Solve the following three exercises:

Exercise 1.

In scalar QED the tree+one-loop effective potential for the charged scalar ϕ in a class of R_ξ gauges reads

$$V_{eff}(\rho) = \frac{\lambda}{4!}\rho^4 + \frac{\rho^4}{64\pi^2}\Big(3e^4 + \frac{5}{18}\lambda^2 + \frac{1}{3}\xi\lambda e^2\Big)\log\frac{\rho^2}{\mu^2} + \dots, \tag{1}$$

where $\dots$ are irrelevant scheme-dependent terms and ρ is the radial scalar field. Compute the leading one-loop order β-function $\beta_\lambda = \mu\, d\lambda/d\mu$ associated to the coupling λ by knowing that the anomalous dimension of ρ reads

$$\gamma_\rho \equiv \frac{\mu}{\rho}\frac{d\rho}{d\mu} = \frac{e^2}{16\pi^2}(\xi+3)\,. \tag{2}$$

Hint: As useful check, recall that β_λ is gauge-invariant.

Exercise 2.

Prove the Goldstone Theorem using the currents.

Exercise 3.

Consider a neutral scalar field φ with the bare Lagrangian:

$$\mathscr{L} = \frac{1}{2}\sum_{\mu=0}^{d-1}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2M^2}(\partial_d^2\varphi)^2 - \frac{c}{2}(\partial_d\varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4\,, \tag{3}$$

where M is a mass scale, $\lambda > 0$ and c are dimensionless parameters. Discuss the spatio-temporal symmetries of this scalar theory, which for a suitable value of c describes the so-called uniaxial Lifshitz point. *On the basis of power-counting, show that this theory is renormalizable in $d = d^* = 3 + \frac{1}{2}$ spatial dimensions.*

Hint 1: For simplicity, you can set in the proof $m_0 = c = 0$.

Hint 2: Compute the superficial degree of divergence in terms of the number of loops of a diagram, given by $L = I - V + 1$, where I and V are the number of propagators and vertices in the diagram, respectively.