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Topological Strings and Quantum Mechanics

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Abstract: Lectures on TS and QM.

Contents

1 Introduction and motivation

In its standard formulation, string theory is defined in terms of a perturbative series, i.e. as a sum over genera. Each term in this sum corresponds to an amplitudes evaluated on a Riemann surface of genus g. For example, the "free energy" of string theory, as a function of the "moduli" $\boldsymbol{t}$ and the string coupling constant g_s has the structure

$$F(\boldsymbol{t}, g_s) \sim \sum_{g \geq 0} F_g(\boldsymbol{t}) g_s^{2g-2}. \tag{1.1}$$

In some cases, like topological string theory, the genus g free energies $F_g(\boldsymbol{t})$ can be computed in closed form. Defining an object through a power series is not necessarily a bad thing. The problem with the series in the r.h.s. is that it has a *zero radius of convergence*: many examples and calculations, as well as general arguments, indicate that, for fixed $\boldsymbol{t}$,

$$F_g(\boldsymbol{t}) \sim (2g)! \tag{1.2}$$

Therefore, string theory is fundamentally ill-defined. This is typically encoded in the motto "there is no non-perturbative definition of string theory". A non-perturbative definition of string theory should provide *well-defined functions*, whose *asymptotic expansion* at small g_s lead to the perturbative series above.

One of the basic consequences of the gauge/string duality is to solve this problem. In these dualities, the string coupling constant can be regarded as the Planck constant (or coupling constant) of a well-understood quantum system. In a typical gauge/string duality, one has roughly a correspondence of the type,

$$g_{\rm YM}^2 \sim g_s, \qquad g_{\rm YM}^2 N \sim t \tag{1.3}$$

(we are assuming for simplicity that the gauge theory has a single modulus). Under such a correspondence, the string theory perturbative expansion corresponds to the large N 't Hooft expansion of the gauge theory. Quantities in string theory are then identified with quantities in the gauge theory. For example, the free energy of the string theory corresponds to an appropriately well-defined free energy in the gauge theory, and one has

$$F(g_{\rm YM}^2, N) = F(t, g_s). \tag{1.4}$$

Working out this correspondence beyond tree level/planar limit has been challenging.

One possibility to push this further is to consider simpler versions of string theory, e.g. topological string theory. We expect that in this case one will have a simpler counterpart instead of a gauge theory. In recent years, evidence has accumulated that there exists a simpler version of the gauge/string duality for topological string theory on toric Calabi–Yau (CY) manifolds, where the dual theory is simply a quantum mechanical system (one can regard it as a non-interacting Fermi gas of N particles with a non-trivial density matrix determined by the CY geometry). This leads to a non-perturbative definition of topological string theory on these manifolds, as well as to a new family of exactly solvable quantum mechanical problems in one dimension.

2 Topological string theory

2.1 Perturbative definition

Let us consider holomorphic maps from a Riemann surface of genus g to the CY X,

$$f : \Sigma_g \to X. \tag{2.1}$$

Let $[S_i] \in H_2(X, \mathbb{Z})$, $i = 1, \cdots, s$, be a basis for the two-homology of X, with $s = b_2(X)$. The maps (2.1) are classified topologically by the homology class

$$f_*[(\Sigma_g)] = \sum_{i=1}^{s} d_i [S_i] \in H_2(X, \mathbb{Z}), \tag{2.2}$$

where d_i are integers called the *degrees* of the map. We will put them together in a degree vector $\mathbf{d} = (d_1, \cdots, d_s)$. The *Gromov–Witten invariant* at genus g and degree $\mathbf{d}$, which we will denote by $N_g^{\mathbf{d}}$, "counts" (in an appropriate way) the number of holomorphic maps of degree $\mathbf{d}$ from a Riemann surface of genus g to the CY X. Due to the nature of the moduli space of maps, these invariants are in general rational, rather than integer, numbers; see for example [1] for rigorous definitions and examples.

The Gromov–Witten invariants at fixed genus g but at all degrees can be put together in generating functionals $F_g(\mathbf{t})$, usually called *genus g free energies*. These are formal power series in e^{-t_i}, $i = 1, \cdots, s$, where t_i are the *Kähler parameters* of X. More precisely, the t_i are flat coordinates on the moduli space of Kähler structures of X. It is convenient to add to these generating functionals polynomial terms which appear naturally in the study of mirror symmetry and topological strings. In this way, we have, at genus zero,

$$F_0(\mathbf{t}) = \frac{1}{6}\sum_{i,j,k=1}^{s} a_{ijk} t_i t_j t_k + \sum_{\mathbf{d}} N_0^{\mathbf{d}} \mathrm{e}^{-\mathbf{d}\cdot\mathbf{t}}. \tag{2.3}$$

In the case of a compact CY threefold, the numbers a_{ijk} are interpreted as triple intersection numbers of two-classes in X. At genus one, one has

$$F_1(\mathbf{t}) = \sum_{i=1}^{s} b_i t_i + \sum_{\mathbf{d}} N_1^{\mathbf{d}} \mathrm{e}^{-\mathbf{d}\cdot\mathbf{t}}. \tag{2.4}$$

In the compact case, the coefficients b_i are related to the second Chern class of the CY manifold [2]. At higher genus one finds

$$F_g(\mathbf{t}) = C_g + \sum_{\mathbf{d}} N_g^{\mathbf{d}} \mathrm{e}^{-\mathbf{d}\cdot\mathbf{t}}, \qquad g \geq 2, \tag{2.5}$$

where C_g is a constant, called the constant map contribution to the free energy [3]. It turns out that these functionals have a physical interpretation as the free energies at genus g of the so-called type A topological string on X. Roughly speaking, this free energy can be computed by considering the path integral of a string theory on Riemann surfaces or "worldsheets" of genus g. This makes it possible to use a large amount of methods and ideas of physics in order to shed light on these quantities.

Although the above generating functionals are in principle formal generating functions, they have a common region of convergence near the large radius point $t_i \to \infty$. The total free energy of the topological string is formally defined as the sum,

$$F^{\mathrm{TS}}(\mathbf{t}, g_s) = \sum_{g\geq 0} g_s^{2g-2} F_g(\mathbf{t}) = F^{(\mathrm{p})}(\mathbf{t}, g_s) + F^{\mathrm{WS}}(\mathbf{t}, g_s), \tag{2.6}$$

where

$$\begin{aligned} F^{(\mathrm{p})}(\mathbf{t}, g_s) &= \frac{1}{6g_s^2}\sum_{i,j,k=1}^{s} a_{ijk} t_i t_j t_k + \sum_{i=1}^{s} b_i t_i + \sum_{g\geq 2} C_g g_s^{2g-2}, \\ F^{\mathrm{WS}}(\mathbf{t}, g_s) &= \sum_{g\geq 0}\sum_{\mathbf{d}} N_g^{\mathbf{d}} \mathrm{e}^{-\mathbf{d}\cdot\mathbf{t}} g_s^{2g-2}, \end{aligned} \tag{2.7}$$

The superscript "WS" refers to worldsheet instantons, which are counted by this generating functional. The variable g_s, called the *topological string coupling constant*, is in principle a formal variable keeping track of the genus. However, in string theory this constant has a physical meaning, and measures the strength of the string interaction. When g_s is very small, only Riemann surfaces of low genus contribute to a given quantum observable. On the contrary, if g_s is large, the contribution of higher genus Riemann surfaces becomes very important.

2.2 Gopakumar–Vafa invariants

As we mentioned in the introduction, there is strong evidence that, for fixed t_i in the common region of convergence, the numerical series $F_g(\mathbf{t})$ diverges factorially, as $(2g)!$. Therefore, the total free energy (2.6) does *not* define in principle a function of g_s and t_i. In a remarkable paper

[4], Gopakumar and Vafa pointed out that the series $F^{\rm WS}(\mathbf{t}, g_s)$ can be however *resummed* order by order in $\exp(-t_i)$, at all orders in g_s. This resummation involves a new set of enumerative invariants, the so-called *Gopakumar–Vafa invariants* $n_g^{\mathbf{d}}$. Out of these invariants, one constructs the generating series

$$F^{\rm GV}(\mathbf{t}, g_s) = \sum_{g\geq 0} \sum_{\mathbf{d}} \sum_{w=1}^{\infty} \frac{1}{w} n_g^{\mathbf{d}} \left(2 \sin \frac{w g_s}{2}\right)^{2g-2} {\rm e}^{-w \mathbf{d}\cdot \mathbf{t}}, \tag{2.8}$$

and one has, as an equality of formal series,

$$F^{\rm WS}(\mathbf{t}, g_s) = F^{\rm GV}(\mathbf{t}, g_s). \tag{2.9}$$

The Gopakumar–Vafa invariants turn out to be integers, in contrast to the original Gromov–Witten invariants. One can obtain one set of invariants from the other by comparing (2.6) to (2.9), but there exist direct mathematical constructions of the Gopakumar–Vafa invariants as well, see [5].

One could think that (2.9) provides a non-perturbative definition of the topological string free energy. In order to see if this is the case, one has to analyze the convergence properties of the formal series defined by (2.9). Let us focus on the one-modulus case for simplicity, and let us write this series as

$$F^{\rm GV}(t, g_s) = \sum_{\ell\geq 1} a_\ell(g_s) {\rm e}^{-\ell t}. \tag{2.10}$$

The first thing one notices is that the coefficients $a_\ell(g_s)$ have poles at values of g_s of the form

$$g_s = \frac{p}{q} 2\pi, \qquad p, q \in \mathbb{Z}. \tag{2.11}$$

This is a dense set of poles on the real line, so $F^{\rm GV}(t, g_s)$ fails to define a function when $g_s \in \mathbb{R}$. In addition, one finds [6] that, if $g_s \in \mathbb{C}\backslash\mathbb{R}$,

$$\log |a_\ell(g_s)| \sim \ell^2, \qquad \ell \gg 1. \tag{2.12}$$

Therefore, the generating functional (2.8) is in general ill-defined and, as it stands, it can not be used as a non-perturbative definition of the topological string free energy. We should point out that, in some geometries, (2.8) can be resummed by using instanton calculus, and one obtains a convergent function if $g_s \in \mathbb{C}\backslash\mathbb{R}$ [7]. However, the resulting function still has a dense set of poles on the real line.

2.3 Mirror symmetry

The mirror manifold $\widehat{X}$ to X has s complex deformation parameters z_i, $i = 1, \cdots, s$, which are related to the Kähler parameters of X by the so-called *mirror map* $t_i(\boldsymbol{z})$. At genus zero, the theory of deformation of complex structures of $\widehat{X}$ can be formulated as a theory of periods for the holomorphic $(3, 0)$ form of the CY $\widehat{X}$, which vary with the complex structure moduli z_i. As first noted in [8], this theory is remarkably simple. In particular, one can determine a function $F_0(\boldsymbol{z})$ depending on the complex moduli, called in this context the *prepotential*, which is the generating functional of genus zero Gromov–Witten invariants (2.3), once the mirror map is used. The theory of deformation of complex structures also has a physical realization in terms of the so-called type B topological string. However, for general CY manifolds, a precise definition of this theory at higher genus is still lacking. In practice, one often uses the description of the B-model provided in [3]. This provides a set of constraints for a series of functions $F_g(\boldsymbol{z})$, $g \geq 1$, known as holomorphic anomaly equations. After using the mirror map, these functions become the generating functionals (2.5).

One of the consequences of mirror symmetry and the B-model is the existence of many possible descriptions of the topological string, related by symplectic transformations. The existence of these descriptions can be regarded as a generalization of electric-magnetic duality. In this way, one finds different "frames" in which the topological string amplitudes can be expressed. Although these frames are in principle equivalent, some of them might be more convenient, depending on the region of moduli space we are looking at. The topological string amplitudes $F_g(\mathbf{t})$ written down above, in terms of Gromov–Witten invariants, correspond to the so-called large radius frame in the B-model, and they are appropriate for the so-called large radius limit $\mathrm{Re}(t_i) \gg 1$. The topological string free energies in different frames are related by a formal Fourier–Laplace transform, as explained in [9].

Although mirror symmetry was formulated originally for compact CY threefolds, one can extend it to the so-called "local" case [10, 11]. In local mirror symmetry, the CY X is taken to be a toric CY manifold, which is necessarily non-compact. It can be described as a symplectic quotient

$$X = \mathbb{C}^{k+3}//G, \tag{2.13}$$

where $G = U(1)^k$. Alternatively, X may be viewed physically as the moduli space of vacua for the complex scalars ϕ_i, $i = 0, \ldots, k+2$ of chiral superfields in a 2d gauged linear, $(2,2)$ supersymmetric σ-model [12]. These fields transform as

$$\phi_i \to \mathrm{e}^{\mathrm{i}Q_i^\alpha \theta_\alpha} \phi_i, \qquad Q_i^\alpha \in \mathbb{Z}, \quad \alpha = 1, \ldots, k \tag{2.14}$$

under the gauge group $U(1)^k$. Therefore, X is determined by the D-term constraints

$$\sum_{i=0}^{k+2} Q_i^\alpha |X_i|^2 = r^\alpha, \quad \alpha = 1, \ldots, k \tag{2.15}$$

modulo the action of $G = U(1)^k$. The r^α correspond to the Kähler parameters. The CY condition $c_1(TX) = 0$ holds if and only if the charges satisfy [12]

$$\sum_{i=0}^{k+2} Q_i^\alpha = 0, \qquad \alpha = 1, \ldots, k. \tag{2.16}$$

The mirrors to these toric CYs were constructed by [13], extending [10, 14]. They involve $3+k$ dual fields Y^i, $i = 0, \cdots, k+2$, living in $\mathbb{C}^*$. The D-term equation (2.15) leads to the constraint

$$\sum_{i=0}^{k+2} Q_i^\alpha Y^i = \log z_\alpha, \qquad \alpha = 1, \ldots, k. \tag{2.17}$$

Here, the z_α are moduli parametrizing the complex structures of the mirror $\widehat{X}$, which is given by

$$w^+ w^- = W_X, \tag{2.18}$$

where

$$W_X = \sum_{i=0}^{k+2} \mathrm{e}^{Y_i}. \tag{2.19}$$

The constraints (2.17) have a three-dimensional family of solutions. One of the parameters correspond to a translation of all the fields

$$Y^i \to Y^i + c, \qquad i = 0, \cdots, k+2, \tag{2.20}$$

which can be used for example to set one of the Y^is to zero. The remaining fields can be expressed in terms of two variables which we will denote by x, p. The resulting parametrization has a group of symmetries given by transformations of the form [15],

$$\begin{pmatrix} x \\ p \end{pmatrix} \to G \begin{pmatrix} x \\ p \end{pmatrix}, \qquad G \in \mathrm{SL}(2, \mathbb{Z}). \tag{2.21}$$

After solving for the variables Y^i in terms of the variables x, p, one finds a function

$$W_X(\mathrm{e}^x, \mathrm{e}^p). \tag{2.22}$$

Note that, due to the translation invariance (2.20) and the symmetry (2.21), the function $W_X(\mathrm{e}^x, \mathrm{e}^p)$ in (2.23) is only well-defined up to an overall factor of the form $\mathrm{e}^{\lambda x + \mu p}$, $\lambda, \mu \in \mathbb{Z}$, and a transformation of the form (2.21). It turns out [16, 17] that all the perturbative information about the B-model topological string on $\widehat{X}$ is encoded in the equation

$$W_X(\mathrm{e}^x, \mathrm{e}^p) = 0, \tag{2.23}$$

which can be regarded as the equation for a Riemann surface Σ_X embedded in $\mathbb{C}^* \times \mathbb{C}^*$.

An important class of examples is given by toric CY manifolds X in which Σ_X has genus one, i.e. it is an elliptic curve. The most general class of such manifolds are toric del Pezzo CYs, which are defined as the total space of the canonical bundle on a del Pezzo surface[1] S,

$$\mathcal{O}(K_S) \to S. \tag{2.24}$$

These manifolds can be classified by reflexive polyhedra in two dimensions (see for example [11, 18] for a review of this and other facts on these geometries). The polyhedron Δ_S associated to a surface S is the convex hull of a set of two-dimensional vectors

$$\nu^{(i)} = \left(\nu_1^{(i)}, \nu_2^{(i)}\right), \qquad i = 1, \cdots, k+2. \tag{2.25}$$

The extended vectors

$$\begin{aligned} \overline{\nu}^{(0)} &= (1, 0, 0), \\ \overline{\nu}^{(i)} &= \left(1, \nu_1^{(i)}, \nu_2^{(i)}\right), \qquad i = 1, \cdots, k+2, \end{aligned} \tag{2.26}$$

satisfy the relations

$$\sum_{i=0}^{k+2} Q_i^\alpha \overline{\nu}^{(i)} = 0, \tag{2.27}$$

where Q_i^α is the vector of charges characterizing the geometry in (2.15). Note that the two-dimensional vectors $\nu^{(i)}$ satisfy,

$$\sum_{i=1}^{k+2} Q_i^\alpha \nu^{(i)} = 0. \tag{2.28}$$

[1] Sometimes a distinction is made between del Pezzo surfaces and almost del Pezzo surfaces. Since our results apply to both of them, we will call them simply del Pezzo surfaces.

It turns out that the complex moduli of the mirror $\widehat{X}$ are of two types: one of them, which we will denote $\tilde{u}$ as in [18, 19], is a "true" complex modulus for the elliptic curve Σ, and it is associated to the compact four-cycle S in X. The remaining moduli, which will be denoted as m_i, should be regarded as parameters. For local del Pezzos, there is a canonical parametrization of the curve (2.23), as follows. Let

$$\begin{aligned} Y^0 &= \log \kappa, \\ Y^i &= \nu_1^{(i)} x + \nu_2^{(i)} p + f_i(m_j), \qquad i = 1, \cdots, k+2. \end{aligned} \tag{2.29}$$

Due to (2.28), the terms in x, p cancel, as required to satisfy (2.17). In addition, we find the parametrization

$$\log z_\alpha = \log \kappa^{Q_0^\alpha} + \sum_{i=1}^{k+2} Q_i^\alpha f_i(m_j), \tag{2.30}$$

which can be used to solve for the functions $f_i(m_j)$, up to reparametrizations. We then find the equation for the curve,

$$W_X = \mathcal{O}_X(x,p) + \kappa = 0, \tag{2.31}$$

where

$$\mathcal{O}_X(x,p) = \sum_{i=1}^{k+2} \exp\left(\nu_1^{(i)} x + \nu_2^{(i)} p + f_i(m_j)\right). \tag{2.32}$$

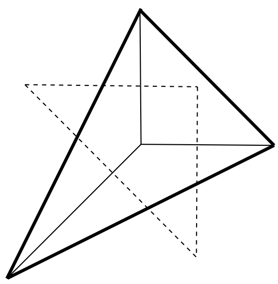

Figure 1. The vectors (2.33) defining the local $\mathbb{P}^2$ geometry, together with the polyhedron $\Delta_{\mathbb{P}^2}$ (in thick lines) and the dual polyhedron (in dashed lines).

Example 2.1. *Local* $\mathbb{P}^2$. In order to illustrate this procedure, let us consider the well-known example of local $\mathbb{P}^2$. In this case, we have $k = 1$ and the toric CY is defined by a single charge vector $Q = (-3, 1, 1, 1)$. The corresponding polyhedron Δ_S for $S = \mathbb{P}^2$ is obtained as the convex hull of the vectors

$$\nu^{(1)} = (1,0), \qquad \nu^{(2)} = (0,1), \qquad \nu^{(3)} = (-1,-1). \tag{2.33}$$

In the mirror, the variables Y^i satisfy

$$-3Y^0 + Y^1 + Y^2 + Y^3 = -3\log\kappa, \tag{2.34}$$

and the canonical parametrization is given by

$$Y^0 = \log\kappa, \qquad Y^1 = x, \qquad Y^2 = p, \qquad Y^3 = -x-p, \tag{2.35}$$

so that

$$W_{\mathbb{P}^2}(\mathrm{e}^x, \mathrm{e}^p) = \mathrm{e}^x + \mathrm{e}^p + \mathrm{e}^{-x-p} + \kappa \tag{2.36}$$

and

$$\mathcal{O}_{\mathbb{P}^2}(\mathrm{e}^x, \mathrm{e}^p) = \mathrm{e}^x + \mathrm{e}^p + \mathrm{e}^{-x-p}. \tag{2.37}$$

□

Local mirror symmetry is considerably simpler than full-fledged mirror symmetry. On the A-model side, the enumerative invariants can be computed algorithmically in various ways, either by localization [20–22], or by using the so-called topological vertex [23]. At the same time, the theory of deformation of complex structures of $\widehat{X}$ can be simplified very much. At genus zero, one should consider the periods of the differential

$$\lambda = y(x)\mathrm{d}x \tag{2.38}$$

on the curve (2.23) [10, 11]. The mirror map and the genus zero free energy $F_0(\mathbf{t})$ in the large radius frame are determined by making an appropriate choice of cycles on the curve, α_i, β_i, $i = 1, \cdots, s$, and one finds

$$t_i = \oint_{\alpha_i} \lambda, \qquad \frac{\partial F_0}{\partial t_i} = \oint_{\beta_i} \lambda, \qquad i = 1, \cdots, s. \tag{2.39}$$

In general, $s \geq g_\Sigma$, where g_Σ is the genus of the mirror curve. Also, in the local case, the type B topological string can be formulated in a more precise way, by using the topological recursion of [24], in terms of periods and residues of meromorphic forms on the curve (2.23) [16, 17]. In particular, mirror symmetry can be proved to all genera [25, 26] by comparing the definition of the B model in [16, 17] with localization computations in the A model.

Example 2.2. *Local* $\mathbb{P}^2$. In the case of the local $\mathbb{P}^2$ CY, the genus zero free energy can be obtained by mirror symmetry as follows. The complex deformation parameter in the mirror curve is the parameter κ appearing in (2.36), or equivalently the parameter $z = \kappa^{-3}$. Let us introduce the power series,

$$\begin{aligned} \widetilde{\varpi}_1(z) &= \sum_{j\geq 1} 3\frac{(3j-1)!}{(j!)^3}(-z)^j, \\ \widetilde{\varpi}_2(z) &= \sum_{j\geq 1} \frac{18}{j!}\frac{\Gamma(3j)}{\Gamma(1+j)^2}\left\{\psi(3j) - \psi(j+1)\right\}(-z)^j, \end{aligned} \tag{2.40}$$

where $\psi(z)$ is the digamma function, as well as

$$\begin{aligned} \varpi_1(z) &= \log(z) + \widetilde{\varpi}_1(z), \\ \varpi_2(z) &= \log^2(z) + 2\widetilde{\varpi}_1(z)\log(z) + \widetilde{\varpi}_2(z). \end{aligned} \tag{2.41}$$

The mirror map is given by

$$t = -\varpi_1(z), \tag{2.42}$$

where $\varpi_1(z)$ is the power series in (2.41). Then, $F_0(t)$ is defined, up to a constant, by

$$\frac{\partial F_0}{\partial t} = \frac{\varpi_2(z)}{6}, \tag{2.43}$$

where $\varpi_2(z)$ is the other period in (2.41). One then finds, after fixing the integration constant appropriately,

$$F_0(t) = \frac{t^3}{18} + 3\mathrm{e}^{-t} - \frac{45}{8}\mathrm{e}^{-2t} + \frac{244}{9}\mathrm{e}^{-3t} + \cdots . \tag{2.44}$$

From here one can read the very first Gromov–Witten invariants, like for example $N_0^{d=1} = 3$, see [11] for more details. □

2.4 Refined topological strings and the NS limit

Another interesting feature of the local case is that it is possible to define a more general set of enumerative invariants, and consequently a more general topological string theory, known sometimes as the *refined* topological string. This refinement has its roots in the instanton partition functions of Nekrasov for supersymmetric gauge theories [27]. Different aspects of the refinement have been worked out in for example [28–30]. In particular, one can generalize the Gopakumar–Vafa invariants to the so-called *refined BPS invariants*. Precise mathematical definitions can be found in [31, 32]. These invariants, which are also integers, depend on the degrees $\mathbf{d}$ and on two non-negative half-integers, j_L, j_R, or "spins". We will denote them by $N^{\mathbf{d}}_{j_L,j_R}$. Given this invariants, one can defined a refined topological string free energy as follows. One first introduces two parameters $\epsilon_{1,2}$, as well as the combinations,

$$\epsilon_L = \frac{\epsilon_1 - \epsilon_2}{2}, \qquad \epsilon_R = \frac{\epsilon_1 + \epsilon_2}{2}, \tag{2.45}$$

and their exponentiated versions,

$$q_{1,2} = \mathrm{e}^{\mathrm{i}\epsilon_{1,2}}, \qquad q_{L,R} = \mathrm{e}^{\mathrm{i}\epsilon_{L,R}}. \tag{2.46}$$

We also need the $SU(2)$ character for the spin j,

$$\chi_j(q) = \frac{q^{2j+1} - q^{-2j-1}}{q - q^{-1}} \tag{2.47}$$

Then, the refined topological string free energy is given by

$$F_{\mathrm{ref}}(\boldsymbol{t}, \epsilon_1, \epsilon_2) = \sum_{j_L, j_R \geq 0} \sum_{w \geq 1} \sum_{\mathbf{d}} \frac{1}{w} N^{\mathbf{d}}_{j_L,j_R} \frac{\chi_{j_L}(q_L^w)\chi_{j_R}(q_R^w)}{\left(q_1^{w/2} - q_1^{-w/2}\right)\left(q_2^{w/2} - q_2^{-w/2}\right)} \mathrm{e}^{-w \boldsymbol{d}\cdot\boldsymbol{t}}. \tag{2.48}$$

The standard topological string is recovered when

$$\epsilon_1 = -\epsilon_2 = g_s. \tag{2.49}$$

In particular, the Gopakumar–Vafa invariants are particular combinations of these refined BPS invariants, and one has the following relationship,

$$\sum_{j_L, j_R} \chi_{j_L}(q)(2j_R + 1) N^{\mathbf{d}}_{j_L,j_R} = \sum_{g \geq 0} n^{\mathbf{d}}_g (-1)^{g-1} \left(q^{1/2} - q^{-1/2}\right)^{2g}, \tag{2.50}$$

where q is a formal variable We note that the sums in (2.50) are well-defined, since for given degrees $\mathbf{d}$ only a finite number of j_L, j_R, g give a non-zero contribution.

There is another limit which is particularly important, the Nekrasov–Shatashvili (NS) limit [33], in which one considers

$$\epsilon_1 = \hbar, \qquad \epsilon_2 \to 0. \tag{2.51}$$

The NS free energy (more precisely, its "instanton" part) is defined as

$$
\begin{aligned}
F_{\mathrm{inst}}^{\mathrm{NS}}(\mathbf{t}, \hbar) &= \mathrm{i} \lim_{\epsilon_2 \to 0} \epsilon_2 F_{\mathrm{ref}}(\boldsymbol{t}, \epsilon_1, \epsilon_2) \\
&= \sum_{j_L, j_R} \sum_{w, \mathbf{d}} N_{j_L, j_R}^{\mathbf{d}} \frac{\sin \frac{\hbar w}{2}(2j_L + 1) \sin \frac{\hbar w}{2}(2j_R + 1)}{2w^2 \sin^3 \frac{\hbar w}{2}} \mathrm{e}^{-w \mathbf{d} \cdot \mathbf{t}}.
\end{aligned} \tag{2.52}
$$

One then defines the "full" NS free energy as

$$
F^{\mathrm{NS}}(\mathbf{t}, \hbar) == \frac{1}{6\hbar} \sum_{i,j,k=1}^{s} a_{ijk} t_i t_j t_k + \sum_{i=1}^{s} b_i^{\mathrm{NS}} t_i \hbar + F_{\mathrm{inst}}^{\mathrm{NS}}(\mathbf{t}, \hbar). \tag{2.53}
$$

By expanding (2.53) in powers of $\hbar$, we find the NS free energies at order n, $F_n^{\mathrm{NS}}(\mathbf{t})$, as

$$
F^{\mathrm{NS}}(\mathbf{t}, \hbar) = \sum_{n=0}^{\infty} F_n^{\mathrm{NS}}(\mathbf{t}) \hbar^{2n-1}. \tag{2.54}
$$

The expression (2.52) can be regarded as a Gopakumar–Vafa-like resummation of the series in (2.54). An important observation is that the first term in this series, $F_0^{\mathrm{NS}}(\mathbf{t})$, is equal to $F_0(\mathbf{t})$, the standard genus zero free energy. Note that the term involving the coefficients b_i^{NS} contributes to $F_1^{\mathrm{NS}}(\mathbf{t})$.

Note that the standard topological string and the NS limit give two *a priori* different one-parameter deformations of the "classical" topological string theory. The second one turns out to be a quantization deformation, as we will review next. The first one is not clearly so. However, as we will following [34], it also has a quantum-mechanical interpretation.

3 Quantum curves

3.1 The all-orders WKB method

One interesting aspect of mirror symmetry at genus zero, i.e. the formulae (2.38) and (2.39), is that it looks like a WKB approximation. Let us recall some basic facts of this approximation. Let us assume that we have a classical Hamiltonian of the form,

$$
H(x, p) = \frac{p^2}{2} + V(x), \tag{3.1}
$$

where y is interpreted as the momentum, and $V(x)$ is a potential. In the WKB method one consider the following plane curve, or *WKB curve*,

$$
H(x, p) = E, \tag{3.2}
$$

where E is interpreted as the energy of the system, and a differential on this curve given by

$$
\lambda = p(x) \mathrm{d}x. \tag{3.3}
$$

If the curve (3.2) has genus g, one can choose a symplectic basis of $2g$ cycles, A_i and B_i, $i = 1, \cdots, g$, and define periods

$$
\begin{aligned}
t_i(E) &= \oint_{A_i} \lambda, \\
\frac{\partial F_0}{\partial t_i} &= \oint_{B_i} \lambda, \qquad i = 1, \cdots, g.
\end{aligned} \tag{3.4}
$$

This defines a "genus zero free energy" $F_0(\boldsymbol{t})$. Therefore, the geometry of the leading order WKB method defines a setting which is very similar to genus zero mirror symmetry.

What happens to the quantum corrections? The all-orders WKB method leads to a deformation of λ, as follows. Let us consider the Schrödinger equation corresponding to the Hamiltonian (3.1),

$$\hbar^2\psi''(x) + p^2(x)\psi(x) = 0, \qquad p(x) = \sqrt{2(E - V(x))}, \tag{3.5}$$

The WKB ansatz for the wavefunction is

$$\psi(x) = \exp\left[\frac{\mathrm{i}}{\hbar}\int^x Q(x')\mathrm{d}x'\right]. \tag{3.6}$$

The Schrödinger equation for $\psi(x)$ becomes a Riccati equation for $Q(x)$,

$$Q^2(x) - \mathrm{i}\hbar\frac{\mathrm{d}Q(x)}{\mathrm{d}x} = p^2(x), \tag{3.7}$$

which can be solved in power series in $\hbar$:

$$Q(x) = \sum_{k=0}^{\infty} Q_k(x)\hbar^k. \tag{3.8}$$

The functions $Q_k(x)$ can be computed recursively as

$$\begin{aligned} Q_0(x) &= p(x), \\ Q_{n+1}(x) &= \frac{1}{2Q_0(x)}\left(\mathrm{i}\frac{\mathrm{d}Q_n(x)}{\mathrm{d}x} - \sum_{k=1}^{n} Q_k(x)Q_{n+1-k}(x)\right). \end{aligned} \tag{3.9}$$

If we split the formal power series in (3.8) into even and odd powers of $\hbar$,

$$Q(x) = Q_{\mathrm{odd}}(x) + P(x), \tag{3.10}$$

one finds that $Q_{\mathrm{odd}}(x)$ is a total derivative,

$$Q_{\mathrm{odd}}(x) = \frac{\mathrm{i}\hbar}{2}\frac{P'(x)}{P(x)} = \frac{\mathrm{i}\hbar}{2}\frac{\mathrm{d}}{\mathrm{d}x}\log P(x). \tag{3.11}$$

Therefore, only the even part $P(x)$ contributes to the periods of $Q(x)\mathrm{d}x$. We define then the differential

$$\lambda(\hbar) = P(x)\mathrm{d}x \tag{3.12}$$

which can be regarded as a deformation of λ. Note that $P(x)$ is defined as a *formal power series* in $\hbar^2$. We also define the *quantum periods* as

$$\begin{aligned} t_i(E;\hbar) &= \oint_{A_i}\lambda(\hbar), \\ \frac{\partial F_{\mathrm{NS}}}{\partial t_i}(E;\hbar) &= \frac{1}{\hbar}\oint_{B_i}\lambda(\hbar), \end{aligned} \tag{3.13}$$

and this defines the *quantum free energy* $F_{\mathrm{NS}}(\boldsymbol{t},\hbar)$, which has the power series expansion

$$F_{\mathrm{NS}}(\boldsymbol{t},\hbar) = \sum_{n\geq 0} F_n(\boldsymbol{t})\hbar^{2n-1}, \tag{3.14}$$

where $F_0(\boldsymbol{t})$ is defined by the leading WKB method. We emphasize that this can be done in any one-dimensional quantum-mechanical model, see e.g. [35] for explicit formulae in relevant examples.

In conclusion, the quantum corrections to $F_0(\boldsymbol{t})$ are obtained by a *quantization* of the classical curve (3.2), in which we promote the variables x, p to Heisenberg operators,

$$x \to \mathsf{x}, \qquad p \to \mathsf{p}, \tag{3.15}$$

with

$$[\mathsf{x}, \mathsf{p}] = \mathrm{i}\hbar. \tag{3.16}$$

The equation for the classical curve

$$H(x, y) - E = 0 \tag{3.17}$$

is then promoted to the stationary Schrödinger equation

$$H(\mathsf{x}, \mathsf{p})|\psi\rangle = E|\psi\rangle. \tag{3.18}$$

3.2 Quantum curves in supersymmetric gauge theory

Although the above considerations were made for a curve of the form (3.1), it turns out that one can consider more general curves, involving in particular *difference* (not differential) operators. For example, let us consider the SW curve for the pure $\mathcal{N} = 2$ Yang–Mills theory with gauge group $SU(N)$:

$$\Lambda^N \left(\mathrm{e}^p + \mathrm{e}^{-p}\right) + W_N(x) = 0, \tag{3.19}$$

where

$$W_N(x) = \sum_{k=0}^{N} (-1)^k x^{N-k} h_k. \tag{3.20}$$

The differential on this curve is still (3.3), and the periods of this curve are precisely the SW periods a_i, $a_{D,i}$, $i = 1, \cdots, g$. The genus zero function $F_0(a_i)$ is the famous SW prepotential. By promoting x, p to operators we obtain a *difference operator*

$$\mathsf{H}_N = \Lambda^N \left(\mathrm{e}^{\mathsf{p}} + \mathrm{e}^{-\mathsf{p}}\right) + V_N(\mathsf{x}), \tag{3.21}$$

where

$$V_N(x) = \sum_{k=0}^{N-1} (-1)^k x^{N-k} h_k. \tag{3.22}$$

Notice that we have removed the last term, $(-1)^N h_N$, since we will interpret it as an eigenvalue. By acting on wavefunctions, we have

$$\langle x|\mathsf{H}_N|\psi\rangle = \Lambda^N \left(\psi(x + \mathrm{i}\hbar) + \psi(x - \mathrm{i}\hbar)\right) + V_N(x)\psi(x). \tag{3.23}$$

This is a difference operator. The eigenvalue problem for this operator is

$$\mathsf{H}_N|\psi\rangle = (-1)^{N-1} h_N|\psi\rangle. \tag{3.24}$$

In [36] it was suggested that the quantization of curves appearing in supersymmetric gauge theory, non-critical strings and topological string theory makes it possible to compute quantum corrections to genus zero quantities. These quantum corrections were tentatively identified with "stringy corrections" in the string coupling constant $\epsilon_1 = -\epsilon_2 = g_s$. However, motivated by [33], Mironov and Morozov in [37, 38] applied the WKB method to the SW curve, in order to obtain "quantum corrections" to the SW prepotential. They computed the quantum periods in (3.13) and determined the quantum free energy (3.14) at the very first orders. They found that this agrees with the NS limit of the Nekrasov free energy for this theory. In particular, by performing a perturbative, WKB quantization of the curve, one does *not* obtain the standard topological string deformation $\epsilon_1 = -\epsilon_2 = g_s$, but rather the NS deformation $\epsilon_1 = \hbar$, $\epsilon_2 = 0$.

3.3 Quantum mirror curves

The next step is to consider the quantization of *mirror curves* of the form (2.23). Here, we have to face a standard issue in quantization which didn't occur in (3.1) nor in (3.19), namely ordering ambiguities. Upon quantization, the classical term

$$\mathrm{e}^{ax+bp} \tag{3.25}$$

leads to different answers depending on the quantization procedure. The most natural prescription is *Weyl quantization*, which is simply

$$\mathrm{e}^{ax+bp} \to \mathrm{e}^{a\mathsf{x}+b\mathsf{p}}. \tag{3.26}$$

Let us consider for simplicity curves (of genus one) of the form (2.31). Weyl quantization leads to an operator

$$\mathcal{O}_X(x,p) \to \mathsf{O}_X(\mathsf{x},\mathsf{p}) \tag{3.27}$$

and we will interpret κ as (minus) its eigenvalue, as in the case of the SW curve. We can now look for solutions of

$$\mathsf{O}_X(\mathsf{x},\mathsf{p})|\psi\rangle = -\kappa|\psi\rangle \tag{3.28}$$

by using the WKB method, and calculate the corresponding deformation of $F_0(\boldsymbol{t})$. This was done in [39] with the same result obtained in [37, 38] for the SW curve: the deformation gives the NS free energy.

Remark 3.1. In the literature on quantum curves it is sometimes argued that the WKB wavefunction can be constructed with the Eynard–Orantin recursion, which corresponds to the standard topological string $\epsilon_1 = -\epsilon_2 = g_s$. This is clearly in tension with the results of [37, 39], and it is probably true only when the curve has genus zero.

Remark 3.2. The above result shows that, in the case of the NS limit of the refined topological string, the mirror computation of the amplitudes *is* the WKB method as applied to the mirror curve. For the general refined amplitudes, no explicit mirror formulation has been found so far.

3.4 Spectral theory and quantum curves

The WKB approach of [37, 39] is very nice but it does not lead us beyond a perturbative approach. It does not say anything about the standard topological string, which was the original goal of [36]. In order to go beyond perturbation theory in $\hbar$, we have to look at the exact spectral problem defined by the quantum operators obtained by quantizing the mirror curve (or the SW curve).

In pursuing this approach, one has to abandon momentarily the realm of complex variables. Mirror curves are complex curves, and all variables appearing there (x, p, κ, ...) are in principle complex. However, in spectral theory (or Quantum Mechanics) we need to impose reality conditions (e.g. we want operators to be self-adjoint). In fact, spectral properties are very sensitive to *positivity conditions*. Once we understand a spectral problem in some range of values, one can try to extend it to more general values of the parameters. As we will see, in spite of all these reality conditions, we will eventually recover to a large extent the original, complex realm of topological string theory.

The first reality choice we have to make concerns the quantization of the curve itself. Namely, we have assumed that the variables x, p appearing in the curve become self-adjoint Heisenberg operators on $L^2(\mathbb{R})$, so that e^{x}, e^{p} act as

$$\langle x|\mathrm{e}^{\mathsf{x}}|\psi\rangle = \mathrm{e}^{x}\psi(x), \qquad \langle x|\mathrm{e}^{\mathsf{p}}|\psi\rangle = \psi(x-\mathrm{i}\hbar). \tag{3.29}$$

In particular, the domain of the operator e^{p} consists of functions which admit an analytic continuation to the strip

$$\mathcal{S}_{-\hbar} = \{x - \mathrm{i}y \in \mathbb{C} : 0 < y < \hbar\}\,, \tag{3.30}$$

such that $\psi(x-\mathrm{i}y) \in L^2(\mathbb{R})$ for all $0 \leq y < \hbar$, and the limit

$$\psi\left(x - \mathrm{i}\hbar + \mathrm{i}0\right) = \lim_{\epsilon\to 0^+} \psi\left(x - \mathrm{i}\hbar + \mathrm{i}\epsilon\right) \tag{3.31}$$

exists in the sense of convergence in $L^2(\mathbb{R})$. In addition, we will assume that $\hbar \in \mathbb{R}_{>0}$. Finally, many mirror curves involve a "mass parameter". For example, in local $\mathbb{F}_0$, the curve involves the function

$$\mathcal{O}_X = \mathrm{e}^{x} + m_{\mathbb{F}_0}\mathrm{e}^{-x} + \mathrm{e}^{p} + \mathrm{e}^{-p} \tag{3.32}$$

where $m_{\mathbb{F}_0}$ is in principle a complex parameter, associated to a Kähler parameter of the CY. However, as we will see, it is useful to start the analysis in the case $m_{\mathbb{F}_0} > 0$.

Once this has been settled, we can ask about the spectral properties of these operators. Take for example the operator associated to local $\mathbb{P}^2$, which is the Weyl quantization of (2.37):

$$\mathsf{O}_{\mathbb{P}^2} = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{\mathsf{p}} + \mathrm{e}^{-\mathsf{x}-\mathsf{p}}. \tag{3.33}$$

To have a first approach to the spectrum, we can use numerical methods, as pointed out in [40]. First, one should choose an appropriate orthonormal basis of functions $\{\phi_i\}_{i=0,1,\cdots}$ of $L^2(\mathbb{R})$, in the domain of $\mathsf{O}_{\mathbb{P}^2}$ (for example, the eigenfunctions of the harmonic oscillator will do). Then, the e^{E_n} are the eigenvalues of the infinite-dimensional matrix

$$M_{ij} = (\phi_i, \mathsf{O}_{\mathbb{P}^2}\phi_j), \qquad i,j = 0,1,\cdots. \tag{3.34}$$

They can be obtained numerically by truncating the matrix to very large sizes. Boundary effects can be partially eliminated with standard extrapolation methods, and one can find very accurate values for the E_n. For $\hbar = 2\pi$, the results for the very first eigenvalues are listed in Table 1.

Another approach that we can follow is the WKB method that we presented above. although our quantum operator (3.33) is not of the standard form (3.1), we can still use the Bohr–Sommerfeld condition. The classical counterpart of the operator $\mathsf{O}_{\mathbb{P}^2}$ is the function (2.37) on the phase space $\mathbb{R}^2$. The analogue of the hyperelliptic curve (3.2) is now,

$$\mathcal{O}_{\mathbb{P}^2}(x,y) = \mathrm{e}^{E}, \tag{3.35}$$

and the function (2.37) defines a compact region in phase space

$$\mathcal{R}(E) = \{(x,y) \in \mathbb{R}^2 : \mathcal{O}_{\mathbb{P}^2}(x,y) \leq \mathrm{e}^{E}\}, \tag{3.36}$$

n	E_n
0	2.56264206862381937
1	3.91821318829983977
2	4.91178982376733606
3	5.73573703542155946
4	6.45535922844299896

Table 1. Numerical spectrum of the operator (3.33) for $n = 0, 1, \cdots, 4$, and $\hbar = 2\pi$.

The Bohr–Sommerfeld quantization condition is that

$$\mathrm{vol}_0(E) = 2\pi\hbar\left(n + \frac{1}{2}\right), \qquad n = 0, 1, 2, \cdots \tag{3.37}$$

where $\mathrm{vol}_0(E)$ is the volume of the region $\mathcal{R}(E)$. This volume can be computed explicitly in terms of special functions, but since the resulting quantization condition is only valid at large energies, it is worth to simplify it, as follows. At large E, the region $\mathcal{R}(E)$ has a natural mathematical interpretation: it is the region enclosed by the tropical limit of the curve

$$\mathrm{e}^x + \mathrm{e}^y + \mathrm{e}^{-x-y} - \mathrm{e}^E = 0, \tag{3.38}$$

and limited by the lines

$$x = E, \qquad y = E, \qquad x + y + E = 0, \tag{3.39}$$

see Fig. 2 for an illustration when $E = 15$. In this polygonal limit, we find

$$\mathrm{vol}_0(E) \approx \frac{9E^2}{2}, \tag{3.40}$$

and we obtain from (3.37) the following approximate behavior for the eigenvalues, at large quantum numbers,

$$E_n \approx \frac{2}{3}\sqrt{\pi\hbar}\, n^{1/2}, \qquad n \gg 1. \tag{3.41}$$

It can be seen, by direct examination of the numerical spectrum, that this rough WKB estimate gives a good approximation to the eigenvalues of the operator (3.33) when n is large. The estimate (3.41) has been rigorously proved in [41].

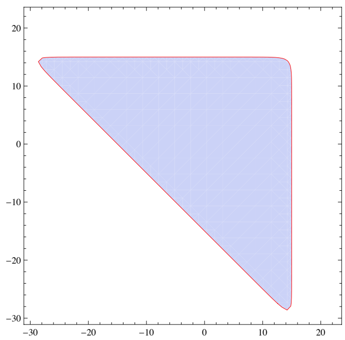

Figure 2. The region $\mathcal{R}(E)$ for $E = 15$.

The above analysis shows that the spectrum of $\mathsf{O}_{\mathbb{P}^2}$ is discrete, just as that of a confining potential in conventional QM. This also suggests that $\rho_{\mathbb{P}^2} = \mathsf{O}_{\mathbb{P}^2}^{-1}$ is a compact operator. In fact, more is true. Semiclassically, we have

$$\mathrm{Tr}\rho_{\mathbb{P}^2} = \int \frac{\mathrm{d}x\mathrm{d}p}{2\pi} \frac{1}{\mathrm{e}^x + \mathrm{e}^p + \mathrm{e}^{-x-p}} < +\infty \tag{3.42}$$

which suggests that $\rho_{\mathbb{P}^2}$ is a *trace class operator*. We recall that an operator ρ is trace class if

$$\mathrm{Tr}\rho < +\infty. \tag{3.43}$$

If ρ is of trace class, it is in particular compact and Hilbert–Schmidt, and all its powers are trace class as well. Therefore, all its *spectral traces* are finite

$$\mathrm{Tr}\rho^\ell < +\infty, \qquad \ell = 1, 2, \cdots \tag{3.44}$$

The fact that $\rho_{\mathbb{P}^2}$ is trace class was proved in [42], by an explicit calculation of the integral kernel of $\rho_{\mathbb{P}^2}$. In fact, in [34] we proposed that, for local del Pezzo (2.24), and provided the mass parameters are positive, the operator

$$\rho_S = \mathsf{O}_S^{-1} \tag{3.45}$$

is of trace class (this can be extended to the general toric case, as done in [43]).

Remark 3.3. The operators O_S can be constructed as well as Hamiltonians of cluster integrable systems [44] for mirror curves of genus one.

3.5 Fredholm determinants

Trace class operators occupy a central rôle in spectral theory, and they are in a sense the "best" possible operators one can consider: they are compact, therefore they have a discrete spectrum, and in addition are all their spectral traces are finite. Given a trace class operator ρ, its *spectral* or *Fredholm determinant* is defined by an infinite determinant

$$\Xi(\kappa) = \det\left(1 + \kappa\rho\right). \tag{3.46}$$

The trace class property guarantees [45, 46] that these determinant exists and is an entire function of κ.

One *conceptual* reason to introduce the Fredholm determinant is the following. We want to relate the world of spectral theory to the world of topological strings. However, the natural mathematical objects in these worlds are very different. Topological string free energies are complex functions, analytic in a domain of the complex plane. Spectral theory deals with discrete spectra. If topological strings are to emerge from quantum mechanics, we need a way of obtaining smooth objects from discrete objects. The Fredholm determinant encodes the spectrum but at the same time is an analytic function of its parameter and then offers a possibility to cross that bridge.

The Fredholm determinant has as a power series expansion around $\kappa = 0$, of the form

$$\Xi(\kappa) = 1 + \sum_{N=1}^{\infty} Z(N)\kappa^N, \tag{3.47}$$

where $Z(N)$ is the *fermionic spectral trace*, given by

$$Z(N) = \mathrm{Tr}\left(\Lambda^N(\rho)\right), \qquad N = 1, 2, \cdots \tag{3.48}$$

In this expression, the operator $\Lambda^N(\rho)$ is defined by $\rho^{\otimes N}$ acting on $\Lambda^N\left(L^2(\mathbb{R})\right)$. A theorem of Fredholm asserts that, if $\rho(p_i, p_j)$ is the kernel of ρ, the fermionic spectral trace can be computed as a multi-dimensional integral,

$$Z(N) = \frac{1}{N!}\int \det\left(\rho(p_i, p_j)\right) \mathrm{d}^N p. \tag{3.49}$$

In turn, the logarithm of the Fredholm determinant can be regarded as a generating functional of the spectral traces of ρ, since

$$\mathcal{J}(\kappa) = \log \Xi(\kappa) = -\sum_{\ell=1}^{\infty} \frac{(-\kappa)^\ell}{\ell} \mathrm{Tr}\rho^\ell. \tag{3.50}$$

The Fredholm determinant (3.46) has the infinite product representation

$$\Xi(\kappa) = \prod_{n=0}^{\infty}\left(1 + \kappa \mathrm{e}^{-E_n}\right), \tag{3.51}$$

where e^{-E_n} are the eigenvalues of ρ. Therefore, one way of obtaining the spectrum of ρ is to look for the zeroes of Ξ, which occur at

$$\kappa = -\mathrm{e}^{E_n}, \qquad n \geq 0. \tag{3.52}$$

There are very few operators on $L^2(\mathbb{R})$ for which the Fredholm determinant can be written down explicitly. One family of examples which has been studied in some detail are Schrödinger operators with homogeneous potentials $V(x) = |x|^s$, $s = 1, 2, \cdots$ [47, 48]. In this case, the Fredholm determinant is defined as a regularized version of

$$D(E) = \prod_{n \geq 0}\left(1 + \frac{E}{E_n}\right), \tag{3.53}$$

see for example [49] for a detailed treatment. For these potentials, the function $D(E)$ is known to satisfy certain functional equations and it is captured by integral equations of the TBA type [48].

3.6 A conjecture for the spectral determinant

One of the surprising (conjectural) results of [34] is that the Fredholm determinant of the operators ρ_S can be computed *exactly and explicitly* in terms of the enumerative invariants of X.

The conjectural expression of [34] for the Fredholm determinant of the operator ρ_S requires the two generating functionals of enumerative invariants considered before, (2.8) and (2.53). In order to state the result, we identify the parameter κ appearing in the Fredholm determinant, with the geometric modulus of X appearing in (2.31). We will also write κ in terms of the "chemical potential" μ

$$\kappa = \mathrm{e}^{\mu}. \tag{3.54}$$

In addition to the modulus κ, we have the "mass parameters" ξ_j. The flat coordinates for the Kähler moduli space, t_i, are related to κ and ξ_j by the mirror map, and at leading order, in the large radius limit, we have

$$t_i \approx c_i \mu - \sum_{j=1}^{s-1} \alpha_{ij} \log \xi_j, \quad i = 1, \cdots, s, \tag{3.55}$$

where c_i, α_{ij} are constants. As shown in [39], the WKB approach makes it possible to define a *quantum mirror map* $t_i(\hbar)$, which in the limit $\hbar \to 0$ agrees with the conventional mirror map. It is a function of μ, ξ_j and $\hbar$ and it can be computed as an A-period of a quantum corrected version of the differential (2.38). This quantum correction is simply obtained by using the all-orders, perturbative WKB method of Dunham.

We now introduce two different functions of μ. The first one is the WKB grand potential,

$$\begin{aligned} \mathsf{J}_S^{\mathrm{WKB}}(\mu, \boldsymbol{\xi}, \hbar) = & \sum_{i=1}^{s} \frac{t_i(\hbar)}{2\pi} \frac{\partial F^{\mathrm{NS}}(\mathbf{t}(\hbar), \hbar)}{\partial t_i} + \frac{\hbar^2}{2\pi} \frac{\partial}{\partial \hbar} \left(\frac{F^{\mathrm{NS}}(\mathbf{t}(\hbar), \hbar)}{\hbar} \right) \\ & + \sum_{i=1}^{s} \frac{2\pi}{\hbar} b_i t_i(\hbar) + A(\boldsymbol{\xi}, \hbar). \end{aligned} \tag{3.56}$$

In this equation, $F^{\mathrm{NS}}(\mathbf{t}, \hbar)$ is given by the expression (2.53). In the second term of (3.56), the derivative w.r.t. $\hbar$ does not act on the implicit dependence of t_i. The coefficients b_i appearing in the last line are the same ones appearing in (2.4). The function $A(\boldsymbol{\xi}, \hbar)$ is not known in closed form for arbitrary geometries, although detailed conjectures for its form exist in various examples. It is closely related to a all-genus resummed version of the constant map contribution C_g appearing in (2.5). The second function is the "worldsheet" grand potential, which can be obtained from the generating functional (2.8),

$$\mathsf{J}_S^{\mathrm{WS}}(\mu, \boldsymbol{\xi}, \hbar) = F^{\mathrm{GV}} \left(\frac{2\pi}{\hbar} \mathbf{t}(\hbar) + \pi \mathrm{i} \mathbf{B}, \frac{4\pi^2}{\hbar} \right). \tag{3.57}$$

In this formula, $\mathbf{B}$ is a constant vector ("B-field") which depends on the geometry under consideration. This vector should satisfy the following requirement: for all $\mathbf{d}$, j_L and j_R such that the refined BPS invariant $N^{\mathbf{d}}_{j_L, j_R}$ is non-vanishing, we must have

$$(-1)^{2j_L + 2j_R + 1} = (-1)^{\mathbf{B} \cdot \mathbf{d}}. \tag{3.58}$$

For local del Pezzo CY threefolds, the existence of such a vector was established in [6]. Note that the effect of this constant vector is to introduce a sign

$$(-1)^{w \mathbf{d} \cdot \mathbf{B}} \tag{3.59}$$

in the generating functional (2.8). An important remark is that, in (3.57), the topological string coupling constant g_s appearing in (2.6) and (2.8) is related to the Planck constant appearing in the spectral problem by,

$$g_s = \frac{4\pi^2}{\hbar}. \tag{3.60}$$

Therefore, the regime of weak coupling for the topological string coupling constant, $g_s \ll 1$, corresponds to the strong coupling regime of the spectral problem, $\hbar \gg 1$, and conversely, the semiclassical limit of the spectral problem corresponds to the strongly coupled topological string. We therefore have a *strong-weak coupling duality* between the spectral problem and the conventional topological string.

The *total grand potential* is the sum of these two functions,

$$\mathsf{J}_S(\mu, \boldsymbol{\xi}, \hbar) = \mathsf{J}_S^{\mathrm{WKB}}(\mu, \boldsymbol{\xi}, \hbar) + \mathsf{J}_S^{\mathrm{WS}}(\mu, \boldsymbol{\xi}, \hbar), \tag{3.61}$$

and it was first considered in [6]. It has the structure

$$\mathsf{J}_S(\mu, \boldsymbol{\xi}, \hbar) = \frac{1}{12\pi\hbar} \sum_{i,j,k=1}^{s} a_{ijk} t_i t_j t_k + \sum_{i=1}^{s} \left(\frac{2\pi b_i}{\hbar} + \frac{\hbar b_i^{\rm NS}}{2\pi} \right) t_i + \mathcal{O}\left(\mathrm{e}^{-t_i}, \mathrm{e}^{-2\pi t_i/\hbar} \right), \tag{3.62}$$

where the last term stands for a formal power series in e^{-t_i}, $\mathrm{e}^{-2\pi t_i/\hbar}$, whose coefficients depend explicitly on $\hbar$. Note that the trigonometric functions appearing in (2.52) and (2.8) have double poles when $\hbar$ is a rational multiple of π. However, as shown in [6], the poles *cancel* in the sum (3.61). This HMO cancellation mechanism was first discovered in [50], in a slightly different context, and it was first advocated in [51] in the study of quantum curves.

A natural question is whether the formal power series in (3.61) converges, at least for some values of its arguments. Although we do not have rigorous results on this problem, the available evidence suggests that, for real $\hbar$, $\mathsf{J}_S(\mu, \boldsymbol{\xi}, \hbar)$ *converges* in a neighbourhood of the large radius point $t_i \to \infty$.

We can now state the main conjecture of [34].

Conjecture 3.4. *The Fredholm determinant of the operator ρ_S is given by*

$$\Xi_S(\kappa, \boldsymbol{\xi}, \hbar) = \sum_{n\in\mathbb{Z}} \exp\left(\mathsf{J}_S(\mu + 2\pi \mathrm{i} n, \boldsymbol{\xi}, \hbar) \right). \tag{3.63}$$

The sum over n defines a *quantum theta function* $\Theta_S(\mu, \boldsymbol{\xi}, \hbar)$,

$$\Xi_S(\kappa, \boldsymbol{\xi}, \hbar) = \mathrm{e}^{\mathsf{J}_S(\mu,\boldsymbol{\xi},\hbar)} \Theta_S(\mu, \boldsymbol{\xi}, \hbar). \tag{3.64}$$

The reason for this name is that, when $\hbar = 2\pi$, the quantum theta function becomes a classical, conventional theta function, as we will see in a moment. The vanishing locus of the Fredholm determinant, as we have seen in (3.52), gives the spectrum of the trace class operator. Given the form of (3.64), this is the vanishing locus of the quantum theta function.

It would seem that the above conjecture is very difficult to test, since it gives the Fredholm determinant as a formal, infinite sum. However, it is easy to see that the r.h.s. of (3.63) has a series expansion at large μ in powers of $\mathrm{e}^{-\mu}$, $\mathrm{e}^{-2\pi\mu/\hbar}$. In addition, it leads to an integral representation for the fermionic spectral trace which is very useful in practice: if we write $Z_S(N, \hbar)$ as a contour integral around $\kappa = 0$, simple manipulations give the expression [34, 50]

$$Z_S(N, \hbar) = \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}} \mathrm{e}^{\mathsf{J}_S(\mu,\boldsymbol{\xi},\hbar) - N\mu} \mathrm{d}\mu, \tag{3.65}$$

where $\mathcal{C}$ is a contour going from $\mathrm{e}^{-\mathrm{i}\pi/3}\infty$ to $\mathrm{e}^{\mathrm{i}\pi/3}\infty$. Note that this is the standard contour for the integral representation of the Airy function and it should lead to a convergent integral, since $\mathsf{J}_S(\mu, \boldsymbol{\xi}, \hbar)$ is given by a cubic polynomial in μ, plus exponentially small corrections. Finally, as we will see in a moment, in some cases the r.h.s. of (3.63) can be written in terms of well-defined functions.

What is the interpretation of the total grand potential that we introduced in (3.61)? The WKB part takes into account the perturbative corrections (in $\hbar$) to the spectral problem defined by the operator ρ_S. In fact, it can be calculated order by order in the $\hbar$ expansion, by using standard techniques in Quantum and Statistical Mechanics (see for example [52, 53]). The expression (3.56) provides a resummation of this expansion at large μ. However, the WKB piece is insufficient to solve the spectral problem, since as we mentioned above, (3.56) is divergent for a dense set of values of $\hbar$ on the real line. Physically, the additional generating functional (3.57) contains the contribution of *complex* instantons, which are non-perturbative in $\hbar$ [52] and cancel the poles in the all-orders WKB contribution. Surprisingly, the full instanton contribution in this spectral problem is simply encoded in the standard topological string partition function.

The formulae (3.56), (3.57), (3.63) are relatively complicated, in the sense that they involve the full generating functionals (2.52) and (2.8). There is however an important case in which they simplify considerably, namely, when $\hbar = 2\pi$. This was called in [34] the "maximally supersymmetric case," since in the closely related spectral problem of ABJM theory, it occurs when there is enhanced $\mathcal{N} = 8$ supersymmetry [54]. In this case, as it can be easily seen from the explicit expressions for the generating functionals, many contributions vanish. For example, in the Gopakumar–Vafa generating functional, all terms involving $g \geq 2$ are zero. A simple calculation shows that the function (3.61) is given by

$$\mathsf{J}_S(\mu, \boldsymbol{\xi}, 2\pi) = \frac{1}{8\pi^2} \sum_{i,j=1}^{s} t_i t_j \frac{\partial^2 \widehat{F}_0}{\partial t_i \partial t_j} - \frac{1}{4\pi^2} \sum_{i=1}^{s} t_i \frac{\partial \widehat{F}_0}{\partial t_i} + \frac{1}{4\pi^2} \widehat{F}_0(\mathbf{t}) + \widehat{F}_1(\mathbf{t}) + \widehat{F}_1^{\rm NS}(\mathbf{t}), \tag{3.66}$$

where $\widehat{F}_0(\mathbf{t})$, $\widehat{F}_1(\mathbf{t})$ and $\widehat{F}_1^{\rm NS}(\mathbf{t})$ are the generating functions appearing in (2.3), (2.4) and (2.54), with the only difference that one has to include as well the sign (3.59) in the expansion in e^{-t_i}. It can be also seen that, for this value of $\hbar$, the quantum mirror map becomes the classical mirror map, up to a change of sign $\kappa \to -\kappa$. One can now use (3.66) to compute the quantum theta function appearing in (3.64). In all cases computed so far, the quantum theta function becomes a classical theta function. We will now illustrate this simplification in the case of local $\mathbb{P}^2$.

Example 3.5. Local $\mathbb{P}^2$ has one single Kähler parameter, and no mass parameters. The function $A(\hbar)$ appearing in (3.56) has been conjectured in closed form, and it is given by

$$A(\hbar) = \frac{3A_{\rm c}(\hbar/\pi) - A_{\rm c}(3\hbar/\pi)}{4}, \tag{3.67}$$

where

$$A_{\rm c}(k) = \frac{2\zeta(3)}{\pi^2 k}\left(1 - \frac{k^3}{16}\right) + \frac{k^2}{\pi^2}\int_0^\infty \frac{x}{\mathrm{e}^{kx} - 1}\log(1 - \mathrm{e}^{-2x})\mathrm{d}x. \tag{3.68}$$

This function was first introduced in [52], and determined in integral form in [55, 56]. It can be obtained by an appropriate all-genus resummation of the constants C_g appearing in (2.5). In the "maximally supersymmetric case" $\hbar = 2\pi$, one can write the Fredholm determinant in closed form. The standard topological string genus zero free energy is given in (2.43), (2.44). The genus one free energies are given by (see e.g. [30, 57]):

$$\begin{aligned} F_1(t) &= \frac{1}{2}\log\left(-\frac{\mathrm{d}z}{\mathrm{d}t}\right) - \frac{1}{12}\log\left(z^7(1 + 27z)\right), \\ F_1^{\rm NS}(t) &= -\frac{1}{24}\log\left(\frac{1 + 27z}{z}\right). \end{aligned} \tag{3.69}$$

We identify

$$z = \mathrm{e}^{-3\mu} = \frac{1}{\kappa^3}. \tag{3.70}$$

Then, it follows from the conjecture above that

$$\begin{aligned}\Xi_{\mathbb{P}^2}(-\kappa, 2\pi) &= \det\left(1-\kappa\rho_{\mathbb{P}^2}\right)\\ &= \exp\left\{A(2\pi)+\frac{1}{4\pi^2}\left(F_0(t)-t\partial_t F_0(t)+\frac{t^2}{2}\partial_t^2 F_0(t)\right)+F_1(t)+F_1^{\rm NS}(t)\right\}\\ &\times \mathrm{e}^{\pi\mathrm{i}/8}\vartheta_2\left(\xi-\frac{1}{4},\tau\right),\end{aligned} \tag{3.71}$$

where

$$\xi=\frac{3}{4\pi^2}\left(t\partial_t^2F_0(t)-\partial_tF_0(t)\right), \qquad \tau=\frac{2\mathrm{i}}{\pi}\partial_t^2F_0(t), \tag{3.72}$$

and $\vartheta_2(z,\tau)$ is the Jacobi theta function. The τ appearing here is the standard modulus of the genus one mirror curve of local $\mathbb{P}^2$. In particular, one has that $\mathrm{Im}(\tau)>0$. The formulae that we have written down here are slightly different from the ones listed earlier in this section, since we are changing the sign of κ in the Fredholm determinant, but they can be easily derived from them. We can now perform a power series expansion around $\kappa=0$, by using analytic continuation of the topological string free energies to the so-called *orbifold point*, which corresponds to the limit $z\to\infty$. This is a standard exercise in topological string theory (see for example [9]), and one finds [34]

$$\Xi_{\mathbb{P}^2}(\kappa,2\pi)=1+\frac{\kappa}{9}+\left(\frac{1}{12\sqrt{3}\pi}-\frac{1}{81}\right)\kappa^2+\mathcal{O}(\kappa^3), \tag{3.73}$$

provided some non-trivial identities are used for theta functions. This predicts the values of the very first fermionic spectral traces $Z_{\mathbb{P}^2}(N,2\pi)$, for $N=1,2$. Interestingly, these traces can be calculated directly in spectral theory, and the values appearing in the expansion (3.73) have been verified in this way in [42, 58].

3.7 Non-perturbative topological strings

Perhaps one of the most compelling consequences of the conjectures in [34] is that it provides a completely well-defined non-perturbative definition of the standard topological string partition function (one should be aware that many of the proposals in the literature for such a non-perturbative completion are not even well-defined as functions, and/or they do not reproduce manifestly the perturbation series that one is supposed to complete).

Again, we face here the problem of how to reconstruct a continuous, geometric world from the discrete world of quantum-mechanical spectra. We have seen that the spectral determinant provides a bridge between these two worlds. Let us then consider the following 't Hooft like double-scaling limit:

$$\hbar\to\infty, \qquad \mu\to\infty, \qquad \frac{\mu}{\hbar}=\zeta \text{ fixed.} \tag{3.74}$$

For simplicity, we will also assume that the mass parameters ξ_j scale in such a way that

$$m_j=\xi_j^{2\pi/\hbar}, \qquad j=1,\cdots,s-1, \tag{3.75}$$

are fixed (although other limits are possible, see [59] for a detailed discussion). It is easy to see that, in this limit, the quantum mirror map becomes trivial. The grand potential entering into the spectral determinant has then the asymptotic expansion,

$$\mathsf{J}_S^{\text{'t Hooft}}(\zeta,\boldsymbol{m},\hbar)\sim\sum_{g=0}^{\infty}\mathsf{J}_g^S(\zeta,\boldsymbol{m})\,\hbar^{2-2g}, \tag{3.76}$$

where

$$\begin{aligned}
\mathsf{J}_0^S(\zeta, \boldsymbol{m}) &= \frac{1}{16\pi^4}\left(\widehat{F}_0(\boldsymbol{t}) + 4\pi^2 \sum_{i=1}^{s} b_i^{\rm NS} t_i + 14\pi^4 A_0(\boldsymbol{m})\right), \\
\mathsf{J}_1^S(\zeta, \boldsymbol{m}) &= A_1(\boldsymbol{m}) + \widehat{F}_1(\boldsymbol{t}), \\
\mathsf{J}_g^S(\zeta, \boldsymbol{m}) &= A_g(\boldsymbol{m}) + (4\pi^2)^{2g-2}\left(\widehat{F}_g(\boldsymbol{t}) - C_g\right), \qquad g \geq 2.
\end{aligned} \tag{3.77}$$

In these equations,

$$t_i = 2\pi c_i \zeta - \sum_{j=1}^{s-1} \alpha_{ij} \log m_j, \tag{3.78}$$

and $\widehat{F}_g(\boldsymbol{t})$ are the standard topological string free energies as a function of the standard Kähler parameters $\boldsymbol{t}$, after turning on the B-field as in (3.59). We have also assumed that the function $A(\boldsymbol{\xi}, \hbar)$ has the expansion

$$A(\boldsymbol{\xi}, \hbar) = \sum_{g=0}^{\infty} A_g(\boldsymbol{m}) \hbar^{2-2g}, \tag{3.79}$$

and this assumption can be tested in examples. We conclude that, in this scaling limit, the grand potential has an asymptotic expansion given by the topological string free energies in the large radius frame. It follows from (3.64) that the logarithm of the spectral determinant has an asymptotic expansion given also by the topological strings, and in addition "modulated" by the contribution of the quantum theta function. A similar type of asymptotics, involving generalized theta functions, has already appeared in the theory of random matrices [60].

This behavior has also another implication, in which the "continuous" physics of the topological string appears through a large N limit of a discretized quantity, as it happens in large N dualities. Indeed, let us consider the corresponding 't Hooft limit of the fermionic spectral traces,

$$\hbar \to \infty, \qquad N \to \infty, \qquad \frac{N}{\hbar} = \lambda \ \text{fixed}. \tag{3.80}$$

Then, it follows from (3.65) that one has the following asymptotic expansion,

$$\log Z_S(N, \hbar) = \sum_{g \geq 0} \mathcal{F}_g(\lambda) \hbar^{2-2g}, \tag{3.81}$$

where $\mathcal{F}_g(\lambda)$ can be obtained by evaluating the integral in the right hand side of (3.65) in the saddle-point approximation and using just the 't Hooft expansion of $\mathsf{J}_S(\mu, \boldsymbol{\xi}, \hbar)$. At leading order in $\hbar$, this is just a Legendre transform, and one finds

$$\mathcal{F}_0(\lambda) = \mathsf{J}_0^S(\zeta, \boldsymbol{m}) - \lambda\zeta, \tag{3.82}$$

evaluated at the saddle-point given by

$$\lambda = \frac{\partial \mathsf{J}_0^S}{\partial \zeta}. \tag{3.83}$$

The higher order corrections can be computed systematically. In fact, the formalism of [9] gives a nice geometric description of the integral transform (3.65) in the saddle-point approximation: the functions $\mathcal{F}_g(\lambda)$ are simply the free energies of the topological string on X, but on a different frame. This frame corresponds to the so-called *conifold* point of the geometry. In particular, λ turns out to be a vanishing period at the conifold point.

Therefore, the conjecture (3.63), in its form (3.65), provides a precise prediction for the 't Hooft limit of the fermionic spectral traces: they are encoded in the standard topological string free energy, but evaluated at the conifold frame. It turns out that this prediction can be tested in detail. The reason is that, at least in some cases, the fermionic spectral traces can be expressed as random matrix integrals, and their 't Hooft limit can be studied by using various techniques developed for matrix models at large N. This solves in fact the long-standing problem of finding matrix model representations for topological strings on toric CY manifolds. For example, for local $\mathbb{P}^2$ and other geometries, one finds a matrix model of the form [59],

$$Z(N,\hbar) = \frac{1}{N!}\int_{\mathbb{R}^N} \frac{\mathrm{d}^N u}{(2\pi)^N} \frac{\prod_{i<j} 4\sinh\left(\frac{u_i-u_j}{2}\right)^2}{\prod_{i,j} 2\cosh\left(\frac{u_i-u_j}{2}+\mathrm{i}\pi C\right)} \prod_{i=1}^{N} \mathrm{e}^{-V(u_i,\hbar)}, \tag{3.84}$$

where the potential function $V(u,\hbar)$ is explicitly known in many examples and it can be expressed in terms of Faddeev's quantum dilogarithm. (3.84) can be regarded as a generalization of the $O(2)$ matrix model [61–63].

One finds in this way a description of the all-genus topological string free energy of local $\mathbb{P}^2$ and other geometries as the asymptotic 't Hooft expansion of matrix integral of the form (3.84). Detailed calculations show that the expansion obtained in this way agrees exactly with the predictions of (3.65). In fact, the conjecture "explains" many aspects of topological string theory at the conifold point, for many toric CY geometries. For example, it has been known for some time that the leading terms in the expansion of the conifold free energies are universal [64], and that they agree with the all-genus free energy of the Gaussian matrix model. Since matrix models like (3.84) are deformations of the Gaussian one, this observation is a consequence of our conjecture (or, equivalently, this observation must hold in order for the conjecture to be true).

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