpha\beta}(z) \rangle | 0 \rangle = 0 \tag{5.7}$$

In these equations $K(x)$ denote the differential operator acting on the energy-momentum tensor in the rhs of (5.1). It is understood that the Lorentz part of $b{\cdot}K(x)$ acts on the indices $\mu\nu$ only, $b{\cdot}K(y)$ on the indices $\lambda\rho$ and $b{\cdot}K(z)$ on $\alpha\beta$ alone.

Due to translational invariance we can set $y = 0$ in (5.6) and $z = 0$ in (5.7). These equations become

$$b{\cdot}K(x) \langle 0 | \mathcal{T} T_{\mu\nu}(x) T_{\lambda\rho}(0) \rangle | 0 \rangle = 0 \tag{5.8}$$

and

$$(b{\cdot}K(x) + b{\cdot}K(y)) \langle 0 | \mathcal{T} T_{\mu\nu}(x) T_{\lambda\rho}(y) T_{\alpha\beta}(0) \rangle | 0 \rangle = 0 \tag{5.9}$$

with the previous prescription for the Lorentz part.

5.1.1 General WI's for sct's

Given a Poincaré invariant correlators for primary fields $\mathcal{O}_i$ of conformal weight Δ_i, $i = 1, \ldots, n$, for it to be conformally covariant it must satisfy the WI for scaling transformation

$$\sum_{i=1}^{n} \left(\Delta_i + x_i^\mu \frac{\partial}{\partial x_i^\mu} \right) \langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle = 0, \tag{5.10}$$

and the WI for sct's

$$\sum_{i=1}^{n}\left(b\cdot\tilde{K}(x_i)+b\cdot L(x_i)\right)\langle\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle=0 \tag{5.11}$$

where $b{\cdot}K(x_i)$ is split into two parts: $b\cdot\tilde{K}(x_i)$ is a differential operator, independent of the tensor structure of the correlator, and $b{\cdot}L(x_i)$ a multiplication operator, both linear in b. For scalar operators the latter is absent, while the former coincides with $b\cdot K(x_i)$:

$$b\cdot K(x_i)=2\,\Delta\,b\cdot x_i+2\,b\cdot x_i\,x_i\cdot\frac{\partial}{\partial x_i}-x_i^2\,b\cdot\frac{\partial}{\partial x_i} \tag{5.12}$$

A simple example. For a scalar field of weight Δ the two point function is $\langle\Phi(x)\Phi(y)\rangle\sim\frac{1}{(x-y)^{2\Delta}}$. The Ward identity reads

$$(b\cdot K(x)+b\cdot K(y))\frac{1}{(x-y)^{2\Delta}}=\frac{1}{(x-y)^{2\Delta}}\,b\cdot(x-y)(2\Delta-2\Delta)=0$$

5.2 The null-cone method

We have already remarked that the Lie algebra (1.13) is isomorphic to the Lie algebra of the group $SO(d,2)$. When the metric is Euclidean the relevant group is $SO(d+1,1)$. The latter is the Lorentz group in $d+2$ dimensions. Let us be more explicit. In d dimension we use the coordinates x^μ, $\mu=1,\dots,d$ with (Euclidean) square line element

$$ds_d^2=\sum_{\mu=1}^{d}(dx^\mu)^2 \tag{5.13}$$

In the $d{+}2$-dimensional space we use the coordinates X^M, $M=1,\dots,d,d{+}1,d{+}2$ with square line element

$$ds_{d+2}^2=\sum_{M=1}^{d+1}(dX^M)^2-(dX^{d+2})^2=\sum_{M=1}^{d}(dX^M)^2-dX^+dX^- \tag{5.14}$$

where $X^\pm=X^{d+2}\pm X^{d+1}$. The Lorentz algebra in $d+2$ dim is given as follows, in terms of the generators J_{MN},

$$[J_{MN},J_{RS}]=i\,(\eta_{NS}J_{MR}-\eta_{NR}J_{MS}-\eta_{MS}J_{NR}+\eta_{MR}J_{NS}) \tag{5.15}$$

The isomorphism with the CA is made explicit by the following identifications:

- $J_{\mu\nu}$ with $L_{\mu\nu}$,
- $J_{\mu+}$ with P_μ,
- $J_{\mu-}$ with K_μ,
- J_{+-} with D

Since the CA in d dimensions is isomorphic to the Lie algebra of the Lorentz group in $d+2$ dimensions, any conformal covariant correlator in d dimensions should be obtainable from Lorentz covariant expressions in $d+2$ dimensions via some kind of dimensional reduction procedure. Needless to say it is extremely simpler to construct Lorentz covariant expressions than conformal covariant ones. Therefore the problem is: once we have constructed Lorentz covariant expressions in $d+2$ dimensions how do we descend to d dimensions without breaking the covariance? This can be done as follows.

We notice first that the null light-cone (or *null cone*) $X^2 = 0$ is Lorentz invariant in $d+2$ dimensions. Therefore reducing a Lorentz covariant expression to the null cone does not break covariance and eliminates one coordinate, say X^- in favor of the remaining ones. In order to eliminate another, say X^+, we choose a section of the null-cone specified by: $X^+ = f(x)$ (referred to hereafter as the Euclidean section) $f(x)$, which we suppose meets each light ray in one point, associates to any point in d dimension a point in the null-cone. That is we implement the following restriction

$$(X^\mu, X^+, X^-) \longrightarrow (x^\mu, f(x), x^2/f(x)) \tag{5.16}$$

and represent it as a map

$$F(x^\mu) = (x^\mu, f(x), x^2/f(x)) \tag{5.17}$$

from the d-dimensional space into the null cone. In fact we will make the simple choice $f(x) = 1$, so that the Euclidean section is

$$F(x^\mu) = (x^\mu, 1, x^2) \tag{5.18}$$

with the projective property

$$F(\lambda(x)x^\mu) = \lambda(x)\left(x^\mu, 1, x^2\right) \tag{5.19}$$

The Lorentz group, acting on the point in the null-cone with the above coordinate, say P, will move it to another point in the null-cone with other coordinates and, in general, outside the section, say P'. However, suppose via a conformal transformation in d dimension we can move P' to a point P'' lying on the same light ray passing through P, then P in the section is uniquely identified. So, the question is to verify whether the two points P' and P'' can be connected via conformal transformations in d dimension: only in that case will not conformal covariance be broken by the reduction.

For instance, let us consider a rotation in the (X^1, X^{d+1}) plane by an (infinitesimal) angle θ. Let us set, for simplicity, $X^1 = t$, and let us denote the coordinates X^i, with $i = 2, \ldots, d$ by $\vec{x}$. The rotation leads to the transformations

$$\begin{aligned} \delta t &= \frac{\theta}{2}(1 - x^2) \\ \delta X^+ &= -t\theta \\ \delta X^- &= t\theta \end{aligned} \tag{5.20}$$

while $\vec{x}$ remains unchanged. The initial point gets transformed to

$$(t, \vec{x}, 1, x^2) \longrightarrow (t + \frac{\theta}{2}(1 - x^2), \vec{x}, 1 - t\theta, x^2 + t\theta) \tag{5.21}$$

which is outside the section. We will show that we can take it back to the section by a suitable rescaling in $d + 2$ dimensions, up to a translation and a s.c.t in d dimensions.

Let us consider a s.c.t. in the t direction, that is a s.c.t. with parameter $b^\mu = (-\frac{\theta}{2}, 0, \ldots, 0)$. That is

$$\begin{aligned} t &\longrightarrow t + \frac{\theta}{2}x^2 - \theta t^2 \\ \vec{x} &\longrightarrow \vec{x}\,(1 - \theta t) \\ x^2 &\longrightarrow x^2(1 - 2\theta t) \end{aligned} \tag{5.22}$$

Next we perform a translation in t: $t \longrightarrow t + \frac{\theta}{2}$. The result is

$$\begin{aligned} (t + \frac{\theta}{2}(1 - x^2), \vec{x}, 1 - t\theta, x^2 + t\theta) &\longrightarrow (t(1 - \theta t), \vec{x}\,(1 - \theta t), (1 - \theta t), x^2(1 - \theta t)) \\ &= (1 - \theta t)(t, \vec{x}, 1, x^2) \end{aligned} \tag{5.23}$$

The final point is in the light ray passing through $(t, \vec{x}, 1, x^2)$, which is what we wanted to show.

The rotations in the (X^1, X^{d+2}) plane have a similar effect. Finally a rotation in the (X^{d+1}, X^{d+2}) plane leads to a constant rescaling (see below).

To conclude we notice that the above choice of section keeps the metric flat, for, restricting the line element to the Euclidean section, implies

$$\begin{aligned} ds^2_{d+2} &= \eta_{MN} dX^M dX^N \longrightarrow \eta_{MN} \frac{\partial X^M}{\partial x^\mu} \frac{\partial X^N}{\partial x^\nu} dx^\mu dx^\nu \\ &= \left(\frac{\partial x^\lambda}{\partial x^\mu} \frac{\partial x_\lambda}{\partial x^\nu} - \frac{\partial X^+}{\partial x^\mu} \frac{\partial X^-}{\partial x^\nu} \right) dx^\mu dx^\nu = \eta_{\mu\nu} dx^\mu dx^\nu = ds^2_d \end{aligned} \tag{5.24}$$

5.2.1 Applications

We wish to reproduce the properties of a scalar primary field $\varphi(x)$ of weight Δ, according to the just outlined approach. To this end we start from a field $\Phi(X)$ in $d+2$ dimensions with the following property

$$\Phi(\lambda X) = \lambda^{-\Delta}\tilde{\Phi}(X) \tag{5.25}$$

and we identify

$$\varphi(x) = \Phi(X)\big|_{\text{section}} \tag{5.26}$$

A constant rescaling in $d+2$ dimensions on a point in the Euclidean section

$$(x^\mu, 1, x^2) \to (\lambda x^\mu, \lambda, \lambda x^2) \tag{5.27}$$

takes the point out of the Euclidean section. However a hyperbolic rotation in the (d+1,d+2) plane leads back to the section. In fact such a rotation can be represented as

$$X^\pm \to X^\pm(\cosh\chi \pm \sinh\chi) \approx X^\pm(1\pm\chi) \tag{5.28}$$

for χ infinitesimal. Taking for simplicity $\lambda \approx 1+\epsilon$ and applying such a transformation to the RHS of (5.27) gives

$$((1+\epsilon)x^\mu, (1+\epsilon), (1+\epsilon)x^2) \to ((1+\epsilon)x^\mu, 1, (1+2\epsilon)x^2) \tag{5.29}$$

provided we choose $\chi = -\epsilon$. The point in the RHS of (5.29) is in the Euclidean section. That is we have done a (permitted) conformal transformation and

$$\varphi(\lambda x) = \Phi(\lambda X)\big|_{\text{section}} \tag{5.30}$$

Scalar 2-point function

We are now ready to use this formalism to compute correlators. Let us derive the 2-point function of φ inserted at the points x and y. We start from the 2-point function of Φ in d+2 dimensions inserted at the points X and Y, which projects to x and y according to the above recipe. The only possibility to construct a two point function with the required properties is

$$\langle\Phi(X)\Phi(Y)\rangle = \frac{c}{(X\cdot Y)^\Delta}, \tag{5.31}$$

where c is an undetermined constant, because $X^2 = Y^2 = 0$ on the null-cone. (5.31) is Lorentz invariant and has the correct dimension. What remains for us to do is to project it to the Euclidean section, i.e. to set

$$X^M = (x^\mu, 1, x^2), \qquad Y^M = (y^\mu, 1, y^2)$$

This implies

$$X\cdot Y = X^\mu \eta_{\mu\nu} Y^\nu - \frac{1}{2}\left(X^+Y^- + X^-Y^+\right) = xy - \frac{1}{2}(y^2+x^2) = -\frac{1}{2}(x-y)^2 \quad (5.32)$$

I.e.

$$\langle\varphi(x)\varphi(y)\rangle = \frac{c'}{(x-y)^{2\Delta}} \quad (5.33)$$

From the previous construction it is clear that the two point function of two scalar fields $\varphi_1(x)$ and $\varphi_2(y)$, with *different* weights Δ_1 and Δ_2, is bound to vanish. For in $d+2$ dimensions we should be able to construct a Lorentz invariant object of weight Δ_1 in X and Δ_2 in Y, but such objects do not exist on the null cone.

Scalar 3-point function

Let us move now to the three-point function of three scalar fields $\varphi_i(x)$ with weight Δ_i, $i=1,2,3$. Introducing the corresponding fields Φ_i in $d+2$ dimensions, the only appropriate object we can construct is

$$\langle\Phi_1(X_1)\Phi_2(X_2)\Phi_3(X_3)\rangle = \frac{c_{123}}{(X_1\cdot X_2)^{a_{123}}(X_1\cdot X_3)^{a_{132}}(X_2\cdot X_3)^{a_{231}}} \quad (5.34)$$

where

$$a_{123}+a_{132}=\Delta_1, \qquad a_{123}+a_{321}=\Delta_2, \qquad a_{132}+a_{231}=\Delta_3 \quad (5.35)$$

The solution is

$$a_{123} = \frac{\Delta_1+\Delta_2-\Delta_3}{2}, \quad a_{132} = \frac{\Delta_1+\Delta_3-\Delta_2}{2}, \quad a_{231} = \frac{\Delta_2+\Delta_3-\Delta_1}{2} \quad (5.36)$$

So finally

$$\langle\varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\rangle = \frac{c'_{123}}{(x_1-x_2)^{2a_{123}}(x_1-x_3)^{2a_{132}}(x_2-x_3)^{2a_{231}}} \quad (5.37)$$

Scalar 4-point function

Proceeding in the same way for the four point function of a scalr field of weight Δ one finds in $d+2$ dimensions:

$$\langle\Phi(X_1)\Phi(X_2)\Phi(X_3)\Phi(X_4)\rangle = f(u,v)\frac{1}{(X_1\cdot X_2)^\Delta(X_3\cdot X_4)^\Delta} \quad (5.38)$$

where f is an undetermined function and u, v are the cross-ratios

$$u = \frac{(X_1 \cdot X_2)(X_3 \cdot X_4)}{(X_1 \cdot X_3)(X_2 \cdot X_4)}, \qquad v = \frac{(X_1 \cdot X_4)(X_2 \cdot X_3)}{(X_1 \cdot X_3)(X_2 \cdot X_4)} \tag{5.39}$$

Upon reduction to d dimension the latter become

$$u = \frac{(x_1 - x_2)^2(x_3 - x_4)^2}{(x_1 - x_3)^2(x_2 - x_4)^2}, \qquad v = \frac{(x_1 - x_4)^2(x_2 - x_3)^2}{(x_1 - x_3)^2(x_2 - x_4)^2} \tag{5.40}$$

So, finally,

$$\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle = g(u, v)\frac{1}{(x_1 - x_2)^{2\Delta}(x_3 - x_4)^{2\Delta}} \tag{5.41}$$

where g is proportional to f by a numerical factor. We could have proceeded in another way, singling out instead the factor $\frac{1}{(X_1 \cdot X_4)^\Delta (X_2 \cdot X_3)^\Delta}$ in (5.38). But since the correlator is symmetric in the exchange of the entries we must have

$$\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle = g(\tilde{u}, \tilde{v})\frac{1}{(x_1 - x_4)^{2\Delta}(x_2 - x_3)^{2\Delta}} \tag{5.42}$$

where $\tilde{u} = v, \tilde{v} = u$. But (5.41) and (5.42) coincide. Therefore

$$g(u, v) = g(v, u)\left(\frac{u}{v}\right)^\Delta$$

5.2.2 Correlators of tensorial operators

Suppose we want to compute a conformal covariant correlator involving a tensorial operator $\varphi_{\mu\nu\lambda\ldots}$. It is natural to introduce its ancestor in $d + 2$ dimensions in the form $\Phi_{MNL\ldots}$. When going back again to d dimension we have for each index two more components than needed. How do we eliminate the exceeding ones? One first condition is transversality, i. e.

$$X^M \Phi_{MNL\ldots} = 0 \tag{5.43}$$

which can be imposed without breaking Lorentz covariance. It eliminates one component for each index. The remaining superfluous component is related to a sort of 'gauge invariance' generated by transversality: if we add to $\Phi_{MNL\ldots}$ a tensor proportional to X_M the transversality condition suppresses it. We will eliminate this gauge freedom by 'fixing the gauge', that is by making a precise choice in order to guarantee covariance in d dimensions.

The tensor $\varphi_{\mu\nu\lambda\ldots}$ in d dimensions will be obtained by projecting $\Phi_{MNL\ldots}$ to the Euclidean section. This implies

$$\Phi^{MNL\ldots}(X)\Big|_{section} = \frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu}\frac{\partial X^L}{\partial x^\lambda}\ldots\varphi^{\mu\nu\lambda\ldots}(x) \tag{5.44}$$

where $X^M = (x^\mu, 1, x^2)$, so

$$\frac{\partial X^M}{\partial x^\sigma} = (\delta^\mu_\sigma, 0, 2x_\sigma) \tag{5.45}$$

Now let us suppose that $\Phi^{MNL\ldots}$ is traceless: $\Phi_M{}^{ML\ldots} = 0$. It follows that also $\varphi^\mu_{\mu\lambda\ldots} = 0$. Indeed, taking the trace in (5.44), on the Euclidean section we get

$$0 = \Phi_M{}^{ML\ldots}(X)\Big|_{section} = \eta_{MN}\frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu}\frac{\partial X^L}{\partial x^\lambda}\ldots\varphi^{\mu\nu\lambda\ldots}(x)$$

The factor

$$\eta_{MN}\frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu} = \eta_{\lambda\rho}\frac{\partial x^\lambda}{\partial x^\mu}\frac{\partial x^\rho}{\partial x^\nu} - \frac{\partial X^+}{\partial x^\mu}\frac{\partial X^-}{\partial x^\nu} = \eta_{\mu\nu}, \tag{5.46}$$

from which the claim follows.

The $d+2$ tensor field $\Phi_{MNL\ldots}(X)$ will of course transform in accordance with the Lorentz group $SO(d+1,1)$. In addition, if the tensor field $\varphi_{\mu\nu\lambda\ldots}$ has conformal weight Δ, we will require

$$\Phi_{MNL\ldots}(\lambda X) = \lambda^{-\Delta}\Phi_{MNL\ldots}(X) \tag{5.47}$$

To see that this is the right behavior let us consider for simplicity a vector field Φ_M and let us form the one-form $\mathbf{\Phi} = \Phi_M dX^M$. From the point of view of the Lorentz group in $d+2$ dimension $\mathbf{\Phi}$ behaves like a scalar, which, under rescaling, transforms with weight $\Delta - 1$[1] . We can apply now the same kind of argument as for the scalar field Φ above and deduce that, as consequence of (5.47), the 1-form $\varphi_\mu dx^\mu$ will be multiplied by the conformal factor $\lambda^{1-\Delta}$. This is what is expected of φ_μ.

The 2-point function of a current

Let us compute the two-point correlator of a vector field $j_\mu(x)$ of weight Δ. We introduce the vector field $J_M(X)$ in $d+2$ dimensions and form the most general

[1] In general when a general coordinate transformation is involved one should take into account also a factor $(ds)^{\Delta-1}$, i.e. $\mathbf{\Phi} = \Phi_M dX^M (ds)^{\Delta-1}$, but in this case we have $d^2 s_{d+2}|_{section} = d^2 s_d$, therefore this factor can be disregarded.

Lorentz covariant

$$\langle J_M(X)J_N(Y)\rangle \;=\; \frac{c}{(X\cdot Y)^\Delta}\left(\eta_{MN}+\frac{\alpha X_M X_N+\beta X_M Y_N+\gamma Y_M X_N+\delta Y_M Y_N}{X\cdot Y}\right) \tag{5.48}$$

$c,\alpha,\beta,\gamma,\delta$ are undetermined constants. Now, saturating (5.48) with X and Y, transversality and null cone condition requires $\alpha=\delta=0$ and $\gamma=-1$. However if we restrict such result to the Euclidean section we obtain an expression which is not translational invariant. We have to start instead from

$$\langle J_M(X)J_N(Y)\rangle = \frac{c}{(X\cdot Y)^\Delta}\left(\eta_{MN}+\frac{(X_M-Y_M)(X_N-Y_N)}{X\cdot Y}\right) \tag{5.49}$$

which is compatible with transversality and null cone condition. Then, restricting to the Euclidean section we get

$$\langle j_\mu(x)j_\nu(y)\rangle = \frac{c'}{(x-y)^{2\Delta}}I_{\mu\nu}(x-y), \qquad I_{\mu\nu}(x)=\eta_{\mu\nu}-2\frac{x_\mu x_\nu}{x^2} \tag{5.50}$$

This is the same result we computed above in a free field theory, see (4.26).

A more limpid derivation of this result is obtained by means of covariant transformation formulas, instead of contravariant ones like (5.44). To obtain covariant formulas one must consider the relevant tensor components as intrisic components of forms: $\Phi=\Phi_{MNL\ldots}dX^M dX^N dX^L\ldots$. In this case the restricted form to the Euclidean section, i.e. $\varphi_{\mu\nu\lambda\ldots}dx^\mu dx^\nu dx^\lambda\ldots$, is the pull-back of Φ by the F map (5.17). That is, $\varphi=F^*\Phi$. Instead of (5.44) we have

$$\varphi_{\mu\nu\lambda\ldots} = \frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu}\frac{\partial X^L}{\partial x^\lambda}\ldots\Phi_{MNL\ldots} \tag{5.51}$$

Starting from the most general pure Lorentz invariant expression in $d{+}2$ dimension, (5.48), using (5.45) and

$$\frac{\partial X^M}{\partial x^\mu}X_M=0,\quad \frac{\partial Y^M}{\partial y^\mu}Y_M=0,\quad \frac{\partial X^M}{\partial x^\mu}Y_M=y^\mu-x^\mu,\quad \frac{\partial Y^M}{\partial y^\mu}X_M=x^\mu-y^\mu \tag{5.52}$$

one obtains again (5.50).

Current conservation

If the previous operator j_μ is a conserved current, its weight is canonical, $\Delta=d-1$, we obtain the same result we computed above in a free field theory for $d=4$, see(4.26).

When $\Delta = d - 1$, one can easily verify that

$$\partial^\mu \langle j_\mu(x) j_\nu(y)\rangle = 0 \tag{5.53}$$

In general current conservation can be verified in the original d spacetime. It does not seem to be possible to represent this property in simple form in $d+2$ dimensions.

All tensorial two-point functions turn out to be constructed with $I_{\mu\nu}$. No new conformal structure appears in two-point correlators. For instance for the e.m. tensor the two-point function is well-known

$$\begin{aligned}&\langle T_{\mu\nu}\,(x)\,T_{\rho\sigma}\,(y)\rangle\\ &= \frac{c/2}{(x-y)^{2d}}\left(I_{\mu\rho}\,(x-y)\,I_{\nu\sigma}\,(x-y) + I_{\nu\rho}\,(x-y)\,I_{\mu\sigma}\,(x-y) - \frac{2}{d}\eta_{\mu\nu}\eta_{\rho\sigma}\right)\end{aligned}$$

This correlator is conserved and traceless (for $x \neq y$), see below.

5.3 Conformal field theory in momentum space

So far we have considered CFT in configuration space. It is however possible, and sometime convenient, to formulate a CFT in momentum space. If we Fourier transform the generators of the confomal algebra we get (a tilde represents the transformed generator and $\tilde{\partial} = \frac{\partial}{\partial k}$)

$$\begin{aligned}&\tilde{P}_\mu = -k_\mu\\ &\tilde{D} = i(d + k^\mu \tilde{\partial}_\mu)\\ &\tilde{L}_{\mu\nu} = i(k_\mu \tilde{\partial}_\nu - k_\nu \tilde{\partial}_\mu)\\ &\tilde{K}_\mu = 2d\;\tilde{\partial}_\mu + 2k_\nu \tilde{\partial}^\nu \tilde{\partial}_\mu - k_\mu \tilde{\Box}\end{aligned}$$

Notice that $\tilde{P}_\mu$ is a multiplication operator and $\tilde{K}_\mu$ is a quadratic differential operator. The Leibniz rule does not hold for $\tilde{K}_\mu$ and $\tilde{P}_\mu$ with respect to the ordinary product. However it does hold for the convolution product:

$$\tilde{K}_\mu(\tilde{f} \star \tilde{g}) = (\tilde{K}_\mu \tilde{f}) \star \tilde{g} + \tilde{f} \star (\tilde{K}_\mu\;\tilde{g})$$

where $(\tilde{f} \star \tilde{g})(k) = \int dp\, f(k-p) g(p)$.

Nevertheless these generators form a closed algebra under commutator

$$
\begin{aligned}
&[\tilde{D}, \tilde{P}_\mu] = i\tilde{P}_\mu \\
&[\tilde{D}, \tilde{K}_\mu] = i\tilde{K}_\mu \\
&[\tilde{K}_\mu, \tilde{K}_\nu] = 0 \\
&[\tilde{K}_\mu, \tilde{P}_\nu] = i(\eta_{\mu\nu}\tilde{D} - \tilde{L}_{\mu\nu}) \\
&[\tilde{K}_\lambda, \tilde{L}_{\mu\nu}] = i(\eta_{\lambda\mu}K_\nu - \eta_{\lambda\nu}K_\mu) \\
&[\tilde{P}_\lambda, \tilde{L}_{\mu\nu}] = i(\eta_{\lambda\mu}P_\nu - \eta_{\lambda\nu}P_\mu) \\
&[\tilde{L}_{\mu\nu}, \tilde{L}_{\lambda\rho}] = i(\eta_{\nu\lambda}\tilde{L}_{\mu\rho} + \eta_{\mu\rho}\tilde{L}_{\nu\lambda} - \eta_{\mu\lambda}\tilde{L}_{\nu\rho} - \eta_{\nu\rho}\tilde{L}_{\mu\lambda})
\end{aligned}
$$

Notice also that they do not generate infinitesimal transformations in momentum space.

For instance the special conformal transformation of a symmetric two-tensor $\tilde{T}_{\lambda\rho}$is given by

$$
\begin{aligned}
\tilde{K}_\mu \tilde{T}_{\lambda\rho}(k) \;\; = \;\; & (2(\Delta - d)\tilde{\partial}_\mu - 2k\cdot\tilde{\partial}\,\tilde{\partial}_\mu + k_\mu\tilde{\Box})\tilde{T}_{\lambda\rho} \\
& +2(\tilde{\partial}^\alpha\tilde{T}_{\alpha\rho}\eta_{\mu\lambda} - \tilde{\partial}_\lambda\tilde{T}_{\mu\rho} + \tilde{\partial}^\alpha\tilde{T}_{\lambda\alpha}\eta_{\mu\rho} - \tilde{\partial}_\rho\tilde{T}_{\lambda\mu})
\end{aligned}
\tag{5.54}
$$

For the e.m. tensor $\Delta = d$ and the first term drops.

5.3.1 A few examples

In momentum representation the CFT correlators must be annihilated by $b{\cdot}\tilde{K}$. For instance the 2-pt function of a scalar field of weight Δ is $\sim (k^2)^{\Delta-\frac{d}{2}}$ and

$$
\tilde{K}_\mu (k^2)^{\Delta-\frac{d}{2}} = (2\Delta - d)\cdot 0 \cdot (k^2)^{\Delta-\frac{d}{2}-1} = 0 \tag{5.55}
$$

in any dimension. A less trivial, but still simple, example is the 2-pt function of two currents in 3d

$$
\langle \tilde{j}_i(k)\tilde{j}_j(-k)\rangle = \frac{\delta_{ij}k^2 - k_i k_j}{|k|} \tag{5.56}
$$

Working out

$$
\begin{aligned}
&\Big(2(b\cdot\tilde{\partial}) - (b\cdot k\tilde{\Box} - 2k\cdot\tilde{\partial}\, b\cdot\tilde{\partial})\Big)\langle \tilde{j}_i(k)\tilde{j}_j(-k)\rangle \\
&\qquad + 2(b^l\partial_i - b_i\tilde{\partial}^l)\langle \tilde{j}_l(k)\tilde{j}_j(-k)\rangle + 2(b^l\partial_j - b_j\tilde{\partial}^l)\langle \tilde{j}_l(k)\tilde{j}_l(-k)\rangle
\end{aligned}
\tag{5.57}
$$

one can see that it is 0.

The 2-pt function of the energy momentum tensor in 3d has three possible (conserved) tensorial structures, which are given by the expression

$$\begin{aligned}\langle T_{\mu\nu}(k)\, T_{\rho\sigma}(-k)\rangle &= -\frac{i\,\tau}{|k|}\left(k_\mu k_\nu - \eta_{\mu\nu}k^2\right)\left(k_\rho k_\sigma - \eta_{\rho\sigma}k^2\right)\\ &\quad -\frac{i\,\tau'}{|k|}\left[\left(k_\mu k_\rho - \eta_{\mu\rho}k^2\right)\left(k_\nu k_\sigma - \eta_{\nu\sigma}k^2\right) + \mu \leftrightarrow \nu\right] \\ &+\frac{\kappa}{192\pi}\left[\epsilon_{\mu\rho\tau}\, k^\tau\left(k_\nu k_\sigma - \eta_{\nu\sigma}k^2\right) + \epsilon_{\mu\sigma\tau}k^\tau\left(k_\nu k_\rho - \eta_{\nu\rho}k^2\right) + \mu \leftrightarrow \nu\right].\end{aligned} \tag{5.58}$$

where τ, τ', κ are model-dependent constants. In Appendix B one can find a detailed proof that they satisfy the SCT WI.

The third tensorial structure, unlike the other two, is a polynomial in momentum space. In fact (5.64) defines a covariant integrated 0-cocycle (action term). Fourier anti-transforming it we get,

$$F_{\mu\nu\lambda\rho}(x, y) \sim \epsilon_{\mu\lambda\sigma}\partial^\sigma\left(\partial_\nu\partial_\rho - \eta_{\nu\rho}\Box\right)\delta^{(3)}(x-y) + \begin{pmatrix}\mu \leftrightarrow \nu\\ \lambda \leftrightarrow \rho\end{pmatrix} \tag{5.59}$$

Saturating it with $h^{\mu\nu}(x)$ and $h^{\lambda\rho}(y)$ and integrating over x and y (according to the formula (5.2)), one gets

$$\sim \int \epsilon_{\mu\lambda\sigma}\left(\partial^\sigma h^{\mu\nu}\partial_\nu\partial_\rho h^{\lambda\rho} - \partial^\sigma h^{\mu\nu}\Box h^\lambda_\nu\right) \tag{5.60}$$

This represents the 3d CS action to lowest order of approximation. Actually the two terms in the RHS of (5.64) are separately invariant under a SCT. What determines the relative - sign is the em tensor conservation.

Remark. The examples of CFT correlators we have met before (5.59) were polynomials of the coordinates divided by powers of the relative distances between the insertion points (or their Fourier transforms). Eq. (5.59) and Eqs. (5.64) below represent a new kind of correlators, which correspond to polynomials of the momenta in momentum space and to completely localized expressions in coordinate space, that is expressions made solely of delta functions and derivatives of the latter. Such expressions are called *contact terms*. The previous ones, like the even parity structures in $3d$, are called *nonlocal terms*. In CFT there appear also *partially local correlators* in which both characteristics are present. Non-local structures contain singularities when the insertion points coincide, which require a regularization. Unregulated correlators are also referred to as *bare*. Regularizing such bare correlators may lead to the appearance of contact terms.

5.3.2 Appendix B. SCT WI in momentum space

In this appendix the tensorial structures (5.58) are shown to satisfy the SCT WI in momentum space.

For the even structures we have

$$\begin{aligned} b\cdot K\, \frac{k_\mu k_\nu k_\lambda k_\rho}{|k|} &= (d-3)\frac{b_\mu k_\nu k_\lambda k_\rho + k_\mu b_\nu k_\lambda k_\rho + k_\mu k_\nu b_\lambda k_\rho + k_\mu k_\nu k_\lambda b_\rho}{|k|} \\ &\quad -(d-3) b\cdot k\, \frac{k_\mu k_\nu k_\lambda k_\rho}{|k|^3}, \end{aligned} \tag{5.61}$$

$$b\cdot K\, \frac{k_\mu k_\nu k^2}{|k|} = (d-3)(b_\mu k_\nu + k_\mu b_\nu)|k| + (d-3)\frac{b\cdot k}{|k|} k_\mu k_\nu \tag{5.62}$$

and

$$b\cdot K\, |k|^3 = 3(d-3) b\cdot k\, |k| \tag{5.63}$$

Therefore the even (nonlocal) tensorial structures (5.58) satisfy the SCT WI (in $d=3$).

The third tensorial structure in 3d is parity-odd and local

$$\langle \tilde T_{\mu\nu}(k) \tilde T_{\lambda\rho}(-k)\rangle \sim \epsilon_{\mu\lambda\sigma} k^\sigma \left(k_\nu k_\rho - \eta_{\nu\rho} k^2\right) + \begin{pmatrix} \mu \leftrightarrow \nu \\ \lambda \leftrightarrow \rho \end{pmatrix} \equiv \tilde F_{\mu\nu\lambda\rho}(k) \tag{5.64}$$

Acting on it with $b\cdot K$ we find

$$\begin{aligned} b\cdot K\, \tilde F_{\mu\nu\lambda\rho} &= \left(-2k\cdot\tilde\partial\, b\cdot\tilde\partial + b\cdot k \tilde\Box\right) \tilde F_{\mu\nu\lambda\rho} + 2(b_\mu \tilde\partial^\tau - b^\tau \tilde\partial_\mu)\tilde F_{\tau\nu\lambda\rho} \\ &\quad + 2(b_\nu \tilde\partial^\tau - b^\tau\tilde\partial_\nu)\tilde F_{\mu\tau\lambda\rho} \\ &= -2(d-2)\, b\cdot k\, \epsilon_{\mu\lambda\sigma} k^\sigma \eta_{\nu\rho} - 2b^\sigma \epsilon_{\sigma\mu\lambda}(k_\nu k_\rho - \eta_{\nu\rho}k^2) \\ &\quad +2(d-2) k^\sigma \epsilon_{\sigma\mu\lambda} b_\nu k_\rho + 2\, b^\tau k^\sigma \epsilon_{\lambda\tau\sigma}(k_\rho \eta_{\mu\nu} + k_\nu \eta_{\mu\rho}) \\ &\quad +4 b^\tau k^\sigma \epsilon_{\tau\lambda\sigma} k_\mu \eta_{\nu\rho} + \begin{pmatrix} \mu \leftrightarrow \nu \\ \lambda \leftrightarrow \rho \end{pmatrix} \end{aligned} \tag{5.65}$$

This vanishes thanks to the identities

$$b^\sigma \epsilon_{\sigma\mu\lambda} k_\nu - b_\nu \epsilon_{\tau\mu\lambda} k^\tau + b^\tau \epsilon_{\tau\lambda\sigma} k^\sigma \eta_{\mu\nu} - b^\tau \epsilon_{\tau\mu\sigma} k^\sigma \eta_{\nu\lambda} = 0 \tag{5.66}$$

$$b^\sigma \epsilon_{\sigma\mu\lambda} k^2 + b^\sigma b^\sigma \epsilon_{\sigma\lambda\tau} k_\mu k^\tau - b^\sigma \epsilon_{\sigma\mu\tau} k_\tau k_\lambda - b\cdot k\, k^\tau \epsilon_{\tau\mu\lambda} = 0 \tag{5.67}$$

which are consequences of

$$\eta_{\mu\nu}\epsilon_{\lambda\rho\sigma} - \eta_{\mu\lambda}\epsilon_{\nu\rho\sigma} + \eta_{\mu\rho}\epsilon_{\nu\lambda\sigma} - \eta_{\mu\sigma}\epsilon_{\nu\lambda\rho} = 0$$

Chapter 6

OPE and bootstrap

CFT in $2d$ is strongly constrained, so constrained that in many cases it can be solved exactly, that is all the correlators can be determined. In $d \geq 3$ the symmetry group is finite and the theory less constrained. However there still exist constraints that allow to extract significant information. They are known as *conformal bootstrap* or *OPE associativity* or, also, *crossing symmetry*. This chapter will first introduce and illustrate the properties of OPE in $d \geq 3$, then discuss the consistency condition or bootstrap.

6.1 OPE

We have already seen the operator product expansion at work in $2d$ and what powerful tool it is. This is true also in higher dimensions. The notion of OPE has been introduced at the beginning, (1.19). It was pointed out there that the series in (1.19) in ordinary field theories may be asymptotic (non convergent). In a CFT such series are instead convergent. From now on let us suppose that the fields $\varphi_1(x)$ and $\varphi_2(y)$ are primary with conformal weight Δ_1 and Δ_2. The sequence of fields φ_i on the RHS includes both primary and secondary fields. In CFT, given the latter, it is possible to determine exactly the coefficients $C_n(x, y)$ belonging to a given conformal family. Let us denote the primaries by $\mathcal{O}_n$. We will write

$$\varphi_1(x)\varphi_2(0) = \sum_n C_n(x, \partial)\mathcal{O}_n(0) \tag{6.1}$$

where $C_n(x, \partial)$ represents in a compact form the full OPE for the family of $\mathcal{O}_n$. Let us see how we can explicitly compute $C_n(x, \partial)$. Pick in the RHS one particular primary $\mathcal{O}$ with weight $\Delta_{\mathcal{O}}$. We suppose for simplicity that φ_1, φ_2 and $\mathcal{O}$ are scalar fields. So

$$\varphi_1(x)\varphi_2(0)|0\rangle = \frac{\lambda_{\mathcal{O}}}{|x|^\delta}(\mathcal{O}(0) + \ldots)|0\rangle \tag{6.2}$$

We wish to determine δ. To this end we apply D to both sides and use (4.1). On the LHS of (6.2) we have

$$iD\left(\varphi_1(x)\varphi_2(0)|0\rangle\right) = i\left([D,\varphi_1(x)]\varphi_2(0)+\varphi_1(x)[D,\varphi_2(0)]\right)|0\rangle \tag{6.3}$$
$$= (\Delta_1+\Delta_2)\left(\varphi_1(x)\varphi_2(0)+x\cdot\partial\varphi_1(x)\varphi_2(0)\right)|0\rangle$$

and, inserting (6.2) in the RHS,

$$= (\Delta_1+\Delta_2-\delta)\frac{\lambda_{\mathcal{O}}}{|x|^{\delta}}(\mathcal{O}(0)+\ldots)|0\rangle \tag{6.4}$$

On the other hand applying D to the RHS of (6.2) we get

$$iD\left(\frac{\lambda_{\mathcal{O}}}{|x|^{\delta}}\mathcal{O}(0)|0\rangle\right) = \Delta_{\mathcal{O}}\frac{\lambda_{\mathcal{O}}}{|x|^{\delta}}(\mathcal{O}(0)|0\rangle \tag{6.5}$$

Therefore

$$\delta = \Delta_1+\Delta_2-\Delta_{\mathcal{O}} \tag{6.6}$$

Going to the next order we will have

$$\varphi_1(x)\varphi_2(0)|0\rangle = \frac{\lambda_{\mathcal{O}}}{|x|^{\delta}}(\mathcal{O}(0)+b\,x\cdot\partial\mathcal{O}(0)+\ldots)|0\rangle \tag{6.7}$$

Now we apply K_μ to both sides. On the LHS we obtain

$$iK_\mu\left(\varphi_1(x)\varphi_2(0)|0\rangle\right) = (2\Delta_1-\delta)x_\mu\left(\frac{\lambda_{\mathcal{O}}}{|x|^{\delta}}(\mathcal{O}(0)+b\,x\cdot\partial\mathcal{O}(0)+\ldots\right)|0\rangle \tag{6.8}$$

while on the RHS

$$iK_\mu\left(\frac{\lambda_{\mathcal{O}}}{|x|^{\delta}}(\mathcal{O}(0)+bx\cdot\partial\mathcal{O}(0)+\ldots)|0\rangle\right) = \left(\frac{\lambda_{\mathcal{O}}}{|x|^{\delta}}\,2b\,x_\mu\,\Delta_{\mathcal{O}}\,\mathcal{O}(0)+\ldots\right)|0\rangle \tag{6.9}$$

It follows that $b=\frac{\Delta_{\mathcal{O}}+\Delta_1-\Delta_2}{2\Delta_{\mathcal{O}}}$. One can calculate the subsequent coefficients by repeatedly applying (4.1). The constant $\lambda_{\mathcal{O}}$ cannot be determined in this way. It is characteristic of the family $\mathcal{O}$ and one of the parameters that define the theory. As we will see it corresponds to the normalization of the two-point function of $\mathcal{O}$.

Let us apply the previous results to the three-point function $\langle\Phi(y)\varphi_1(x)\varphi_2(0)\rangle$ where Φ is a primary with weight Δ_Φ. Using OPE in the product of the last two fields

$$\langle\Phi(y)\varphi_1(x)\varphi_2(0)\rangle = \sum_n C_n(x,\partial)\langle\Phi(y)\mathcal{O}_n(0)\rangle \tag{6.10}$$

The two-point functions in the RHS is nonvanishing only when $\mathcal{O}_n = \Phi$, i.e.

$$\langle \Phi(y)\varphi_1(x)\varphi_2(0)\rangle = \lambda_\Phi C_\Phi(x,\partial)\langle \Phi(y)\Phi(0)\rangle \tag{6.11}$$

where it is understood that

$$\langle \Phi(y)\Phi(x)\rangle = \frac{1}{|y-x|^{2\Delta_\Phi}} \tag{6.12}$$

In other words, singling out λ_Φ in (6.11), we have chosen to normalize to 1 the 2-pt function in (6.12).

The expression $C_\Phi(x,\partial)$ in (6.11) is calculated as above. However now we have a more convenient way to evaluate it. Since we know the general expression for the three-point function, (4.31) , and thus a conformal covariant expression for $\langle \Phi(y)\varphi_1(x)\varphi_2(0)\rangle$, we can replace this on the LHS and rewrite the latter as

$$\langle \Phi(y)\varphi_1(x)\varphi_2(0)\rangle = \frac{c_{\Phi 12}}{|x|^{\Delta_1+\Delta_2-\Delta_\Phi}}\frac{1}{|y|^{2\Delta_\Phi}}\frac{|y|^{\Delta_\Phi+\Delta_1-\Delta_2}}{|y-x|^{\Delta_\Phi+\Delta_1-\Delta_2}} \tag{6.13}$$

Then we replace (6.12) in the RHS of (6.11) and expand around y. It is clear that $c_{\Phi 12} = \lambda_\Phi$, and equating the homogeneous terms, we can determine in this way all the coefficients of $C_\Phi(x,\partial)$.

Remark. It is clear that, inserting the OPE of two adjacent fields in a correlator of n primaries, the latter can be reduced, with the help $C_{\mathcal{O}_n}(x,\partial)$, to a superposition of correlators with n-1 primaries. On the other hand $n-1$-order correlators can be reduced to $n-2$-order ones and so forth, down to three-point correlators. Now suppose we give the following data for a theory: the primaries, that is their weights and two-point function normalizations $\lambda_{\mathcal{O}}$, and so all the OPE coefficients (the so-called *conformal data*). Proceeding in the way just outlined we can calculate all the correlators and so completely determine the theory. The natural question is: can these data be arbitrary? The answer is no, because certain consistency conditions must be satisfied. This is what we are going to discuss next.

6.2 OPE associativity or conformal bootstrap

Let us consider the four-point function of primary fields φ_i, $i = 1,...,4$:

$$\langle \varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\varphi_4(x_4)\rangle. \tag{6.14}$$

We can replace the first and second products with their OPE's

$$\varphi_1(x_1)\varphi_2(x_2) = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}} C_{\mathcal{O}}(x_{12}, \partial_y)\mathcal{O}(y) \tag{6.15}$$

$$\varphi_3(x_3)\varphi_4(x_4) = \sum_{\mathcal{O}'} \lambda_{34\mathcal{O}'} C_{\mathcal{O}'}(x_{34}, \partial_z)\mathcal{O}'(z) \tag{6.16}$$

where $x_{ij} = x_i - x_j$ and $y = x_2, z = x_4$. This is a simplified notation: we understand possible tensor indices in the primary operators $\mathcal{O}$. So, using the same argument as above,

$$\begin{aligned}&\langle\varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\varphi_4(x_4)\rangle\\ &= \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}\Big(C_{\mathcal{O}}(x_{12}, \partial_y)C_{\mathcal{O}}(x_{34}, \partial_z)\langle\mathcal{O}(y)\mathcal{O}(z)\rangle\Big)\end{aligned} \tag{6.17}$$

We could have done the OPE in another way, 13 and 24, and obtained

$$\begin{aligned}&\langle\varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\varphi_4(x_4)\rangle\\ &= \sum_{\mathcal{O}} \lambda_{13\mathcal{O}}\lambda_{24\mathcal{O}}\Big(C_{\mathcal{O}}(x_{13}, \partial_w)C_{\mathcal{O}}(x_{24}, \partial_z)\langle\mathcal{O}(w)\mathcal{O}(z)\rangle\Big)\end{aligned} \tag{6.18}$$

where $w = x_3$. Or even

$$\begin{aligned}&\langle\varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\varphi_4(x_4)\rangle\\ &= \sum_{\mathcal{O}} \lambda_{14\mathcal{O}}\lambda_{23\mathcal{O}}\Big(C_{\mathcal{O}}(x_{14}, \partial_z)C_{\mathcal{O}}(x_{23}, \partial_w)\langle\mathcal{O}(z)\mathcal{O}(w)\rangle\Big)\end{aligned} \tag{6.19}$$

These expansions are reminiscent of the s-, u- and t-channel pictures in S-matrix theory, so they may be called s-, u- and t-channel expansions.

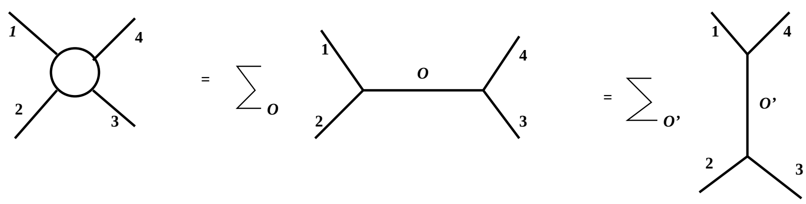

Figure 6.1: *s- and t- channel expansions.*

The expressions in $\Big(\quad\Big)$ brackets in (6.17) and (6.18) are known as conformal partial waves (CPWs).

Of course (6.17,6.18) and (6.19) must be the same, and such set of relations for all correlators of a theory are strong constraints on the conformal data. Similar constraints must be satisfied also for higher order correlators. Such constraints are called *OPE associativity* or conformal bootstrap. An important result is the following:

Theorem. If OPE associativity is satisfied for four-point functions, it is satisfied for all higher order correlators.

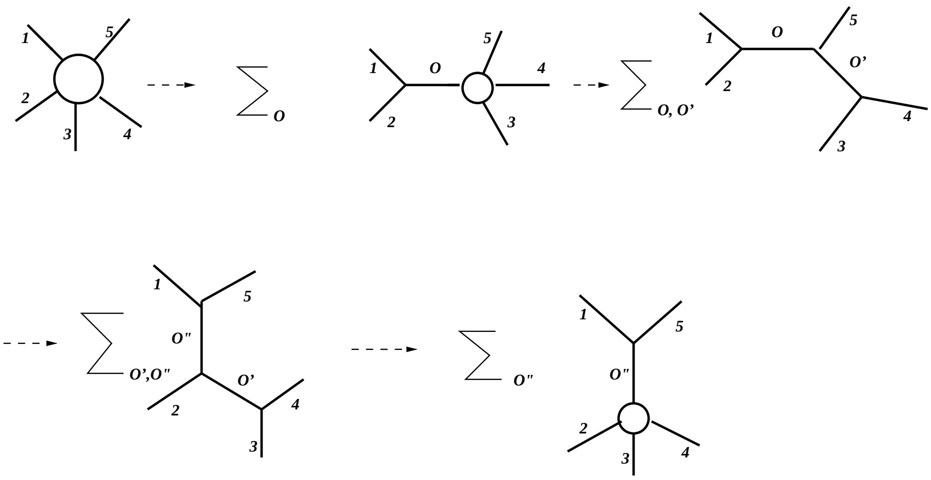

Figure 6.2: *An illustration of the Theorem for five-point functions.*

Fig.(6.2) is an explicit example of the arguments used to prove it. The first step understands the OPE of the first two fields like in the first step of Fig.(6.1), which amounts to an expansion over the set of operators $\mathcal{O}$. In the second step one does the OPE of each $\mathcal{O}$ with the field 5, which amounts to an expansion over another set of operators $\mathcal{O}'$. Then one uses the equivalence illustrated in Fig.(6.1) and replaces the sum over $\mathcal{O}, \mathcal{O}'$ with a sum over $\mathcal{O}', \mathcal{O}''$. Finally one does the inverse of the first step for the fields $\mathcal{O}'', 2, 3, 4$.

In the discussion of bootstrap another concept is useful, that of *conformal block* (CB). Let us consider the four-point function (6.14), where for simplicity all primaries have weight δ. We can write it as follows

$$\langle \varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\varphi_4(x_4)\rangle = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}} \frac{G_{\mathcal{O}}(u,v)}{x_{12}^{2\delta}x_{34}^{2\delta}}$$

The dimensionless quantity $G_{\mathcal{O}}(u, v)$ is called *conformal block*.

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Part III

2d conformal field theory and string theory

String theory is based on conformal symmetry. It is in fact a particular conformal field theory whose distinctive feature is the vanishing central charge. In its simplest formulation bosonic string theory is a theory of free scalar fields and (Faddeev-Popov) ghosts. Each scalar field is interpreted as a coordinate in the target space-time and the ghosts guarantee the vanishing of the central charge in 26 dimensions. This third part of the lectures contain a simple introduction to bosonic string theory (both open and closed). The lectures start with the Polyakov action and its properties, in which the vanishing e.m. tensor is an on-shell constraint. Gauge fixing it leads to a simplified formulation as a theory of free bosonic fields with such a constraint, and this in turn leads to the old covariant quantization: the string oscillators become creation and annihilation operators, which acting on the vacuum create the Fock space of states. Many of the latter turn out to have negative norm and the problem arises as to whether the constraint, which in the quantum theory is translated into the 'physicality conditions', can eliminate such negative norm states. This turns out to be possible only if the spacetime dimension is 26. In d=26 the spectrum of open string theory contains a tachyon, a Maxwell vector state and an infinite tower of massive states, while the closed string contains a tachyon, a gravity multiplet and an infinite tower of massive states.

The old covariant quantization is an incomplete presentation of string theory. The method of BRST quantization shows that a complete formulation of the theory requires, beside the 26 bosonic fields, also the presence of the $b-c$ ghost system. Its central charge exactly matches minus the central charge of the bosonic fields, so that the overall central charge of the theory vanishes. This is important and marks the difference with the CFT models discussed in the previous part. In those model the central charge is nonvanishing: a nonvanishing central charge implies a nontrivial trace anomaly (proportional by the central charge to the 2d curvature, see part IV). However such trace anomalies are effective only in presence of a nontrivial background metric. If the background metric is flat, as it is the case for the above models, the anomaly vanishes. In the case of string theory, instead, nontrivial background metrics play a crucial role and the theory makes sense as a gauge theory only when the trace anomaly vanishes, i.e. when the central chage vanishes.

Chapter 7

The Polyakov action

To introduce the string action we need two spaces: the proper two-dimensional space spacetime Σ, with local coordinate τ and σ, endowed with a metric $h_{\alpha\beta}$; and a D-dimensional target spacetime M with local coordinates x^μ, $\mu = 0, 1, \ldots, D-1$. For the time being we will assume M to be a flat Minkowski spacetime, with metric $\eta_{\mu\nu}$. The signature of both metrics is of the type (–,+,...,+). The action for the bosonic string is the action of maps ϕ from Σ to M.

$$S = -\frac{T}{2}\int_\Sigma d^2\sigma\,\sqrt{h}\,h^{\alpha\beta}(\tau,\sigma)\partial_\alpha X^\mu(\tau,\sigma)\,\partial_\beta X^\nu(\tau,\sigma)\,\eta_{\mu\nu} \tag{7.1}$$

where $d^2\sigma = d\tau d\sigma$. T is the string tension, it has the dimension of a square mass and measures the scale of the energy density distributed along the string. The indices α and β take value 0 and 1, where zero represents τ and 1 represents σ; we will also use the notation $\sigma^0 \equiv \tau$, $\sigma^1 \equiv \sigma$; h represents (the absolute value of) the determinant of the matrix $h_{\alpha\beta}(\tau,\sigma)$. In this action the fields $X^\mu(\sigma,\tau)$ are the coordinates of the image of the point in Σ with coordinates (σ,τ) under the map $\phi : \Sigma \longrightarrow M$, i.e. $X^\mu(\sigma,\tau) = x^\mu(\phi(\sigma,\tau))$.

The action (7.1) is invariant under the following local gauge transformations: 1) diffeomorphisms, whose infinitesimal version is specified by

$$\begin{aligned}\delta_{_D} X^\mu &= \xi^\alpha\partial_\alpha X^\mu \\ \delta_{_D} h_{\alpha\beta} &= \xi^\gamma\partial_\gamma h_{\alpha\beta} + \partial_\alpha\xi^\gamma h_{\gamma\beta} + \partial_\beta\xi^\gamma h_{\alpha\gamma}\end{aligned} \tag{7.2}$$

where $\xi^\alpha = \xi^\alpha(\tau,\sigma)$ are the infinitesimal parameters of a group which will be denoted by $\mathcal{D}$. The transformations (7.2) correspond to the infinitesimal coordinate transformations

$$\sigma^\alpha \to \tilde\sigma^\alpha = \sigma^\alpha + \xi^\alpha$$

2) local Weyl transformations, whose infinitesimal version is

$$\begin{aligned} \delta_W X^\mu &= 0 \\ \delta_W h_{\alpha\beta} &= 2\omega h_{\alpha\beta}, \end{aligned} \tag{7.3}$$

where $\omega = \omega(\tau, \sigma)$ is the infinitesimal parameter of an Abelian group denoted by $\mathcal{W}$.

The overall local symmetry group is the semidirect product of the two, $\mathcal{D} \text{ o } \mathcal{W}$.

The Poincaré invariance of the target spacetime translates into a rigid symmetry of the action, represented by the infinitesimal transformations:

$$\delta_P X^\mu = a^\mu{}_\nu X^\nu + b^\mu, \qquad \delta_P h_{\alpha\beta} = 0 \tag{7.4}$$

where $a_{\mu\nu} = -a_{\nu\mu}$ is a constant antisymmetric matrix and b_μ are constant parameters.

Concerning Σ, we will consider two cases:

a) an infinite strip: $-\infty < \tau < +\infty, \quad 0 \le \sigma \le \pi$ in the open string case;

b) an infinite cylinder: $-\infty < \tau < +\infty, \quad 0 \le \sigma \le \pi$ and the identification of $\sigma = 0$ with $\sigma = \pi$ in the closed string case.

7.1 The energy-momentum tensor

The energy-momentum tensor for the bosonic string action is

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{h}}\frac{\delta S}{\delta h^{\alpha\beta}} = \left(\partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X^\nu\right) \eta_{\mu\nu} \tag{7.5}$$

To derive this one has to use in particular $\frac{\delta}{\delta h^{\alpha\beta}}\sqrt{h} = -\frac{1}{2} h_{\alpha\beta}\sqrt{h}$. It is easy to verify that

$$h^{\alpha\beta} T_{\alpha\beta} = 0 \tag{7.6}$$

This is due to Weyl invariance, for $0 = \delta_W S \sim \int T_{\alpha\beta}\, \delta h^{\alpha\beta} = -2 \int \omega\, T_\alpha{}^\alpha$. It is to be remarked that (7.6) holds off-shell.

As a consequence of the definition (7.5), the equation of motion (EOM) corresponding to the $\delta h_{\alpha\beta}$ variation is

$$T_{\alpha\beta} = 0 \tag{7.7}$$

This EOM is actually a constraint, as a consequence of the fact that $h_{\alpha\beta}$ appears in (7.1) as a non-dynamical variable.

The connection of the Polyakov action with the Nambu-Goto action is explained in Appendix C.

Notation

A dot, $\dot{\varphi}$, denotes a derivative with respect to τ. A prime, X', denotes a derivative with respect to σ. A central dot $X \cdot Y$ denotes contraction: $X \cdot Y = X^\mu Y^\nu \eta_{\mu\nu}$, and $X^2 = X \cdot X$.

7.2 Classical string theory

This section is a preparation of the so-called old covariant quantization (OCQ). The quantization referred to as *old covariant* is a heuristic way of quantizing the field theory of strings in 2D. It consists in fixing the gauge freedom by hand (without taking into account the functional Jacobian factor implied in such operation, see the BRST quantization) and carrying out the canonical quantization on the resulting classical theory. This section is devoted to the classical background underlying the OCQ

7.2.1 Classical background

To fix the gauge in (7.1) we trade the three gauge degrees of freedom ξ^α and ω for the three metric degrees of freedom $h_{\alpha\beta}$ by fixing the latter as follows

$$h_{\alpha\beta} = \eta_{\alpha\beta}, \tag{7.8}$$

where $\eta_{\alpha\beta}$ is the 2D Minkowski metric. As a result the action becomes

$$S_{gf} = -\frac{T}{2}\int d^2\sigma\, \eta^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X = -\frac{T}{2}\int d^2\sigma\, \left(X' \cdot X' - \dot{X} \cdot \dot{X}\right) \tag{7.9}$$

Setting to zero the variation with respect to X^μ one gets

$$0 = \delta S_{gf} = T\int d^2\sigma\, \eta^{\alpha\beta}\partial_\alpha\partial_\beta X \cdot \delta X - T\int d\tau X' \cdot \delta X\Big|_{\partial\Sigma}$$

where $\partial\Sigma$ denotes the boundary of Σ (i.e. $\sigma = 0, \pi$). This implies the *Euler-Lagrange equation of motion*

$$\Box X^\mu = \eta^{\alpha\beta}\partial_\alpha\partial_\beta X^\mu = \left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu = 0 \tag{7.10}$$

plus the *boundary condition*

$$\int d\tau X' \cdot \delta X|_{\sigma=0}^{\sigma=\pi} = \int d\tau X' \cdot \delta X\Big|_{\sigma=\pi} - \int d\tau X' \cdot \delta X\Big|_{\sigma=0} = 0 \tag{7.11}$$

Of course we understand that the analogous condition for τ are trivial: X^μ is supposed to vanish at infinite positive or negative τ.

We will consider two different ways of satisfying (7.11).

- Closed string boundary condition (CSbc).

  Closed string theory requires periodic X^μ:

$$X^\mu(\tau, 0) = X^\mu(\tau, \pi) \tag{7.12}$$

  This satisfies (7.11).

- Open string boundary conditions (OSbc).

  The endpoints of open strings independent of each other, therefore one can consider at least two possiblities:

$$\text{either} \qquad X'_\mu\big|_{\partial\Sigma} = 0, \qquad \text{(N)} \tag{7.13}$$
$$\text{or} \qquad X_\mu\big|_{\partial\Sigma} = const, \qquad \text{(D)} \tag{7.14}$$

  They both satisfy (7.11). They are referred to as Neumann (N) and Dirichlet (D) boundary conditions, respectively. The latter can be expressed also as $\dot{X}^\mu\big|_{\partial\Sigma} = 0$. Neumann boundary conditions express a continuity requirement: there cannot be an outgoing or incoming flow of matter at the endpoints of a string. In the Dirichlet case the endpoints are fixed at constant values.

  Since the boundary consists of two disconnected elements, $\sigma = 0$ and $\sigma = \pi$, one can actually envisage four possibilities, instead of two as above,

$$X'_\mu\big|_{\sigma=0} = 0 = X'_\mu\big|_{\sigma=\pi}, \qquad \text{(NN)} \tag{7.15}$$
$$X'_\mu\big|_{\sigma=0} = 0 = \dot{X}_\mu\big|_{\sigma=\pi}, \qquad \text{(ND)} \tag{7.16}$$
$$\dot{X}_\mu\big|_{\sigma=0} = 0 = X'_\mu\big|_{\sigma=\pi}, \qquad \text{(DN)} \tag{7.17}$$
$$\dot{X}_\mu\big|_{\sigma=0} = 0 = \dot{X}_\mu\big|_{\sigma=\pi} \qquad \text{(DD)} \tag{7.18}$$

Beside the *equation of motion* and the appropriate *boundary conditions*, we have the *constraint equations* (7.7). These are the three elements that characterize classical strings. Solving the equation of motion and the constraint equations, subject to the appropriate boundary conditions, completely determine the admissible string configurations.

7.3 Classical solution

The equation of motion (7.10) takes an even simpler form if we use 2D light cone coordinates $\sigma^\pm = \tau \pm \sigma$. We have, for instance,

$$\partial_\pm \equiv \frac{\partial}{\partial\sigma^\pm} \equiv \frac{1}{2}\left(\frac{\partial}{\partial_\tau} \pm \frac{\partial}{\partial_\sigma}\right).$$

Then the equation of motion becomes

$$\partial_+\partial_- X^\mu = 0 \tag{7.19}$$

The generic solution to this wave equation is well-known, it is the sum of one ingoing and one outgoing wave. Therefore the most general solution can be written as

$$X^\mu(\tau,\sigma) = X_+^\mu(\sigma_+) + X_-^\mu(\sigma_-) \tag{7.20}$$

where $X_\pm^\mu$ are generic (smooth) functions of their arguments. Together with the equation of motion and the boundary conditions, also the constraints equations must be satisfied. In the gauge fixed version they take the form

$$T_{01} = T_{10} \equiv \dot{X}\cdot X' = 0 \tag{7.21}$$

$$T_{00} = T_{11} \equiv \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0 \tag{7.22}$$

and in l.c. coordinates

$$T_{++} = \frac{1}{2}\left(T_{00} + T_{01}\right) \equiv \partial_+ X\cdot\partial_+ X = 0 \tag{7.23}$$

$$T_{--} = \frac{1}{2}\left(T_{00} - T_{01}\right) \equiv \partial_- X\cdot\partial_- X = 0 \tag{7.24}$$

The equation

$$T_{+-} = T_{-+} = \frac{1}{4}\left(T_{00} - T_{11}\right) \equiv -\frac{1}{4}T_\alpha{}^\alpha = 0 \tag{7.25}$$

is satisfied even off-shell, as remarked above, so it is not really a constraint.

Beside the (off shell) vanishing of the energy-momentum trace we have also the following conservation laws

$$\partial_- T_{++} = 0, \qquad \partial_+ T_{--} = 0 \tag{7.26}$$

which hold *on shell*. These are the consequence of a *residual symmetry*, which will be discussed shortly.

7.3.1 Residual symmetry

As it often happens in field theory the gauge fixing condition (7.8) we have chosen does not completely destroy the gauge symmetry. There remains a residual symmetry, which manifests itself through the existence (on shell) of the conserved laws (7.26).

Let us describe this residual symmetry in detail. Since the theory is invariant under both diffeomorphisms and Weyl transformations, one can imagine that there exist diffeomorphisms that compensate the effect of a Weyl transformation. This is actually the case. If, for simplicity, we start from the flat metric $\eta_{\alpha\beta}$, such transformations are determined by

$$\delta_{_D}\eta_{\alpha\beta} = \partial_\beta\xi_\alpha + \partial_\alpha\xi_\beta = -2\omega\,\eta_{\alpha\beta} = -\delta_{_W}\eta_{\alpha\beta} \tag{7.27}$$

where $\xi_\alpha = \eta_{\alpha\beta}\xi^\beta$. This leaves the action (7.9) unchanged *on shell*. Such ξ_α satisfy the following equations

$$\begin{aligned}\partial_\tau\xi_0 + \partial_\sigma\xi_1 = 0\\ \partial_\tau\xi_1 + \partial_\sigma\xi_0 = 0\end{aligned}$$

Defining

$$\xi^\pm = \xi^1 \pm \xi^0,$$

such transformations are therefore seen to satisfy

$$\partial_+\xi^- = 0, \qquad \partial_-\xi^+ = 0 \tag{7.28}$$

In other words $\xi^+ = \xi^+(\sigma^+)$ and $\xi^- = \xi^-(\sigma^-)$. These are the *conformal* transformations.

7.3.2 The symplectic structure

Another important ingredient, which is needed for the canonical quantization, is the symplectic structure of the two-dimensional string theory. The conjugate momentum for X^μ is

$$\Pi^\mu = \frac{\delta S}{\delta \dot{X}_\mu} = T\dot{X}^\mu \tag{7.29}$$

Thus the equal time canonical Poisson brackets are

$$\begin{aligned}&[\dot{X}^\mu(\tau,\sigma), X^\nu(\tau,\sigma')]_{_{PB}} = T^{-1}\eta^{\mu\nu}\delta(\sigma-\sigma')\\ &[X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')]_{_{PB}} = 0, \qquad [\dot{X}^\mu(\tau,\sigma), \dot{X}^\nu(\tau,\sigma')]_{_{PB}} = 0\end{aligned} \tag{7.30}$$

These will be used shortly.

7.4 Closed string oscillators

The reason why a particle physics interpretation of string theory is possible is due to string oscillators. Any string oscillator in the quantized theory will be interpreted as creation or annihilation operator of some kind of particle fluctuation. Mathematically string oscillators are coefficients of the Fourier expansion of solutions to the equation of motion. For a reason that will become clear, it is simpler to start with closed strings. We consider separately the X_+^μ and X_-^μ of a solution to the equation of motion. They are continuous functions of σ_+ and σ_-, respectively. They can therefore be expanded on a complete basis of eigenfunctions $e^{-2in\sigma_\pm}$, $n \in \mathbb{Z}$, periodic in σ. We write

$$X_+^\mu(\sigma_+) = \frac{1}{2}x_+^\mu + \frac{\ell^2}{2}p_+^\mu\sigma_+ + \frac{i}{2}\ell\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-2in\sigma_+} \tag{7.31}$$

$$X_-^\mu(\sigma_-) = \frac{1}{2}x_-^\mu + \frac{\ell^2}{2}p_-^\mu\sigma_- + \frac{i}{2}\ell\sum_{n\neq 0}\frac{\tilde{\alpha}_n^\mu}{n}e^{-2in\sigma_-} \tag{7.32}$$

where $\ell = \frac{1}{\sqrt{\pi T}}$ has the dimension of a length. In this way all the α_n^μ and $\tilde{\alpha}_n^\mu$ coefficients are dimensionless; $x_\pm^\mu$ has the dimension of a length and $p_\pm^\mu$ the dimension of a mass. For consistency we will set $x_+^\mu = x_-^\mu = x^\mu$ and $p_+^\mu = p_-^\mu = p^\mu$. To be precise, (7.31,7.32) are not periodic in σ due to the linear term, which is naturally annihilated by the $\Box$ operator. However the complete expansion of X^μ is periodic

$$X^\mu(\tau,\sigma) = x^\mu + \ell^2 p^\mu\tau + \frac{i}{2}\ell\sum_{n\neq 0}\left(\frac{\alpha_n^\mu}{n}e^{-2in(\tau+\sigma)} + \frac{\tilde{\alpha}_n^\mu}{n}e^{-2in(\tau-\sigma)}\right) \tag{7.33}$$

Integrating this in σ from 0 to π we get

$$\int_0^\pi d\sigma\, X^\mu(\tau,\sigma) = \pi\, x^\mu + \pi\,\ell^2 p^\mu\tau \tag{7.34}$$

Therefore x^μ and p^μ can be interpreted as the center of mass position and momentum of the string.

Reality condition

Requiring X^μ to be real implies that x^μ and p^μ are real and

$$(\alpha_n^\mu)^* = \alpha_{-n}^\mu, \qquad (\tilde{\alpha}_n^\mu)^* = \tilde{\alpha}_{-n}^\mu \tag{7.35}$$

It is often convenient to set

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu \equiv \frac{\ell}{2}p^\mu \tag{7.36}$$

Poisson brackets for closed string modes

The Poisson brackets for oscillators can be derived from the symplectic structure (7.30). The result is

$$\begin{aligned}
&[\alpha^\mu_m, \alpha^\nu_n]_{PB} = i\eta^{\mu\nu} m\, \delta_{m+n,0} = [\tilde\alpha^\mu_m, \tilde\alpha^\nu_n]_{PB} \\
&[\alpha^\mu_m, \tilde\alpha^\nu_n]_{PB} = 0, \qquad [p^\mu, x^\nu]_{PB} = \eta^{\mu\nu}
\end{aligned} \tag{7.37}$$

To establish the correspondence one needs the following representation for the delta function in (7.30):

$$\delta(\sigma - \sigma') = \frac{1}{\pi} \sum_{n\in\mathbb{Z}} e^{2in(\sigma-\sigma')}$$

7.5 Open string oscillators

Let us expand now the open string solution X^μ in oscillator modes. We will limit ourselves here to a solution that satisfies the Neumann boundary conditions (7.13). We cannot proceed as in the closed string case, because the solution is not periodic, but we can utilize the *doubling trick*. We define a new solution

$$\hat X^\mu(\tau,\sigma) = \begin{cases} X^\mu(\tau,\sigma) & 0 \le \sigma \le \pi \\ X^\mu(\tau,-\sigma) & -\pi \le \sigma \le 0 \end{cases} \tag{7.38}$$

This is periodic in the interval $-\pi \le \sigma \le \pi$ since $\hat X^\mu(\tau,-\pi) = \hat X^\mu(\tau,\pi)$. Moreover it is continuous at the junction $\sigma = 0$ with its first derivative due the Neumann boundary condition. This allows us to expand $\hat X^\mu$ as we have done for X^μ in the closed string case. We can write

$$\hat X^\mu(\tau,\sigma) = x^\mu + \ell^2 p^\mu \tau + \frac{i}{2}\ell \sum_{n\neq 0}\left(\frac{\alpha^\mu_n}{n} e^{-in(\tau+\sigma)} + \frac{\tilde\alpha^\mu_n}{n} e^{-in(\tau-\sigma)}\right) \tag{7.39}$$

One can verify that the Neumann condition is satisfied only if $\alpha^\mu_n = \tilde\alpha^\mu_n$. Therefore for open strings we have only one set of independent oscillators and the general solution can be written

$$X^\mu(\tau,\sigma) = x^\mu + \ell^2 p^\mu \tau + i\ell \sum_{n\neq 0} \frac{\alpha^\mu_n}{n}\, e^{-in\tau}\, \cos n\sigma \tag{7.40}$$

Reality condition

Requiring X^μ to be real implies that x^μ and p^μ are real and

$$(\alpha^\mu_n)^* = \alpha^\mu_{-n}, \tag{7.41}$$

To make the formulas more compact it will be convenient to set

$$\alpha^\mu_0 \equiv \ell p^\mu \tag{7.42}$$

Poisson brackets for open string modes

As in the closed string case the Poisson brackets for oscillators can be derived from the symplectic structure (7.30). The result is

$$[\alpha_m^\mu, \alpha_n^\nu]_{PB} = i\eta^{\mu\nu} m\, \delta_{m+n,0}, \qquad [p^\mu, x^\nu]_{PB} = \eta^{\mu\nu} \tag{7.43}$$

7.6 Global conserved charges

The global symmetry (7.4) (Poincaré spacetime symmetry) give rise to currents

$$\mathcal{P}_\alpha^\mu = T\, \partial_\alpha X^\mu \tag{7.44}$$
$$\mathcal{M}_\alpha^{\mu\nu} = T\,(X^\mu \partial_\alpha X^\nu - X^\nu \partial_\alpha X^\mu) \tag{7.45}$$

which are conserved on shell

$$\partial_\alpha \mathcal{P}^{\alpha\mu} = 0, \qquad \partial_\alpha \mathcal{M}^{\alpha\mu\nu} = 0$$

These are the conserved density of momentum and angular momentum. The corresponding charges are the string (total) momentum

$$P^\mu = \int_0^\pi d\sigma\, \mathcal{P}_0^\mu = T \int_0^\pi d\sigma\, \dot{X}^\mu = \pi T \ell^2 p^\mu = p^\mu \tag{7.46}$$

and angular momentum

$$M^{\mu\nu} = \int_0^\pi d\sigma\, \mathcal{M}_0^{\mu\nu} = T \int_0^\pi d\sigma\, \left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu\right) \tag{7.47}$$

These results hold for both open and closed strings.

One can find some explicit examples of classical strings in Appendix D.

7.7 Classical symmetry algebras

We have shown above that both the T_{++} and T_{--} components of the energy-momentum tensor are conserved. We can Fourier analyze them and obtain an infinite sequence of conserved quantities. On shell we have $T_{\pm\pm} = T_{\pm\pm}(\sigma_\pm)$. **In the closed string case** we can thus set

$$T_{++}(\sigma_+) = 2\ell^2 \sum_{n\in\mathbb{Z}} L_n e^{-2in\sigma_+}, \qquad T_{--}(\sigma_-) = 2\ell^2 \sum_{n\in\mathbb{Z}} \tilde{L}_n e^{-2in\sigma_-} \tag{7.48}$$

which imply

$$L_n = \frac{T}{2}\int_0^\pi d\sigma\, T_{++}e^{2in\sigma_+}, \qquad \tilde{L}_n = \frac{T}{2}\int_0^\pi d\sigma\, T_{--}e^{2in\sigma_-} \tag{7.49}$$

Using (7.23) and $\partial_+ X^\mu = \ell \sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-2in\sigma_+}$, we can easily obtain the Fourier moments in terms of oscillators

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha_k \cdot \alpha_{n-k} \tag{7.50}$$

Similarly

$$\tilde{L}_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{\alpha}_k \cdot \tilde{\alpha}_{n-k} \tag{7.51}$$

The constraints equations for closed strings (7.21,7.22) can now be written

$$L_n = 0, \qquad \tilde{L}_n = 0, \qquad \forall n \in \mathbb{Z} \tag{7.52}$$

Using (7.37) it is easy to see that these moments satisfy the Poisson algebra

$$[L_n, L_m]_{PB} = i(n-m)L_{n+m} \tag{7.53}$$
$$[\tilde{L}_n, \tilde{L}_m]_{PB} = i(n-m)\tilde{L}_{n+m} \tag{7.54}$$
$$[L_n, \tilde{L}_m]_{PB} = 0$$

The first and second brackets define a copy of the so-called Witt or classical Virasoro algebra.

This is a good point to recall that the Hamiltonian for closed strings is

$$\begin{aligned} H &= \int_0^\pi d\sigma\,\left(\dot{X}\cdot\Pi_0 - L\right) = \frac{T}{2}\int_0^\pi d\sigma\,\left(\dot{X}^2 + X'^2\right) \\ &= \frac{1}{2}\sum_{n\in\mathbb{Z}}(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n) = L_0 + \tilde{L}_0 \end{aligned} \tag{7.55}$$

L is the Lagrangian.

For open strings the construction of the analogous classical conserved generators is slightly more complicated due to the lack of periodicity of both T_{++} and T_{--}. We will proceed as above for X^μ. We define the new object

$$\hat{T}(\tau,\sigma) = \begin{cases} T_{++}(\sigma_+) & 0 \le \sigma \le \pi \\ T_{--}(\sigma_-) & -\pi \le \sigma \le 0 \end{cases}$$

which is periodic in the interval $-\pi \le \sigma \le \pi$. Now we can set

$$\begin{aligned} L_n &= T\int_{-\pi}^{\pi} e^{in\sigma_+}\,\hat{T}(\tau,\sigma) = 2T\int_0^{\pi} e^{in\sigma_+}\,T_{++}(\sigma_+) \\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha_k\cdot\alpha_{n-k} \end{aligned} \tag{7.56}$$

Such L_n obey the same algebra as (7.53)

$$[L_n, L_m]_{PB} = i(n-m)L_{n+m} \tag{7.57}$$

In the same way as for closed strings one can prove that the Hamiltonian for open strings is

$$H = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot\alpha_n = L_0 \tag{7.58}$$

This depends on the fact that, for open strings, $\int_0^\pi d\sigma\,\dot{X}\cdot X' = 0$.

The constraints equations for open strings can be written

$$L_n = 0, \qquad \forall n\in\mathbb{Z} \tag{7.59}$$

In conclusion, *we can say that a classical string theory solution is defined by an infinite set of numbers (the oscillators) subject to the constraints (7.52) for closed strings and (7.59) for open strings.*

Chapter 8

Old covariant quantization

The old covariant quantization is the quantization of the classical theory developed in the previous chapter. This is done by promoting the Poisson bracket to Dirac brackets

$$[\quad,\quad]_{PB} \longrightarrow i\,[\quad,\quad]$$

Thus, at the level of oscillators, for closed strings we are left with the following basic commutators

$$\begin{aligned} &[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = \eta^{\mu\nu} m\, \delta_{m+n,0} \\ &[\alpha_m^\mu, \tilde{\alpha}_n^\nu] = 0, \qquad [\hat{p}^\mu, \hat{x}^\nu] = -i\eta^{\mu\nu} \end{aligned} \tag{8.1}$$

The oscillators are promoted to creation and annihilation operators in a Fock space we will define in a moment. We have chosen to put a hat over p and x as operators, to distinguish them from their respective eigenvalues. The reality condition becomes

$$(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu, \qquad (\tilde{\alpha}_n^\mu)^\dagger = \tilde{\alpha}_{-n}^\mu$$

where a † denotes hermitean conjugation.

Similarly, for open strings, the quantum commutators are

$$[\alpha_m^\mu, \alpha_n^\nu] = \eta^{\mu\nu} m\, \delta_{m+n,0}, \qquad [\hat{p}^\mu, \hat{x}^\nu] = -i\,\eta^{\mu\nu} \tag{8.2}$$

with the following hermiticity relation

$$(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu$$

Let us continue, for the moment, with open string theory.

8.1 Old covariant quantization for open strings

We split the set of oscillators into creation operators α_n^μ with $n < 0$, and annihilation operators with $n \geq 0$. The latter annihilate the vacuum $|0\rangle$

$$\alpha_n^\mu |0\rangle = 0, \qquad n \geq 0$$

Acting with the creation operators on the vacuum we create the states of the Fock space. The generic Fock space state is a linear superposition of basis states

$$|\phi_{n_1,n_2,\ldots,n_s}^{\mu_1,\mu_2,\ldots,\mu_s}\rangle = (a_{n_1}^{\mu_1})^\dagger (a_{n_2}^{\mu_2})^\dagger \ldots (a_{n_s}^{\mu_s})^\dagger |0\rangle, \qquad n_1,\ldots,n_s > 0 \tag{8.3}$$

This space has a natural inner product defined by the algebra (8.2), by the hermiten conjugation and by

$$\langle 0| = (|0\rangle)^\dagger, \qquad \langle 0|0\rangle = 1 \tag{8.4}$$

An example will suffice. Let us consider the inner product of $(a_n^\mu)^\dagger|0\rangle$ with itself (that is its 'norm')

$$\|(a_n^\mu)^\dagger|0\rangle\|^2 = \langle 0|a_n^\mu (a_n^\mu)^\dagger|0\rangle = \langle 0|[a_n^\mu, (a_n^\mu)^\dagger]|0\rangle = \langle 0|n\,\eta^{\mu\mu}|0\rangle = n\eta^{\mu\mu}$$

(no μ summation understood). From this simple example a serious problem is already evident. The 'norm' of the chosen state is in general non-positive-definite, as it is negative in the case $\mu = 0$. The Fock space inner product is non-positive definite (so, in fact, it is not a norm, but we will keep speaking about vanishing norm and negative norm states). A non-positive norm state cannot represent a physical state, since such a norm has to be interpreted as probability. Eventually the problem will be to find out under what conditions the quantum theory we are constructing contains only positive norm states. Before proceeding in this discussion we need to introduce other elements.

Physical states carry momentum. The next ingredient we need to introduce is therefore momentum. Classically the string momentum is $p^\mu \sim a_0^\mu$. The latter, as a quantum operator $\hat{p}^\mu$, commutes with all the a_n^μ, while it does not commute with $\hat{x}^\mu$. We can assign a momentum k^μ to a state by tensorially multiplying it by an eigenstate $|k\rangle$ of $\hat{p}$. Since the states in question will be asymptotic states, we can represent $|k\rangle$ by a plane wave e^{ikx}, which is an eigenstate of $\hat{p}^\mu$ in the coordinate representation of the latter:

$$\hat{p}^\mu = -i\frac{\partial}{\partial x^\mu}, \qquad \hat{p}^\mu|k\rangle = k^\mu|k\rangle.$$

We will use the shorthand notation

$$|\phi_{n_1,n_2,\ldots,n_s}^{\mu_1,\mu_2,\ldots,\mu_s}; k\rangle = |\phi_{n_1,n_2,\ldots,n_s}^{\mu_1,\mu_2,\ldots,\mu_s}\rangle \otimes |k\rangle \tag{8.5}$$

Another ingredient is *normal ordering.* Expressions, such as L_n, are ill-defined when classical oscillators are replaced by quantum creation and annihilation operators. To eliminate the ambiguity it is necessary to specify which type of operator come first in a given expression. Any quantum expression will therefore appear in normal ordered form, meaning that in any monomial involving creation and annihilation operators the former will always feature to the left of the latter:

$$: \alpha_n^\mu \alpha_{-n}^\nu :=: \alpha_{-n}^\nu \alpha_n^\mu := \alpha_{-n}^\nu \alpha_n^\mu, \qquad n > 0 \tag{8.6}$$

It is evident that normal ordering is irrelevant for all the L_n, except L_0. For the latter we have

$$L_0 = \frac{1}{2}\sum_k : \alpha_k \cdot \alpha_{-k} := \frac{1}{2}\alpha_0^2 + \sum_{k=1} \alpha_k^\dagger \cdot \alpha_k. \tag{8.7}$$

Normal ordering has dramatic consequences on the quantum algebra corresponding to the Witt algebra. This becomes in fact

$$[L_m, L_n] = (m-n)L_{n+m} + \frac{D}{12}(m^3 - m)\,\delta_{n+m,0} \tag{8.8}$$

and is known as the *Virasoro algebra* for open strings. The additional term with respect to the Witt algebra (7.57) is the central term and D is referred to as the *central charge.* This term is generated by the fact that even though L_n and L_m are normal ordered the commutator $[L_n, L_m]$ does not automatically give rise to normal ordered terms. Reshuffling the latter one obtains both $(m-n)L_{n+m}$ and the central term. It is easy to see that each field X^μ contributes 1 to the central charge. This is universal: in 2D the central charge of a free bosonic field is 1.

8.1.1 The physicality conditions

The Fock space of states generated by the quantum oscillators contains in general also non-positive norm states. This is physically unacceptable. However so far we have not yet used a crucial ingredient of the classical theory: the constraints, which come from the property of the 2D metric $h_{\alpha\beta}$ not being dynamical. In the open string case they boil down to $L_n = 0$. First of all we remark that conditions like $L_n = 0$ cannot be imposed in the quantum theory, even though the L_n are normal ordered, because constraints for operators do not make sense. They are rather to be interpreted as restrictions on the Fock space states, i.e. a selection criterion for those states $|\phi\rangle$ that satisfy $L_n|\phi\rangle = 0$. But these conditions would be too restrictive, in view of (8.8) they would lead to a contradiction. The best we can do is to impose such type of condition on the largest possible subalgebra of the Virasoro algebra. A maximal subalgebra is spanned by all generators L_n

with $n \geq -1$. But the rearrangements of terms on L_0, see eq.(8.7), gives rise to an ambiguity, so that one cannot a priori impose the condition $L_0|\phi\rangle = 0$, but rather one has to allow for a relation $L_0|\phi\rangle = a\,|\phi\rangle$, where a is a constant to be determined from self-consistency of the theory. In conclusion the most general quantum constraints are

$$L_n|\phi;k\rangle = 0, \qquad n > 0 \tag{8.9}$$
$$L_0|\phi;k\rangle = a\,|\phi;k\rangle \tag{8.10}$$

These are referred to as *physicality conditions*. They will select a subset of Fock space states whose norm will turn out to be non-negative, at least in specific situations.

8.1.2 Analysis of the open string spectrum

Now the stage is set for analyzing the spectrum of physical states in open string theory. Let us introduce the *number operator* $N = L_0 - \frac{1}{2}\alpha_0^2 = \sum_{k=1}^{\infty} \alpha_k^\dagger \cdot \alpha_k$. It satisfies

$$[N, a_k^\dagger] = k a_k^\dagger$$

therefore it is diagonal on a basis of states such as (8.5). Its eigenvalue is precisely the sum of the lower labels of (8.5). We will call this number the *level*. It will be a useful tool to classify the Fock space states.

Using (8.3,7.42) and the definition $\hat{p}^2 = -M^2$, we can rewrite the physicality condition (8.10) as follows

$$M^2 = \frac{1}{\alpha'}(N - a) \tag{8.11}$$

which defines the square mass operator (for the sake of simplicity we will often use such equations among operators, but they have always to be understood as applied to states).

Let us start the analysis of the states according to this classification. The first level is $N = 0$. There is only one state, the vacuum with momentum $|t(k)\rangle = |0;k\rangle$. Its mass is

$$M^2 = -\frac{a}{\alpha'} \tag{8.12}$$

and its norm

$$\langle t(k)|t(k')\rangle = \delta^{(D)}(k,k')$$

with coefficient 1 in front of the delta function (we shall simply say that the norm is positive). It is easy to see that all the other physicality conditions (8.9) are satisfied. Therefore $|0;k\rangle$ is the first allowed state.

At level 1 we find D states $(\alpha_1^\mu)^\dagger|0;k\rangle$. Thus the most general state takes the form

$$|A(\zeta,k)\rangle = \eta_{\mu\nu}\zeta^\mu(\alpha_1^\nu)^\dagger|0;k\rangle = \zeta\cdot\alpha_1^\dagger|0;k\rangle$$

The vector ζ^μ is referred to as polarization. According to (8.11) this state has square mass $M^2 = \frac{1-a}{\alpha'}$. The constraint $L_1|A(\zeta,k)\rangle = 0$ is satisfied if and only if

$$\zeta\cdot k = 0, \tag{8.13}$$

while the other constraints are identically satisfied. The norm of this state is $\zeta^*\cdot\zeta$.

We could continue with level 2, but before let us determine the value of the parameter a. In spacetime the state $|A(\zeta,k)\rangle$ is an asymptotic vector state satisfying (8.13). The latter is the transversality condition between momentum and polarization typical of a massless vector boson. Since its square mass is proportional to $1-a$, it is natural to assume that

$$a = 1 \tag{8.14}$$

which seems to render everything consistent. Therefore let us assume (8.14). We will justify it later on. The consequence of this choice is that the level 0 state has square mass $-\frac{1}{\alpha'}$, and is therefore a tachyon. The level 1 state is a massless vector state, i.e. the excitation of a Maxwell field. The condition (8.13) corresponds to the covariant gauge condition $\partial^\mu A_\mu = 0$ of electrodynamics. Choosing $k^\mu = (k^0, k^1, 0, \ldots, 0)$, we see that $k^0 = |k^1|$. Thus the condition (8.13) yields $|\zeta^0| = |\zeta^1|$, which implies that the norm of $|A(\zeta,k)\rangle$ is ≥ 0 (it is zero when $\zeta^i = 0$ for $i = 2,\ldots,D-1$, i.e. when the state is longitudinal). Therefore the physical vector states have only transverse components (we call transverse the components labeled by $\mu = 2,\ldots,D-1$) and transform according to the little subgroup $SO(D-2)$ of the Lorentz group for massless states.

This is a partial manifestation of a gauge symmetry underlying string theory, which becomes fully manifest only via the BRST analysis.

It is worth remarking that other choices of a are possible. If we choose $a > 1$ the vector state is also a tachyon and $k^2 > 0$. In this case we can chose k^μ to have only space components, but, due to (8.13), in such a case we could choose ζ so that $\zeta^*\cdot\zeta < 0$, that is we would have a negative norm state (a *physical* ghost). On the other hand if $a < 1$, then the vector state has $k^2 < 0$ and k^μ can be chosen to have only the time component; in this case $\zeta^*\cdot\zeta > 0$. In conclusion a necessary condition for the absence of ghost states is

$$a \leq 1 \tag{8.15}$$

In the next level (=2) the most general state is $|B(\zeta,\theta,k)\rangle = (\zeta_\mu \alpha^\mu_{-2} + \theta_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1}|0;k\rangle$. ζ and θ are, respectively, a vector and a symmetric tensor polarizations. The square mass is $M^2 = \frac{1}{\alpha'}$. The L_1 and L_2 constraints imply the conditions:

$$\zeta^\mu + \theta^{\mu\nu}k_\nu = 0, \qquad \zeta \cdot k + \theta^\mu_\mu = 0, \tag{8.16}$$

respectively. The other constraints are identically satisfied. The norm of $|B(\zeta,\theta,k)\rangle$ is $\zeta\cdot\zeta + \theta\cdot\theta$.

It is clear that the same analysis can be carried out in principle at all levels, but complexity increases dramatically and one had better use more sophisticated tools. What is elementary to establish is the mass spectrum. There will be an infinite tower of states with square mass

$$M^2 = \frac{n}{\alpha'}, \qquad n \geq -1 \tag{8.17}$$

No-ghost theorem

We can collect additional useful suggestions about the spectrum if we limit ourselves to the dimension $D = 26$. In a $D = 26$-dimensional spacetime the number of null states definitely increases, suggesting that it is the fittest to accommodate a gauge theory. This has been confirmed by the so-called *no-ghost* theorem. It has been proved that for $D = 26$ with the choice $a = 1$ the spectrum does not contain negative norm states: there are plenty of zero norm states, the remaining ones being transverse. This is true also for $D \leq 26$ and $a \leq 1$, but in this case in general also longitudinal components appear in the spectrum and the number of zero norm states is strongly reduced.

8.2 Old covariant quantization for closed strings

The old covariant quantization is carried out much in the same way as for open string theory. The main macrosopic difference is that now we have two sets of creation and annihilation operators α^μ_n and $\tilde{\alpha}^\mu_n$, defined by

$$\alpha^\mu_n|0\rangle = 0, \qquad \tilde{\alpha}^\mu_n|0\rangle = 0 \qquad n \geq 0$$

The generic Fock space states are linear combinations of the basis formed by

$$(a^{\mu_1}_{m_1})^\dagger (a^{\mu_2}_{m_2})^\dagger \ldots (a^{\mu_s}_{m_s})^\dagger (\tilde{a}^{\nu_1}_{n_1})^\dagger (\tilde{a}^{\nu_2}_{n_2})^\dagger \ldots (\tilde{a}^{\nu_s}_{n_s})^\dagger |0;k\rangle, \quad n_1,\ldots,n_s > 0 \tag{8.18}$$

with $m_1,\ldots,m_r,n_1,\ldots,n_r > 0$. That means that the Fock space is the tensor product of two copies of the open string Fock space. In many respects closed string theory is a tensor product of two copies of open string theory, the latter being variously referred to as left(-handed) and right(-handed) or holomorphic and antiholomorphic sectors of the theory. Of course, the closed string Fock space has non-positive norm states. Once again we have to resort to the constraints in order to be able to exclude them. After introducing normal ordering exactly as in the open string case, we can verify that the L_n and $\tilde{L}_n$ generators form two independent copies of the Virasoro algebra

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{n+m} + \frac{D}{12}(m^3 - m)\,\delta_{n+m,0} \\
[\tilde{L}_m, \tilde{L}_n] &= (m-n)\tilde{L}_{n+m} + \frac{D}{12}(m^3 - m)\,\delta_{n+m,0} \\
[L_n, \tilde{L}_m] &= 0
\end{aligned} \tag{8.19}$$

Once again we cannot impose the same constraints as in the classical case, (7.52), they would contradict this algebra. By the same arguments as for open strings we impose the following physicality conditions on the generic state $|\phi; k\rangle$:

$$L_n|\phi; k\rangle = \tilde{L}_n|\phi; k\rangle = 0, \qquad n > 0 \tag{8.20}$$

$$(L_0 - a)|\phi; k\rangle = (\tilde{L}_0 - a)|\phi; k\rangle = 0 \tag{8.21}$$

Now, recalling (7.36), let us write $L_0 = \frac{\alpha'}{4}\hat{p}^2 + N$ and $\tilde{L}_0 = \frac{\alpha'}{4}\hat{p}^2 + \tilde{N}$, where

$$N = \sum_{k=1}^{\infty} \alpha_n^\dagger \cdot \alpha_n, \qquad \tilde{N} = \sum_{k=1}^{\infty} \tilde{\alpha}_n^\dagger \cdot \tilde{\alpha}_0$$

Taking the sum and difference of (8.21) we can rewrite the latter as follows

$$\frac{\alpha'}{2}M^2 = N + \tilde{N} - 2a \tag{8.22}$$

$$N - \tilde{N} = 0 \tag{8.23}$$

The first equation defines the square mass operator. The second equation is called the *level matching condition*, and tells us that in any physical state the eigenvalue of the number operator in the left and right sector must be the same.

An analysis parallel to the one for open strings (by resorting, for instance, to the so-called light-cone quantization) leads us to conclude that $a = 1$ and $D = 26$ guarantee unitarity of the theory. Waiting for the BRST quantization, we will assume here this result and proceed to the analysis of the spectrum.

- At level $N = \tilde{N} = 0$ we have again a tachyon $|0; k\rangle$ with square mass $M^2 = -\frac{4}{\alpha'}$.

- At level $N = \tilde{N} = 1$ we have a tensor state
$$\omega_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0;k\rangle, \tag{8.24}$$
whose square mass $M^2 = 0$. The L_1 and $\tilde{L}_1$ physicality conditions (8.20) impose on the polarization $\omega_{\mu\nu}$ the transversality conditions
$$\omega_{\mu\nu}k^{\mu} = \omega_{\mu\nu}k^{\nu} = 0 \tag{8.25}$$
The remaining conditions (8.20) are identically satisfied.
- At level $N = \tilde{N} = 2$ there is a massive state with $M^2 = \frac{4}{\alpha'}$; it is the generic combination of states created by α^{μ}_{-2} and $\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}$ on the left and $\tilde{\alpha}^{\lambda}_{-2}$ and $\tilde{\alpha}^{\lambda}_{-1}\tilde{\alpha}^{\rho}_{-1}$ on the right.

and so on. We have thus an infinite tower of states with square mass
$$M^2 = \frac{4n}{\alpha'}, \qquad n \geq -1 \tag{8.26}$$

The massless state (8.24) is most interesting. It has a reducible structure in terms of Lorentz group representations. The irreducible representations correspond to the symmetric, antisymmetric and trace components
$$s_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0;k\rangle, \qquad a_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0;k\rangle, \qquad \eta_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0;k\rangle \tag{8.27}$$
where $s_{\mu\nu} = s_{\nu\mu}$ and $a_{\mu\nu} = -a_{\nu\mu}$. The symmetric state lends itself to be interpreted as the excitation of the metric in $D = 26$ spacetime dimension and (8.27) goes under the name of gravity multiplet. The appearance of gravity (the metric $G_{\mu\nu}$) in a multiplet with an antisymmetric field (always denoted $B_{\mu\nu}$) and a scalar (the dilaton) is a hallmark of string theory.

The interpretation of the (8.27) multiplet as the gravity multiplet was a turning point in the history of string theory.

8.3 The field theory limit

It is time to remark that both in open and closed string theory, in the limit $\alpha' \to 0$ all the massive modes become infinitely massive and can therefore be disregarded, at least dynamically. The only surviving degrees of freedom are the massless one. Since $\sqrt{\alpha'}$ measures the length of the string, $\alpha' \to 0$ is the limit in which strings can be considered point-like. In this limit it is natural to expect an ordinary field theory. In the open string case this is the theory of a Maxwell field. In closed string theory it is gravity theory in which also a massless antisymmetric field and a massless dilaton field are present. The analysis of the string amplitudes in this limit will confirm this interpretation.

8.4 The complex variable notation

One of the tools that have brought about more progress in string theory is no doubt the adoption of the complex variable notation, which allows one to use methods and results of one complex variable analysis. We pass to complex variable by means of a Wick rotation in the string world-sheet, so that the metric there becomes Euclidean, and a conformal map to the complex plane. More precisely, in the closed string case we Wick-rotate $\tau \to i\tau = t$ and map the infinite cylinder (which is the relevant world-sheet) to the z–plane by means of the map

$$z = e^{2(t+i\sigma)}$$

This maps a string configuration at constant τ to a circle in the complex z–plane. $t = -\infty$ is mapped to the origin of the z–plane and $t = +\infty$ to the circle at infinity in the z–plane. Therefore a string freely propagating from $\tau = -\infty$ to $\tau = -\infty$ is mapped to circles around the origin departing from the origin with increasing radius e^{2t}.

The action (7.9) is mapped to

$$-\frac{T}{2}\int d^2\sigma \left(X'\cdot X' - \dot{X}\cdot\dot{X}\right) \to i\,T\int dzd\bar{z}\,\partial_z X\cdot\partial_{\bar{z}}X$$

and (7.31,7.32) become

$$X_+^\mu(z) = \frac{1}{2}x^\mu - \frac{i}{4}\ell^2 p^\mu \ln z + \frac{i}{2}\ell\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}z^{-n} \tag{8.28}$$

$$X_-^\mu(\bar{z}) = \frac{1}{2}x^\mu - \frac{i}{4}\ell^2 p^\mu \ln \bar{z} + \frac{i}{2}\ell\sum_{n\neq 0}\frac{\tilde{\alpha}_n^\mu}{n}\bar{z}^{-n} \tag{8.29}$$

The left-handed T_{++} and right-handed T_{--} components of the energy-momentum tensor are mapped to

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \qquad \bar{T}(z) = \sum_{n\in\mathbb{Z}}\tilde{L}_n\bar{z}^{-n-2}, \tag{8.30}$$

respectively. In order to obtain the latter one has to take into account the fact that $T_{\alpha\beta}$ are geometrically the intrinsic components of a symmetric two-differential.

For future use we recall that the complex z-plane introduced above, via a stereographic projection, can (and will) be identified with the Riemann sphere, with the South pole corresponding to the origin and the North pole to the circle at ∞. It is clear from the above that these two points are distinct points of the sphere, because X^μ is allowed to have a singularity of arbitrary order there. In accordance with the mathematical terminology these two points will be referred to as *punctures*.

For open strings the world–sheet is an infinite strip. After the same Wick rotation as above, we map it to the upper half z plane via the map

$$z = e^{t+i\sigma}$$

In this case an open string configuration at constant τ in the strip is mapped to a semicircle centered at the origin in the upper half plane. A freely propagating open string is represented in the upper half plane by semi-circles exiting from the origin with increasing radius. The most general string solution (7.40) becomes

$$X^\mu(z,\bar{z}) = x^\mu - i\ell^2 p^\mu \ln|z| + i\ell \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} \frac{z^{-n} + \bar{z}^{-n}}{2} \tag{8.31}$$

It can be represented more economically by means of the holomorphic expression

$$X^\mu(z) = x^\mu - i\ell^2 p^\mu \ln z + i\ell \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} z^{-n} \tag{8.32}$$

provided we understand that it is inserted on the real axis.

Chapter 9

The BRST quantization

The original sin of old covariant was fixing the gauge by hand. This does not permit one of the most important ingredients of modern quantization, the Faddeev-Popov ghosts, to emerge. The third type of quantization we consider is the BRST quantization, the most accurate method we know of quantizing a theory. We need to start from the path integral formulation of string theory. The path integral in question is

$$Z = \int \mathcal{D}h(z,\bar{z})\mathcal{D}X^\mu(z,\bar{z})e^{-S[h,X]} \tag{9.1}$$

where the action is the transcription of the string action (7.1) in complex coordinates

$$S[h,X] = T\int dz d\bar{z}\,\sqrt{h}\,h^{\alpha\beta}\,\partial_\alpha X\cdot\partial_\beta X \tag{9.2}$$

Here α, β take the values $1, 2$, corresponding to $z, \bar{z}$ (the old +,-).

9.1 The Faddeev-Popov procedure

The transformations (7.2,7.3) generate an (infinitesimal) orbit in the space $\mathcal{M}$ of metrics $h_{\alpha\beta}$, so that $\mathcal{M}$ is foliated by such orbits. The space of orbits is called the *moduli space* of metrics. In order to fix the gauge symmetries (7.2,7.3) in the action (9.2), we need to choose a gauge slice, that is a family of metrics, each metric corresponding to one single orbit. Fortunately, in the present case, the moduli space (the moduli space of a sphere or of the upper half plane with two punctures) is trivial: it reduces to a point. So the slice reduces to the choice of one metric, the *fiducial metric* $\hat{h}(z,\bar{z})$. From $\hat{h}(z,\bar{z})$ any metric $h(z,\bar{z})$ is generated via a suitable gauge transformation $\zeta(z,\bar{z})$, in symbols $h = \hat{h}^\zeta$.

The Faddeev-Popov procedure consists in replacing the integration over metrics by an integration over the symmetry group. Since the action of the symmetry group spans the whole space of metrics, the only problem is to compute the Jacobian factor necessary in order to shift in the path integral from the field variables $h_{\alpha\beta}$ to the symmetry group variables. This is done as follows. Let us denote by $\mathcal{G}$ the infinitesimal version of the group $\mathcal{D}$ o $\mathcal{W}$. Let us consider the following identity

$$1 = \int_{\mathcal{O}} \mathcal{D}h \, \delta(h - \hat{h}) \tag{9.3}$$

where $\mathcal{O}$ represents the orbit of $\mathcal{G}$, which, in the present case, coincides with $\mathcal{M}$. Let us represent collectively the variables in $\mathcal{G}$ by ζ. As we have already remarked, any metric h can be written as $\hat{h}^\zeta$. We can therefore rewrite the previous identity as

$$1 = \int_{\mathcal{G}} \mathcal{D}\zeta \, \frac{\mathcal{D}h}{\mathcal{D}\zeta} \, \delta(\hat{h}^\zeta - \hat{h}) \tag{9.4}$$

We shift in this way from integration over metrics to integration over $\mathcal{G}$. The significant memory of $\mathcal{M}$ is contained in the Jacobian factor. Introducing (9.4) in the path integral we get

$$Z = \int \mathcal{D}\zeta \mathcal{D}h \mathcal{D}X^\mu \, \frac{\mathcal{D}h}{\mathcal{D}\zeta} \, \delta(\hat{h}^\zeta - \hat{h}) \, e^{-S[\hat{h}^\zeta, X]}. \tag{9.5}$$

Next we make the replacement $X \to X^\zeta$ and suppose that this replacement produces a trivial Jacobian[1]. Then we notice that $S[\hat{h}^\zeta, X^\zeta] = S[\hat{h}, X]$ from the invariance of the action. Now we can easily integrate over $h = \hat{h}^\zeta$,

$$\begin{aligned} Z &= \int \mathcal{D}\zeta \mathcal{D}h \mathcal{D}X^\mu \, \frac{\mathcal{D}\hat{h}^\zeta}{\mathcal{D}\zeta} \, \delta(h - \hat{h}) \, e^{-S[\hat{h}, X]} \\ &= \int \mathcal{D}\zeta \mathcal{D}X^\mu \left(\frac{\mathcal{D}\hat{h}^\zeta}{\mathcal{D}\zeta} \right) \Big|_{\zeta = id} e^{-S[\hat{h}, X]}. \end{aligned}$$

There is no dependence left on ζ in the integrand, and $\int \mathcal{D}\zeta = Vol(\mathcal{G})$, the volume of the symmetry group, which is infinite. Since we are interested in amplitudes, which are always normalized with respect to the functional integral Z, we can formally drop this infinite factor and write

$$Z = \int \mathcal{D}X^\mu \left(\frac{\mathcal{D}\hat{h}^\zeta}{\mathcal{D}\zeta} \right) \Big|_{\zeta = id} e^{-S[\hat{h}, X]} \tag{9.6}$$

[1]This is true in $D = 26$. For $D \neq 26$ the Jacobian produces a factor that can be lifted to the Liouville action.

It remains for us to write in more explicit form the Jacobian factor. The remarkable occurrence is that it can be lifted to the exponent and form an additional local piece of the action. The action of $\mathcal{G}$ on the metric is

$$\begin{aligned} \delta h_{\alpha\beta} &= \omega h_{\alpha,\beta} - \nabla_\alpha \xi_\beta - \nabla_\beta \xi_\alpha \\ &= (\omega - \nabla_\gamma \xi^\gamma) h_{\alpha\beta} - (\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha - h_{\alpha\beta} \nabla_\gamma \xi^\gamma) \end{aligned} \tag{9.7}$$

where $\xi_\alpha = h_{\alpha\beta}\xi^\beta$. The second piece is a first order differential operator that maps ξ^α, that is a vector field, to a traceless symmetric tensor. The Jacobian factor is simply the determinant of this operator. The first piece is a rescaling of the metric, so its determinant is even simpler. We can write schematically

$$\frac{\mathcal{D}\hat{h}}{\mathcal{D}\zeta} = \det\left(\frac{\mathcal{D}\hat{h}}{\mathcal{D}\omega}\right) \det\left(\frac{\mathcal{D}\hat{h}}{\mathcal{D}\xi}\right)$$

however we do not need compute explicitly such determinants. We can do better. The determinant of a differential operator in a path integral can be lifted to the exponent by means of anticommuting fields. This can be done as follows. In the RHS of (9.7) we replace the vector field ξ^α by an anticommuting field c^α and the parameter ω by an anticommuting field λ. Then we contract the so–obtained expression at the RHS of (9.7) with another anticommuting field $b_{\alpha\beta}$ and write the (formal) equivalence

$$\left(\frac{\mathcal{D}\hat{h}^\zeta}{\mathcal{D}\zeta}\right)\Big|_{\zeta=id} = \int \mathcal{D}b_{\alpha\beta}\mathcal{D}c^\alpha\mathcal{D}\lambda\, e^{\frac{1}{2\pi}\int d^2z\,\sqrt{\hat{h}}\, b_{\alpha\beta}\left[\left(\lambda - \hat{\nabla}_\gamma c^\gamma\right)\hat{h}^{\alpha\beta} - 2\left(\hat{P}c\right)^{\alpha\beta}\right]}$$

where $(Pc)_{\alpha\beta} = \frac{1}{2}\left(\nabla_\alpha c_\beta + \nabla_\beta c_\alpha - h_{\alpha\beta}\nabla_\gamma c^\gamma\right)$ and a hat means that we are evaluating the hatted quantity at the fiducial metric. The dependence on λ in the exponent is linear, we can integrate over it and get the constraint: $b_{\alpha\beta}\hat{h}^{\alpha\beta} = 0$, which means that b is traceless. Since this is a non–dynamical condition we can replace it into the action and get rid of the corresponding term. Now we are left with

$$\left(\frac{\mathcal{D}\hat{h}^\zeta}{\mathcal{D}\zeta}\right)\Big|_{\zeta=id} = \int \mathcal{D}b_{\alpha\beta}\mathcal{D}c^\alpha\, e^{-\frac{1}{\pi}\int d^2z\,\sqrt{\hat{h}}\, b_{\alpha\beta}\left(\hat{P}c\right)^{\alpha\beta}} \tag{9.8}$$

After some simple algebra one gets

$$b_{\alpha\beta}\left(\hat{P}\,c\right)^{\alpha\beta} = b_{\alpha\beta}\,\hat{h}^{\alpha\gamma}\hat{\nabla}_\gamma c^\beta$$

In conclusion the functional integral becomes

$$Z = \int \mathcal{D}X^\mu \mathcal{D}b_{\alpha\beta}\mathcal{D}c^\alpha\, e^{-S[\hat{h},X] - \frac{1}{\pi}\int d^2z\,\sqrt{\hat{h}}\, b_{\alpha\beta}\,\hat{h}^{\alpha\gamma}\,\hat{\nabla}_\gamma c^\beta} \tag{9.9}$$

From now on it is convenient to choose a definite fiducial metric, the conformal one:

$$\hat{h}_{\alpha\beta} = \frac{e^\phi}{2}\eta_{\alpha\beta} \tag{9.10}$$

where $\phi(z, \bar{z})$ is an arbitrary function; e^ϕ is often referred to as the conformal factor.

Conformal tensor calculus

The conformal gauge (9.10) allows us a remarkable notational simplification. Let us set

$$\hat{h}_{\alpha\beta} = \frac{e^\phi}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \text{i.e.} \quad \hat{h}_{zz} = \hat{h}_{\bar{z}\bar{z}} = 0, \qquad \hat{h}_{z\bar{z}} = h_{\bar{z}z} = \frac{e^\phi}{2} \tag{9.11}$$

Then

$$ds^2 = \hat{h}_{\alpha\beta} dz^\alpha dz^\beta = e^\phi \, dz d\bar{z}$$

We remark that under a holomorphic transformation $z \to f(z)$, we have $e^\phi \to |f'(z)|^2 e^\phi$, so that the gauge is preserved. Lowering and raising of the indices take the following form. Let V_α be the intrinsic component of a 1-differential:

$$V_\alpha = \hat{h}_{\alpha\beta} V^\beta, \qquad V_z = \frac{e^\phi}{2} V^{\bar{z}}, \qquad V_{\bar{z}} = \frac{e^\phi}{2} V^z$$

Covariant derivatives take a simplified form. Let us consider $\hat{\nabla}_\alpha V_\beta = \partial_\alpha V_\beta - \hat{\Gamma}^\gamma_{\alpha\beta} V_\gamma$. It si easy to prove that the only non-trivial components of the Christoffel symbols $\hat{\Gamma}^\gamma_{\alpha\beta}$ are

$$\hat{\Gamma}^z_{zz} = \partial_z \phi, \qquad \hat{\Gamma}^{\bar{z}}_{\bar{z}\bar{z}} = \partial_{\bar{z}} \phi \tag{9.12}$$

Using this result the following formulas for the components of an n-th order symmetric tensor V are easy to prove

$$\begin{aligned}
\hat{\nabla}_z V_{zz\ldots z} &= (\partial_z - n\partial_z\phi) V_{zz\ldots z} \\
\hat{\nabla}_{\bar{z}} V_{zz\ldots z} &= \partial_{\bar{z}} V_{zz\ldots z} \\
\hat{\nabla}_z V^{zz\ldots z} &= (\partial_z + n\partial_z\phi) V^{zz\ldots z} \\
\hat{\nabla}_{\bar{z}} V^{zz\ldots z} &= \partial_{\bar{z}} V^{zz\ldots z}
\end{aligned}$$

Moreover we can derive a useful formula

$$[\hat{\nabla}_{\bar{z}}, \hat{\nabla}_z] V_{zz\ldots z} = -4n \, e^\phi \, R^{(2)} V_{zz\ldots z}, \qquad R^{(2)} = 4e^{-\phi}\partial_z\partial_{\bar{z}}\phi \tag{9.13}$$

where $R^{(2)}$ is the scalar curvature of the conformal metric.

9.2 The BRST symmetry

Using the just introduced notation we can rewrite (recall that $b_{z\bar{z}} = 0$)

$$b_{\alpha\beta}\,\hat{h}^{\alpha\gamma}\hat{\nabla}_\gamma c^\beta = b_{zz}\hat{\nabla}^z c^z + b_{\bar{z}\bar{z}}\hat{\nabla}^{\bar{z}} c^{\bar{z}} = 2e^{-\phi}\left(b_{zz}\partial_{\bar{z}} c^z + b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}}\right)$$

So, finally, (9.9) becomes

$$Z = \int \mathcal{D}X^\mu \mathcal{D}b_{\alpha\beta}\mathcal{D}c^\alpha\, e^{-S[\hat{h},X]-\frac{1}{\pi}\int d^2z(c^{\bar{z}}\partial_z b_{\bar{z}\bar{z}}+c^z\partial_{\bar{z}} b_{zz})} \tag{9.14}$$

The anticommuting vector field c^α and traceless symmetric tensor field $b_{\alpha\beta}$ are the well-known Fadeedv-Popov (FP) ghost and antighost fields, respectively.

From the previous derivation we can deduce the ghost gauge-unfixed action:

$$S_{gh} = \frac{1}{\pi}\int d^2z\sqrt{h}h^{\alpha\beta}c^\gamma\nabla_\alpha b_{\beta\gamma} \tag{9.15}$$

Taking the variation with respect to the metric we can derive the ghost energy momentum tensor. After gauge fixing the two holomorphic and antiholomorphic components are (the trace is 0):

$$T^{(g)}_{zz} = c^z\partial_z b_{zz} + 2\left(\partial_z c^z\right) b_{zz} \tag{9.16}$$

$$T^{(g)}_{\bar{z}\bar{z}} = c^{\bar{z}}\partial_{\bar{z}} b_{\bar{z}\bar{z}} + 2\left(\partial_{\bar{z}} c^{\bar{z}}\right) b_{\bar{z}\bar{z}} \tag{9.17}$$

9.2.1 The BRST transformations

In the BRST quantization the original gauge symmetry is lost, because of the gauge fixing. But it is replaced by a new field-dependent invariance, the BRST symmetry, which has the remarkable property of being represented by idempotent transformations. The BRST trasformations (denoted s) are obtained from the original gauge transformations by replacing the gauge parameters with corresponding ghost and anti–ghost fields. The transformation of the latter is determined in such a way as to close over an idempotent algebra. Proceeding in this way from (7.2) we find

$$sX^\mu = c^\alpha\partial_\alpha X^\mu = c^z\partial_z X^\mu + c^{\bar{z}}\partial_{\bar{z}} X^\mu \tag{9.18}$$

The s–transformation of the ghost field in gauge theories is dictated by the Maurer-Cartan equation on the gauge group. In the present case we have

$$sc^z = c^\alpha\partial_\alpha c^z, \qquad sc^{\bar{z}} = c^\alpha\partial_\alpha c^{\bar{z}} \tag{9.19}$$

Finally, the transformation of the antighost field is defined in such a way that the transformation algebra be idempotent. In our case we have

$$sb_{zz} = \frac{1}{\ell^2}T_{zz}, \qquad sb_{\bar{z}\bar{z}} = \frac{1}{\ell^2}T_{\bar{z}\bar{z}} \tag{9.20}$$

T_{zz} and $T_{\bar{z}\bar{z}}$ being the components of the *total* e.m. tensor (matter+ghosts). For instance

$$T_{zz} = T_{zz}^{(X)} + \ell^2 T_{zz}^{(g)} = -\frac{1}{2}\partial_z X \cdot \partial_z X + \ell^2 \left(c^z \partial_z b_{zz} - 2b_{zz}\partial_z c^z\right) \tag{9.21}$$

It is not hard to prove that for the above transformations $s^2 = 0$ and that the quantum action

$$S = T \int d^2z \left[\partial_z X \cdot \partial_{\bar{z}} X + 2\ell^2 \left(c^{\bar{z}} \partial_z b_{\bar{z}\bar{z}} + c^z \partial_{\bar{z}} b_{zz}\right)\right] \tag{9.22}$$

is invariant under them.

Naturally there is a conserved current corresponding to the BRST symmetry, the BRST current, whose components are

$$\begin{aligned} J_z^B &= c^z \left(T_{zz}^{(X)} + \frac{1}{2} T_{zz}^{(g)}\right) + 3\partial_z^2 c^z, \\ J_{\bar{z}}^B &= c^{\bar{z}} \left(T_{\bar{z}\bar{z}}^{(X)} + \frac{1}{2} T_{\bar{z}\bar{z}}^{(g)}\right) + 3\partial_{\bar{z}}^2 c^{\bar{z}} \end{aligned} \tag{9.23}$$

The corresponding charges are defined by

$$Q_z = \frac{1}{2\pi i} \oint dz\, J_z^B, \qquad Q_{\bar{z}} = \frac{1}{2\pi i} \oint d\bar{z}\, J_{\bar{z}}^B \tag{9.24}$$

Via the symplectic structure of the theory these charges generate the BRST transformations.

9.3 b-c systems

The FP ghost fields c and b (geometrically, a vector field and a traceless symmetric 2-tensor) are an example of b-c system. The action of an integral order b-c system is

$$S_{bc} = \frac{1}{\pi} \int d^2z \, \left(c^{\bar{z}\ldots\bar{z}} \partial_z b_{\bar{z}\ldots\bar{z}} + c^{z\ldots z} \partial_{\bar{z}} b_{z\ldots z}\right) \tag{9.25}$$

The number λ of indices in $b_{z\ldots z}$ is called the order or weight of the b-c system. We will denote it by $h(b) = \lambda$. The number of indices of $c^{z\ldots z}$ is $\lambda - 1$ and its order or weight is $h(c) = 1 - \lambda$. The holomorphic e.m. tensor is

$$T_{zz}^{(bc)} = -\lambda b_{z\ldots z}\, \partial_z c^{z\ldots z} + (1-\lambda)\partial_z b_{z\ldots z}\, c^{z\ldots z} \tag{9.26}$$

A similar expression holds for the antiholomorphic component.

The equations of motion are

$$\partial_{\bar z} b_{\bar z \ldots \bar z} = 0, \qquad \partial_{\bar z} c^{z \ldots z} = 0$$
$$\partial_z b_{z \ldots z} = 0, \qquad \partial_z c^{\bar z \ldots \bar z} = 0$$

The case of the FP ghosts for the bosonic string corresponds to $\lambda = 2$. For simplicity we will continue with them *for the case of closed string theory.* The equation of motion simply means that we can expand the b-c fields as follows

$$c^z = \sum_{n \in \mathbb{Z}} c_n z^{-n+1}, \qquad c^{\bar z} = \sum_{n \in \mathbb{Z}} \tilde c_n \bar z^{-n+1}, \tag{9.27}$$

$$b_{zz} = \sum_{n \in \mathbb{Z}} b_n z^{-n-2}, \qquad b_{\bar z \bar z} = \sum_{n \in \mathbb{Z}} \tilde b_n \bar z^{-n-2} \tag{9.28}$$

Notice the +1 and -2 addends in the exponents of $z, \bar z$, which are the opposite of the relevant field weights. They are added in accord with the conformal transformation properties of the b-c fields (see next chapter).

Using the symplectic structure

$$[b_{zz}(z), c^z(z')]_+ = 2\pi i\, \delta(z, z'), \qquad [b_{\bar z \bar z}(\bar z), c^{\bar z}(\bar z')]_+ = 2\pi i\, \delta(\bar z, \bar z'), \tag{9.29}$$

we can extract the canonical anticommutation relations:

$$[c_n, b_m]_+ = \delta_{n+m,0}, \qquad [c_n, c_m]_+ = 0 = [b_n, b_m]_+ \tag{9.30}$$
$$[\tilde c_n, \tilde b_m]_+ = \delta_{n+m,0}, \qquad [\tilde c_n, \tilde c_m]_+ = 0 = [\tilde b_n, \tilde b_m]_+$$

For the open strings we have only one set of bosonic oscillators b_n and c_n and the canonical commutators are given by the first line of (9.30).

In the quantum theory $c_n, b_n, \tilde c_n, \tilde c_n$ obey the hermitean conjugation properties:

$$c_n^\dagger = c_{-n}, \qquad b_n^\dagger = b_{-n}, \qquad \tilde c_n^\dagger = \tilde c_{-n}, \qquad \tilde b_n^\dagger = \tilde b_{-n} \tag{9.31}$$

The ghost vacuum is a rather complex object and will be discussed in detail further on. For the time being suffice it to give the following definitions

$$c_n |0\rangle_g = 0, \qquad n \geq 2, \qquad\qquad b_n |0\rangle_g = 0, \quad n \geq -1 \tag{9.32}$$

which define simultaneously the annihilation operators. This allows us to define normal ordering.

We can now introduce the normal ordered generators for the Virasoro algebra

$$L_n^{(g)} = \sum_{k \in \mathbb{Z}} (n-k) : b_{n+k} c_{-k} : \tag{9.33}$$

for the open string case. They obey the Lie algebra brackets

$$[L_m^{(g)}, L_n^{(g)}] = (m-n)L_{n+m}^{(g)} - \frac{13}{6}(m^3-m)\delta_{m+n,0} \tag{9.34}$$

For the closed string case we have two copies both of (9.33) and (9.34), one for each sector.

From (9.34) we conclude that the central charge of the weight 2 b-c system is $c^{(bc)} = -26$.

In terms of the Virasoro generators the (holomorphic) BRST charge Q is given by

$$Q = \sum_{n\in\mathbb{Z}} : \left(L_n^{(X)} + \frac{1}{2}L_n^{(g)} - a\delta_{n,0}\right) c_{-n} : \tag{9.35}$$

where a is the usual constant which comes from reordering L_0. Now, setting $a = 1$, one can compute

$$Q^2 = \frac{1}{2}\{Q, Q\} = \frac{1}{2}\sum_{n,m} : ([L_m, L_n] - (m-n)L_{n+m})\, c_n c_m : \tag{9.36}$$

Here $L_n = L_n^{(X)} + L_n^{(g)}$ are the total Virasoro generators. Putting together (8.8) and (9.34) we get

$$[L_m, L_n] = (m-n)L_{n+m} + \frac{D-26}{12}(m^3-m)\delta_{n+m,0} \tag{9.37}$$

Therefore Q is nilpotent only when $D = 26$. Moreover, using the Hermitean conjugation properties of c_n and L_n one can easily prove that Q is hermitean

$$Q^\dagger = Q \tag{9.38}$$

One can show that

$$[Q, L_n] = 0, \qquad [Q, b_n^\dagger]_+ = L_n \tag{9.39}$$

In the closed string case we have the same properties also for the antiholomorphic charge $\bar{Q}$.

The open string ghost Fock space will be generated by acting on the vacuum with the b and c creation operators. The most general Fock space state will be generated by applying to the vacuum both ghost and matter creation operators. Any such state will be characterized by the *ghost number*. The latter will be determined by assigning to each c creation operator acting on the vacuum the ghost number 1 and to each b creation operator the ghost number -1, and taking the sum. Extension to closed string is obvious.

One final comment is in order. Deriving (9.34,9.36) requires a sizable amount of algebra. We will describe later on a much simpler and elegant method, based on OPE, by which all these and many other results can be obtained with remarkably less effort.

9.3.1 Generalized b-c system

We have introduced above the action and e.m. tensor of a generalized b-c system of weight $h(b) = \lambda$. The generalization of the previous definitions and properties are straightforward. For instance the Virasoro generators are

$$L_n^{(\lambda)} = \sum_{k\in\mathbb{Z}} (n(1-\lambda)+k) : b_{n+k}c_{-k} : \tag{9.40}$$

They satisfy Virasoro algebra with central charge

$$c^{(\lambda)} = -2(6\lambda^2 - 6\lambda + 1) \tag{9.41}$$

So far we have considered b-c systems with integral weight, but the weight can also be a half-integer. In the latter case eq.(9.41) remains the same.

9.4 BRST analysis of the string spectrum

The BRST quantization allows us to formulate in a mathematically elegant way the physicality conditions on the string spectrum. For definiteness let us concentrate from now on in this chapter on the *open string theory*. The existence of the nilpotent operator Q acting on the Fock space states, suggests the use of cohomology. This is in fact the right mathematical tool in this context. As it turns out physical states will correspond to cohomology classes. States that do not belong to any such class are not physical.

9.4.1 The BRST cohomology

The nilpotent operator Q defines a cohomology problem. Q is a coboundary operator acting on the Fock space (the differential space or space of cochains). States closed under the action of Q, i.e states $|\phi\rangle$ such that $Q|\phi\rangle = 0$ form the space of *cocycles*. States which are exact, i.e. states $|\phi\rangle$ such that $|\phi\rangle = Q|\chi\rangle$, for some other state $|\chi\rangle$, are *coboundaries*. They are also called *spurious* states because they have vanishing norm, for

$$\langle\chi|Q^\dagger Q|\chi\rangle = \langle\chi|Q^2|\chi\rangle = 0$$

due to the hermiticity and nilpotency of Q. Cocycles are defined up to coboundaries. Therefore cocycles split into classes, the cohomology classes. The cohomology classes form a vector space, also called the cohomology group. Let us denote by $\mathcal{H}_{\text{closed}}$ the space of cocycles, by $\mathcal{H}_{\text{exact}}$ the space of coboundaries, the physical Hilbert (states with positive norm) identifies with the cohomology group

$$\mathcal{H}_{\text{phys}} = \frac{\mathcal{H}_{\text{closed}}}{\mathcal{H}_{\text{exact}}} = \frac{\text{ker}Q}{\text{Im}Q}$$

Let us apply this abstract scheme to analyze the open string spectrum. To start with we have to discuss the ghost vacuum. The fittest ghost vacuum to study the perturbative string spectrum is not the one we have introduced above, $|0\rangle_g$, but the one discussed hereafter.

The $|\downarrow\rangle$ ghost vacuum

We define the 'down' vacuum $|\downarrow\rangle$ as

$$|\downarrow\rangle = c_1|0\rangle_g \tag{9.42}$$

It follows that

$$c_n|\downarrow\rangle = 0, \quad n > 0, \qquad \text{and} \qquad b_n|\downarrow\rangle = 0, \quad n \geq 0$$

It is easy to see that

$$L_n^{(g)}|\downarrow\rangle = 0, \qquad n \geq 0$$

One can also introduce the 'up' vacuum

$$|\uparrow\rangle = c_0|\downarrow\rangle \tag{9.43}$$

so that

$$c_n|\uparrow\rangle = 0, \quad n \geq 0, \qquad \text{and} \qquad b_n|\uparrow\rangle = 0, \quad n > 0$$

$|\downarrow\rangle$ and $|\uparrow\rangle$ form a doublet representation of the algebra

$$[c_0, b_0]_+ = 1$$

The hermitean conjugate vacuum of $|\downarrow\rangle$ will be denoted $\langle\downarrow|$. We will show later on that consistency requires

$$\langle\downarrow|c_0|\downarrow\rangle = 1 \tag{9.44}$$

The classification of the Fock space states will be based from now on on the $|\downarrow\rangle$ vacuum.

9.4.2 Connection with OCQ

As a first step we will establish the link with the old covariant quantization. The vacuum is the tensor product of the matter and ghost vacuum: $|0,\downarrow\rangle = |0\rangle \otimes |\downarrow\rangle$.

Let us see the action of Q on it. We get

$$\begin{aligned} Q|0,\downarrow\rangle &= \sum_{n\geq 0} c_{-n}\left(L_n^{(X)} + \frac{1}{2}L_n^{(g)} - \delta_{n,0}\right)|0,\downarrow\rangle \\ &\quad + \sum_{n<0}\left(L_n^{(X)} + \frac{1}{2}L_n^{(g)}\right) c_{-n}|0,\downarrow\rangle \\ &= c_0\left(L_0^{(X)} - 1\right)|0,\downarrow\rangle = -c_0|0,\downarrow\rangle \end{aligned} \tag{9.45}$$

Similarly, to any matter state $|\phi,k\rangle$ of the OCQ we associate

$$|\phi,k\rangle \longrightarrow |\phi,k,\downarrow\rangle = |\phi,k\rangle\otimes|\downarrow\rangle. \tag{9.46}$$

Then we notice that

$$L_n^{(g)}|\phi,k,\downarrow\rangle = 0, \qquad n\geq 0$$

Thus, if $|\phi,k\rangle$ happens to satisfy the physicality conditions, since $c_n|\downarrow\rangle = 0$ for $n>0$, we have

$$Q|\phi,k,\downarrow\rangle = 0 \tag{9.47}$$

that is, $|\phi,k,\downarrow\rangle$ is closed.

On the other hand, if a state $|\phi,k,\downarrow\rangle = |\phi,k\rangle\otimes|0,\downarrow\rangle$, where $|\phi,k\rangle$ is a pure matter state, is annihilated by Q, then since $c_n|\downarrow\rangle = 0$ for $n>0$, we must have

$$c_{-n}\left(L_n^{(X)} - \delta_{n,0}\right)|\phi,k,\downarrow\rangle = 0, \qquad n\geq 0$$

But these are precisely the physicality conditions of the OCQ. Thus, under the correspondence (9.46), to any physical state in the OCQ there corresponds a closed state (which identifies a cohomology class) and to any closed state with the same form as the RHS of (9.46) there corresponds a physical state in the OCQ.

9.5 String theory and conformal field theory

String theory is a particular two-dimensional conformal field theory. It is a gauge theory. Its total central charge (matter+ghost) must vanish. It is nevertheless a conformal field theory and we are entitled to apply to it the formalism introduced above.

The total holomorphic energy-momentum tensor was given in eq.(9.21). Setting, for simplicity, in this section, $\ell = 1$, we rewrite it here in simplified form

$$T(z) = -\frac{1}{2}\partial X\cdot\partial X + c\,\partial b - 2\,b\,\partial c \tag{9.48}$$

The (holomorphic) propagators are as follows:

$$\langle X^\mu(z)X^\nu(w)\rangle = -\eta^{\mu\nu}\ln(z-w), \qquad \langle c(z)b(w)\rangle = \langle b(z)c(w)\rangle = \frac{1}{z-w} \tag{9.49}$$

Using these and the Wick theorem for normal products, one easily finds

$$\begin{aligned}
T(z)\,X(w) &= \frac{1}{z-w}\,\partial_w X(w) + \dots \\
T(z)\,\partial_w X(w) &= \frac{1}{(z-w)^2}\,\partial_w X(w) + \frac{1}{z-w}\,\partial_w^2 X(w) + \dots \\
T(z)\,c(w) &= -\frac{1}{(z-w)^2} + \frac{1}{z-w}\,\partial_w c(w) + \dots \\
T(z)\,b(w) &= \frac{2}{(z-w)^2} + \frac{1}{z-w}\,\partial_w b(w) + \dots
\end{aligned}$$

where ellipses denote regular terms. Thus $\partial X, c, b$ are primary fields of weight 1,-1 and 2, respectively. X is not itself a primary field. It is sometime said that it has logarthmic weight, because, as will be seen, exponentials of X have finite weight.

The BRST transformation of a generic field $A(z)$ is given by the general formula (let us set $Q = Q_z$ and $J^B = J^B_z$)

$$sA(z) = [Q, A(z)] = \frac{1}{2\pi i}\oint dw\, J^B(z)\, A(z) \tag{9.50}$$

The relavant OPE are

$$\begin{aligned}
J^B(z)\,X(w) &= \frac{1}{z-w}\,c(w)\,\partial_w X(w) + \dots \\
J^B(z)\,c(w) &= \frac{1}{z-w}\,c(w)\,\partial_w c(w) + \dots \\
J^B(z)\,b(w) &= \frac{1}{z-w}\,T(w) + \frac{1}{(z-w)^2}\,c(w)\,b(w) + \frac{3}{(z-w)^3} + \dots
\end{aligned}$$

Inserting these into the RHS of (9.50) one recovers the (holomorphic) BRST transformations (9.18,9.19,9.20).

Another application of the CFT formalism is the computation of the critical dimension. Since $Q = \frac{1}{2\pi i}\oint dz J^B(z)$, in order to compute the square of Q we can proceed by computing first the OPE of J^B with itself. The result is

$$\begin{aligned}
J^B(z)J^B(w) \;\; &= \;\; -\frac{D-18}{2(z-w)^3}c\partial c(w) - \frac{D-18}{4(z-w)^2}c\partial^2 c(w) \\
&\quad\; -\frac{D-26}{12(z-w)}c\partial^3 c(w) + \mathrm{Reg}
\end{aligned} \tag{9.51}$$

Integrating first with respect to z the regular terms and the first two terms in the RHS drop out. The third term gives a non-trivial contribution, which vanishes if and only if $D = 26$.

9.5.1 The b-c ghost vacuum

We recall the mode-expansion for the (holomorphic outside the origin and ∞) b and c fields:

$$c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n+1}, \qquad b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-2}$$

We require them to be regular at the origin when applied to the vacuum, which implies that

$$c_n|0\rangle_g = 0, \quad n \geq 2, \qquad b_n|0\rangle_g = 0, \quad n \geq -1 \tag{9.52}$$

This fixes the state-operator correspondence

$$c(0)|0\rangle_g = c_1|0\rangle_g = |\downarrow\rangle, \qquad b(0)|0\rangle_g = b_{-2}|0\rangle_g$$

We recall that $c(z)$ and $b(z)$ are ghost fields that geometrically correspond to a vector field and a quadratic differential, respectively. This dictates the transformation rules when we change coordinate system, in particular in the definition of the dual vacuum. From the conformal transformation property of the c field we get that

$$\tilde{c}(w) = \frac{1}{z^2}c(z)$$

must be regular at $w = 0$. The mode expansion near this point is

$$c(w) = \sum_{n\in\mathbb{Z}} \tilde{c}_n w^{-n+1}$$

Equating the expansion in z and w at regular points we find that $\tilde{c}_n = (-1)^{n-1}c_{-n}$. Moreover

$${}_g\langle 0|\tilde{c}_n = 0, \quad n > 1,$$

This implies

$${}_g\langle 0|c_n = 0, \quad n < -1, \tag{9.53}$$

Similarly one can prove that

$${}_g\langle 0|b_n = 0, \quad n < 2, \tag{9.54}$$

The definitions (9.52,9.53,9.54) imply, in particular, that the only non-contradictory position about the inner product of the ghost vacuum with itself is, up to normalization,

$${}_g\langle 0|c_{-1}c_0c_1|0\rangle_g = 1 \tag{9.55}$$

Any other choice contradicts the algebra (9.30).

The oscillators c_{-1}, c_0, c_1 correspond to the zero modes of the operator $\bar{\partial}$. This means that the field $c_0(z) = c_{-1} + c_0 z + c_1 z^2$ satisfies the equation $\partial_{\bar{z}} c_0(z) = 0$, and the transformed field $\tilde{c}_0(w) = \tilde{c}_{-1} + \tilde{c}_0 w + \tilde{c}_1 w^2$ under the map $z \to w = -\frac{1}{z}$, satisfies $\partial_{\bar{w}} \tilde{c}_0(w) = 0$ (in other words the property of being a zero mode is a global property). This is strictly connected with the property (9.55), as will be seen later on.

Let us define

$$
{}_g\langle\tilde{0}| =_g \langle 0|c_{-1}c_0c_1
$$

It is clear that when computing correlators involving ghost fields the adjoint vacuum must be ${}_g\langle\tilde{0}|$. For instance, for the ghost propagator we have

$$
\begin{aligned}
\langle c(z)b(w)\rangle &= {}_g\langle\tilde{0}|c(z)b(w)|0\rangle_g, \qquad \text{for} \quad |z| > |w| \\
&= \sum_{n>1,m<-1} {}_g\langle\tilde{0}|c_n b_m|0\rangle_g z^{-n+1} w^{-m-2} \\
&= \sum_{n>1} \left(\frac{w}{z}\right)^{n-1} \frac{1}{w} = \frac{1}{z-w}
\end{aligned}
$$

A similar expression is obtained for $|z| < |w|$, so, finally, the ghost propagator is

$$
\langle c(z)b(w)\rangle = \frac{1}{z-w} \tag{9.56}
$$

We record also the following amplitude, which turns out to be important in string amplitudes:

$$
\begin{aligned}
{}_g\langle 0|c(z_1)c(z_2)c(z_3)|0\rangle_g &= {}_g\langle 0|c_0(z_1)c_0(z_2)c_0(z_3)|0\rangle_g \\
&= (z_1 - z_2)(z_3 - z_1)(z_2 - z_3)
\end{aligned} \tag{9.57}
$$

where $c_0(z)$ represents the zero mode part of $c(z)$. (9.57) follows immediately from (9.55).

9.5.2 State-operator correspondence in string theory

The state-operator correspondence in string theory can be made very explicit by the use of vertex operators. The vertex operators corresponding to the first states of the open string spectrum are as follows. The tachyon state is generated by

$$
V_t(k,z) =: e^{ik_\mu X^\mu(z)} :, \qquad V_t(k,0)|0\rangle = |0\rangle e^{ikx} \tag{9.58}
$$

with $k^2 = 2$, where z is understood to be a point of the real axis in the UHP. The massless vector state is generated by

$$V_A(k,\zeta,z) = i : \zeta_\mu \frac{dX^\mu(z)}{dz} e^{ik_\mu X^\mu(z)} :, \qquad V_A(k,\zeta,0)|0\rangle = \zeta_\mu a^\mu_{-1}|0\rangle e^{ikx} \tag{9.59}$$

where $k^2 = 0, k\cdot\zeta = 0$. The normal ordered exponential in (9.58) has to be understood as

$$: e^{ik_\mu X^\mu(z)} := e^{ik\cdot x}\, e^{ik\cdot p \ln z}\, e^{-\sum_{n<0} \frac{k\cdot\alpha_n}{n} z^{-n}}\, e^{-\sum_{n>0} \frac{k\cdot\alpha_n}{n} z^{-n}} \tag{9.60}$$

Then the second equation in (9.58) follows immediately. In the same way one gets the second relation in (9.59).

One can easily prove that if $|z_j| < |z_i|$,

$$e^{-\sum_{n>0} \frac{k_i\cdot\alpha_n}{n} z_i^{-n}}\, e^{-\sum_{m<0} \frac{k_j\cdot\alpha_m}{m} z_j^{-m}} = e^{-\sum_{m<0} \frac{k_j\cdot\alpha_m}{m} z_j^{-m}}\, e^{-\sum_{n>0} \frac{k_i\cdot\alpha_n}{n} z_i^{-n}}\, e^{k_i\cdot k_j \ln \frac{z_i - z_j}{z_i}} \tag{9.61}$$

and

$$e^{ik_i\cdot p \ln z_i}\, e^{ik_j\cdot x} = e^{ik_j\cdot x}\, e^{ik_i\cdot p \ln z_i}\, e^{k_i\cdot k_j\ \ln z_i} \tag{9.62}$$

Using these relations one can easily get

$$\langle 0|V_t(k_1,z_1)\ldots V_t(k_n,z_n)|0\rangle = \prod_{i<j}(z_i - z_j)^{k_i\cdot k_j} \langle 0|e^{i(k_1\cdot x + \ldots k_n\cdot x)}|0\rangle \tag{9.63}$$

Inserting $\frac{1}{2\pi}\oint dz\, \frac{dX^\mu}{dz} = p^\mu$ in $\langle 0|e^{i(k_1\cdot x+\ldots k_n\cdot x)}|0\rangle$, from the left, using that it annihilates the vacuum and commuting it to the right, one finds

$$\langle 0|e^{i(k_1\cdot x+\ldots+k_n\cdot x)}|0\rangle = \delta(k_1 + \ldots + k_n) \tag{9.64}$$

So, finally,

$$\langle 0|V_t(k_1,z_1)\ldots V_t(k_n,z_n)\rangle 0\rangle = \prod_{i<j}(z_i - z_j)^{k_i\cdot k_j}\, \delta(\sum_{i=1}^{n} k_i) \tag{9.65}$$

There is no need to stress the similarity of all this with Coulomb gas approach in section 2.5. The only difference is that in the present case there is no background charge and the final result is obtained via the oscillator algebra (but it could have been obtained as well by means of the Wick theorem with the propagators (9.49)). The formalism is clearly the same.

9.5.3 Open string amplitudes

In string theory the points of the real axis in the UHP where the vertex operators are inserted are non-physical and they have to be integrated over. Therefore, for instance, the tachyon amplitudes are

$$\mathcal{A}(k_1,\ldots,k_n) = g_o^{n-2}\int\prod_{i=1}^{n} dz_i\theta(z_{i-1}-z_i)\prod_{i<j}(z_i-z_j)^{k_i\cdot k_j} \tag{9.66}$$

where g_o is the open string coupling and θ is the step function. This amplitude is infinite. The reason is that the integral is invariant under the fractional transformations

$$z \to \frac{az+b}{cz+d}, \qquad ad-bc=1 \tag{9.67}$$

Using the fact that $k_i^2 = 2$, one can easily prove that the transformation of the integrand compensates for the transformation of the measure. Therefore the integral is infinite because the symmetry group is noncompact. This is similar to the problem of fixing the gauge in the Polyakov action. One can do it by fixing three points out of the n z_i, say $z_A = z_A^0$, $z_B = z_B^0$ and $z_C = z_C^0$. The price to pay is to introduce the Jacobian of the transformation between these points and the group parameters. The Jacobian is precisely (9.57), the 3-point amplitudes of the c ghost. After getting rid of a multiplicative infinite factor (the group volume), the amplitude becomes

$$\begin{aligned}\mathcal{A}(k_1,\ldots,k_n) &= g_o^{n-2}(z_A-z_A^0)(z_B-z_B^0)(z_C-z_C^0)\\ &\quad\cdot\int\prod_{i=1,i\neq A,B,C}^{n} dz_i\theta(z_{i-1}-z_i)\prod_{i<j}(z_i-z_j)^{k_i\cdot k_j}\end{aligned} \tag{9.68}$$

The Veneziano amplitude is the four-point tachyon amplitude obtained in this way. One sets $z_A^0=\infty, z_B^0=1$, $z_3=z$ and $z_C^0=0$. The result is:

$$\mathcal{A}(k_1,k_2,k_3,k_4) = g_o^2\int dz_0^1\, z^{-\frac{1}{2}s-2}(1-z)^{-\frac{1}{2}t-2} \tag{9.69}$$

where $s=-(k_1+k_2)^2, t=-(k_2+k_3)^2$.

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Appendix C. The Nambu-Goto action

This Appendix is devoted to the Nambu-Goto action. One can derive it by putting partially on shell the Polyakov action (7.1). Defining $A_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$ and $A = |\det A_{\alpha\beta}|$, eq.(7.7) becomes

$$A_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} A_{\gamma\delta}$$

whose determinant is $A \equiv \det A = \frac{1}{4} h \left(h^{\gamma\delta} A_{\gamma\delta} \right)^2$. Therefore

$$\frac{1}{2} \int_\Sigma d^2\sigma \sqrt{h} h^{\alpha\beta} A_{\alpha\beta} = \int_\Sigma d^2\sigma \sqrt{A} \tag{9.70}$$

$d\tau d\sigma \sqrt{A}$ is the inverse image of the area element spanned by the string in the target spacetime. To see this let us make a Wick rotation in M and introduce the compact notation $\vec{X}$ to represents $\{X^\mu\}$. The image (push-forward) of an infinitesimal vector in the σ direction is given by $X'^\mu \frac{\partial}{\partial x^\mu}$ and the analogous for τ is $\dot{X}^\mu \frac{\partial}{\partial x^\mu}$, where a dot denotes a derivative with respect to τ and a prime a derivative with respect to σ. The area element spanned by the two vectors is given by the modulus of their exterior product $\dot{\vec{X}} \wedge \vec{X}'$. Now $|\dot{\vec{X}} \wedge \vec{X}'| = |\dot{\vec{X}}||\vec{X}'|\sin\theta$, where θ is the angle between $\dot{\vec{X}}$ and $\vec{X}'$. It follows that $|\dot{\vec{X}} \wedge \vec{X}'|^2 = \dot{\vec{X}}^2 \vec{X}'^2 - \left(\dot{\vec{X}} \cdot \vec{X}' \right)^2$. Then

$$\sqrt{A} = \left(\dot{\vec{X}}^2 \vec{X}'^2 - \left(\dot{\vec{X}} \cdot \vec{X}' \right)^2 \right)^{\frac{1}{2}} = |\dot{\vec{X}} \wedge \vec{X}'|$$

which shows that the area element spanned by the string in spacetime is given by $\sqrt{A}$.

Appendix D. Examples of classical strings

Let the spacetime be 3-dimensional, $D = 3$, and set

$$\begin{aligned} t(\tau,\sigma) &\equiv X^0(\tau,\sigma) = 2\,R\,\tau \\ x(\tau,\sigma) &\equiv X^1(\tau,\sigma) = R\,\cos 2\sigma\,\cos 2\tau \\ y(\tau,\sigma) &\equiv X^2(\tau,\sigma) = R\,\sin 2\sigma\,\cos 2\tau \end{aligned} \tag{9.71}$$

where R is a constant length. It is easy to verify that these expressions satisfy both the equation of motion and the constraints. Moreover they are periodic in σ with period π. Therefore they represent a closed string. It is a pulsating string, a circle in the (x, y) plane with a radius varying periodically from a minimum $= 0$ to a maximum $= R$.

Next let us consider, in the same spacetime,

$$t(\tau,\sigma) = A\,\tau, \qquad x(\tau,\sigma) = A\,\cos\tau\,\cos\sigma, \qquad y(\tau,\sigma) = A\,\sin\tau\,\cos\sigma \tag{9.72}$$

where A is a constant length. This represents a bar of length $2A$ rotating about its center of mass located at the origin in the (x, y) plane with constant angular velocity (rigid rotor). It is easy to verify that (9.72) satisfy the equation of motion and the constraints. Moreover they satisfy the Neumann boundary conditions for open strings.

In the case of the pulsating closed string the total energy is

$$E = P^0 = T \int_0^\pi d\sigma\,\dot{t} = 2\pi RT$$

Therefore $T = \frac{E}{2\pi R}$ is the energy per unit length of the string at its maximal extension.

In the case of the open string as a rigid rotor the total energy is $E = \pi AT$. On the other hand the total angular momentum is

$$J = M^{xy} = \int_0^\pi d\sigma\,M_0^{xy} = T\int_0^\pi d\sigma\;(\dot{x}y - \dot{y}x) = \frac{\pi}{2}TA^2 = \frac{E^2}{2\pi T} \tag{9.73}$$

This string configuration maximizes the total angular momentum per unit energy. The ratio

$$\alpha' = \frac{\text{total angular momentum}}{\text{squared energy}} = \frac{1}{2\pi T} \tag{9.74}$$

is known as the *Regge slope*. We often use α' instead of T.

$$T = \frac{1}{2\pi\alpha'}$$

Part IV

CFT and related subjects

Chapter 10

Conserved, traceless correlators and anomalies

Correlators that contains insertion of a conserved current or of the energy-momentum tensor are expected to be conserved in any covariant theory. If the theory is conformal invariant the correlators that contain the e.m. tensor should also be traceless. There may however be anomalies. Appearance of anomalies is signaled by the violation of the relevant WI's by *contact terms*. In this chapter we would like to illustrate the emergence of trace and diffeomorphisms anomalies. What highlights this phenomenon is the coupling of the current to an external gauge field and of the e.m. tensor to a metric.

Contact terms: actions and anomalies

In quantum field theories there are two types of contact terms. The first type appear in correlators of currents and/or e.m. tensors (for instance, the third line of (5.58)). The second type appear in correlators involving the divergence of a current or the e.m. tensor, or the trace of the latter. In momentum space they are, in both cases, polynomials of the (external) momenta, so that in configuration space both types have a completely local structure, containing only delta functions and derivatives of delta functions (and no other local functions). But there is a fundamental difference between the two.

- type I. In configuration space they can be integrated to yield action terms (see, for instance, (5.59)). They satisfy the relevant Ward indenties.
- type II. They are similar to type I, but they break some Ward indentity, and, once integrated over spacetime, they yield local anomalies (gauge, Weyl and/or diffeomorphisms anomalies). Although they break a Ward identity, they satisfy a sort of second order Ward identity, the consistency condition.

In most cases they originate from the regularization of *bare* correlators at singular points.

Examples of type II will be shown presently.

10.1 Ward Identities for currents and e.m. tensors

Suppose we wish to describe correlators of a non-Abelian current J_μ^a, which is classically covariantly conserved:

$$(DJ)^a = (\partial^\mu \delta^{ac} + f^{abc} A^{b\mu}) J_\mu^c = 0 \tag{10.1}$$

where $D_\mu^{ab} = \partial_\mu \delta^{ab} + f^{abc} A_\mu^c$ is the covariant gauge derivative. The generating functional of the connected Green functions is given by

$$\begin{aligned} W[A] &= W[0] \qquad\qquad (10.2)\\ &+\sum_{n=1}^{\infty} \frac{i^{n-1}}{n!} \int \prod_{i=1}^{n} d^d x_i A^{a_1\mu_1}(x_1)\dots A^{a_n\mu_n}(x_n) \langle 0|\mathcal{T} J_{\mu_1}^{a_1}(x_1)\dots J_{\mu_n}^{a_n}(x_n)|0\rangle \end{aligned}$$

The full one-loop one-point function of J_μ^a in the presence of the source $A^{a\mu}$ is

$$\begin{aligned} \langle\!\langle J_\mu^a(x) \rangle\!\rangle = \frac{\delta W[A]}{\delta A^{a\mu}(x)} &= \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \prod_{i=1}^{n} d^d x_i A^{a_1\mu_1}(x_1)\dots A^{a_n\mu_n}(x_n)\\ &\times \langle 0|\mathcal{T} J_\mu^a(x) J_{\mu_1}^{a_1}(x_1)\dots J_{\mu_n}^{a_n}(x_n)|0\rangle \end{aligned} \tag{10.3}$$

Assuming invariance of (10.2) under the trasformation $\delta A_\mu^a = \partial_\mu \lambda^a + f^{abc} A_\mu^b \lambda^c$ we get the full 1-loop conservation

$$(D_\mu \langle\!\langle J_\mu(x) \rangle\!\rangle)^a = \partial^\mu \langle\!\langle J_\mu^a(x) \rangle\!\rangle + f^{abc} A_\mu^b(x) \langle\!\langle J^{\mu c}(x) \rangle\!\rangle = 0 \tag{10.4}$$

This splits into infinite many equations for correlators of different orders, which are obtained by subsequently differentiating (10.4) with respect to A. The simplest ones are obtained by differentiating once and twice with respect to the gauge field.

$$\partial_x^\mu \langle 0|\mathcal{T} J_\mu^a(x) J_\nu^b(y)|0\rangle = 0 \tag{10.5}$$

$$\begin{aligned} &\partial_x^\mu \langle 0|\mathcal{T} J_\mu^a(x) J_\nu^b(y) J_\lambda^c(z)|0\rangle \qquad\qquad (10.6)\\ &= i f^{abc'} \delta(x-y) \langle 0|\mathcal{T} J_\nu^{c'}(x)|0\rangle J_\lambda^c(z)|0\rangle + f^{acc'} \delta(x-z) \langle 0|\mathcal{T} J_\lambda^{c'}(x)|0\rangle J_\nu^b(y)|0\rangle \end{aligned}$$

We can do the same for correlators of the e.m. tensor, with some minor but significant changes. First we couple the e.m. tensor to an external symmetric tensor $h_{\mu\nu}$ (to be interpreted as the metric fluctuation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$) via an action term

$$-\frac{1}{2}\int d^d x\, h^{\mu\nu}(x) T_{\mu\nu}(x) \tag{10.7}$$

to be added to the (scalar, fermion,...) free field theory. In covariant theories like (4.15), (4.16) and (4.21) such coupling is already included in the action. As we have seen, in such cases the relevant classical covariant conservation law for diffeomorphisms is

$$\nabla^\mu T_{\mu\nu}(x) = 0 \tag{10.8}$$

The generating functional $W[h]$ of the connected Green functions is (5.2) from which one can derive the full one-loop one-point function of $T_{\mu\nu}$

$$\begin{aligned} &\langle\langle T_{\mu\nu}(x)\rangle\rangle \qquad\qquad (10.9)\\ &= \sum_{n=1}^{\infty} \frac{(-i)^{n+1}}{2^n n!} \int \prod_{i=1}^{n} dx_i \; h^{\mu_i\nu_i}(x_i)\, \langle 0|\mathcal{T} T_{\mu\nu}(x) T_{\mu_1\nu_1}(x_1)\dots T_{\mu_n\nu_n}(x_n)|0\rangle_c. \end{aligned}$$

Assuming invariance of $W[h]$ under the diffeomorphism $\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ one can immediately derive the full one-loop conservation law

$$\nabla^\mu \langle\langle T_{\mu\nu}\rangle\rangle = 0 \tag{10.10}$$

As above, this splits into infinite many equations for correlators of different order, which are obtained by differentiating the appropriate number of times with respect to h. The two lowest ones are

$$\partial_x^\mu \langle 0|\mathcal{T} T_{\mu\nu}(x) T_{\lambda\rho}(y)|0\rangle = 0 \tag{10.11}$$

and

$$\begin{aligned} &\partial_x^\mu \langle 0|\mathcal{T} T_{\mu\nu}(x) T_{\lambda\rho}(y) T_{\alpha\beta}(z)|0\rangle \qquad\qquad (10.12)\\ &= -i\Bigg\{ 2\frac{\partial}{\partial x^{(\alpha}} \left[\delta(x-z)\langle 0|\mathcal{T} T_{\beta)\nu}(x) T_{\lambda\rho}(y)|0\rangle\right]\\ &+2\frac{\partial}{\partial x^{(\lambda}} \left[\delta(x-y)\langle 0|\mathcal{T} T_{\rho)\nu}(x) T_{\alpha\beta}(z)|0\rangle\right]\\ &-\frac{\partial}{\partial x_\tau}\delta(x-z)\eta_{\alpha\beta}\langle 0|\mathcal{T} T_{\tau\nu}(x) T_{\lambda\rho}(y)|0\rangle - \frac{\partial}{\partial x_\tau}\delta(x-y)\eta_{\lambda\rho}\langle 0|\mathcal{T} T_{\tau\nu}(x) T_{\alpha\beta}(z)|0\rangle\\ &+\frac{\partial}{\partial x^\nu}\delta(x-z)\langle 0|\mathcal{T} T_{\lambda\rho}(y) T_{\alpha\beta}(x)|0\rangle + \frac{\partial}{\partial x^\nu}\delta(x-y)\langle 0|\mathcal{T} T_{\lambda\rho}(x) T_{\alpha\beta}(z)|0\rangle \Bigg\}, \end{aligned}$$

In these equations we use the time-ordering symbol $\mathcal{T}$. In a Euclidean theory this should be replaced by the radial ordering symbol $\mathcal{R}$.

In a similar way we can proceed for the Weyl symmetry $\delta g_{\mu\nu}(x) = 2\omega(x) g_{\mu\nu}(x)$, to obtain the traceless WI for the full one-loop one-point

$$\langle\langle T^{\mu}_{\mu}(x)\rangle\rangle \equiv g^{\mu\nu}(x)\langle\langle T_{\mu\nu}(x)\rangle\rangle = 0 \tag{10.13}$$

Differentiating as above we obtain WI's for correlators of different order. Assuming that the vacumm expectation value of $T_{\mu\nu}(x)$ vanishes, the two lowest order WI's are

$$\langle 0|\mathcal{T} T^{\mu}_{\mu}(x) T_{\lambda\rho}(y)|0\rangle = 0 \tag{10.14}$$

and

$$\langle 0|\mathcal{T} T^{\mu}_{\mu}(x) T_{\lambda\rho}(y) T_{\alpha\beta}(z)|0\rangle = i\left(\delta(x-y) + \delta(x-z)\right)\langle 0|\mathcal{T} T_{\lambda\rho}(y) T_{\alpha\beta}(z)|0\rangle \tag{10.15}$$

A comment is in order concerning the terms that appear in the RHS of (10.12) and (10.15). This terms are delta functions or derivatives thereof multiplying 2-point functions of the em tensor. Since the latter are generically non-local, it is natural to call such terms *semi-local*.

10.2 Anomalies

The classical conservation laws (10.4,10.10) and (10.13) may all be violated in a quantum theory by anomalies:

- gauge anomalies,

$$(D_\mu \langle\langle J_\mu(x)\rangle\rangle)^a \equiv X^a(x) W[A] = \mathcal{A}^a[A](x), \tag{10.16}$$

  where

$$X^a(x) = \partial_\mu \frac{\delta}{\delta A^a_\mu(x)} + f^{abc} A^b_\mu(x) \frac{\delta}{\delta A^c_\mu(x)}; \tag{10.17}$$

- diffeomorphisms anomalies,

$$\nabla^\mu \langle\langle T_{\mu\nu}\rangle\rangle \equiv \nabla_\mu \frac{\delta}{\delta g_{\mu\nu}(x)} W[g] = \mathcal{A}_\nu[g](x), \tag{10.18}$$

- trace anomalies,

$$\langle\langle T^{\mu}_{\mu}\rangle\rangle \equiv g_{\mu\nu}(x) \frac{\delta}{\delta g_{\mu\nu}(x)} W[g] = -T[g](x). \tag{10.19}$$

The minus sign in (10.16) and (10.18) is to keep track of the partial integration which is necessary in order to write the WI in that form.

To avoid a proliferation of symbols from now on we will denote in the same way the variation of a field and the corresponding functional operator

$$\begin{aligned}
&\delta_\lambda A^a = d\lambda^a + f^{abc}A^b\lambda^c, \qquad \delta_\lambda = \int dx\, \delta_\lambda A^a(x)\frac{\delta}{\delta A^a(x)} = -\int dx\, \lambda^a(x)X^a(x)\\
&\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \qquad \delta_\xi = \int dx\, \delta_\xi g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}(x)} \qquad (10.20)\\
&\delta_\omega g_{\mu\nu} = \omega g_{\mu\nu}, \qquad \delta_\omega = \int dx\, \delta_\omega g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}(x)}
\end{aligned}$$

Using this and integrating over spacetime both sides of (10.16,10.18) and (10.19) multiplied by their respective gauge parameters $\lambda^a(x), \xi^\mu(x)$ and $\omega(x)$, we obtain the integrated anomalous WI's:

$$\delta_\lambda W[A] = \mathcal{A}_\lambda, \qquad \mathcal{A}_\lambda = \int d^dx\, \sqrt{g}\lambda^a(x)\mathcal{A}^a[A](x) \qquad (10.21)$$

$$\delta_\xi W[g] = \mathcal{A}_\xi, \qquad \mathcal{A}_\xi = \int d^dx\, \sqrt{g}\xi^\mu(x)\mathcal{A}_\mu[g](x) \qquad (10.22)$$

$$\delta_\omega W[g] = T_\omega, \qquad T_\omega = \int d^dx\, \sqrt{g}\omega(x)T[g](x) \qquad (10.23)$$

Anomalies must satisfy consistency conditions which comes from the group theoretical nature of the symmetry they violate. Stated more precisely, we can operate another transformation (say λ', ξ', ω') on both sides of (10.21,10.22,10.23), and we can repeat the transformations in reverse order: then the way the RHS's of these equations are related under such reversing is dictated by the formal rules of the LHS's. For instance for gauge anomalies

$$\delta_{\lambda_1}\mathcal{A}_{\lambda_2} - \delta_{\lambda_2}\mathcal{A}_{\lambda_2} = \mathcal{A}_{[\lambda_1,\lambda_2]}, \qquad (10.24)$$

and likewise for the other cocycles. If the group is Abelian the RHS vanishes. This is the case for Weyl transformations (which are Abelian). There is an elegant way to express these otherwise very complicated relations by promoting the gauge parameters to *anticommuting fields* (or *ghost*) and endowing them with their own transformation property. If we assume

$$\delta_\xi \xi^\mu = \xi^\nu\partial_\nu\xi^\mu, \qquad \delta_\xi\lambda^a = \xi^\nu\partial_\nu\lambda^a, \qquad \delta_\xi\omega = \xi^\nu\partial_\nu\omega, \qquad (10.25)$$

$$\delta_\lambda\xi^\mu = 0, \qquad \delta_\lambda\lambda^a = -f^{abc}\lambda^b\lambda^c, \qquad \delta_\lambda\omega = 0, \qquad (10.26)$$

$$\delta_\omega\xi^\mu = 0, \qquad \delta_\omega\lambda^a = 0, \qquad \delta_\omega\omega = 0, \qquad (10.27)$$

one can easily verify that

$$(\delta_\xi + \delta_\lambda + \delta_\omega)^2 = 0 \tag{10.28}$$

by applying the LHS to any field in the game. This means $\delta_\xi + \delta_\lambda + \delta_\omega$ is a coboundary operator and defines a cohomology problem. It follows that

$$(\delta_\lambda + \delta_\xi + \delta_\omega)(\mathcal{A}_\lambda + \mathcal{A}_\xi + T_\omega) = 0, \tag{10.29}$$

i.e. the anomaly is a cocycle of $\delta_\xi + \delta_\lambda + \delta_\omega$. Eq.(10.29) is also called (Wess-Zumino) *consistency condition*. The LHS is bilinear in the various ghost fields. Thus (10.29) splits into separate homogeneous consistency conditions. The most relevant for us are

$$\delta_\lambda \mathcal{A}_\lambda = 0 \tag{10.30}$$
$$\delta_\xi \mathcal{A}_\xi = 0, \qquad \delta_\omega \mathcal{A}_\xi + \delta_\xi T_\omega = 0, \qquad \delta_\omega T_\omega = 0 \tag{10.31}$$

10.3 Cohomological analysis of anomalies

In quantum field theory anomalies are met when regularizing the logarithmic d ivergent parts of the relevant WIs, an operation which gives rise to local expressions. Very often the approach is perturbative, most of the times by means of Feynman diagrams. Perturbative calculations are usually very handy at the lowest order of approximation, but they become soon unprac- tical if one wants to proceed to the next orders. It is therefore of tremendous help to know in advance the form anomalies can take, so that knowing the first order can already identify them. This is provided by the cohomological analysis. The first advantage of the cohomological analysis is that, given the sym- metries of a theory, it provides a classification of its possible anomalies. For the sake of simplicity let us consider two such symmetries, which we shall refer to as R and S (for instance vector and axial gauge transformations, or diffeomorphisms and Weyl transformations). Let δR and δS be the corre- sponding coboundary operators. They are functional differential operators linear in the relevant ghost fields, and satisfy

$$(\delta_R + \delta_S)^2 = 0 \tag{10.32}$$

which splits into

$$\delta_R^2 = 0 \qquad \delta_S^2 = 0, \qquad \delta_R\delta_S + \delta_S\delta_R = 0 \tag{10.33}$$

A cocycle of $\delta_R + \delta_S$ is the sum of two integrated local expressions Δ_R and Δ_S

$$(\delta_R + \delta_S)\, W = \Delta_R + \Delta_S \tag{10.34}$$

Therefore

$$(\delta_R + \delta_S)(\Delta_R + \Delta_S) = 0 \tag{10.35}$$

This equation split into three distinct ones

$$\delta_R \Delta_R = 0, \qquad \delta_S \Delta_R + \delta_R \Delta_S = 0, \qquad \delta_S \Delta_S = 0 \tag{10.36}$$

Focusing on Δ_R, we are faced with two possibilities. Either it is a non-trivial R-cocycle or it is an R-coboundary. In the latter case $\Delta_R = \delta_R \mathcal{C}$, where $\mathcal{C}$ is a local expression of the fields excluding ghosts. Then we can redefine $W \to W' = W - \mathcal{C}$, so that

$$\delta_R W' = \Delta'_R = 0 \tag{10.37}$$
$$\delta_S W' = \Delta_S - \delta_S \mathcal{C} \equiv \Delta'_S \tag{10.38}$$

We have, of course,

$$\delta_S \Delta'_S = 0 \tag{10.39}$$

and, using the middle equation in eq.(10.36)

$$0 = \delta_S \Delta_R + \delta_R \Delta_S = \delta_R \Delta_S + \delta_S \delta_R \mathcal{C} = \delta_R (\Delta_S - \delta_S \mathcal{C}) = \delta_R \Delta'_S \tag{10.40}$$

This means that the cocycle Δ'_S is R-invariant. It might happen instead that Δ_S is an S-coboundary, i.e. $\Delta_S = delta_S \mathcal{C}'$. We could repeat the previous steps and come to the conclusion that $\Delta'_S = 0$ and Δ'_R is S-invariant. One further possibility is that, simultaneously Δ_R is an R-coboundary and Δ_S is an S-coboundary. We get $\Delta'_R = 0$ and $\Delta'_S = 0$ provided the counterterms satisfy the condition

$$\delta_S \delta_R \mathcal{C} + \delta_R \delta_S \mathcal{C}' = 0 \tag{10.41}$$

which is verified, for instance, if $\mathcal{C} = \mathcal{C}'$ or if $\mathcal{C}$ is S-invariant and $\rfloor'$ is R- invariant, but not in general. I.e. the fact that Δ_R and Δ_S are separately trivial, does not necessarily imply that the sum i$\Delta_R + \Delta_S$ is a coboundary of the cohomology of $\delta_R + \delta_S$. One last possibility is that neither Δ_R is an R-coboundary nor Δ_S is an S-coboundary.

10.4 Construction of non-trivial cocycles

As already pointed out it is important to know in advance explicit expressions of non-trivial cocycles, i.e. of local expressions that satisfy the relevant consistency conditions and cannot be absorbed into a redefinition of the effec- tive action. Solving in general the consistency conditions is not an easy task, and has been done so far only in a few particular cases. But it is perhaps not such an important problem. From a physical point of view there is an obvious hierarchy: if a gauge theory has (non-trivial) chiral gauge anomalies satisfying the Wess-Zumino consistency conditions it is a doomed theory, and it does not make much sense to search for possible anomalies of other (if any) classical symmetries of the theory; the same can be said for diffeomorphism anomalies in theories coupled to gravity: if diffeomorphisms are irremediably anomalous the theory is inconsistent and it does not make sense to look for, say, Weyl anomalies. Therefore the foremost cases to be considered are the chiral gauge anomalies and the diffeomorphism anomalies. The method to produce the corresponding cocycles is, in principle, very simple: one writes down all the possible 1-cochains, i.e. local expressions of the fields involved together with their derivatives, linear in the appropriate ghost field and of the right canonical dimensions; then one applies to them the coboundary operator and determines the combinations that satsify the consistency conditions (cocy- cles); finally one writes down all the 0-cochains, i.e. local expressions without ghosts, applies to them the coboundary operator and identifies the cocycles that are coboundaries. The trouble is that this method is far from efficient, especially when we deal with diffeomorphisms or with larger dimensions. For- tunately a most welcome shortcut in the case of gauge and diffeomorphisms anomalies is provided by a general formula due to S. S. Chern. This is the first and, probably, most important case in which we have compact formulas for non-trivial cocycles. Another important instance is when one can use (gauge or diffeomorphism) covariance in order to construct cocycles for another symmetry, for instance Weyl cocycles. We are going to illustrate these two cases in turn.

10.4.1 The Chern formula and descent equations

Let us consider a generic gauge theory, with connection $A_\mu^a T^a$, valued in a Lie algebra $\mathfrak{g}$ with anti-hermitean generators T^a , such that $[T^a, T^b] = f^{abc}T^c$. In the following it is convenient to use the more compact form notation and represent the connection as a one-form $\mathbf{A} = A_\mu^a T^a dx^\mu$, so that the gauge transformation becomes

$$\delta_\lambda \mathbf{A} = \mathbf{d}\lambda + [\mathbf{A}, \lambda] \tag{10.42}$$

with $\lambda(x) = \lambda^a(x)T^a$ and $\mathbf{d} = dx^\mu \frac{\partial}{\partial x^\mu}$. As explained above the mathematical problem is better formulated if we promote the gauge parameter λ to an anticommuting ghost field $c = c^a T$ and define the BRST transform as

$$s\mathbf{A} = \mathbf{d}c + [\mathbf{A}, c], \qquad sc = -\frac{1}{2}[c, c] \tag{10.43}$$

The operation s is nilpotent. We represent with the same symbol s the corresponding functional operator, i.e.

$$s = \int d^d x \left(s\mathbf{A}^a(x) \frac{\partial}{\partial \mathbf{A}^a(x)} + sc^a(x) \frac{\partial}{\partial c^a(x)} \right) \tag{10.44}$$

To construct the descent equations we start from a symmetric polynomial in the Lie algebra of order n, $P_n(T^{a_1}, ..., T^{a_n})$, invariant under the adjoint transformations (see Appendix B):

$$P_n([X, T^{a_1}], ..., T^{a_n}) + \ldots + P_n(T^{a_1}, ..., [X, T^{a_n}]) = 0 \tag{10.45}$$

for any element X of $\mathfrak{g}$. In many cases these polynomials are symmetric traces of the generators in the corresponding representation

$$P_n(T^{a_1}, ..., T^{a_n}) = Str(T^{a_1} ... T^{a_n}) \tag{10.46}$$

(Str denotes the symmetric trace). With this one can construct the 2n-form

$$\Delta_{2n}(\mathbf{A}) = P_n(\mathbf{F}, \mathbf{F}, \ldots \mathbf{F}) \tag{10.47}$$

where $\mathbf{F} = \mathbf{dA} + \frac{1}{2}[\mathbf{A}, \mathbf{A}]$. It is easy to prove that

$$P_n(\mathbf{F}, \mathbf{F}, \ldots \mathbf{F}) = \mathbf{d} \left(n \int_0^1 dt\, P_n(\mathbf{A}, \mathbf{F}_t, \ldots, \mathbf{F}_t) \right) = \mathbf{d} \Delta^{(0)}_{2n-1}(\mathbf{A}) \tag{10.48}$$

where we have introduced the symbols $\mathbf{A}_t = t\mathbf{A}$ and its curvature $\mathbf{F}_t = \mathbf{dA}_t + \frac{1}{2}[\mathbf{A}_t, \mathbf{A}_t]$, where $0 \leq t \leq 1$. In the above expressions the product of forms is understood to be the exterior product. It is important to recall that to prove eq.(10.48) one uses in an essential way the symmetry of P_n and the graded communtativity of the exterior product of forms. Eq.(10.48) is the first of a sequence of equations that can be proven

$$\Delta_{2n}(\mathbf{A}) - \mathbf{d}\Delta^{(0)}_{2n-1}(\mathbf{A}) = 0 \tag{10.49}$$

$$s\Delta^{(0)}_{2n-1}(\mathbf{A}) - \mathbf{d}\Delta^{(1)}_{2n-2}(\mathbf{A}, c) = 0 \tag{10.50}$$

$$s\Delta^{(1)}_{2n-2}(\mathbf{A}, c) - \mathbf{d}\Delta^{(2)}_{2n-3}(\mathbf{A}, c) = 0 \tag{10.51}$$

$$\ldots\ldots$$

$$s\Delta^{(2n-1)}_{0}(c) = 0 \tag{10.52}$$

All the expressions $\Delta^{(p)}_k(A, c)$ are polynomials of $\mathbf{d}, c$ and $\mathbf{A}$. The lower index k is the form order and the upper one p is the ghost number, i.e. the number of c factors. The last polynomial $\Delta^{(2n-1)}_0(c)$ is a 0-form and clearly a function only of c. All these polynomials have explicit compact form. For instance, the next interesting case after eq.(10.49) is

$$s\,\Delta_{2n-1}(\mathbf{A}) = \mathbf{d}\left(n(n-1)\int_0^1 dt(1-t)P_n(\mathbf{d}c,\mathbf{A},\mathbf{F}_t,\dots\mathbf{F}_t)\right) \tag{10.53}$$

This means in particular that integrating $\Delta_{2n1}(\mathbf{A})$ over spacetime in d = 2n 1 dimensions we obtain an invariant local expression. This gives the gauge CS action in any odd dimension. But what matters here is that the RHS contains the general expression of the consistent gauge anomaly in d = 2n 2 dimension, for, integrating (3.53) over spacetime, one gets

$$\begin{aligned} s\,\boldsymbol{\mathcal{A}}[c,\mathbf{A}] &= 0 \qquad\qquad (10.54)\\ \boldsymbol{\mathcal{A}}[c,\mathbf{A}] &= \int d^dx\,\Delta_d^1(c,\mathbf{A}), \qquad \text{where}\\ \Delta_d^1(c,\mathbf{A}) &= n(n-1)\int_0^1 dt(1-t)P_n(\mathbf{d}c,\mathbf{A},\mathbf{F}_t,\dots\mathbf{F}_t) \end{aligned}$$

$\boldsymbol{\mathcal{A}}[c,\mathbf{A}]$ identifies the anomaly up to an overall numerical coefficient.

Thus the existence of chiral gauge anomalies relies on the existence of the adjoint-invariant polynomials P_n. The properties of the latter are reviewed in Appendix E, below. One may wonder if the so-obtained cocycles are non-trivial. We can show that they are with a it reductio ad absurdum argument. Let us suppose that (10.54) is trivial. Then we can write

$$\Delta_d^1(c,\mathbf{A}) = s\,C_{2n-2}^{(0)}(\mathbf{A},c) + \mathbf{d}C_{2n-3}^{(1)}(\mathbf{A},c) \tag{10.55}$$

where here and below $C_k^{(p)}(\mathbf{A},c)$ denotes a polynomial k-form of ghost num- ber p. Applying s to (10.55) and using (10.51) we get

$$d\,s\,C_{2n-3}^{(1)}(\mathbf{A},c) - \mathbf{d}\Delta_{2n-3}^{(2)}(\mathbf{A},c) = 0 \tag{10.56}$$

Applying the local Poincaré theorem this implies

$$\Delta_{2n-3}^{(2)}(\mathbf{A},c) = s\,C_{2n-3}^{(1)}(\mathbf{A},c) + \mathbf{d}C_{2n-4}^{(2)}(\mathbf{A},c) \tag{10.57}$$

Repeating the procedure down to the 0-form order we will eventually find that there must exist a 0-form $C_0^{(2n-2)}(c)$ such that

$$\Delta_0^{(2n-1)}(c) = s\,C_0^{(2n-2)}(c) \tag{10.58}$$

However this is impossible, for the expression for $\Delta_0^{(2n-1)}(c)$ is

$$\Delta_0^{(2n-1)}(c) \sim P_n(c,[c,c]_+,\dots,[c,c]_+) \tag{10.59}$$

and the only possibility for $C_0^{(2n-2)}(c)$ to satisfy (10.58) is to have the form

$$C_0^{(2n-2)}(c) \sim P_n(c, c, [c, c]_+, \dots, [c, c]_+) \tag{10.60}$$

which, however, vanishes due to the symmetry of P_n and the anticommuta- tivity of c.

Remark If a nontrivial background metric $g_{\mu\nu}$ is present we have to insert a $\sqrt{g} \equiv \sqrt{\det(g_{\mu\nu})}$ factor in the integrand in order to guarantee diffeomorphism invariance

$$\boldsymbol{\mathcal{A}}[c, \mathbf{A}] \;=\; \int d^d x\, \sqrt{g} \Delta_d^1(c, \mathbf{A}) \tag{10.61}$$

The epression $\Delta_d^1(c, \mathbf{A})$ is a d-form, because $\mathbf{A}$ is a 1-form, $\mathbf{F}_t$ is a 2-form, while c is geometrically a scalar (i.e. $\delta_\xi c = \xi \cdot \partial c$), so that $\mathbf{d}c$ is a 1-form too. Under a diffeomorphism, which is the action of a vector field $\xi = \xi^\mu \frac{\partial}{\partial x^\mu}$, any form ϕ transform as $\delta_\xi \Phi = (\mathbf{i}_\xi \mathbf{d} + \mathbf{d}\mathbf{i}_\xi)\Phi$. Therefore, using (10.20)

$$\begin{aligned} \delta_\xi \boldsymbol{\mathcal{A}}[c, \mathbf{A}] \;&=\; \int d^d x \sqrt{g} \left(\nabla_\mu \xi^\mu \Delta_d^1(c, \mathbf{A}) + \mathbf{i}_\xi \mathbf{d} \Delta_d^1(c, \mathbf{A}) \right) \qquad (10.62) \\ &=\; \int d^d x \sqrt{g} \left(\nabla_\mu \xi^\mu - \nabla_\mu \xi^\mu \right) \Delta_d^1(c, \mathbf{A}) = 0 \end{aligned}$$

which follows from replacing the ordinary derivative in $\mathbf{d}$ by the covariant derivative, which is correct when $\mathbf{d}$ is applied to forms, and integrating by parts.

10.4.2 Diffeomorphism and Lorentz cocycles

In theories including gravity symmetry under dieomorphism is a fundamental invariance. If the theory involves fermions also the local Lorentz symmetry enters the game. They are both fundamental symmetries which are not allowed to be broken by anomalies, the price being the inconsistency of the theory. The study of the corresponding anomalies is thus of utmost importance.

Anomalies of the local Lorentz symmetry are not formally different from the local gauge anomalies analysed so far. The role of gauge connection is played by the spin connection

$$\boldsymbol{\omega} = \omega_\mu dx^\mu, \qquad \omega_\mu = \omega_\mu^{ab} \Sigma_{ab} \tag{10.63}$$

whose transformation law is

$$\delta_\Lambda \boldsymbol{\omega} = \mathbf{d}\Lambda + [\boldsymbol{\omega}, \Lambda], \qquad \Lambda = \Lambda^{ab} \Sigma_{ab} \tag{10.64}$$

where $\Sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$ are the generetors of the Lorentz Lie algebra. The curvature of $\boldsymbol{\omega}$ is the Riemann two form

$$\mathbf{R} = \mathbf{d}\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}], \qquad \mathbf{R} = \mathbf{R}^{ab}\Sigma_{ab} \tag{10.65}$$

Thus the general cocycle in dimension $d = 2n-2$ is obtained from formula (10.54) by simply making the replacements $c \to \Lambda$, $\mathbf{A} \to \boldsymbol{\omega}$ and $\mathbf{F} \to \mathbf{R}$:

$$\Delta_d^1(\Lambda, \boldsymbol{\omega}) \;\; = \;\; n(n-1)\int_0^1 dt(1-t)P_n(\mathbf{d}\Lambda, \boldsymbol{\omega}, \mathbf{R}_t, \dots \mathbf{R}_t)$$

In this case the anomaly is (up to the overall coefficient)

$$\boldsymbol{\mathcal{A}}_L[\Lambda, \boldsymbol{\omega}] = \int d^dx\,\sqrt{g}\Delta_d^1(\Lambda, \boldsymbol{\omega}), \qquad \delta_\Lambda \boldsymbol{\mathcal{A}}_L[\Lambda, \boldsymbol{\omega}] = 0 \tag{10.66}$$

The $\sqrt{g}$ is introduced in order to guarantee diffeomorphism invariance, as explained in the above remark:

$$\delta_\xi \boldsymbol{\mathcal{A}}_L[\Lambda, \boldsymbol{\omega}] = 0 \tag{10.67}$$

Let us next come to the cocycles of diffeomorphisms. In order to exploit the paralellism with the gauge case we introduce for the Christoffel symbols the matrix-form notation

$$\boldsymbol{\Gamma} \equiv \{\boldsymbol{\Gamma}_\nu{}^\lambda\}, \qquad \boldsymbol{\Gamma}_\nu{}^\lambda = dx^\mu\,\Gamma_{\mu\nu}{}^\lambda \tag{10.68}$$

and for the Riemannian curvature

$$\boldsymbol{\mathcal{R}} = \mathbf{d}\boldsymbol{\Gamma} + \boldsymbol{\Gamma}^2, \qquad \boldsymbol{\Gamma}_t, \qquad \boldsymbol{\mathcal{R}}_t = \mathbf{d}\boldsymbol{\Gamma}_t + \boldsymbol{\Gamma}_t^2 = t\boldsymbol{\mathcal{R}} + (t^2-t)\boldsymbol{\Gamma}^2 \tag{10.69}$$

The product between adjacent entries is the matrix product: $(XY)_\lambda{}^\rho = X_\lambda{}^\nu Y_\nu{}^\rho$. More explicitly,

$$\begin{aligned} \boldsymbol{\mathcal{R}} &= \{\boldsymbol{\mathcal{R}}_\lambda{}^\rho\}, \qquad \boldsymbol{\mathcal{R}}_\lambda{}^\rho = \frac{1}{2}dx^\mu \wedge dx^\nu R_{\mu\nu\lambda}{}^\rho \\ R^\rho{}_{\lambda\mu\nu} &= \partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} - \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\lambda} \end{aligned} \tag{10.70}$$

Now we can adapt the previous cocycle formulas to this case

$$\Delta_d^1(\xi, \boldsymbol{\Gamma}) \;\; = \;\; n(n-1)\int_0^1 dt(1-t)Str(\mathbf{d}\Xi\boldsymbol{\Gamma}\boldsymbol{\mathcal{R}}_t, \dots \boldsymbol{\mathcal{R}}_t)$$

where $\Xi_\lambda{}^\rho = \partial_\lambda \xi^\rho$, and Str denotes the symmetric trace of the matrix entries. This formula is justied because, using (10.20) and (10.25), one gets

$$\begin{aligned}
\delta_\xi \mathbf{\Gamma}_\lambda{}^\rho &= (\mathbf{i}_\xi \mathbf{d} + \mathbf{d}\mathbf{i}_\xi)\mathbf{\Gamma}^\rho_\lambda + \mathbf{d}\Xi_\lambda{}^\rho + [\mathbf{\Gamma}, \Xi]_\lambda{}^\rho && (10.71)\\
\delta_\xi \boldsymbol{\mathfrak{R}}_\lambda{}^\rho &= (\mathbf{i}_\xi \mathbf{d} + \mathbf{d}\mathbf{i}_\xi)\boldsymbol{\mathfrak{R}}^\rho_\lambda + [\boldsymbol{\mathfrak{R}}, \Xi]_\lambda{}^\rho && (10.72)\\
\delta_\xi \mathbf{d}\Xi_\lambda{}^\rho &= (\mathbf{i}_\xi \mathbf{d} + \mathbf{d}\mathbf{i}_\xi)\mathbf{d}\Xi_\lambda{}^\rho + [\mathbf{d}\Xi, \Xi]_\lambda{}^\rho && (10.73)
\end{aligned}$$

The corresponding anomaly (up to an overall coefficient) is:

$$\boldsymbol{\mathcal{A}}_D[\xi, \mathbf{\Gamma}] = \int d^dx \, \sqrt{g} \Delta^1_d(\xi, \mathbf{\Gamma}) \qquad (10.74)$$

which is obviously local Lorentz invariant.

Remark. The relation between $\mathbf{R}$ and $\boldsymbol{\mathfrak{R}}$ is

$$\boldsymbol{\mathfrak{R}}^\alpha{}_\beta = e^\alpha_a \, \mathbf{R}^{ab} \, e_{b\beta} \qquad (10.75)$$

10.4.3 Cohomological analysis of trace anomalies

After the pedagogical introduction of the previous chapter, we have now to tackle the same problem in higher dimensions, notably in 4d. From the 2d example it is clear that it is not advisable to preceed blindly with perturbative calculations. It is a tremendous help to know in advance the form trace anomalies can take. This is provided by the cohomological analysis.

The approach we will follow has been outlined in sec. 10.3. In the trace anomaly case we are not so lucky as to have a general formula like the Chern one for gauge anomalies, which is valid in any dimension, incorporates the relevant descent equations, and contain, in particular, the consistency conditions. Therefore we have to proceed in a more down-to-earth way. The context where trace anomalies become important is in theories that preserve diffeomorphisms. So we will set out to determine diffeomorphism invariant cocycles of the nilpotent Weyl coboundary operator δ_ω, defined by (10.20) with the associated transformation rules 10.25 and 10.27). In a spacetime of dimension d the procedure consists in

1. listing all the diff invariant 1-cochains (i.e. cochains linear in ω) and all the diff invariant 0-cochains (i.e. cochains independent of it) having d mass dimension;

2. upon acting with δ_ω on them, determining the combinations of 1-cochains which are annihilated by it (cocycles);

3. selecting among the latter those that can be obtained by acting with δ_ω on 0-cochains (coboundaries) and those that cannot (non-trivial cocycles).

To carry out this program we will need the following transformation formulas

$$\delta_\omega \Gamma^\lambda_{\mu\nu} = \partial_\mu\omega\,\delta^\lambda_\nu + \partial_\nu\omega\,\delta^\lambda_\mu - g^{\lambda\rho}\partial_\rho\omega\, g_{\mu\nu} \tag{10.76}$$

$$\delta_\omega R_{\mu\nu\lambda}{}^\rho = -\nabla_\mu\partial_\lambda\omega\,\delta^\rho_\nu + \nabla_\nu\partial_\lambda\omega\,\delta^\rho_\mu + g^{\rho\sigma}\nabla_\mu\partial_\sigma\omega\, g_{\nu\lambda} - g^{\rho\sigma}\nabla_\nu\partial_\sigma\omega\, g_{\mu\lambda} \tag{10.77}$$

$$\delta_\omega R_{\mu\nu} = (2-d)\nabla_\mu\nabla_\nu\omega - \Box\omega\, g_{\mu\nu} \tag{10.78}$$

$$\delta_\omega R = 2(1-d)\,\Box\omega\, R - 2\omega\, R \tag{10.79}$$

$$\delta_\omega\sqrt{g} = d\,\sqrt{g} \tag{10.80}$$

Weyl cocycles in d=2

In 2d the only diff invariant 1-cochain is $\Delta^{(1)}[\omega,g] = \int d^2x\,\sqrt{g}\,\omega\, R$, and the only diff-invariant 0-cochain is $\Delta^{(0)}[g] = \int d^2x\,\sqrt{g}\, R$. From (10.79) it is easy to prove that $\delta_\omega\Delta^{(1)}[\omega,g] = 0$ and $\delta_\omega\Delta^{(1)}[\omega,g] = 0$. Therefore $\Delta^{(1)}[\omega,g]$ is a non-trivial cocycle.

In 2d there is no possibility to construct a diff-invariant odd parity 1- or 0-cochain of dimension 2.

Weyl cocycles in d=4

In 4d the relevant 1-cochains are listed below together with their Weyl transforms

i	$\Delta^{(1)}_i[\omega,g]$	$\delta_\omega\Delta^{(1)}_i[\omega,g]$
1	$\int d^4x\,\sqrt{g}\,\omega\, R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}$	$4\int d^4x\sqrt{g}\;R\,\omega\Box\omega$
2	$\int d^4x\,\sqrt{g}\,\omega\, R_{\mu\nu}R^{\mu\nu}$	$4\int d^4x\,\sqrt{g}\,R\,\omega\Box\omega$
3	$\int d^4x\,\sqrt{g}\,\omega\, R^2$	$12\int d^4x\,\sqrt{g}\,R\,\omega\Box\omega$
4	$\int d^4x\,\sqrt{g}\,\omega\,\Box R$	0
5	$\int d^4x\,\sqrt{g}\,\omega\,\epsilon^{\mu\nu\mu'\nu'}R_{\mu\nu\lambda\rho}R_{\mu'\nu'}{}^{\lambda\rho}$	0

It is clear that $\Delta^{(1)}_4$ and $\Delta^{(1)}_5$ are cocycles, as is the combination $\sum_{i=1}^3 a_i\,\Delta^{(1)}_i$ provided $a_1 + a_2 + 3a_3 = 0$.

In order to see whether they are trivial or not let us consider the list of 0-cocycles with relative Weyl transformations

i	$\mathcal{C}_i[g]$	$\delta_\omega\mathcal{C}_i[g]$
1	$\int d^4x\,\sqrt{g}\,R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}$	$4\int d^4x\sqrt{g}\,\omega\,\Box R$
2	$\int d^4x\,\sqrt{g}\,R_{\mu\nu}R^{\mu\nu}$	$4\int d^4x\,\sqrt{g}\,\omega\,\Box R$
3	$\int d^4x\,\sqrt{g}\,\omega\,R^2$	$12\int d^4x\,\sqrt{g}\omega\,\Box R$

(10.81)

Notice that we have not included $\int d^4x\,\sqrt{g}\,\Box R$ and $\int d^4x\,\sqrt{g}\,\omega\,\epsilon^{\mu\nu\mu'\nu'}R_{\mu\nu\lambda\rho}R_{\mu'\nu'}{}^{\lambda\rho}$ among the 0-cocycles, because on a trivial spacetime manifold without boundary they vanish identically.

From this it follows that $\Delta_4^{(1)}$ is a coboundary, while $\Delta_5^{(1)}$ as well as the combination $\sum_{i=1}^{3} a_i\, \Delta_i^{(1)}$ with $a_1 + a_2 + 3a_3 = 0$, are non-trivial cocycles. The latter therefore contains two independent linear combinations, which are chosen to be defined by the quadratic Weyl density

$$\mathcal{W}^2 = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 \tag{10.82}$$

and the Gauss-Bonnet (or Euler) density,

$$E = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \tag{10.83}$$

while $\Delta_5^{(1)}$ is characterized by the Pontryagin density

$$P = \frac{1}{2}\left(\epsilon^{\mu\nu\mu'\nu'}\mathcal{R}_{\mu\nu\lambda\rho}\mathcal{R}_{\mu'\nu'}{}^{\lambda\rho}\right) \tag{10.84}$$

So that the trace of the energy-momentum tensor in 4d is expected to take the form

$$T_\mu{}^\mu = aE + c\mathcal{W}^2 + eP \tag{10.85}$$

The coefficients a, c and e are model-dependent.

10.4.4 Split and non-split anomalies

Before ending this chapter an important specification is in order for anomalies in fermionic field theories: it is the distinction between *split* and *non-split* anomalies. Split anomalies have opposite sign for opposite fermion chiralities. Non-split anomalies have the same sign for opposite chiralities. An example of the first are the consistent chiral gauge or gravity anomalies. They may of course arise only in the presence of a chiral asymmetry. These anomalies undermine the consistency of theories in which they are present, and, as a consequence, they have been used as an exclusion criterion. An example of non-split anomalies are the covariant gauge or gravity anomalies, such as the Kimura-Delbourgo-Salam anomaly or the anomaly that is utilized to explain the decay of a π^0 into two γ's. But the examples are manifold. In the family of trace anomalies, the even ones are non-split, while the odd trace anomaly is split.

Split and non-split anomalies differ also for the difficulties one comes across when computing them. While there are several tested techniques to compute non-split anomalies, the calculation of the split ones is rather non-trivial. In many of the latter cases one may avail oneself of such a powerful tool as the family index theorem (for instance for consistent gauge and gravity anomalies). But, like for the odd trace anomaly, this is not always so, and, in any case, it is important for us to be able to derive such anomalies with independent field-theoretical methods. If one resorts to path integral methods, one has to integrate out the fermion field(s), in which case the origin of the difficulties resides in the functional measure. Now, a basic ingredient for the calculation is the functional integration measure which, for chiral fermions, is not well-defined. On the other hand, to get the correct result, it is imperative to preserve throughout the calculation the information that the fermion field that is being integrated out, which has a definite chirality. One is then obliged to either use indirect methods or to elude a direct intrusion of the functional measure in the calculation. The second possibility refers to the use of Feynman diagrams, in which case the chirality of fermions is preserved by vertices containing the appropriate chiral projector. The former alternative is to use the indirect method of calculation first used by Bardeen for chiral gauge anomalies. He considered a theory of Dirac fermions coupled to two external non-Abelian (vector V_μ and axial A_μ) gauge potentials. Clearly this poses no problems from the point of view of the functional measure and the derivation of the anomaly goes through without difficulties. Eventually one takes the collapsing limit $V \to \frac{V}{2}$ and $A \to \frac{V}{2}$ and verifies that, in such a limit, the anomaly becomes the desired consistent gauge anomaly.

Appendix E. Adjoint invariant polynomials and anomalies

This Appendix is devoted to a review of the adjoint-invariant symmetric polynomials used to dene chiral anomalies. Let us consider a nite dimensional Lie group G with Lie algebra $\mathfrak{g}$. and let us introduce real-valued symmetric multi-linear mappings with n entries

$$P_n : \underbrace{\mathfrak{g} \times \ldots \times \mathfrak{g}}_{n \quad factors} \to \mathbb{R} \tag{10.86}$$

which are invariant under the adjoint action of G:

$$P_n(ad_g X_1, \ldots, ad_g X_n) = P_n(X_1, \ldots, X_n) \tag{10.87}$$

for any $g \in G$. The innitesimal version of (10.87) is

$$\sum_{i=1}^{n} P_n(X_1, \ldots, [Y, X_i], \ldots, X_n) = 0 \tag{10.88}$$

for any $Y \in \mathfrak{g}$. Let $T^a, (a = 1, \ldots, dim\mathfrak{g})$, be a basis of $\mathfrak{g}$. If we set

$$h^{a_1 \ldots a_n} = P_n(T^{a_1}, \ldots, T^{a_n}) \tag{10.89}$$

and $X_i = X_i^a T^a$, we get $P_n(X_1, \ldots, X_n) = h^{a_1 \ldots a_n} X_1^{a_1} \ldots X_n^{a_n}$ (summation over repeated indices is understood).

Now take n $\mathfrak{g}$-valued forms $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_n$ of order $q_1, \ldots, q_n$ respectively. Then we can write the $q_1 + ... + q_n$-form

$$P_n(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_n) = h^{a_1 \ldots a_n} \boldsymbol{\omega}_1^{a_1} \wedge \ldots \wedge \boldsymbol{\omega}_n^{a_n} \tag{10.90}$$

A well-known realization of completely symmetric tensors is given by

$$d^{a_1 \ldots a_n} = Str(T^{a_1} \ldots T^{a_n}) \tag{10.91}$$

whenever the generators are given in matrix form and Str means symmetrized trace. For instance

$$Tr(T^a T^b) = c_2(R) \delta^{ab} \tag{10.92}$$

$$\frac{1}{2} \mathrm{Tr}(T^a \{T^b, T^c\}) = c_3(R) d^{abc} \tag{10.93}$$

where $c_2(R), c_3(R)$ are representation-dependent numerical coefficients. As these two examples show, the tensors $d^{a_1 \ldots a_n}$ are universal, they are characteristic of the Lie algebra; changing the representation only modifies the numerical coefficients in from of them.

The space of symmetric ad-invariant polynomials of the simple group G is usually denoted by $I(G)$. It is a commutative algebra. If g has rank r, then $I(G)$ has r algebraically independent generators $m_1, ..., m_r$. In other words we have r algebraically independent polynomials P_n , or symmetric tensors $h^{a_1 \ldots a_m}$, of order $m_1, ..., m_r$. The values $m_1, ..., m_r$ for the simple Lie algebra are as follows

Lie algebra	$m_1, \ldots, m_r$
A_r	$2, 3, \ldots r+1$
B_r	$2, 4, \ldots, 2r$
C_r	$2, 4, \ldots, 2r$
D_r	$2, 4, \ldots 2r-2, r$
G_2	$2, 6$
F_4	$2, 6, 8, 12$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$

This table tells a lot about anomalies. If a symmetric tensor is absent in this table, the corresponding cocycle, and thus the corresponding anomaly does not exist.

This is the case, for instance, for $SU(2)$ in 4d. The corresponding Lie algebra is A_1, which has only the the second order symmetric tensor, while in order to construct a consistent chiral anomaly in 4d one needs the third order tensor. Analogously, the third order symmetric tensor does not exist for D_2, which is the Lie algebra of $SO(4)$, the compact version of the Lorentz group in 4d. In fact the local Lorentz symmetry is not anomalous in 4d.

10.5 A 2d example

Here we deal with examples of type II contact terms, arising from a regularization procedure. For reason of conciseness we consider a 2d example, which, though simpler than the higher dimension ones, contains all the characteristic features of type II. It concerns the two-point correlator of the e.m. tensor. Below we regularize this two-point function using the techniques of dierential regularization and we derive the 2d trace anomaly. We also discuss the ambiguities implicit in the regularization procedure which allow us to make manifest the interplay between dieomorphism and trace anomalies. Then we will redo the same derivation by means of Feynman diagrams. In both cases the method is perturbative, i.e. we represent the metric as g = + h, and consider series expansions in h. In this regard a clarication is in order. In (2.2) the generating functional is written as a function of h, W[h], while elsewhere we have denoted it W[g]. There is no disagreement between the two notations. The notation W[h] emphasizes the fact that the RHS of (2.2) is a series in h. But of course once the series is resummed the result must be a functional of g. We will use the symbol h whenever we want to stress the perturbative aspect, and the symbol g whenever we want to indicate the complete result. It should be remarked that anomalies, like in this chapter, are often obtained via perturbative calculations, therefore we have also to adapt the previous cohomological formulation to such a perturbative environment. This is done in Appendix F.

10.6 Differential regularization

The e.m. tensor 2-point function in $2d$ (i.e. the "bare" correlator) is very well-known and is given by

$$\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle = \frac{c/2}{x^4}\left(I_{\mu\rho}(x)\, I_{\nu\sigma}(x) + I_{\nu\rho}(x)\, I_{\mu\sigma}(x) - \eta_{\mu\nu}\eta_{\rho\sigma}\right) \tag{10.94}$$

where c is the central charge of the theory. For $x \neq 0$ it satisfies the Ward identities

$$\partial^\mu \langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle = 0, \tag{10.95}$$

$$\left\langle T^\mu_\mu(x) T_{\rho\sigma}(0)\right\rangle = 0. \tag{10.96}$$

This 2-point function is UV singular for $x \to 0$ and such a divergence has to be dealt with for the correlator to be well-defined everywhere. In this context the most convenient way to regularize it is with the technique of *differential regularization.* The recipe of differential regularization is: given a function $f(x)$ that needs to be regularized at $x = 0$, find the most general function $F(x)$ such that $\mathcal{D}F(x) = f(x)$ for $x \neq 0$, where $\mathcal{D}$ is some differential operator, and such that the Fourier transform of $\mathcal{D}F(x)$ is well-defined (alternatively, $\mathcal{D}F(x)$ has integrable singularities).

In our case we have two guiding principles: the Ward identities and dimensional analysis. Differential regularization tells us that our 2-point function should be some differential operator applied to a function, i.e.

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle = \mathcal{D}_{\mu\nu\rho\sigma}(f(x)), \tag{10.97}$$

while conservation requires that the differential operator $\mathcal{D}_{\mu\nu\rho\sigma}$ be transverse, i.e.

$$\partial^\mu \mathcal{D}_{\mu\nu\rho\sigma} = \cdots = \partial^\sigma \mathcal{D}_{\mu\nu\rho\sigma} = 0.$$

The most general transverse operator with four derivatives, symmetric in μ,ν and in ρ,σ and in the exchange of the couple μ, ν with ρ, σ, one can write is

$$\mathcal{D}_{\mu\nu\rho\sigma} = \alpha \mathcal{D}^{(1)}_{\mu\nu\rho\sigma} + \beta \mathcal{D}^{(2)}_{\mu\nu\rho\sigma}, \tag{10.98}$$

where

$$\mathcal{D}^{(1)}_{\mu\nu\rho\sigma} = \partial_\mu\partial_\nu\partial_\rho\partial_\sigma - (\eta_{\mu\nu}\partial_\rho\partial_\sigma + \eta_{\rho\sigma}\partial_\mu\partial_\nu)\Box + \eta_{\mu\nu}\eta_{\rho\sigma}\Box\Box, \tag{10.99}$$

$$\begin{aligned}\mathcal{D}^{(2)}_{\mu\nu\rho\sigma} &= \partial_\mu\partial_\nu\partial_\rho\partial_\sigma - \frac{1}{2}(\eta_{\mu\rho}\partial_\nu\partial_\sigma + \eta_{\nu\rho}\partial_\mu\partial_\sigma + \eta_{\mu\sigma}\partial_\nu\partial_\rho + \eta_{\nu\sigma}\partial_\mu\partial_\rho)\Box \\ &\quad + \frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma})\Box\Box.\end{aligned} \tag{10.100}$$

Dimensional analysis tells us that the function $f(x)$ in (10.97) can be at most a function of $\log \mu^2 x^2$ since the lhs of (10.97) scales like $1/x^4$ and this scaling is already saturated by the differential operator with four derivatives. We have introduced an arbitrary mass scale μ to make the argument of log dimensionless. Let us write the most general ansatz for (10.97):

$$\begin{aligned}\langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle &= \mathcal{D}^{(1)}_{\mu\nu\rho\sigma}\left[\alpha_1 \log\mu^2x^2 + \alpha_2\left(\log\mu^2x^2\right)^2 + \cdots\right] \\ &\quad + \mathcal{D}^{(2)}_{\mu\nu\rho\sigma}\left[\beta_1 \log\mu^2x^2 + \beta_2\left(\log\mu^2x^2\right)^2 + \cdots\right].\end{aligned} \tag{10.101}$$

Now our task is to fix the coefficients α_i and β_j for (10.101) to match (10.94) for $x \neq 0$. As it turns out we only need terms up to $\log^2$ (otherwise one cannot avoid unwanted logarithmic terms for $x \neq 0$). The matching gives us

$$\alpha_1 + \beta_1 = -\frac{c}{24}, \quad \alpha_2 = -\beta_2 = -\frac{c}{96},$$

Only the sum $\alpha_1 + \beta_1$ is fixed. So, to simplify, we can set, for instance $\beta_1 = 0$. Finally

$$\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle = -\frac{c}{24}\mathcal{D}^{(1)}_{\mu\nu\rho\sigma}\left(\log \mu^2 x^2\right) - \frac{c}{96}\left(\mathcal{D}^{(1)}_{\mu\nu\rho\sigma} - \mathcal{D}^{(2)}_{\mu\nu\rho\sigma}\right)\left(\log \mu^2 x^2\right)^2 . \tag{10.102}$$

Eq.(10.102) gives back (10.94) for $x \neq 0$. But if we take the trace we find that

$$\left\langle T^{\mu}_{\mu}(x)\, T_{\rho\sigma}(0)\right\rangle = -\frac{c}{24}\eta^{\mu\nu}\mathcal{D}^{(1)}_{\mu\nu\rho\sigma}\left(\log \mu^2 x^2\right) = -\frac{c}{24}\left(\partial_\rho\partial_\sigma - \eta_{\rho\sigma}\Box\right)\Box \log \mu^2 x^2 .$$

These terms have support only at $x = 0$, for in $2d$ the d'Alembertian of a log is a delta function, more precisely

$$\Box \log \mu^2 x^2 = -4\pi\delta^2(x) . \tag{10.103}$$

Therefore we find the anomalous Ward identity

$$\left\langle T^{\mu}_{\mu}(x)\, T_{\rho\sigma}(y)\right\rangle = c\frac{\pi}{6}\left(\partial_\rho\partial_\sigma - \eta_{\rho\sigma}\Box\right)\delta^2(x-y) , \tag{10.104}$$

Now, if we insert this result in (5.2), i.e. saturate with $h^{\rho\sigma}(y)$ and integrate over y, we get

$$\langle\!\langle T^{\mu}_{\mu}\rangle\!\rangle = c\frac{\pi}{6}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)h^{\mu\nu}, \tag{10.105}$$

which coincides with the lowest contribution of the expansion in h of the Ricci scalar, i.e.

$$R = (\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)h^{\mu\nu} + \mathcal{O}(h^2). \tag{10.106}$$

Covariance requires the higher order corrections in h to the 'full one-loop' trace of the e.m. tensor in the presence of a background metric g to be such that we recover the covariant expression

$$\langle\!\langle T^{\mu}_{\mu}\rangle\!\rangle = c\frac{\pi}{6}R. \tag{10.107}$$

For instance, for a free Dirac fermion $c = \frac{1}{16\pi^2}$. We are authorized to use the covariant expression (10.107) because the energy-momentum tensor is conserved (there are no diffeomorphism anomalies, see below). We can also say that requiring the regularized correlator to be conserved at $x = 0$ implies the appearance of a trace anomaly. However this is not the end of the story, since there are ambiguities in the regularization process we have so far disregarded.

But before that let us pause to consider the cohomological aspects. It is easy to see that

$$\delta^{(0)}_\omega \int d^2x\, \omega(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)h^{\mu\nu} = 0 \tag{10.108}$$

and, using $\delta_\omega R = -2\omega\, R - 2\Box\omega$,

$$\delta_\omega \int d^2x\, \sqrt{g}\, \omega\, R = 0 \tag{10.109}$$

This is a first simple example of how perturbative cohomology and full cohomology (with the help of covariance) operate. In this case the diffeomorphisms do not enter the game because we have preserved e.m. tensor conservation all through the construction.

10.6.1 Ambiguities

The ambiguity arises from the fact that we can add to (10.102) terms that have support only at $x = 0$. The most general modification of the parity-even part that would affect only its expression for $x = 0$ is given by

$$\begin{aligned} A^{(e)}_{\mu\nu\rho\sigma} \;=\;& A\,(\eta_{\mu\nu}\partial_\rho\partial_\sigma + \eta_{\rho\sigma}\partial_\mu\partial_\nu)\,\Box \log \mu^2 x^2 \\ &+B\,(\eta_{\mu\rho}\partial_\nu\partial_\sigma + \eta_{\nu\rho}\partial_\mu\partial_\sigma + \eta_{\mu\sigma}\partial_\nu\partial_\rho + \eta_{\nu\sigma}\partial_\mu\partial_\rho)\,\Box \log \mu^2 x^2 \\ &+C\,(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma})\,\Box\Box \log \mu^2 x^2 \\ &+D\eta_{\mu\nu}\eta_{\rho\sigma}\Box\Box \log \mu^2 x^2. \end{aligned} \tag{10.110}$$

We remark that this term is in general neither conserved nor traceless

$$\begin{aligned} \partial^\mu A^{(e)}_{\mu\nu\rho\sigma}(x) \;=\;& 4\pi\,((A + 2B)\partial_\nu\partial_\rho\partial_\sigma + (A + D)\eta_{\rho\sigma}\partial_\nu\Box \\ &+ (B + C)\,(\eta_{\rho\nu}\partial_\sigma\Box + \eta_{\sigma\nu}\partial_\rho\Box))\,\delta^{(2)}(x) \end{aligned} \tag{10.111}$$

$$A^{(e)}_\mu{}\mu\rho\sigma(x) = 4\pi\,((2A + 4B)\partial_\rho\partial_\sigma + (A + 2C + 2D)\eta_{\rho\sigma}\Box)\,\delta^{(2)}(x) \tag{10.112}$$

Thus these terms are potential anomalies. For them to be true anomalies they must satisfy the consistency conditions.

We notice that by imposing (10.111) to vanish implies that also (10.112) will vanish. We may wonder whether using this ambiguity we can cancel the trace anomaly. Using eqs.(10.22) and (10.23), let write down these potential anomalies in integrated form.

$$\mathcal{A}_{\xi}^{(e,p)} = -\int d^2x\,\xi^\nu \partial^\mu A_{\mu\nu\rho\sigma}(x)\delta(x-y)h^{\rho\sigma}(y) \tag{10.113}$$

$$\mathcal{A}_{\omega}^{(e,p)} = \int d^2x\,\omega A^\mu_{\mu\rho\sigma}(x)\delta(x-y)h^{\rho\sigma}(y) \tag{10.114}$$

Now, imposing $\delta_\xi^{(0)}\mathcal{A}_\xi^{(e,p)} = 0$ we find the condition $2A+3B+C+D=0$. This means that $\mathcal{A}_\xi^{(e,p)}$ can be rearranged as follows

$$\mathcal{A}_{\xi}^{(e,p)} = -4\pi\int d^2x\,\xi^\nu \Big(E\,\partial_\nu\big(\partial_\rho\partial_\sigma h^{\rho\sigma} - \Box h\big) + F\,\partial_\sigma\big(\partial_\nu\partial_\rho h^{\rho\sigma} - 2\Box h^\sigma_\nu\big)\Big) \tag{10.115}$$

where $E = A+3B+C$ and $F = 2A+2B+D$ are unconstrained parameters.

Next, if we impose $\delta_\omega^{(0)}\mathcal{A}_\omega^{(e,p)} = 0$, we find the condition $3A+4B+2C+2D=0$ and

$$\mathcal{A}_{\omega}^{(e,p)} = 8\pi(A+2B)\int d^2x\,\omega\big(\partial_\rho\partial_\sigma h^{\rho\sigma} - \Box h\big) \tag{10.116}$$

with uncostrained $A+2B$. Finally we have to impose $\delta_\xi^{(0)}\mathcal{A}_\omega^{(e,p)} + \delta_\omega^{(0)}\mathcal{A}_\xi^{e,(p)} = 0$. It is easy to verify that $\delta_\xi^{(0)}\mathcal{A}_\omega^{(e,p)} = 0$ without any constraint, while $\delta_\omega^{(0)}\mathcal{A}_\xi^{(e,p)} = 0$ requires $F=0$, which in turn implies $E = A+2B$.

In conlcusion the ambiguities (10.110) are allowed provided $C=-B$ and $D=-2A-2B$, where A and B are free parameters. They give rise to the trace anomaly (10.116) and to the diff anomaly

$$\mathcal{A}_{\xi}^{(e,p)} = -4\pi(A+2B)\int d^2x\,\xi^\nu\,\partial_\nu\big(\partial_\rho\partial_\sigma h^{\rho\sigma} - \Box h\big) \tag{10.117}$$

The cocycles (10.117) and (10.116) satisfy separately the overall consistency conditions $\left(\delta_\xi + \delta_\omega^{(0)}\right)\left(\mathcal{A}_\xi^{(e,p)}, \mathcal{A}_\omega^{(e,p)}\right) = 0$.

Returning now to the initial question of whether we can cancel the trace anomalyl (10.105), or (10.107), by means of these ambiguities, the answer is yes. We can do it by subtracting a combination of ambiguities that satisfy $C=-B$ and $D=-2A-2B$, and adjusting the overall coefficient. But this operation triggers the diff anomaly (10.117). In other words the anomaly (10.107) is a non-trivial cocycle of the overall symmetry under diffeomorphisms plus Weyl transformations. It may take different forms, either as a pure diffeomorphism anomaly or a pure trace anomaly. In general both components may be nonvanishing. It is obvious that it is preferable to preserve diffeomorphism invariance, so that the cocycle takes the form (10.107).

10.6.2 Parity-odd terms in $2d$

In this section we compute all possible "bare" parity-odd terms in the 2-point function of the energy-momentum tensor in $2d$. To this end we write the most general expression $\mathcal{T}^{\text{odd}}_{\mu\nu\rho\sigma}(x)$ linear in the antisymmetric tensor $\epsilon_{\alpha\beta}$ with the right dimensions, which is symmetric and traceless in μ, ν and ρ, σ separately, is symmetric in the exchange $(\mu, \nu) \leftrightarrow (\rho, \sigma)$, and is conserved. The calculation is tedious but straightforward. The result is as follows. Let us define

$$T_{\mu\nu\rho\sigma} = \frac{1}{x^4}\left(I_{\mu\rho}(x)I_{\nu\sigma}(x) + I_{\mu\sigma}(x)I_{\nu\rho}(x) - \eta_{\mu\nu}\eta_{\rho\sigma}\right), \tag{10.118}$$

which is proportional to the parity-even 2-point function, and

$$\mathcal{T}^{\text{odd}}_{\mu\nu\rho\sigma}(x) = \frac{e}{2}\left(\epsilon_{\alpha\mu}T^{\alpha}{}_{\nu\rho\sigma}(x) + \epsilon_{\alpha\nu}T_{\mu}{}^{\alpha}{}_{\rho\sigma}(x) + \epsilon_{\alpha\rho}T_{\mu\nu}{}^{\alpha}{}_{\sigma}(x) + \epsilon_{\alpha\rho}T_{\mu\nu\rho}{}^{\alpha}(x)\right), \tag{10.119}$$

where e is an undetermined constant. We assume (10.119) to represent the two-point correlator $\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle_{\text{odd}}$. It satisfies all the desired properties (it is traceless and conserved). In order to make sure that it is conformal covariant, we can check that it is chirally split. To this end we introduce the light-cone coordinates $x_{\pm} = x^0 \pm x^1$. It is not hard to verify that

$$\langle T_{++}(x)T_{--}(0)\rangle_{\text{odd}} = 0. \tag{10.120}$$

The task of regularizing the parity-odd terms is very much simplified if we write them in terms of the parity-even part. We can therefore use the same regularization as above. Let us start by the regularization that preserves diffeomorphisms for the parity-even part, eq. (10.102):

$$T_{\mu\nu\rho\sigma}(x) = -\frac{1}{12}\mathcal{D}^{(1)}_{\mu\nu\rho\sigma}\left(\log\mu^2x^2\right) - \frac{1}{48}\left(\mathcal{D}^{(1)}_{\mu\nu\rho\sigma} - \mathcal{D}^{(2)}_{\mu\nu\rho\sigma}\right)\left(\log\mu^2x^2\right)^2. \tag{10.121}$$

Regularizing with (10.119) leads to both a trace anomaly

$$\left\langle T^{\mu}_{\mu}(x)\,T_{\rho\sigma}(0)\right\rangle_{\text{odd}} = \frac{\pi e}{24}\left(\epsilon_{\rho\alpha}\partial^{\alpha}\partial_{\sigma} + \epsilon_{\sigma\alpha}\partial^{\alpha}\partial_{\rho}\right)\delta^2(x), \tag{10.122}$$

and a diffeomorphism anomaly

$$\partial^{\mu}\left\langle T_{\mu\nu}(x)\,T_{\rho\sigma}(0)\right\rangle_{\text{odd}} = \frac{\pi e}{24}\epsilon_{\nu\alpha}\partial^{\alpha}\left(\eta_{\rho\sigma}\Box - \partial_{\rho}\partial_{\sigma}\right)\delta^2(x). \tag{10.123}$$

Inserting these results in (5.2) they give rise to the following lowest order anomalies

$$\mathcal{A}^{(0)}_{\xi} = -\frac{\pi e}{24}\int d^2x\,\xi^{\nu}\epsilon_{\nu\alpha}\partial^{\alpha}\left(\eta_{\rho\sigma}\Box - \partial_{\rho}\partial_{\sigma}\right)h^{\sigma\rho} \tag{10.124}$$

and

$$T^{(0)}_\omega = -\frac{\pi e}{24}\int d^2x\, \omega\, \epsilon_{\lambda\alpha}\partial^\alpha\partial_\rho h^{\lambda\rho} \tag{10.125}$$

It is a useful exercise to verify that

$$\delta^{(0)}_\xi \mathcal{A}^{(0)}_\xi = 0, \qquad \delta^{(0)}_\xi T^{(0)}_\omega = 0 = \delta^{(0)}_\omega \mathcal{A}^{(0)}_\xi, \qquad \delta^{(0)}_\omega T^{(0)}_\omega = 0. \tag{10.126}$$

We see that the diffeomorphism anomaly is accompanied by the a trace anomaly.

It is worth pointing out that (10.123) is the lowest order approximation of the following 'full one-loop' relation

$$\langle\!\langle \nabla_\mu T^{\mu\nu}(x)\rangle\!\rangle \;=\; \frac{\pi e}{24}\epsilon^{\nu\alpha}\partial_\alpha R, \tag{10.127}$$

Eq.(10.127) is the well-known covariant form of the diffeomorphism anomaly. At the lowest order, however, also the consistent anomaly

$$\langle\!\langle \nabla^\mu T_{\mu\nu}(x)\rangle\!\rangle = \frac{\pi e}{12}\frac{\epsilon^{\nu\alpha}}{\sqrt{g}}\partial_\mu\partial_\alpha\Gamma^\alpha_{\rho\nu} = \frac{\pi e}{24}\epsilon_{\alpha\rho}\partial^\alpha\left(\Box h^\rho_\nu - \partial_\nu\partial_\sigma h^{\rho\sigma}\right) + \mathcal{O}\left(h^2\right) \tag{10.128}$$

takes the same form due to the 2d identity

$$2\epsilon_{\mu\nu}\partial^\mu\left(\partial_\alpha\partial_\beta - \eta_{\alpha\beta}\Box\right) = \epsilon_{\mu\alpha}\left(\partial^\mu\partial_\nu\partial_\beta - \eta_{\nu\beta}\partial^\mu\Box + (\alpha\leftrightarrow\beta)\right) \tag{10.129}$$

We recall that saturating (10.128) with the ghost ξ^ν, and integrating it over the 2d spacetime, it can be written in the compact form

$$\sim \int d^2x\, \mathrm{Tr}\big(d\Lambda\,\Gamma\big) \tag{10.130}$$

where Λ is the matrix with components $\Lambda_\rho{}^\sigma = \partial_\rho\xi^\sigma$ and Γ is the matrix one-form $\Gamma^\rho_{\mu\sigma}dx^\mu$. The trace is over the matrix indices. In the form (10.130) proving consistency is a very simple exercise, as was elsewhere.

Neither (10.127) nor (10.128) is invariant under Weyl transformations. Therefore they must be both accompanied by a trace anomaly partner, whose lowest term is given by (10.125).

10.6.3 Ambiguities in the parity-odd part

We know that the regularization used above is not the ultimate one, because there are ambiguities. They entail a modification of the parity-odd part given by

$$A^{(o)}_{\mu\nu\rho\sigma} = \epsilon_{\alpha\mu}A^\alpha{}_{\nu\rho\sigma} + \epsilon_{\alpha\nu}A_\mu{}^\alpha{}_{\rho\sigma} + \epsilon_{\alpha\rho}A_{\mu\nu}{}^\alpha{}_\sigma + \epsilon_{\alpha\rho}A_{\mu\nu\rho}{}^\alpha, \tag{10.131}$$

where the RHS is written in terms of (10.110), which explicitly is

$$\begin{aligned} A^{(o)}_{\mu\nu\rho\sigma} = \ & A\left[\eta_{\mu\nu}\left(\epsilon_{\rho\alpha}\partial^\alpha\partial_\sigma + \epsilon_{\sigma\alpha}\partial^\alpha\partial_\rho\right) + \eta_{\rho\sigma}\left(\epsilon_{\mu\alpha}\partial^\alpha\partial_\nu + \epsilon_{\nu\alpha}\partial^\alpha\partial_\mu\right)\right]\Box\log\mu^2x^2 \\ & + B\left[\epsilon_{\mu\alpha}\left(\eta_{\nu\rho}\partial^\alpha\partial_\sigma + \eta_{\nu\sigma}\partial^\alpha\partial_\rho\right) + \epsilon_{\nu\alpha}\left(\eta_{\mu\rho}\partial^\alpha\partial_\sigma + \eta_{\mu\sigma}\partial^\alpha\partial_\rho\right)\right. \\ & \left. + \epsilon_{\rho\alpha}\left(\eta_{\sigma\mu}\partial^\alpha\partial_\nu + \eta_{\sigma\nu}\partial^\alpha\partial_\mu\right) + \epsilon_{\sigma\alpha}\left(\eta_{\rho\mu}\partial^\alpha\partial_\nu + \eta_{\rho\nu}\partial^\alpha\partial_\mu\right)\right]\Box\log\mu^2x^2. \end{aligned} \tag{10.132}$$

The trace and the divergence of (10.132) are given by:

$$\eta^{\mu\nu}A^{(o)}_{\mu\nu\rho\sigma} = 8\pi\left(A+2B\right)\left(\epsilon_{\rho\alpha}\partial^\alpha\partial_\sigma + \epsilon_{\sigma\alpha}\partial^\alpha\partial_\rho\right)\delta^2\left(x\right), \tag{10.133}$$

$$\begin{aligned} \partial^\mu A^{(o)}_{\mu\nu\rho\sigma} = \ & 4\pi\left(B\eta_{\nu\rho}\Box + \left(A+B\right)\partial_\nu\partial_\rho\right)\epsilon_{\sigma\alpha}\partial^\alpha\delta^2\left(x\right) \\ & + 4\pi\left(B\eta_{\nu\sigma}\Box + \left(A+B\right)\partial_\nu\partial_\sigma\right)\epsilon_{\rho\alpha}\partial^\alpha\delta^2\left(x\right) \\ & + 4\pi\left(A\eta_{\rho\sigma}\Box + 2B\partial_\rho\partial_\sigma\right)\epsilon_{\nu\alpha}\partial^\alpha\delta^2\left(x\right). \end{aligned} \tag{10.134}$$

As before let us define

$$\mathcal{A}^{(o,p)}_\xi = \int d^2x\,\xi^\nu\partial^\mu A^{(o)}_{\mu\nu\rho\sigma}(x)\delta(x-y)h^{\rho\sigma}(y) \tag{10.135}$$

$$\mathcal{A}^{(o,p)}_\omega = -\int d^2x\,\omega A^{(o)\mu\rho\sigma}_\mu(x)\delta(x-y)h^{\rho\sigma}(y) \tag{10.136}$$

It is not hard to see that[1]

$$\begin{aligned} \delta^{(0)}_\xi\mathcal{A}^{(o,p)}_\xi = \ & 0 = \delta^{(0)}_\omega\mathcal{A}^{(o,p)}_\omega \\ \delta^{(0)}_\omega\mathcal{A}^{(o,p)}_\xi = -\delta^{(0)}_\xi\mathcal{A}^{(o,p)}_\omega = \ & -16\pi(A+2B)\int d^2x\,\xi^\nu\epsilon_{\nu\alpha}\partial^\alpha\Box\omega \end{aligned} \tag{10.138}$$

(10.135) and (10.136), coming from the local ambiguity (10.132), satisfy the diff + Weyl consistency conditions, for any A and B.

Using these ambiguities we can modify the previously computed trace anomaly (10.122) and diff anomaly (10.123). In particular it is interesting to see whether we can cancel the latter. To achieve this we have to adjust the coefficient A and B so that (10.135) takes the form (10.123). This is possible only if $A+2B=0$, but this entails the vanishing of both (10.135) and (10.136), as one can verify by means of the identity (10.129). Thus it is impossible to cancel in this way the diffeomorphism anomaly. By suitably adjusting the coefficient we can instead cancel the trace anomaly (10.122) at the expenses of a diff anomaly.

[1]To prove (10.138) one has to use the 2d identity

$$\xi^\nu\epsilon_{\sigma\alpha}\partial_\nu\partial^\alpha\xi^\sigma = \xi^\nu\epsilon_{\alpha\nu}\partial^\alpha\partial_\sigma\Box\xi^\sigma \tag{10.137}$$

which holds due to the anticommutativity of ξ.

10.7 The Feynman diagrams method in $2d$

It is interesting and instructive to derive the previous results using Feynman diagrams. The action for a massless Dirac fermion in 2d is

$$S = \int d^2x\,\sqrt{g}\,i\overline{\psi}\gamma^\mu\left(\partial_\mu + \frac{1}{2}\omega_\mu\right)\psi \tag{10.139}$$

where $g = \det(g_{\mu\nu})$, $\gamma^\mu = e_a^\mu\gamma^a$, ($\mu, \nu, ...$ are world indices, $a, b, ...$ are flat indices) and ω_μ is the spin connection:

$$\omega_\mu = \omega_\mu^{ab}\Sigma_{ab}$$

where $\Sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$ are the Lorentz generators.

Let us recall a few basic definitions concerning 2d gamma matrices:

$$\left\{\gamma^a, \gamma^b\right\} = 2\eta^{ab} \Rightarrow \left(\gamma^0\right)^2 = 1, \quad \left(\gamma^1\right)^2 = -1. \tag{10.140}$$

Clearly, $\gamma^0 = \gamma_0$ and $\gamma^1 = -\gamma_1$. The chirality matrix γ_* is given by $\gamma_* = -\gamma_0\gamma_1$.

It is straightforward to check that the following relations are true:

$$\gamma_a = \epsilon_{ab}\gamma^b\gamma_*, \quad \epsilon_{ab}\gamma^b = \gamma_a\gamma_*, \tag{10.141}$$

where we are using the convention $\epsilon_{01} = 1$. It follows

$$\mathrm{tr}(\gamma_a\gamma_b\gamma_*) = -2\epsilon_{ab}. \tag{10.142}$$

10.7.1 Two-point e.m. correlator for chiral fermions

In two dimensions the spin connection drops out of the action (10.139). This is due to the proportionality between the unique Lorentz generator and the chiral matrix γ_* It is convenient to consider first the case of a left-handed spinor: $\psi_L = P_L\psi$, where $P_L = \frac{1+\gamma_*}{2}$. For ψ_L the action becomes

$$S_L = i\int d^2x\,\sqrt{g}\,\overline{\psi}_L\gamma^\mu\partial_\mu\psi_L \tag{10.143}$$

We can further simplify it by absorbing the $\sqrt{g}$ into a redefinition of ψ: $\psi \to \widetilde{\psi} = g^{\frac{1}{4}}\psi$. In the path integral spirit this is allowed because the Jacobian of this transformation factors out and can be disregarded. We can therefore replace S_L with

$$\widetilde{S}_L = i\int d^2x\,\overline{\widetilde{\psi}}_L\gamma^\mu\partial_\mu\widetilde{\psi}_L = i\int d^2x\,\overline{\widetilde{\psi}}_L\gamma^a e_a^\mu\partial_\mu\widetilde{\psi}_L \tag{10.144}$$

Now let us write $e^\mu_a \approx \delta^\mu_a - \chi^\mu_a$ and make the identification $2\chi^\mu_a = h^\mu_a$ [2], where $h_{\mu\nu}$ is the gravitational fluctuation field: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The fermion propagator is

$$\frac{i}{\not p + i\epsilon} \tag{10.145}$$

and there is only one graviton-fermion-fermion vertex (see figure 10.1) given by

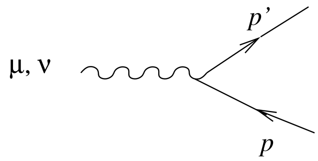

Figure 10.1: *The V_{gff} vertex.*

$$\frac{i}{8}\left[(p+p')_\mu \gamma_\nu + (p+p')_\nu \gamma_\mu\right]\frac{1+\gamma_*}{2}. \tag{10.146}$$

There is only one non-trivial contribution that comes from the bubble diagram with one incoming and one outgoing line with momentum k and an internal momentum p (see figure 10.2).

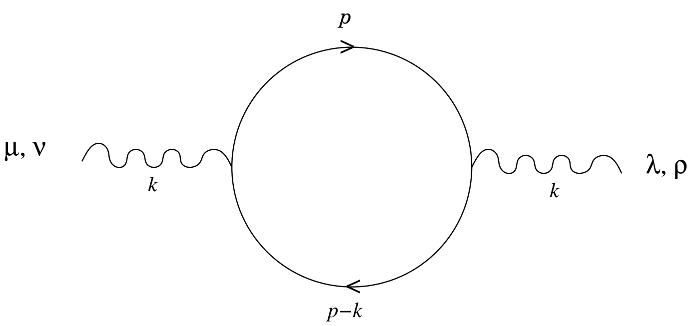

Figure 10.2: *The relevant Feynman diagram for the computation.*

The relevant 2-point function is[3]

$$\langle T_{\mu\nu}(x) T_{\lambda\rho}(y)\rangle = 4\int \frac{d^2k}{(2\pi)^2} e^{-ik(x-y)} \widetilde{\mathcal{T}}_{\mu\nu\lambda\rho}(k) \tag{10.147}$$

[2]This implies a choice of gauge because in this way χ^μ_a is symmetric.

[3]The factor of 4 in (10.147) is produced by the fact that the vertex (10.146) corresponds to the insertion of $-\frac{1}{2}T_{\mu\nu}$, not simply $-T_{\mu\nu}$, in the correlator .

with

$$\widetilde{\mathcal{T}}_{\mu\nu\lambda\rho}(k) = -\frac{1}{64}\int \frac{d^2p}{(2\pi)^2}\mathrm{tr}\left(\frac{1}{\not p}(2p-k)_\mu\gamma_\nu\frac{1+\gamma_*}{2}\frac{1}{\not p-\not k}(2p-k)_\lambda\gamma_\rho\frac{1+\gamma_*}{2}\right) + \left\{\begin{array}{c}\mu\leftrightarrow\nu\\ \lambda\leftrightarrow\rho\end{array}\right\}. \tag{10.148}$$

The last line means that we have to add three more terms like in the first line so as to realize a symmetry under the exchanges $\mu\leftrightarrow\nu$, $\lambda\leftrightarrow\rho$. Moreover we have to symmetrize (with weight 1) with respect to the exchange $(\mu,\nu)\leftrightarrow(\lambda,\rho)$ (bosonic symmetry). This will be understood in the sequel.

Eq.(10.148) would seem to be the same as

$$\widetilde{\mathcal{T}}'_{\mu\nu\lambda\rho}(k) = -\frac{1}{64}\int \frac{d^2p}{(2\pi)^2}\mathrm{tr}\left(\frac{1}{\not p}(2p-k)_\mu\gamma_\nu\frac{1}{\not p-\not k}(2p-k)_\lambda\gamma_\rho\frac{1+\gamma_*}{2}\right). \tag{10.149}$$

But they are both divergent expressions that need to be regularized. It is hardly a surprise that, once this is done, they lead to different results. The difference consists, however, of local terms.

Let us proceed to regularize these expressions, starting from (10.148), with the methd of dimensional regularization. To this end we introduce extra space components of the momentum running around the loop, $p\to p+\ell$ ($\ell=\ell_2,\dots,\ell_{\delta+2}$) . So (10.148) becomes

$$\widetilde{\mathcal{T}}^{(reg)}_{\mu\nu\lambda\rho}(k) = -\frac{1}{64}\int \frac{d^2p\,d^\delta\ell}{(2\pi)^{2+\delta}}\mathrm{tr}\Big(\frac{1}{\not p+\not\ell}(2p-k)_\mu\gamma_\nu\frac{1+\gamma_*}{2}\frac{1}{\not p+\not\ell-\not k} \times (2p-k)_\lambda\gamma_\rho\frac{1+\gamma_*}{2}\Big). \tag{10.150}$$

Now let us recall that $\not p^2 = p^2, \not\ell^2 = -\ell^2$, $\not p\not\ell+\not\ell\not p=0$ and $[\gamma_*,\not\ell]=0$, while $\{\gamma_*,\not p\}=0$. Moreover eq.(10.142) is replaced by

$$\mathrm{tr}(\gamma_\mu\gamma_\nu\gamma_*) = -2^{1+\frac{\delta}{2}}\epsilon_{\mu\nu}. \tag{10.151}$$

Thus (10.150) can be rewritten (for simplicity we drop from now on the $^{(reg)}$

tag)

$$\begin{aligned}
\widetilde{\mathcal{T}}_{\mu\nu\lambda\rho}(k) &= -\frac{1}{64}\int \frac{d^2p\, d^\delta\ell}{(2\pi)^{2+\delta}} \mathrm{tr}\Big(\frac{\not{p}+\not{\ell}}{p^2-\ell^2}(2p-k)_\mu\gamma_\nu \frac{1+\gamma_*}{2}\frac{\not{p}+\not{\ell}-\not{k}}{(p-k)^2-\ell^2} \\
&\times \quad (2p-k)_\lambda\gamma_\rho \frac{1+\gamma_*}{2}\Big). \qquad (10.152) \\
&= -\frac{1}{64}\int \frac{d^2p\, d^\delta\ell}{(2\pi)^{2+\delta}} \mathrm{tr}\Big(\frac{\not{p}}{p^2-\ell^2}(2p-k)_\mu\gamma_\nu \frac{1+\gamma_*}{2}\frac{\not{p}-\not{k}}{(p-k)^2-\ell^2} \\
&\times \quad (2p-k)_\lambda\gamma_\rho \frac{1+\gamma_*}{2}\Big). \qquad (10.153) \\
&= -\frac{1}{64}\int \frac{d^2p\, d^\delta\ell}{(2\pi)^{2+\delta}} \mathrm{tr}\Big(\frac{\not{p}}{p^2-\ell^2}(2p-k)_\mu\gamma_\nu \frac{\not{p}-\not{k}}{(p-k)^2-\ell^2} \\
&\times \quad (2p-k)_\lambda\gamma_\rho \frac{1+\gamma_*}{2}\Big). \qquad (10.154)
\end{aligned}$$

Next one evaluates the gamma matrix traces and introduces a Feynman parameter u, $0 \le u \le 1$ in order to evaluate the p integral. $\mathcal{T}_{\mu\nu\lambda\rho}(k)$ becomes

$$\begin{aligned}
&\widetilde{\mathcal{T}}_{\mu\nu\lambda\rho}(k) = -\frac{1}{64}\int \frac{d^2p\, d^\delta\ell}{(2\pi)^{2+\delta}} \int_0^1 du\, \Big((p+uk)_\nu(p+(u-1)k)_\rho \\
&+(p+uk)_\rho(p+(u-1)k)_\nu - (p+uk)\cdot(p+(u-1)k)\eta_{\nu\rho} \\
&+\epsilon_{\rho\sigma}\big((p+uk)_\nu(p+(u-1)k)^\sigma + (p+uk)^\sigma(p+(u-1)k)_\nu\big)\Big) \\
&\times \quad \frac{(2p+(2u-1)k)_\mu(2p+(2u-1)k)_\lambda}{[p^2+u(1-u)k^2-\ell^2]^2}. \qquad (10.155)
\end{aligned}$$

The integral can be evaluated after a Wick rotation of the momenta $p_0 \to ip_0^E, k_0 \to ik_0^E$, using the results in Appendix G. The result is recorded in Appendix H.

10.7.2 Trace and diff cocycles

From the previous result we can compute the trace and the divergence (k_μ denotes the Euclidean momentum, in particular $\mathsf{k}^2 = -k^2$):

$$\widetilde{\mathcal{T}}^{E\mu}_{\mu\lambda\rho}(\mathsf{k}) = \frac{i}{192\pi}\Big[\mathsf{k}_\lambda\mathsf{k}_\rho + \mathsf{k}^2\eta_{\lambda\rho} + \frac{1}{2}\big(\mathsf{k}_\lambda\epsilon_{\rho\sigma}\mathsf{k}^\sigma + \mathsf{k}_\rho\epsilon_{\lambda\sigma}\mathsf{k}^\sigma\big)\Big] \qquad (10.156)$$

and

$$\begin{aligned}
\mathsf{k}^\mu\widetilde{\mathcal{T}}^E_{\mu\nu\lambda\rho}(\mathsf{k}) &= -\frac{i}{384\pi}\Big[\mathsf{k}_\nu\mathsf{k}_\lambda\mathsf{k}_\rho + \frac{1}{2}\mathsf{k}^2\big(\eta_{\nu\lambda}\mathsf{k}_\rho + \eta_{\nu\rho}\mathsf{k}_\lambda\big) \qquad (10.157) \\
&+\frac{1}{2}\mathsf{k}_\nu\big(\mathsf{k}_\lambda\epsilon_{\rho\sigma} + \mathsf{k}_\rho\epsilon_{\lambda\sigma}\big)\mathsf{k}^\sigma + \frac{1}{2}\mathsf{k}^2\big(\eta_{\nu\lambda}\epsilon_{\rho\sigma} + \eta_{\nu\rho}\epsilon_{\lambda\sigma}\big)\mathsf{k}^\sigma\Big]
\end{aligned}$$

Thus trace and divergence are nonvanishing both for the even and odd part. Using (10.9) and (10.19) we can derive the trace anomaly in the space of coordinates

$$\begin{aligned}\Delta_\omega &= \frac{1}{2}\int d^2x\,\omega(x)\int d^2y\,h^{\lambda\rho}(y)\langle 0|\mathcal{T}T^\mu_\mu(x)T_{\lambda\rho}(y)|0\rangle_c \\ &= 2\int d^2x\,\omega(x)\int d^2y\,h^{\lambda\rho}(y)\int\frac{d^2k}{(2\pi)^2}\,e^{-ik\cdot(x-y)}\widetilde{\mathfrak{T}}^\mu_{\mu\lambda\rho}(k) \\ &= \frac{1}{96\pi}\int d^2x\,\omega(x)\left[\partial_\lambda\partial_\rho h^{\lambda\rho}(x)-\Box h^\lambda_\lambda(x)-\epsilon_{\lambda\sigma}\partial^\sigma\partial_\rho h^{\lambda\rho}\right]\end{aligned}\tag{10.158}$$

where, in the last step, an inverse Wick rotation has been performed in order to insert (10.156).

In the same way the diffeomorphism anomaly can be obtained

$$\begin{aligned}\Delta_\xi &= -\frac{1}{2}\int d^2x\,\xi^\nu(x)\int d^2y\,h^{\lambda\rho}(y),\partial^\mu_x\langle 0|\mathcal{T}T_{\mu\nu}(x)T_{\lambda\rho}(y)|0\rangle_c \\ &= -\frac{1}{2}\int d^2x\,\xi^\nu(x)\int d^2y\,h^{\lambda\rho}(y)(-ik^\mu)e^{-ik\cdot(x-y)}\widetilde{\mathfrak{T}}^E_{\mu\nu\lambda\rho}(k) \\ &= \frac{1}{192\pi}\int d^2x\,\xi^\nu(x)\Big[\partial_\nu\partial_\lambda\partial_\rho h^{\lambda\rho}(x)-\partial_\lambda\Box h^\lambda_\nu(x) \\ &\qquad\qquad +\epsilon_{\rho\sigma}\partial^\sigma\left(\partial_\nu\partial_\lambda-\eta_{\nu\lambda}\Box\right)h^{\lambda\rho}(x)\Big]\end{aligned}\tag{10.159}$$

It is easy to prove that

$$\delta^{(0)}_\omega\Delta_\omega=0,\qquad \delta^{(0)}_\xi\Delta_\xi=0,\qquad \delta^{(0)}_\omega\Delta_\xi+\delta^{(0)}_\xi\Delta_\omega=0.\tag{10.160}$$

Now let us focus on the even parity part. Both trace and diffeomorphism anomalies are nonvanishing. However the diffeomorphism one is trivial. For let us consider the local counterterm

$$\mathcal{C}^{(even)}=\frac{1}{384\pi}\int d^2x\left(h^\nu_\rho(x)\partial_\lambda\partial_\nu h^{\lambda\rho}(x)-h_{\lambda\rho}(x)\Box h^{\lambda\rho}(x)\right)\tag{10.161}$$

It is easy to prove that

$$\mathcal{A}^{(even)}_\xi\equiv\Delta^{(even)}_\xi+\delta^{(0)}_\xi\mathcal{C}^{(even)}=0\tag{10.162}$$

On the other hand the overall even trace anomaly

$$\mathcal{A}^{(even)}_\omega\equiv\Delta^{(even)}_\omega+\delta^{(0)}_\omega\mathcal{C}^{(even)}=\frac{1}{48\pi}\int d^2x\,\omega\left[\partial_\lambda\partial_\rho h^{\lambda\rho}-\Box h^\lambda_\lambda\right]\tag{10.163}$$

Contrary to the even case, the cocycle $\Delta^{(odd)}_\xi$ in non-trivial. Consequently we cannot impose diffeomorphism covariance by subtracting some local counterterm.

With a suitable counterterm we can cancel instead $\Delta_\omega^{(odd)}$, but it is a mere academical exercise.

To conclude: as already noticed (10.163) is the first order approximation of

$$\mathcal{A}_\omega^{(even)} = \frac{1}{48\pi} \int d^2x \sqrt{g}\, \omega\, R \tag{10.164}$$

Comparing with (10.107) we can determine the central charge for a Weyl spinor

$$c = c_W = \frac{1}{8\pi^2} \tag{10.165}$$

In view of the fact that the theory of a single (real) Weyl spinor has a diff anomaly, the result (10.165) is only formal. However, a (real) free Dirac spinor can be seen as a couple of opposite chirality free Weyl spinors[4] . The chirality affects only the odd parity results, which have opposite sign for opposite chirality. As a consequence the odd parity anomalies cancel out, while the even parity ones, having the same sign, will add up. Therefore for a Dirac spinor the diff anomaly vanishes and the even trace anomaly is twice (10.163). So the central charge for a free (real) Dirac spinor is

$$c = c_D = \frac{1}{4\pi^2}, \tag{10.166}$$

Eqs. (10.165) and (10.166) represent unnormalized central charges. In order to normalize them one must consider the fermion propagator in coordinate space

$$\left\langle \psi(x)\, \overline{\psi}(y) \right\rangle = \frac{i}{2\pi} \frac{\gamma \cdot (x-y)}{(x-y)^2}, \tag{10.167}$$

This implies that in eq.(10.94) the parameter c contains the factor $\frac{1}{4\pi^2}$. Stripping off this factor the normalized central charge $\mathsf{c} = 4\pi^2\, c$ for a real Weyl fermion is

$$\mathsf{c}_W = \frac{1}{2} \tag{10.168}$$

and twice as much for a Dirac fermion.

As for the odd-parity part of the anomalies one can repeat the same analysis as in 10.6.2, since the cocycles are the same apart from an overall coefficient, and conclude that one can get rid of the trivial Weyl cocycle proportional to (10.122) by shifting the anomaly to the diffeomorphism sector. The final result is (proportional to) the non-trivial diffeomorphism anomaly (10.127) or (10.130).

[4] We recall that in 2d reality can be imposed on a Weyl spinor, giving rise to a Majorana-Weyl spinor.

10.7.3 Regularization ambiguities

We have mentioned above the possibility to regulate the expression (10.149) instead of (10.148). They look formally identical, but they are unregulated expressions. If we start from (10.149) we get

$$\begin{aligned}\widetilde{\mathcal{T}}^{(reg)'}_{\mu\nu\lambda\rho}(k) &= -\frac{1}{32}\int \frac{d^2p\, d^\delta\ell}{(2\pi)^{2+\delta}}\mathrm{tr}\Big(\frac{1}{\not p+\not\ell}(2p-k)_\mu\gamma_\nu\frac{1}{\not p+\not\ell-\not k}\\ &\times \quad (2p-k)_\lambda\gamma_\rho\frac{1+\gamma_*}{2}\Big). \qquad (10.169)\end{aligned}$$

We recall that here we understand for the moment the symmetrization with respect to the exchanges $\mu\leftrightarrow\nu$, $\lambda\leftrightarrow\rho$ and $(\mu,\nu)\leftrightarrow(\lambda,\rho)$ (bosonic symmetry).

The difference between (10.169) and (10.150)is

$$\begin{aligned}\Delta\widetilde{\mathcal{T}}^{(reg)}_{\mu\nu\lambda\rho}(k) &= \widetilde{\mathcal{T}}^{(reg)'}_{\mu\nu\lambda\rho}(k)-\widetilde{\mathcal{T}}^{(reg)}_{\mu\nu\lambda\rho}(k) \qquad (10.170)\\ &= -\frac{1}{64}(\eta_{\nu\rho}-\epsilon_{\nu\rho})\int\frac{d^2p\, d^\delta\ell}{(2\pi)^{2+\delta}}\,\ell^2\,\frac{(2p-k)_\mu(2p-k)_\lambda}{(p^2-\ell^2)((p-k)^2-\ell^2)}\end{aligned}$$

After a Wick rotation one can carry out the integrals and symmetrize. The result is (dropping the $^{(reg)}$ label)

$$\begin{aligned}\Delta\widetilde{\mathcal{T}}^E_{\mu\nu\lambda\rho}(k) = \frac{i}{768\pi}\Big[&2\mathsf{k}^2(\eta_{\mu\lambda}\eta_{\nu\rho}+\eta_{\nu\lambda}\eta_{\mu\rho})+\eta_{\nu\rho}\mathsf{k}_\mu\mathsf{k}_\lambda \qquad (10.171)\\ &+\eta_{\mu\rho}\mathsf{k}_\nu\mathsf{k}_\lambda+\eta_{\nu\lambda}\mathsf{k}_\mu\mathsf{k}_\rho+\eta_{\mu\rho}\mathsf{k}_\nu\mathsf{k}_\lambda\Big]\end{aligned}$$

Inserted in (10.9) this gives rise to local terms in the effective action. From this result we learn that different regularization procedures may lead to discrepancies in the effective action. These differences however are expressed by local terms. The contribution to the trace from (10.171) is

$$\Delta\widetilde{\mathcal{T}}^{E\mu}_{\mu\lambda\rho}(\mathsf{k}) = \frac{i}{192\pi}\Big[\mathsf{k}_\lambda\mathsf{k}_\rho+\mathsf{k}^2\eta_{\lambda\rho}\Big] \qquad (10.172)$$

and the contribution to the divergence is

$$\mathsf{k}^\mu\Delta\widetilde{\mathcal{T}}^E_{\mu\nu\lambda\rho}(\mathsf{k}) \quad = \quad \frac{i}{768\pi}\Big[\mathsf{k}^2\big(\eta_{\nu\lambda}\mathsf{k}_\rho+\eta_{\nu\rho}\mathsf{k}_\lambda\big)+2\mathsf{k}_\nu\mathsf{k}_\lambda\mathsf{k}_\rho\Big] \qquad (10.173)$$

These additions modify (only) the even parts of both (10.156) and (10.157), in fact they double the even part of (10.156) and exactly cancel the even part of (10.157). This implies diff covariance for even part of the trace anomaly. As for the odd-parity part, it does not change (see end of previous subsection).

10.7.4 Other ambiguities

Instead of calculating the two-point correlator of the energy-momentum tensor and subsequently taking its trace, we could have proceeded directly to the computation of a two-point correlator containing one insertion of the trace of the e.m. tensor. The integral to be regulated is (the symmetrization $\lambda \leftrightarrow \rho$ is understood)

$$\begin{aligned}
&\widetilde{\mathcal{T}}^{\mu}{}_{\mu\lambda\rho}(k) \qquad (10.174)\\
&= -\frac{1}{32}\int \frac{d^2p}{(2\pi)^2}\,\mathrm{tr}\left(\frac{\not p}{p^2}\,(2\not p-\not q)\,\frac{1+\gamma_*}{2}\,\frac{\not p-\not k}{(p-k)^2}(2p-k)_\lambda\gamma_\rho\frac{1+\gamma_*}{2}\right)\\
= \;& -\frac{1}{32}\int \frac{d^2p}{(2\pi)^2}\,\mathrm{tr}\left(\frac{\not p}{p^2}\,(2\not p-\not q)\,\frac{\not p-\not k}{(p-k)^2}(2p-k)_\lambda\gamma_\rho\frac{1+\gamma_*}{2}\right)
\end{aligned}$$

If we regularize it as follows

$$\begin{aligned}
&\widetilde{\mathcal{T}}'^{\mu}{}_{\mu\lambda\rho}(k) = -\frac{1}{32}\int \frac{d^2p\, d^\delta\ell}{(2\pi)^{2+\delta}} \qquad (10.175)\\
&\mathrm{tr}\left(\frac{\not p+\not\ell}{p^2-\ell^2}\,(2\not p-\not k)\,\frac{1+\gamma_*}{2}\,\frac{\not p+\not\ell-\not k}{(p-k)^2-\ell^2}(2p-k)_\lambda\gamma_\rho\frac{1+\gamma_*}{2}\right)\\
= \;& -\frac{1}{64}\int \frac{d^2p\, d^\delta\ell}{(2\pi)^{2+\delta}}\,\mathrm{tr}\left(\frac{\not p}{p^2-\ell^2}\,(2\not p-\not k)\,\frac{\not p-\not k}{(p-k)^2-\ell^2}(2p-k)_\lambda\gamma_\rho\frac{1+\gamma_*}{2}\right)
\end{aligned}$$

we obtain

$$\widetilde{\mathcal{T}}'^{E\mu}{}_{\mu\lambda\rho}(k) = \frac{i}{192\pi}\Big[\mathsf{k}_\lambda\mathsf{k}_\rho + \mathsf{k}^2\eta_{\lambda\rho} + \frac{1}{2}\big(\mathsf{k}_\lambda\epsilon_{\rho\sigma}\mathsf{k}^\sigma + \mathsf{k}_\rho\epsilon_{\lambda\sigma}\mathsf{k}^\sigma\big)\Big] \qquad (10.176)$$

which is the same as (10.156).

It would seem more natural to regularize the second line of (10.174)

$$\begin{aligned}
&\widetilde{\mathcal{T}}''^{\mu}{}_{\mu\lambda\rho}(k) = -\frac{1}{32}\int \frac{d^2p\, d^\delta\ell}{(2\pi)^{2+\delta}} \qquad (10.177)\\
&\times\,\mathrm{tr}\left(\frac{\not p+\not\ell}{p^2-\ell^2}\,(2\not p+2\not\ell-\not k)\,\frac{\not p+\not\ell-\not k}{(p-k)^2-\ell^2}(2p-k)_\lambda\gamma_\rho\frac{1+\gamma_*}{2}\right)
\end{aligned}$$

A direct calculation yields

$$\widetilde{\mathcal{T}}''^{E\mu}{}_{\mu\lambda\rho}(k) = 0. \qquad (10.178)$$

The result does not change if we use the regularization

$$\widetilde{\mathcal{T}}'''^{\mu}{}_{\mu\lambda\rho}(k) = -\frac{1}{32}\int \frac{d^2p\, d^\delta\ell}{(2\pi)^{2+\delta}} \tag{10.179}$$

$$\times \mathrm{tr}\left(\frac{\not p + \not\ell}{p^2-\ell^2}\,\left(2\not p + 2\not\ell - \not k\right)\frac{1+\gamma_*}{2}\frac{\not p + \not\ell - \not k}{(p-k)^2-\ell^2}(2p-k)_\lambda\gamma_\rho\frac{1+\gamma_*}{2}\right)$$

$$= \; -\frac{1}{64}\int \frac{d^2p\, d^\delta\ell}{(2\pi)^{2+\delta}}$$

$$\times \mathrm{tr}\left(\frac{\not p + \not\ell}{p^2-\ell^2}\,\left(2\not p + 2\not\ell - \not k\right)\frac{\not p - \not k}{(p-k)^2-\ell^2}(2p-k)_\lambda\gamma_\rho\frac{1+\gamma_*}{2}\right) = 0$$

Let us now compute directly the divergence of the two-point e.m. tensor. We have to regularize

$$\begin{aligned} k^\mu \widetilde{\mathcal{T}}_{\mu\nu\lambda\rho}(k) \;&=\; -\frac{1}{64}\int \frac{d^2p}{(2\pi)^2}\mathrm{tr}\Big(\frac{\not p}{p^2}(2p{\cdot}k - k^2)\gamma_\nu\frac{1+\gamma_*}{2}\frac{\not p - \not k}{(p-k)^2} \\ &\times\; (2p-k)_\lambda\gamma_\rho\frac{1+\gamma_*}{2} \\ &+\frac{\not p}{p^2}(2p-k)_\nu\not k\,\frac{1+\gamma_*}{2}\frac{\not p - \not k}{(p-k)^2}(2p-k)_\lambda\gamma_\rho\frac{1+\gamma_*}{2}\Big). \end{aligned} \tag{10.180}$$

We can regulate it as follows

$$k^\mu \widetilde{\mathcal{T}}'_{\mu\nu\lambda\rho}(k) = -\frac{1}{64}\int \frac{d^2p\, d^\delta\ell}{(2\pi)^{2+\delta}} \tag{10.181}$$

$$\mathrm{tr}\Big(\frac{\not p + \not\ell}{p^2-\ell^2}(2p{\cdot}k - k^2)\gamma_\nu\frac{1+\gamma_*}{2}\frac{\not p + \not\ell - \not k}{(p-k)^2-\ell^2}(2p-k)_\lambda\gamma_\rho\frac{1+\gamma_*}{2}$$

$$+\frac{\not p + \not\ell}{p^2-\ell^2}(2p-k)_\nu\not k\,\frac{1+\gamma_*}{2}\frac{\not p + \not\ell - \not k}{(p-k)^2-\ell^2}\quad(2p-k)_\lambda\gamma_\rho\frac{1+\gamma_*}{2}\Big)$$

$$= -\frac{1}{64}\int \frac{d^2p\, d^\delta\ell}{(2\pi)^{2+\delta}}\mathrm{tr}\Big[\Big(\frac{\not p}{p^2-\ell^2}(2p{\cdot}k - k^2)\gamma_\nu\frac{\not p - \not k}{(p-k)^2-\ell^2}(2p-k)_\lambda\gamma_\rho$$

$$+\frac{\not p}{p^2-\ell^2}(2p-k)_\nu\not k\,\frac{\not p - \not k}{(p-k)^2-\ell^2}\quad(2p-k)_\lambda\gamma_\rho\Big)\frac{1+\gamma_*}{2}\Big]$$

which gives the same result as (10.157). A regularization closer to (10.177) is

$$
\begin{aligned}
k^\mu \widetilde{\mathcal{T}}''_{\mu\nu\lambda\rho}(k) = -\frac{1}{64}\int \frac{d^2p\, d^\delta \ell}{(2\pi)^{2+\delta}} \qquad\qquad (10.182)\\
\mathrm{tr}\Big(\frac{\not p + \not \ell}{p^2-\ell^2}(2p\cdot k - k^2)\gamma_\nu \frac{\not p + \not \ell - \not k}{(p-k)^2-\ell^2}(2p-k)_\lambda \gamma_\rho \frac{1+\gamma_*}{2}\\
+\frac{\not p + \not \ell}{p^2-\ell^2}(2p-k)_\nu \not k\, \frac{\not p + \not \ell - \not k}{(p-k)^2-\ell^2} \quad (2p-k)_\lambda \gamma_\rho \frac{1+\gamma_*}{2}\Big)
\end{aligned}
$$

This yields

$$
\mathsf{k}^\mu \widetilde{\mathcal{T}}''^E_{\mu\nu\lambda\rho}(\mathsf{k}) \quad = \quad 0 \qquad (10.183)
$$

The latest result looks like a companion result to (10.178).

10.7.5 A discussion about ambiguities

Of course (10.178) and (10.182) cannot be the right results. Different regularization procedures should lead to the same results up to local conterterms. It is not the case here, because both diff and trace anomalies vanish, while we know from the previous differential regularization (as well as from other approaches) that both even-parity trace and odd-parity diff cocycles are non-trivial and in no way can they be eliminated with local counterterms. On the other hand the above calculations that lead to (10.178) and (10.182) make perfect sense. Therefore the apparent contradiction must be a question of interpretation.

Let us notice first that there are two origins to the ambiguities. The first consists in the number of the γ_5 matrices in the integrals to be calculated. As we have remarked, we may get different results if, before introducing the regulator, we simplify the γ matrix algebra by keeping only one chiral projector or we leave all the γ_5 matrices in the original position dictated by the Feynman diagram. The difference however always consists of trivial terms and can be eliminated by local counterterms. The minimal configuration for anomalies (i.e. a configuration in which either diffeomorphism or Weyl invariance is preserved) is when only one chiral projector survives. Therefore to simplify our calculations we can resort to the following heuristic rule.

First heuristic rule. Before introducing any regulator move all the chiral projectors to the rightmost position and simplify.

A second, more important, source of ambiguity depends on whether we introduce the regulator before or after taking the trace or the divergence of the two-point correlator of the em tensor. This is evident comparing the results of subsection 10.7.4 with the ones obtained in subsections 10.7.2 and 10.7.3. The reason is not hard to guess. In subsections 10.7.2 and 10.7.3 we computed first the two-point correlator of the em tensor in which the bosonic symmetry (i.e. the symmetry under the exchange of the two em tensor entries, in particular the exchange $(\mu, \nu) \leftrightarrow (\lambda, \rho)$ in eq.(10.148)) must hold. Only subsequently we contracted two indices or the correlator with k^μ. If, instead, such contraction operations are performed before regularizing there is no guarantee that the bosonic symmetry is implemented in the same form. In fact this may be the origin of a discontinuity in the effective action which can be phrased as follows: differentiating the effective action twice with respect to the metric tensor and then taking the trace, may not be the same as differentiating it with respect to the metric and its trace. An analogous claim holds for the divergence of the em tensor.

We are therefore forced to take into account this possible discontinuity of the effective action. In fact everything becomes clear if we refine the definitions of trace and anomalies, (10.19) and (10.18), as follows.

Second heuristic rule. Instead of (10.19) we set

$$g^{\mu\nu}(x)\langle\langle T_{\mu\nu}(x)\rangle\rangle - \langle\langle g^{\mu\nu}(x)T_{\mu\nu}(x)\rangle\rangle = -T[g](x). \tag{10.184}$$

The first term in the LHS means that we compute first the n-point functions of the em tensor and then saturate the indices μ, ν with the metric, while the second term means that we compute the same n-point functions in which one of the entries is replaced by the trace of the em tensor. The first corresponds to the regularization (10.169) and leads to the result (10.156) corrected by (10.172), the second corresponds to the regularized expression (10.177) or (10.179) and leads to (10.178).

Similarly, instead of (10.18) we define

$$\nabla^\mu\langle\langle T_{\mu\nu}(x)\rangle\rangle - \langle\langle \nabla^\mu T_{\mu\nu}(x)\rangle\rangle = \mathcal{A}_\nu[g](x), \tag{10.185}$$

The first term in the LHS means that we compute first the n-point functions of the em tensor and take the covariant divergence of the latter at the point x, while the second term means that we compute the same n-point functions in which one of the entries is replaced by the covariant divergence of the em tensor. After applying the first heuristic rule, the first corresponds to the regularization (10.150) and leads to the result (10.157) corrected by (10.173) , the second corresponds to the regularized expression (10.182) and leads to the result (10.183).

Of course the conclusions of subsection 10.7.2 do not change. It should be remarked that the two terms in the LHS of both (10.184) and (10.185) tend to the same expression when we remove the regulator. Thus the difference in the LHS of both remove the infinities of the original expressions. These refined definitions of the trace and diff anomalies allows us to get rid also of the simi-local terms. Moreover, as we have noticed, the final results do not depend on the regularization details.

The vanishing of the second terms in the LHS of both (10.184) and (10.185) is not a general feature. There are cases in which an n-point em tensor correlator vanishes, while the same correlator in which one of the entries is replaced by the trace or the divergence of the em tensor does not. In such cases the corresponding of (10.179) and (10.181) would vanish identically, while (10.178) and (10.182) would not.

This is what happens for the odd-parity trace anomaly of a Weyl fermion in 4d. As we shall see, the odd-parity three-point em correlator vanishes, but if one entry is replaced by its trace, it does not. In this case the trace anomaly will come from a formula analogous to (10.177).

Appendix F. Perturbative cohomology

In this Appendix we define the form of local cohomology which is needed in a perturbative approach. Let us start from the gauge transformations.

$$\delta A = d\lambda + [A, \lambda], \qquad \delta\lambda = -\frac{1}{2}[\lambda, \lambda]_+, \qquad \delta^2 = 0, \qquad \lambda = \lambda^a(x) T^a \tag{10.186}$$

To dovetail the perturbative expansion it is useful to split it (say, A and λ infinitesimal), and define the perturbative cohomology

$$\begin{aligned} &\delta^{(0)} A = d\lambda, \qquad \delta^{(0)}\lambda = 0, \qquad (\delta^{(0)})^2 = 0 \\ &\delta^{(1)} A = [A, \lambda], \qquad \delta^{(1)}\lambda = -\frac{1}{2}[\lambda, \lambda]_+ \\ &\delta^{(0)}\delta^{(1)} + \delta^{(1)}\delta^{(0)} = 0, \qquad (\delta^{(1)})^2 = 0 \end{aligned} \tag{10.187}$$

The full coboundary operator for diffeomorphisms is given by the transformations

$$\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\nu, \qquad \delta_\xi \xi^\mu = \xi^\lambda \partial_\lambda \xi^\mu \tag{10.188}$$

with $\xi_\mu = g_{\mu\nu}\xi^\nu$. We can introduce a perturbative cohomology, or graded cohomology, using as grading the order of infinitesimal, as follows

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^\mu_\lambda h^{\lambda\nu} + \dots \tag{10.189}$$

The analogous expansions for the vielbein is

$$e^a_\mu = \delta^a_\mu + \chi^a_\mu + \frac{1}{2}\psi^a_\mu + \dots,$$

Since $e^a_\mu \eta_{ab} e^b_\nu = h_{\mu\nu}$, we can choose

$$\chi_{\mu\nu} = \frac{1}{2} h_{\mu\nu}, \qquad \psi_{\mu\nu} = -\chi^a_\mu \chi_{a\nu} = -\frac{1}{4} h^\lambda_\mu h_{\lambda\nu}, \qquad \dots \tag{10.190}$$

Inserting the above expansions in (10.188) we see that we have a grading in the transformations, given by the order of infinitesimals. So we can define a sequence of transformations

$$\delta_\xi = \delta^{(0)}_\xi + \delta^{(1)}_\xi + \delta^{(2)}_\xi + \dots$$

At the lowest level we find immediately

$$\delta^{(0)}_\xi h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \qquad \delta^{(0)}_\xi \xi_\mu = 0 \tag{10.191}$$

and $\xi_\mu = \xi^\mu$. Since $(\delta^{(0)}_\xi)^2 = 0$ this defines a cohomology problem.

At the next level we get

$$\delta^{(1)}_\xi h_{\mu\nu} = \xi^\lambda \partial_\lambda h_{\mu\nu} + \partial_\mu \xi^\lambda h_{\lambda\nu} + \partial_\nu \xi^\lambda h_{\mu\lambda}, \qquad \delta^{(1)}_\xi \xi^\mu = \xi^\lambda \partial_\lambda \xi^\mu \tag{10.192}$$

One can verify that

$$(\delta^{(0)}_\xi)^2 = 0 \qquad \delta^{(0)}_\xi \delta^{(1)}_\xi + \delta^{(1)}_\xi \delta^{(0)}_\xi = 0, \qquad (\delta^{(1)})^2 = 0 \tag{10.193}$$

Proceeding in the same way we can define an analogous sequence of transformations for the Weyl transformations. From $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $\delta_\omega h_{\mu\nu} = 2\omega g_{\mu\nu}$ we find

$$\delta^{(0)}_\omega h_{\mu\nu} = 2\omega \eta_{\mu\nu}, \qquad \delta^{(1)}_\omega h_{\mu\nu} = 2\omega h_{\mu\nu}, \qquad \delta^{(2)}_\omega h_{\mu\nu} = 0, \dots \tag{10.194}$$

as well as $\delta^{(0)}_\omega \omega = \delta^{(1)}_\omega \omega = 0, \dots.$

Notice that we have $\delta^{(0)}_\xi \omega = 0, \delta^{(1)}_\xi \omega = \xi^\lambda \partial_\lambda \omega$. As a consequence we can extend (10.193) to

$$(\delta^{(0)}_\xi + \delta^{(0)}_\omega)(\delta^{(1)}_\xi + \delta^{(1)}_\omega) + (\delta^{(1)}_\xi + \delta^{(1)}_\omega)(\delta^{(0)}_\xi + \delta^{(0)}_\omega) = 0 \tag{10.195}$$

and $\delta^{(1)}_\xi \delta^{(1)}_\omega + \delta^{(1)}_\omega \delta^{(1)}_\xi = 0$, which together with the previous relations make

$$(\delta^{(0)}_\xi + \delta^{(0)}_\omega + \delta^{(1)}_\xi + \delta^{(1)}_\omega)^2 = 0 \tag{10.196}$$

Appendix G. Regularization formulas in $2d$ and $4d$

In this appendix we collect the regularized integrals that are needed to evaluate the Feynman diagrams in the text both in $2d$ and $4d$. The integrals below are *Euclidean integrals*. They are an intermediate results needed in order to compute the Feynman diagrams in the text. Since the starting points and the final results are Lorentzian, it is understood that one has to do the appropriate Wick rotations in order to be able to use them.

In $2d$, after introducing δ extra dimensions in the internal momentum and a Feynman parameter u ($0 \le u \le 1$), in the limit $\delta \to 0$, we have

$$\begin{aligned}
\int \frac{d^2p}{(2\pi)^2} \int \frac{d^\delta \ell}{(2\pi)^\delta} \frac{\ell^2}{(p^2+\ell^2+\Delta)^2} &= -\frac{1}{4\pi} \\
\int \frac{d^2p}{(2\pi)^2} \int \frac{d^\delta \ell}{(2\pi)^\delta} \frac{\ell^2 p^2}{(p^2+\ell^2+\Delta)^2} &= \frac{1}{4\pi}\Delta
\end{aligned} \tag{10.197}$$

and

$$\begin{aligned}
\int \frac{d^2p}{(2\pi)^2} \int \frac{d^\delta \ell}{(2\pi)^\delta} \frac{1}{(p^2+\ell^2+\Delta)^2} &= \frac{1}{4\pi}\frac{1}{\Delta} \qquad (10.198)\\
\int \frac{d^2p}{(2\pi)^2} \int \frac{d^\delta \ell}{(2\pi)^\delta} \frac{p^2}{(p^2+\ell^2+\Delta)^2} &= \frac{1}{4\pi}\left(-\frac{2}{\delta} - \gamma + \log(4\pi) - \log\Delta\right) \\
\int \frac{d^2p}{(2\pi)^2} \int \frac{d^\delta \ell}{(2\pi)^\delta} \frac{p^4}{(p^2+\ell^2+\Delta)^2} &= \frac{1}{2\pi}\Delta\left(\frac{2}{\delta} - 1 + \gamma - \log(4\pi) + \log\Delta\right)
\end{aligned}$$

where $\Delta = u(1-u)k^2$.

Proceeding in the same way in $4d$, with two Feynman parameters u and v, in the limit $\delta \to 0$, we find

$$\begin{aligned}
\int \frac{d^4p}{(2\pi)^4} \int \frac{d^\delta \ell}{(2\pi)^\delta} \frac{p^2}{(p^2+\ell^2+\Delta)^3} &= \frac{1}{(4\pi)^2}\left(-\frac{2}{\delta} - \gamma + \log(4\pi) - \log\Delta\right)\Delta \\
\int \frac{d^4p}{(2\pi)^4} \int \frac{d^\delta \ell}{(2\pi)^\delta} \frac{p^4}{(p^2+\ell^2+\Delta)^3} &= \frac{\Delta}{2(4\pi)^2}\left(-\frac{2}{\delta} - \gamma + 4 + \log(4\pi) - \log\Delta\right)
\end{aligned} \tag{10.199}$$

and

$$\begin{aligned}
\int \frac{d^4p}{(2\pi)^4} \int \frac{d^\delta \ell}{(2\pi)^\delta} \frac{\ell^2}{(p^2+\ell^2+\Delta)^3} &= -\frac{1}{2(4\pi)^2} \\
\int \frac{d^4p}{(2\pi)^4} \int \frac{d^\delta \ell}{(2\pi)^\delta} \frac{\ell^2 p^2}{(p^2+\ell^2+\Delta)^3} &= \frac{1}{(4\pi)^2}\Delta
\end{aligned} \tag{10.200}$$

where $\Delta = u(1-u)k_1^2 + v(1-v)k_2^2 + 2uv\, k_1 k_2$, with $0 \le u \le 1$ and $0 \le v \le 1-u$.

Appendix H. The $\widetilde{\mathcal{T}}_{\mu\nu\lambda\rho}(\mathsf{k})$ correlator

Here we record the full expression $\widetilde{\mathcal{T}}_{\mu\nu\lambda\rho}(\mathsf{k})$ in terms of the Euclidean momentum. Let us set $a(\delta, \mathsf{k}) = \frac{2}{\delta} + 1 + \gamma + \log\left(\frac{\mathsf{k}^2}{2\pi}\right)$ and separate even and odd parts,

$$\widetilde{\mathcal{T}}_{\mu\nu\lambda\rho}(\mathsf{k}) = \widetilde{\mathcal{T}}^{(even)}_{\mu\nu\lambda\rho}(\mathsf{k}) + \widetilde{\mathcal{T}}^{(odd)}_{\mu\nu\lambda\rho}(\mathsf{k}).$$

Then

$$
\begin{aligned}
768\pi\,\widetilde{\mathfrak{T}}^{(even)}_{\mu\nu\lambda\rho}(\mathsf{k}) \;=\; & \left(a(\delta,\mathsf{k})-\frac{8}{3}\right)\Big[-\big(\mathsf{k}^2\eta_{\mu\nu}\eta_{\lambda\rho}+\eta_{\mu\nu}\mathsf{k}_\lambda\mathsf{k}_\rho+\eta_{\lambda\rho}\mathsf{k}_\mu\mathsf{k}_\nu\big) \\
& +\frac{3}{4}\left(\eta_{\mu\lambda}\mathsf{k}_\nu\mathsf{k}_\rho+\eta_{\nu\lambda}\mathsf{k}_\mu\mathsf{k}_\rho+\eta_{\mu\rho}\mathsf{k}_\nu\mathsf{k}_\lambda+\eta_{\nu\rho}\mathsf{k}_\mu\mathsf{k}_\lambda\right)\Big] \\
& -\frac{1}{4}\left(a(\delta,\mathsf{k})-\frac{14}{3}\right)\left(\eta_{\mu\lambda}\mathsf{k}_\nu\mathsf{k}_\rho+\eta_{\nu\lambda}\mathsf{k}_\mu\mathsf{k}_\rho+\eta_{\mu\rho}\mathsf{k}_\nu\mathsf{k}_\lambda+\eta_{\nu\rho}\mathsf{k}_\mu\mathsf{k}_\lambda\right) \\
& +\frac{1}{2}\left(a(\delta,\mathsf{k})-\frac{5}{3}\right)\mathsf{k}^2\left(\eta_{\mu\lambda}\eta_{\nu\rho}+\eta_{\nu\lambda}\eta_{\mu\rho}\right) \\
& +\frac{2}{\mathsf{k}^2}\mathsf{k}_\mu\mathsf{k}_\nu\mathsf{k}_\lambda\mathsf{k}_\rho+\frac{1}{4}\left(\eta_{\mu\lambda}\mathsf{k}_\nu\mathsf{k}_\rho+\eta_{\nu\lambda}\mathsf{k}_\mu\mathsf{k}_\rho+\eta_{\mu\rho}\mathsf{k}_\nu\mathsf{k}_\lambda+\eta_{\nu\rho}\mathsf{k}_\mu\mathsf{k}_\lambda\right)
\end{aligned}
\tag{10.201}
$$

and

$$
\begin{aligned}
768\pi\,\widetilde{\mathfrak{T}}^{(odd)}_{\mu\nu\lambda\rho}(\mathsf{k}) = & -\frac{1}{4}\left(a(\delta,\mathsf{k})-\frac{8}{3}\right)\Big[\eta_{\mu\nu}\mathsf{k}_\lambda\epsilon_{\rho\sigma}\mathsf{k}^\sigma+\eta_{\mu\nu}\mathsf{k}_\rho\epsilon_{\lambda\sigma}\mathsf{k}^\sigma+\eta_{\lambda\rho}\mathsf{k}_\mu\epsilon_{\nu\sigma}\mathsf{k}^\sigma \\
& +\eta_{\lambda\rho}\mathsf{k}_\nu\epsilon_{\mu\sigma}\mathsf{k}^\sigma+\frac{1}{2}\big(\eta_{\mu\lambda}\mathsf{k}_\nu\epsilon_{\rho\sigma}\mathsf{k}^\sigma+\eta_{\mu\rho}\mathsf{k}_\nu\epsilon_{\lambda\sigma}\mathsf{k}^\sigma+\eta_{\nu\lambda}\mathsf{k}_\mu\epsilon_{\rho\sigma}\mathsf{k}^\sigma+\eta_{\nu\rho}\mathsf{k}_\mu\epsilon_{\lambda\sigma}\mathsf{k}^\sigma \\
& +\eta_{\mu\lambda}\mathsf{k}_\rho\epsilon_{\nu\sigma}\mathsf{k}^\sigma+\eta_{\lambda\nu}\mathsf{k}_\rho\epsilon_{\mu\sigma}\mathsf{k}^\sigma+\eta_{\rho\nu}\mathsf{k}_\lambda\epsilon_{\nu\sigma}\mathsf{k}^\sigma+\eta_{\nu\rho}\mathsf{k}_\lambda\epsilon_{\mu\sigma}\mathsf{k}^\sigma\big)\Big] \\
& +\frac{1}{4}\left(a(\delta,\mathsf{k})-\frac{5}{3}\right)\Big[\eta_{\mu\lambda}\mathsf{k}_\nu\epsilon_{\rho\sigma}\mathsf{k}^\sigma+\eta_{\mu\rho}\mathsf{k}_\nu\epsilon_{\lambda\sigma}\mathsf{k}^\sigma+\eta_{\nu\lambda}\mathsf{k}_\mu\epsilon_{\rho\sigma}\mathsf{k}^\sigma+\eta_{\nu\rho}\mathsf{k}_\mu\epsilon_{\lambda\sigma}\mathsf{k}^\sigma \\
& +\eta_{\mu\lambda}\mathsf{k}_\rho\epsilon_{\nu\sigma}\mathsf{k}^\sigma+\eta_{\lambda\nu}\mathsf{k}_\rho\epsilon_{\mu\sigma}\mathsf{k}^\sigma+\eta_{\rho\nu}\mathsf{k}_\lambda\epsilon_{\nu\sigma}\mathsf{k}^\sigma+\eta_{\nu\rho}\mathsf{k}_\lambda\epsilon_{\mu\sigma}\mathsf{k}^\sigma\Big] \\
& +\frac{1}{2\mathsf{k}^2}\Big[\mathsf{k}_\mu\mathsf{k}_\lambda\mathsf{k}_\rho\epsilon_{\nu\sigma}\mathsf{k}^\sigma+\mathsf{k}_\nu\mathsf{k}_\lambda\mathsf{k}_\rho\epsilon_{\mu\sigma}\mathsf{k}^\sigma+\mathsf{k}_\mu\mathsf{k}_\lambda\mathsf{k}_\nu\epsilon_{\rho\sigma}\mathsf{k}^\sigma+\mathsf{k}_\nu\mathsf{k}_\mu\mathsf{k}_\rho\epsilon_{\lambda\sigma}\mathsf{k}^\sigma\Big]
\end{aligned}
\tag{10.202}
$$

The trace of these two expressions are

$$
\eta^{\mu\nu}\widetilde{\mathfrak{T}}^{(even)}_{\mu\nu\lambda\rho}(\mathsf{k}) \;=\; \frac{i}{768\pi}\big(\mathsf{k}_\lambda\mathsf{k}_\rho+\mathsf{k}^2\eta_{\lambda\rho}\big) \tag{10.203}
$$

$$
\eta^{\mu\nu}\mathfrak{T}^{(odd)}_{\mu\nu\lambda\rho}(\mathsf{k}) \;=\; \frac{i}{1536\pi}\big(\mathsf{k}_\lambda\epsilon_{\rho\sigma}\mathsf{k}^\sigma+\mathsf{k}_\rho\epsilon_{\lambda\sigma}\mathsf{k}^\sigma\big) \tag{10.204}
$$

and the divergence

$$
\mathsf{k}^\mu\widetilde{\mathfrak{T}}^{(even)}_{\mu\nu\lambda\rho}(\mathsf{k}) \;=\; -\frac{i}{1536\pi}\Big[\mathsf{k}_\nu\mathsf{k}_\lambda\mathsf{k}_\rho+\frac{1}{2}\mathsf{k}^2\big(\eta_{\nu\lambda}\mathsf{k}_\rho+\eta_{\nu\rho}\mathsf{k}_\lambda\big)\Big] \tag{10.205}
$$

$$
\mathsf{k}^\mu\mathfrak{T}^{(odd)}_{\mu\nu\lambda\rho}(\mathsf{k}) \;=\; -\frac{i}{3072\pi}\Big[\mathsf{k}_\nu\big(\mathsf{k}_\lambda\epsilon_{\rho\sigma}+\mathsf{k}_\rho\epsilon_{\lambda\sigma}\big)+\mathsf{k}^2\big(\eta_{\nu\lambda}\epsilon_{\rho\sigma}+\eta_{\nu\rho}\epsilon_{\lambda\sigma}\big)\mathsf{k}^\sigma\Big]. \tag{10.206}
$$