SISSA ISAS

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Spring 2002 – Entrance Examination: Statistical and Biological Physics

Solve **one** of the following problems. Write out solutions clearly and concisely. State each approximation used. Diagrams welcome. Number page, problem, and question clearly. Do not write your name on the problem sheet, but use extra envelope.

Problem 1

A classical harmonic oscillator of mass m and spring constant k is known to have a total energy E, but its starting time is completely unknown.

1. Find the probability distribution density $p(x)$, where $p(x)dx$ is the probability to find the mass in the interval $[x, x + dx)$.
2. Calculate $p(x)$ in the ground state of the quantum case.
3. Compare and discuss cases 1 and 2.

Problem 2

Consider a chain of N particles with spin 1/2 whose Hamiltonian is

$$\mathcal{H} = -K \sum_{i=1}^{N} \sigma_i^z \, \sigma_{i+1}^z - \Gamma \sum_{i=1}^{N} \sigma_i^x \tag{1}$$

where σ_i^α ($\alpha = x,\, y,\, z$) are the Pauli matrices and K and Γ are non-negative constants.

1. For the case $\Gamma = 0$ calculate the free energy per site, f,

$$f = -\frac{1}{N\,\beta} \ln \mathrm{Tr}\; e^{-\beta\,\mathcal{H}} \tag{2}$$

   the average energy and the specific heat.

2. For the case $\Gamma \neq 0$ estimate the ground state energy using a trial wave-function

$$|0\rangle = \otimes_i \; |m\rangle_i \tag{3}$$

   where

$$|m\rangle_i = \sqrt{\frac{1+m}{2}}\; |+\rangle_i + \sqrt{\frac{1-m}{2}}\; |-\rangle_i \qquad\qquad |m| \leq 1 \;, \tag{4}$$

   and $|\pm\rangle_i$ are the eigenstates of σ_i^z.

3. What is the physical meaning of the parameter m ? What is the optimal choice of the parameter m and how does it depend on K and Γ?

4. Compare the two cases, 1 and 2.

Definition of Pauli matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5}$$