Geometric derivation of the TKNN equations

Gianluca Panati

Dipartimento di Matematica
Università di Roma " La Sapienza ", Roma

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based on joint work with:
G. De Nittis *(Paris 13)* & F. Faure *(Institut Fourier, Grenoble)*

The team

Giuseppe De Nittis
Institut Galilée
Université Paris 13
Paris, France

Frédéric Faure
Insitute Fourier
Université Joseph Fourier
Grenoble, France

In a milestone paper [TKNN82] Thouless et al. contributed to the understanding of the Quantum Hall effect (QHE) with two important discoveries.

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Quantized Hall Conductance in a Two-Dimensional Periodic Potential

D. J. Thouless, M. Kohmoto,[a] M. P. Nightingale, and M. den Nijs
Department of Physics, University of Washington, Seattle, Washington 98195
(Received 30 April 1982)

The Hall conductance of a two-dimensional electron gas has been studied in a uniform magnetic field and a periodic substrate potential U. The Kubo formula is written in a form that makes apparent the quantization when the Fermi energy lies in a gap. Explicit expressions have been obtained for the Hall conductance for both large and small $U/\hbar\omega_c$.

PACS numbers: 72.15.Gd, 72.20. Mg, 73.90.+b

1. for rational magnetic flux the Hall conductance, defined as a bulk observable by linear response theory (Kubo formula), has a topological meaning, which explains its integer value. This topological structure emerges since the Hamiltonian operator commutes with the magnetic translations.
2. the integers corresponding to the Hall conductance, in the limit of weak and strong magnetic field respectively, satisfy a Diophantine equation, nowadays known as TKNN equation (an equation for each spectral gap).

1. The first intuition has been later dramatically pushed forward [Connes][Bellissard][Bellissard van Elst Schulz-Baldes], by realizing that the notion of topological invariant can be extended to the case of irrational flux, or of a random perturbation, by replacing the algebra generated by magnetic translations with the rotation C*-algebra, alias the algebra of the non-commutative torus. The relation with alternative approaches to the QHE, based either on a quantum pumping mechanism [Laughlin] [Avron Seiler Simon] or on the edge currents [Halperin][Buttiker] [Kellendonc Richter Schulz-Baldes] [Combes Germinet], has been clarified [Schulz-Baldes Kellendonc Richter] [Elgart Graf Schlein].
2. On the contrary, the status and the meaning of the TKNN equations have not been completely clarified. No one of the existing derivations took into account the fact that the integers are Chern numbers of suitable vector bundles over $\mathbb{T}^2$.

Outline

Theoretical model (without randomness)

- Independent electrons, effectively 2-dimensional sample
- periodic potential $V_\Gamma : \mathbb{R}^2 \to \mathbb{R}$, $\Gamma \simeq \mathbb{Z}^2$
- uniform orthogonal magnetic field (symmetric gauge)

The dynamics is described by the Bloch-Landau Hamiltonian

$$H_{\mathrm{BL}} = \underbrace{\tfrac{1}{2}\left(-i\tfrac{\partial}{\partial x} - \tfrac{\varphi_B}{2}\,y\right)^2 + \tfrac{1}{2}\left(-i\tfrac{\partial}{\partial y} + \tfrac{\varphi_B}{2}\,x\right)^2}_{\text{Landau Hamiltonian}} + V_\Gamma(x,y)$$

acting in $L^2(\mathbb{R}^2)$.

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acting in $L^2(\mathbb{R}^2)$.

The competition between the periodic and the magnetic length scales makes a direct study of H_{BL} a formidable task. Thus the need of simpler effective models which capture the physics in suitable asymptotic regimes.

Multiscale structure: separation of scales

A. The **Hofstadter** regime ($\varphi_B \to 0$)

B. The **Landau** regime ($\varphi_B \to \infty$)

The ratio of the **energy scales** considered in [TKNN] is $\hbar\omega_c/|V_\Gamma| \propto \varphi_B$.

The ratio of the typical **time-scales** is

$$\frac{\text{Crystal period}}{\text{Cyclotron period}} = \frac{ma^2/2\pi\hbar}{mc/eB} = \frac{BS_\Gamma}{hc/e} = \frac{\Phi}{\Phi_0} = \varphi_B$$

Here crystal period is estimated as the time needed to travel trough a distance a ($a = \sqrt{S_\Gamma}$ = lattice constant) when the momentum is

$$k = 2\pi\hbar/a$$

Outline

Effective models: Hofstadter regime $\varphi_B \to 0$

In the **Hofstadter regime**, $\varphi_B \to 0$, one considers firstly the unperturbed periodic Hamiltonian $H_{\text{per}} = -\Delta + V_\Gamma$.

The periodicity leads to consider the Bloch-Floquet transform

$$\mathcal{U} : L^2(\mathbb{R}^2) \longrightarrow L^2(\mathbb{B}) \otimes \underbrace{L^2(\mathbb{T}^2_Y)}_{\mathcal{H}_{\text{f}}} \cong L^2(\mathbb{T}^2, \mathcal{H}_{\text{f}})$$

$$(\mathcal{U}\psi)(k, y) = \sum_{\gamma \in \Gamma} \mathrm{e}^{-i(y+\gamma)\cdot k} \; \psi(y + \gamma).$$

Effective models: Hofstadter regime $\varphi_B \to 0$

By using the Bloch-Floquet transform $\mathcal{U}$, one obtains a fibered operator

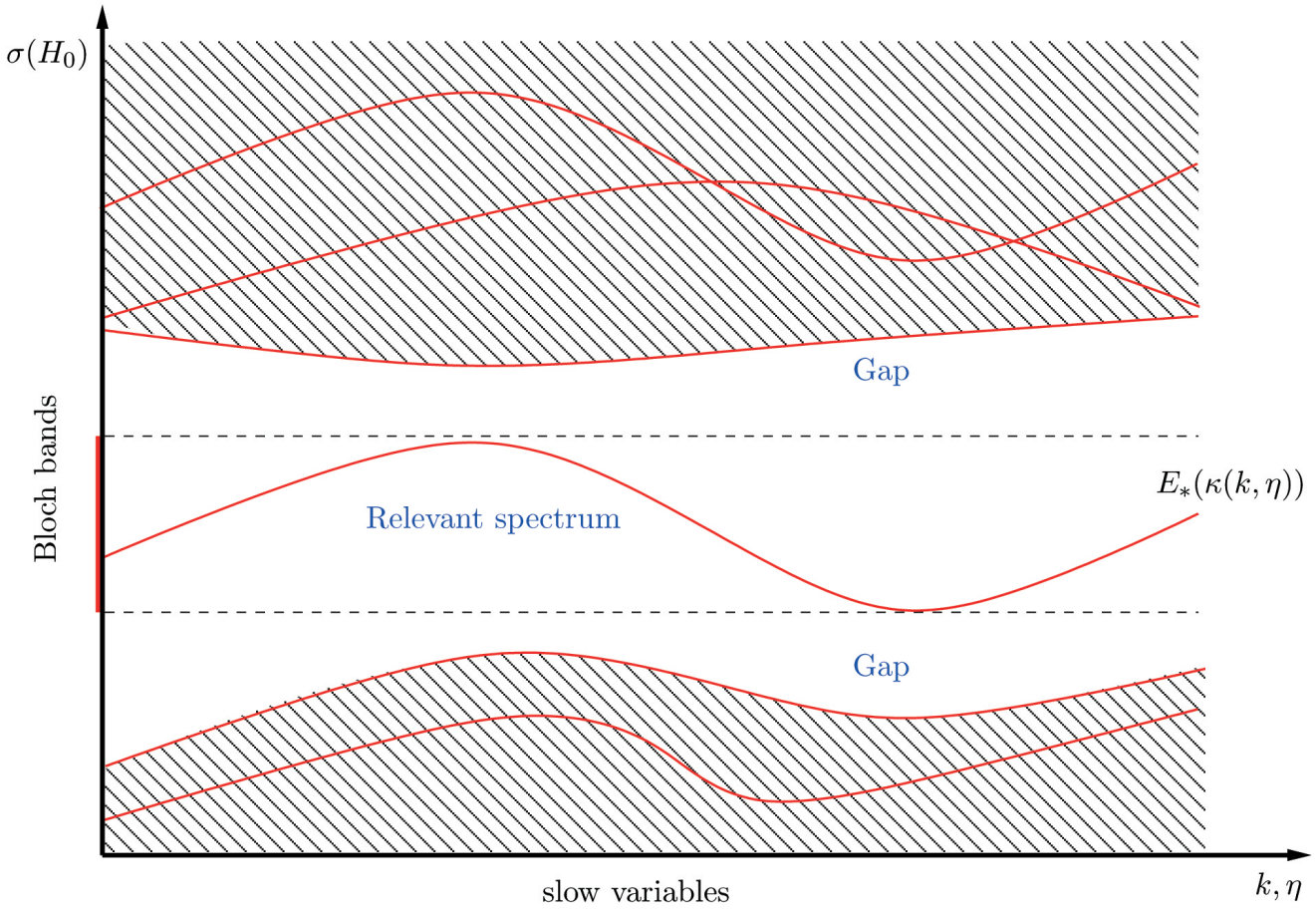
$$\mathcal{U} H_{\text{per}} \mathcal{U}^{-1} = \int_{\mathbb{B}}^{\oplus} H_{\text{per}}(k)\, dk \qquad \text{in } L^2(\mathbb{B}, \mathcal{H}_{\text{f}}),$$

$$H_{\text{per}}(k) = \tfrac{1}{2}(-i\nabla_y + k)^2 + V_\Gamma(y) \qquad \text{acting on } \mathcal{D} \subseteq L^2(\mathbb{T}^d_Y, dy) =: \mathcal{H}_{\text{f}}\,.$$

We assume to know the solution of the eigenvalue problem:

$$H_{\text{per}}(k)\, \chi_n(k,y) = E_n(k)\, \chi_n(k,y)$$

Effective models: Hofstadter regime $\varphi_B \to 0$



Effective models: Hofstadter regime $\varphi_B \to 0$

To any isolated Bloch band $E_*(k_1, k_2) := \sum h_{n,m}\, \mathrm{e}^{i(nk_1+mk_2)}$ corresponds an effective Hamiltonian acting in $L^2(\mathbb{T}^2, dk_1 dk_2)$ as (formally)

$$E_*(K_1, K_2) \qquad \text{with} \begin{cases} K_1 = k_1 + \frac{i}{2}\pi\varphi_B \frac{\partial}{\partial k_2} \\ \\ K_2 = k_2 - \frac{i}{2}\pi\varphi_B \frac{\partial}{\partial k_1} \end{cases} \qquad C^\infty(\mathbb{T}^2) \subset L^2(\mathbb{T}^2)$$

$$[K_1, K_2] = i\pi\varphi_B \mathbb{1}$$

This procedure corresponds to Peierl's substitution [Peierls 33] and it has been mathematically established in [Bellissard 88] [Hellfer Sjöstrand 89] [Nenciu91] [P Spohn Teufel 03] [DeNittis P 10].

Effective models: Hofstadter regime $\varphi_B \to 0$

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$$\begin{aligned} E_*(K_1, K_2) &= \sum h_{n,m}\, \mathrm{e}^{i(nK_1+mK_2)} \\ &= \sum h_{n,m}\, \mathrm{e}^{i\pi nm\varphi_B} {U_0}^m {V_0}^n \end{aligned}$$

$$U_0 = \mathrm{e}^{iK_2}, \qquad V_0 = \mathrm{e}^{iK_1}, \qquad U_0 V_0 = \mathrm{e}^{i2\pi\varphi_B} V_0 U_0$$

Hofstadter model: $E_\star(k_1, k_2) = 2[\cos(k_1) + \cos(k_2)]$.

$$\hat{h}_\star = U_0 + {U_0}^{-1} + V_0 + {V_0}^{-1} =: H_{\mathrm{Hof}}(\varphi_B).$$

Outline

Effective models: Landau regime $\varphi_B \to +\infty$

In the **Landau regime**, $\varphi_B \to \infty$, one considers a perturbation of the Landau Hamiltonian

$$H_{\mathrm{L}} = \tfrac{1}{2}\left[\left(-i\frac{\partial}{\partial x} - \frac{\varphi_B}{2}y\right)^2 + \left(-i\frac{\partial}{\partial y} + \frac{\varphi_B}{2}x\right)^2\right] = \tfrac{1}{2}\left[\tilde{K}_1^2 + \tilde{K}_2^2\right].$$

For $V_\Gamma = 0$ the Hamiltonian is quadratic (Landau levels).
The operators G_1 and G_2, corresponding to the classical coordinates of the center of the cyclotron orbit, commute with the Hamiltonian.

Classical Mechanics	Quantum Mechanics
$\bar{x}_1 = x_1(t) + \dfrac{1}{\varphi_B}v_2(t)$	$G_1 = X_1 + \dfrac{1}{\varphi_B}\tilde{K}_2 = \dfrac{1}{\varphi_B}\left(P_2 + \dfrac{\varphi_B}{2}X_1\right)$
$\bar{x}_2 = x_2(t) - \dfrac{1}{\varphi_B}v_1(t)$	$G_2 = X_2 - \dfrac{1}{\varphi_B}\tilde{K}_1 = \dfrac{1}{\varphi_B}\left(-P_1 + \dfrac{\varphi_B}{2}X_2\right)$

$$[\tilde{K}_1, \tilde{K}_2] = i\varphi_B \mathbb{1} \qquad [G_j, \tilde{K}_i] = 0 \qquad [G_1, G_2] = i\frac{1}{\varphi_B}\mathbb{1}$$

Effective models: Landau regime $\varphi_B \to +\infty$

For $V_\Gamma \neq 0$ to each Landau level corresponds an effective Hamiltonian acting in $L^2(\mathbb{R}, ds)$.

In the representation which diagonalizes G_1 the effective Hamiltonian is given (assuming $V_\Gamma \in C^\infty$) by

$$V_\Gamma(G_1, G_2) \qquad \text{with} \begin{cases} G_1 = s \\ G_2 = -i\frac{1}{\varphi_B}\frac{d}{ds} \end{cases}$$

Here $V_\Gamma(G_1, G_2)$ can be interpreted by means of ordinary ϵ-Weyl quantization, with $\epsilon = \epsilon_\infty = \frac{1}{\varphi_B}$. Since the choice

$$V_\Gamma(p, q) = 2\cos(2\pi p) + 2\cos(2\pi q)$$

yields the Harper operator, we say that $V_\Gamma(G_1, G_2)$ is a Harper-like Hamiltonian.

Effective models: Landau regime $\varphi_B \to +\infty$

For $V_\Gamma \neq 0$ to each Landau level corresponds an effective Hamiltonian acting in $L^2(\mathbb{R}, ds)$.

Explicitly, for $V_\Gamma(q,p) = \sum_{n,m\in\mathbb{Z}} v_{n,m}\, \mathrm{e}^{i2\pi(np+mq)}$,

$$V_\Gamma(G_1,G_2) = \sum v_{n,m}\, \mathrm{e}^{i2\pi(nG_1+mG_2)} = \sum v_{n,m}\, \mathrm{e}^{i\pi nm\varepsilon_\infty} U_\infty{}^m V_\infty{}^{-n}$$

$$U_\infty := \mathrm{e}^{i2\pi G_1}, \quad V_\infty := \mathrm{e}^{-i2\pi G_2}, \quad U_\infty V_\infty := \mathrm{e}^{i2\pi\varepsilon_\infty} V_\infty U_\infty.$$

The validity of V_Γ as effective Hamiltonian for $\varphi_B \to \infty$ is proved in [Bellissard 88] [Hellfer Sjöstrand 89] [DeNittis P 10].

Outline

Conductance in the effective models

How can the value of the **conductance** be recovered? The key idea is that it corresponds to a TQN related to the symmetries of the system [TKNN].

Each of the models has a $\mathbb{Z}^2$-symmetry, provided $\varphi_B = \frac{M}{N} \in \mathbb{Q}$.
For any $n = (n_1, n_2) \in \mathbb{Z}^2$ the symmetry is provided by

$L^2(\mathbb{R}^2)$	H_{BL}	magnetic translations T_1^N, T_2 ($\varphi_B \in \mathbb{Q}$)
$L^2(\mathbb{T}^2) \cong \ell^2(\mathbb{Z}^2)$	$E_*(K_1, K_2)$	$(T_1^N \psi)_{n,m} = \mathrm{e}^{i\pi m N \varphi_B}\, \psi_{n-1,m}$ $(T_2 \psi)_{n,m} = \mathrm{e}^{i\pi n \varphi_B}\, \psi_{n,m+1}$
$L^2(\mathbb{R})$	$V_\Gamma(G_1, G_2)$	$(T_1^M \psi)(x) = \psi(x - M)$ $(T_2 \psi)(x) = \mathrm{e}^{i \frac{2\pi}{\varphi_B} x} \psi(x)$

One can use a **generalized Bloch-Floquet transform** to decompose the Hilbert space with respect to the $\mathbb{Z}^2$-symmetry.

The generalized **Bloch-Floquet transform** is defined as

$$\mathcal{U} : \mathcal{H} \longrightarrow \int_{\mathbb{T}^2}^{\oplus} \mathcal{H}(k)\, dk$$

$$(\mathcal{U}\psi)(k) := \sum_{n\in\mathbb{Z}^2} \mathrm{e}^{-in\cdot k}\, \widetilde{T}_1{}^{n_1}\, \widetilde{T}_2{}^{n_2}\psi, \quad k \in \mathbb{T}^2.$$

where $\widetilde{T}_1, \widetilde{T}_2 \in \mathcal{B}(\mathcal{H})$ are the generators of the $\mathbb{Z}^2$-symmetry mentioned above. Hereafter we assume the rationality condition $\varphi_B = \frac{M}{N}$.

The relevant Hamiltonian H (resp. H_{BL}, H_{Hof} or H_{Har}), after BF transform, becomes a fibered operator i.e.

$$\mathcal{U}\, H\, \mathcal{U}^{-1} = \int_{\mathbb{T}^2}^{\oplus} H(k)\, dk \qquad \text{in} \quad \int_{\mathbb{T}^2}^{\oplus} \mathcal{H}(k)\, dk.$$

The direct integral defines a relevant collection of vector spaces.
Fix $\mu \in \mathbb{R}$ (**chemical potential**) so that $\mu \notin \operatorname{Spec} H(k)$ for all $k \in \mathbb{T}^2$ (physical condition). Denote by

$$P_\mu(k) := \chi_{(-\infty,\mu]}\left(H(k)\right)$$

the spectral projector up to the energy μ and pose

$$F(k) := \operatorname{Ran} P_\mu(k) \subset \mathcal{H}(k)$$

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The collection of such vector spaces is a vector bundle over $\mathbb{T}^2$. The Chern number of this vector bundle is the Topological Quantum Number we are interested in.

Outline

Color-coded quantum butterflies

$$E_*(k_1, k_2) = 2(\cos k_1 + \cos k_2)$$

Hofstadter Hamiltonian

$$V_\Gamma(p, x) = 2(\cos(2\pi p) + \cos(2\pi x))$$

Harper Hamiltonian

Color-coded quantum butterflies

$$\varphi_B = \theta = 1/\varphi'_B$$

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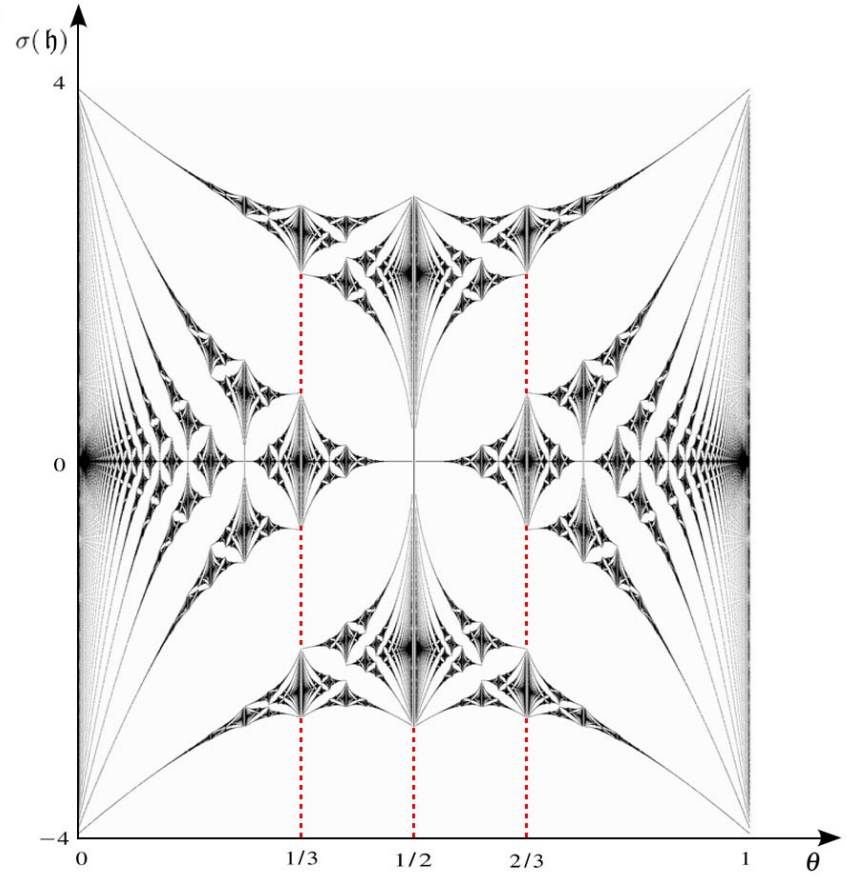

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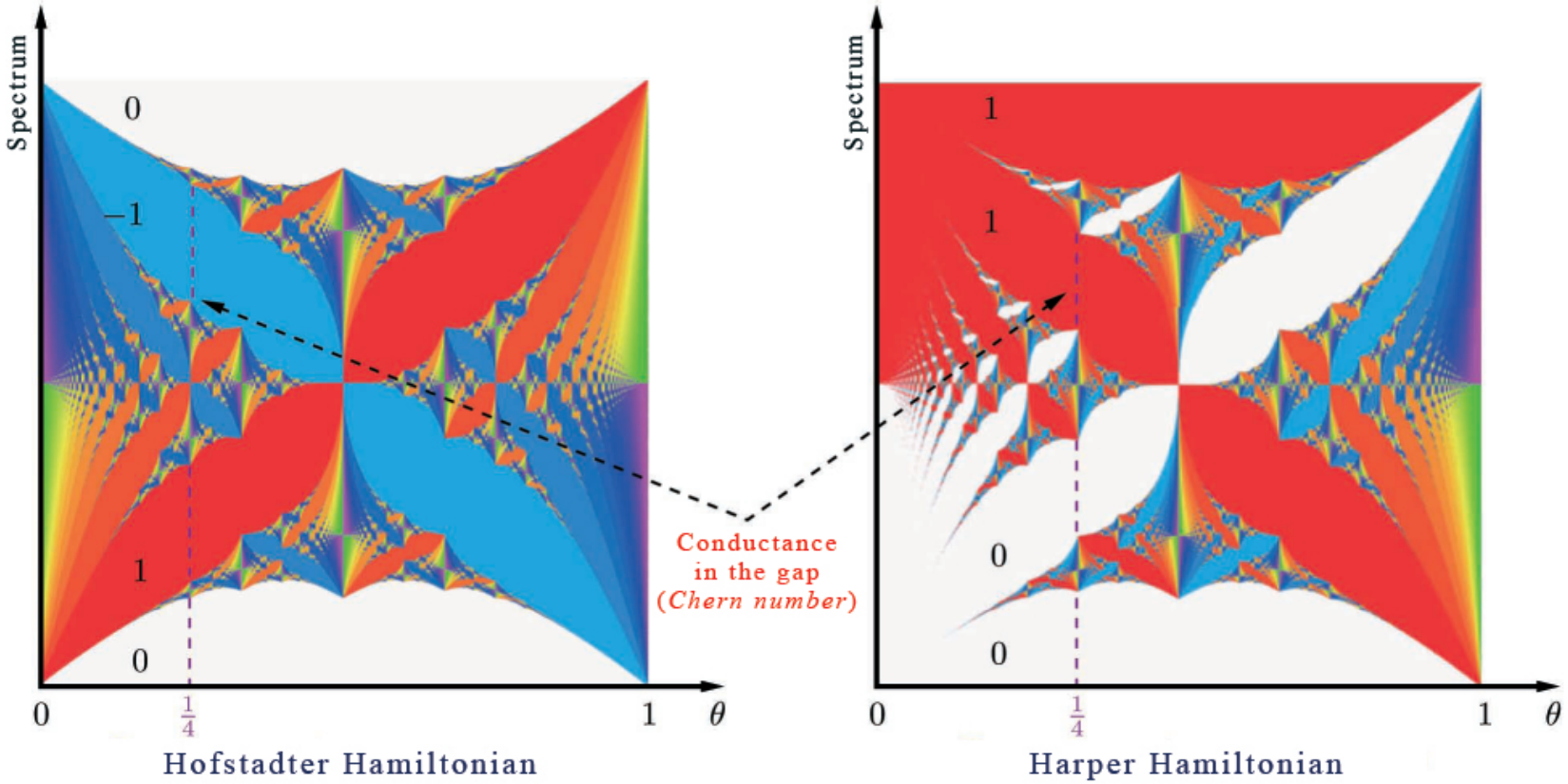

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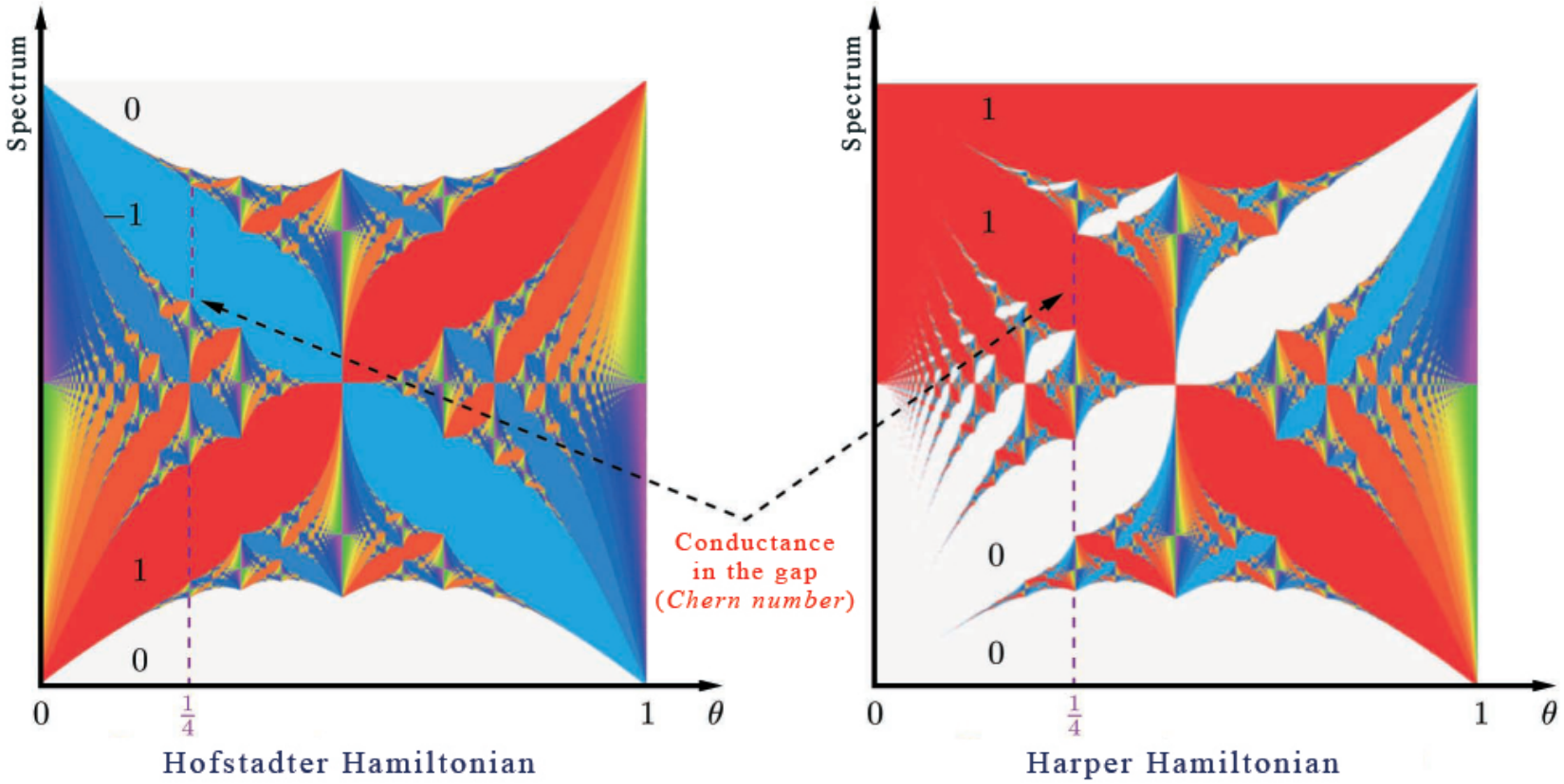

Observation: the colors (Chern numbers) in the limiting models are related by the the TKNN-equation

$$N C_\infty(g) + M C_0(g) = d_g \qquad g \in \{0, 1, \ldots, g_{\max}\} \tag{1}$$

Here the integer d_g is a gap label: it is equal to g for N odd, while for N even is defined by

$$d_g = \begin{cases} g & \text{for } 0 \leq g \leq N/2 - 1 \\ g + 1 & \text{for } N/2 \leq g \leq g_{\max}. \end{cases}$$

Moreover, one has the constraint

$$2|C_0(g)| \leq N.$$

For any $g \in \{0, 1, \ldots, g_{\max}\}$, the TKNN equation is solved by infinite pairs $(t_g, s_g) \in \mathbb{Z}^2$. If the constraint is imposed, the solution is unique (provided M, N are coprime, $M \in \mathbb{Z}$, $N \in \mathbb{N} \setminus \{0\}$).

The twofold way

We obtain the TKNN formula as a corollary of the duality between vector bundles emerging in two inequivalent representations of the same C^*-algebra.

$$
\begin{tikzcd}
 & P_0 \arrow[r, "\mathcal{U}_0 \ldots \mathcal{U}_0^{-1}"] & P_0(\cdot) \arrow[r, "\mathrm{Ran}"] & \mathcal{L}^0_{\mathfrak{p}} \arrow[r, "C"] & C_0(\mathfrak{p}) \\
\mathfrak{p} \in \mathrm{Proj}(\mathfrak{A}_\theta) \arrow[ur, "\Pi_0"] \arrow[dr, "\Pi_\infty"'] & \mathscr{B}(\mathcal{H}) & \Gamma(\mathrm{End}(\mathcal{E})) & \mathcal{E} \to \mathbb{T}^2 & \mathbb{Z} \\
 & P_\infty \arrow[r, "\mathcal{U}_\infty \ldots \mathcal{U}_\infty^{-1}"'] & P_\infty(\cdot) \arrow[r, "\mathrm{Ran}"'] & \mathcal{L}^\infty_{\mathfrak{p}} \arrow[r, "C"'] & C_\infty(\mathfrak{p})
\end{tikzcd}
$$

$C :=$ first Chern number.

What is the relation between $C_0(\mathfrak{p})$ and $C_\infty(\mathfrak{p})$ for a given $\mathfrak{p} \in \mathrm{Proj}(\mathfrak{A}_\theta)$?

Hilbert space: $\mathcal{H}_0 := L^2(\mathbb{T}^2, dk_1\, dk_2)$.

Hofstadter unitaries:

$$U_0 = \mathrm{e}^{iK_2}, \quad V_0 = \mathrm{e}^{iK_1}, \quad U_0 V_0 = \mathrm{e}^{i2\pi\varepsilon_0} V_0 U_0.$$

where

$$K_1 = k_1 + i\pi\varepsilon_0 \frac{\partial}{\partial k_2}, \quad K_2 = k_2 - i\pi\varepsilon_0 \frac{\partial}{\partial k_1}.$$

Hofstadter-like Hamiltonian:

$$h(k_1, k_2) = \sum h_{n,m}\, \mathrm{e}^{i(nk_1 + mk_2)} \qquad \in C^1(\mathbb{T}^2)$$

$\downarrow$ (Peierls substitution)

$$\hat{h}(\varepsilon_0) := \sum h_{n,m}\, \mathrm{e}^{i(nK_1 + mK_2)} = \sum h_{n,m}\, \mathrm{e}^{i\pi nm\varepsilon_0} {U_0}^m {V_0}^n \qquad \in \mathscr{B}(\mathcal{H}_0)$$

Hofstadter model: $h_\star(k_1, k_2) = 2\cos(k_1) + 2\cos(k_2)$.

$$\hat{h}_\star = U_0 + {U_0}^{-1} + V_0 + {V_0}^{-1} =: H_{\mathrm{Hof}}(\varepsilon_0).$$

Hilbert space: $\mathcal{H}_\infty := L^2(\mathbb{R}, dq)$.

Harper unitaries:

$$U_\infty := \mathrm{e}^{i2\pi Q}, \quad V_\infty := \mathrm{e}^{-i2\pi P}, \quad U_\infty V_\infty = \mathrm{e}^{i2\pi\varepsilon_\infty} V_\infty U_\infty.$$

where

$$Q = \text{multiplication by } q, \qquad P = -i\frac{\varepsilon_\infty}{2\pi}\frac{\partial}{\partial q}.$$

Harper-like Hamiltonian:

$$h(p,q) = \sum h_{n,m}\, \mathrm{e}^{i2\pi(np+mq)} \qquad \in C^1_{\mathbb{Z}^2}(\mathbb{R}^2)$$

$\downarrow$ (Weyl quantization)

$$\hat{h}(\varepsilon_\infty) := \sum h_{n,m}\, \mathrm{e}^{i2\pi(nP+mQ)} = \sum h_{n,m}\, \mathrm{e}^{i\pi nm\varepsilon_\infty} {U_\infty}^m {V_\infty}^{-n} \quad \in \mathscr{B}(\mathcal{H}_\infty)$$

Harper model: $h_\star(p,q) = 2\cos(2\pi p) + 2\cos(2\pi q)$.

$$\hat{h_\star} = U_\infty + {U_\infty}^{-1} + V_\infty + {V_\infty}^{-1} =: H_{\mathrm{Har}}(\varepsilon_\infty).$$

Outline

Non-Commutative Torus (NCT) "abstract" C^*-algebra $\mathfrak{A}_\theta$ ($\theta \in \mathbb{R}$ *deformation parameter*) generated by:

$$\mathfrak{u}^* = \mathfrak{u}^{-1}, \qquad \mathfrak{v}^* = \mathfrak{v}^{-1}, \qquad \mathfrak{u}\mathfrak{v} = \mathrm{e}^{i2\pi\theta}\mathfrak{v}\mathfrak{u}$$

with universal norm $\|\mathfrak{a}\| := \sup\{\|\pi(\mathfrak{a})\|_{\mathcal{H}} \mid \pi \text{ rep. of } \mathfrak{A}_\theta \text{ on } \mathcal{H}\}$.

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Hofstadter representation:

$$\Pi_0 : \mathfrak{A}_{\varepsilon_0} \to \mathscr{B}(\mathcal{H}_0), \quad \Pi_0(\mathfrak{u}) := U_0, \quad \Pi_0(\mathfrak{v}) := V_0.$$

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Harper representation:

$$\Pi_\infty : \mathfrak{A}_{\varepsilon_\infty} \to \mathscr{B}(\mathcal{H}_\infty), \quad \Pi_\infty(\mathfrak{u}) := U_\infty, \quad \Pi_\infty(\mathfrak{v}) := V_\infty.$$

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PROPOSITION (isospectrality)

Π_0 and Π_∞ are injective. Let $\varepsilon_0 = \varepsilon_\infty =: \theta$ (duality condition). Then for all $\mathfrak{a} \in \mathfrak{A}_\theta$

$$\sigma(\Pi_0(\mathfrak{a})) = \sigma(\mathfrak{a}) = \sigma(\Pi_\infty(\mathfrak{a})).$$

Let $\mathfrak{h} := \mathfrak{u} + \mathfrak{u}^{-1} + \mathfrak{v} + \mathfrak{v}^{-1} \in \mathfrak{A}_\theta$ and $\theta \in \mathbb{Q}$, [Hof76]:

$\varepsilon_0(B_1) = \varepsilon_\infty(B_2)$ implies $B_1 B_2 \propto 1$ with $B_1 \ll 1 \ll B_2$ (duality)

$$\sigma(H_{\mathrm{Hof}}) = \sigma(\mathfrak{h}) = \sigma(H_{\mathrm{Har}})$$

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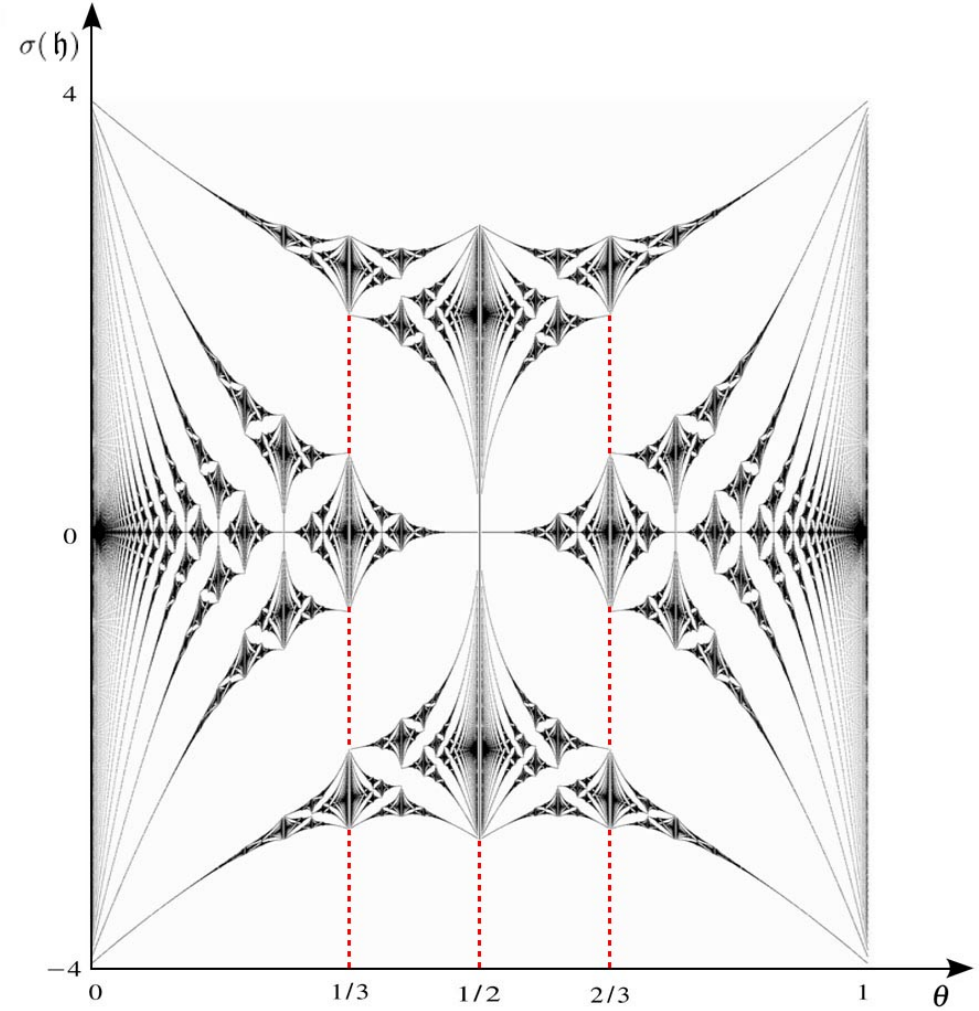

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Outline

Smooth algebra:

$$\mathfrak{A}_\theta^\infty := \left\{ \sum a_{n,m}\, \mathfrak{u}^n \mathfrak{v}^m \mid \{a_{n,m}\} \in S(\mathbb{Z}^2) \right\} \subset \mathfrak{A}_\theta$$

It is a Fréchet algebra.

Canonical derivations (differentiable structure):

$$\mathfrak{d}_j : \mathfrak{A}_\theta^\infty \to \mathfrak{A}_\theta^\infty \qquad j = 1, 2$$

$$\mathfrak{d}_1(\mathfrak{u}^n \mathfrak{v}^m) = i2\pi n(\mathfrak{u}^n \mathfrak{v}^m), \qquad \mathfrak{d}_2(\mathfrak{u}^n \mathfrak{v}^m) = i2\pi m(\mathfrak{u}^n \mathfrak{v}^m).$$

Symmetric ($\mathfrak{d}_j(\mathfrak{a}^*) = \mathfrak{d}_j(\mathfrak{a})^*$), commuting ($\mathfrak{d}_1 \circ \mathfrak{d}_2 = \mathfrak{d}_2 \circ \mathfrak{d}_1$), trace-compatibility ($\mathfrak{f} \circ \mathfrak{d}_j = 0$).

Canonical trace (measure structure):

$$\mathfrak{f} : \mathfrak{A}_\theta \to \mathbb{C} \qquad \mathfrak{f}(\mathfrak{u}^n \mathfrak{v}^m) = \delta_{n,0}\delta_{m,0}.$$

state (linear, positive, normalized), faithful ($\mathfrak{f}(\mathfrak{a}^*\mathfrak{a}) = 0 \Leftrightarrow \mathfrak{a} = 0$), tracial property ($\mathfrak{f}(\mathfrak{a}\mathfrak{b}) = \mathfrak{f}(\mathfrak{b}\mathfrak{a})$), unique if $\theta \notin \mathbb{Q}$.

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Projections:

$$\mathrm{Proj}(\mathfrak{A}_\theta) := \{\mathfrak{p} \in \mathfrak{A}_\theta |\ \mathfrak{p} = \mathfrak{p}^* = \mathfrak{p}^2\}.$$

If $\theta = \frac{M}{N}$ (with $M \in \mathbb{Z}$, $N \in \mathbb{N}^*$ and g.c.d$(M, N) = 1$), then

$$f\colon \mathrm{Proj}(\mathfrak{A}_{M/N}) \to \left\{0, \frac{1}{N}, \ldots, \frac{N-1}{N}, 1\right\}.$$

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"Abstract" (first) Chern number:

$$\mathfrak{Ch} : \mathrm{Proj}(\mathfrak{A}_\theta) \cap \mathfrak{A}_\theta^\infty \to \mathbb{Z}$$

$$\mathfrak{Ch}(\mathfrak{p}) := \tfrac{i}{2\pi}\, f\left(\mathfrak{p}[\partial_1(\mathfrak{p}); \partial_2(\mathfrak{p})]\right).$$

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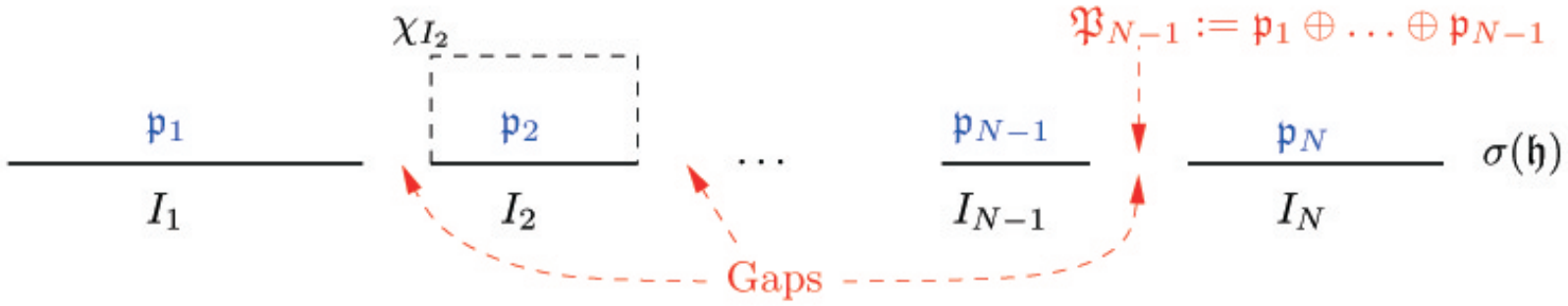
Outline

Band projections and gap band projections

$\mathfrak{h} \in \mathfrak{A}_{M/N}$ selfadjoint ($\mathfrak{h} = \mathfrak{h}^*$). Band spectrum $\sigma(\mathfrak{h}) = \bigcup_{j \in J} I_j$ with $I_j \subset \mathbb{R}$ closed intervals. Gap, open interval between two bands.



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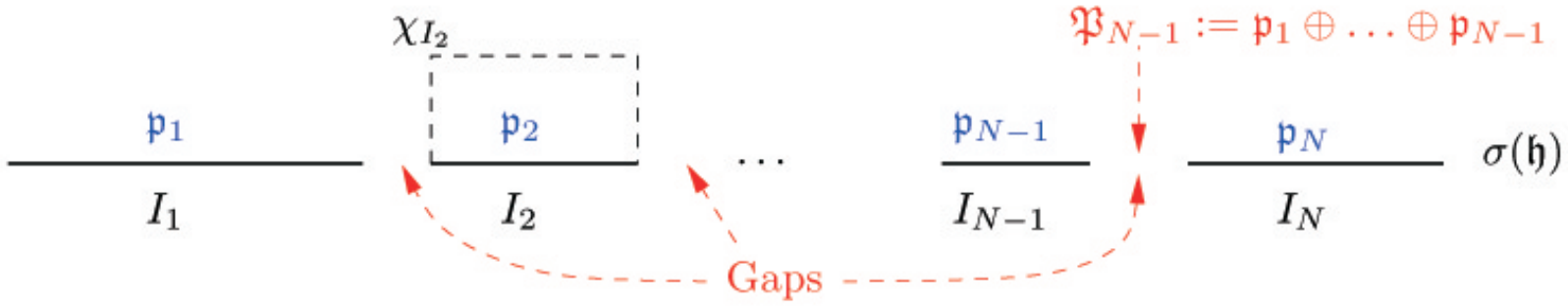

LEMMA (band projection)

Any band $I_j \subset \sigma(\mathfrak{h})$ defines a $\mathfrak{p}_j \in Proj(\mathfrak{A}_\theta) \cap \mathfrak{A}_\theta^\infty$.

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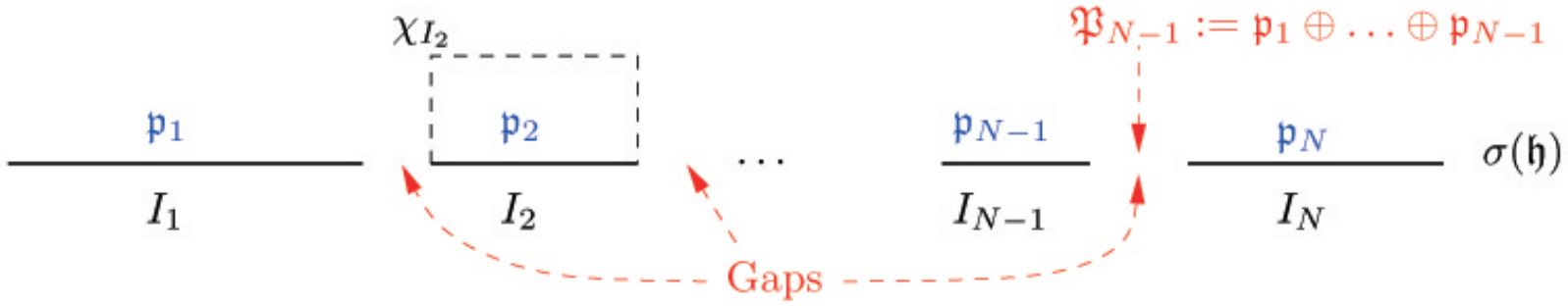

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Any band $I_j \subset \sigma(\mathfrak{h})$ defines a $\mathfrak{p}_j \in Proj(\mathfrak{A}_\theta) \cap \mathfrak{A}_\theta^\infty$.

Gap projection $\mathfrak{P}_j := \bigoplus_{k=1}^{j} \mathfrak{p}_k$. If $\theta = M/N$ then $1 \leqslant j \leqslant N$.

Outline

Let $\Gamma(\mathscr{E})$ be the space of continuous sections of the Hermitian vector bundle $\mathscr{E} \to \mathbb{T}^2$. Any $A \in \Gamma(\mathrm{End}(\mathscr{E}))$ defines a multiplication operator $\widetilde{A}$ acting on $s \in L^2(\mathscr{E})$ by $(\widetilde{A}s)_z = A_z s_z$, thus $\Gamma(\mathrm{End}(\mathscr{E}))$ can be regarded as a subalgebra of $\mathscr{B}(L^2(\mathscr{E}))$.

DEFINITION (Bundle decomposition)

Let $\mathfrak{A} \subset \mathscr{B}(\mathcal{H})$ be a C^*-algebra. We say that $\mathfrak{A}$ admits a bundle decomposition over $\mathbb{T}^2$ if there exist a Hermitian vector bundle $\mathscr{E} \to \mathbb{T}^2$ and a unitary map $\mathcal{F} : \mathcal{H} \to L^2(\mathscr{E})$ such that $\mathcal{F}\,\mathfrak{A}\,\mathcal{F}^{-1} \subset \Gamma(\mathrm{End}(\mathscr{E}))$.

LEMMA

Let $\mathfrak{A} \subset \mathscr{B}(\mathcal{H})$ be a C^*-algebra admitting a bundle decomposition over $\mathbb{T}^2$ with respect to the vector bundle $\iota : \mathscr{E} \to \mathbb{T}^2$. Then any projection $\mathfrak{p} \in \mathfrak{A}$ defines a vector subbundle $\mathscr{L}(\mathfrak{p}) \subset \mathscr{E}$.

THEOREM (Bundle decomposition in Hofstadter and Harper representations)

Let $\mathfrak{A}_{M/N}$ be the the rational NCT-algebra, $\pi_0 : \mathfrak{A}_{M/N} \to \mathscr{B}(\mathcal{H}_0)$ the Hofstadter representation and $\pi_\infty : \mathfrak{A}_{M/N} \to \mathscr{B}(\mathcal{H}_\infty)$ the Harper representation. Then:

(i) *The operator algebra $\pi_0(\mathfrak{A}_{M/N}) \subset \mathscr{B}(\mathcal{H}_0)$ admits a bundle decomposition over $\mathbb{T}^2$ with respect to a (rank N) Hermitian vector bundle $\mathscr{E}_0 \to \mathbb{T}^2$ and a unitary operator $\mathcal{F}_0 : \mathcal{H}_0 \to L^2(\mathscr{E}_0)$. The vector bundle $\mathscr{E}_0$ is trivial. The bundle representation $\mathfrak{a} \mapsto \mathcal{F}_0\, \pi_0(\mathfrak{a})\, {\mathcal{F}_0}^{-1}$ is generated by the endomorphism sections $\mathcal{U}_0(\cdot) := \mathcal{F}_0\, \pi_0(\mathfrak{u})\, {\mathcal{F}_0}^{-1}$ and $\mathcal{V}_0(\cdot) := \mathcal{F}_0\, \pi_0(\mathfrak{v})\, {\mathcal{F}_0}^{-1}$, which are explicitly given in the proof.*

(ii) *The operator algebra $\pi_\infty(\mathfrak{A}_{M/N}) \subset \mathscr{B}(\mathcal{H}_\infty)$ admits a bundle decomposition over $\mathbb{T}^2$ with respect to a (rank N) Hermitian vector bundle $\mathscr{E}_\infty \to \mathbb{T}^2$ and unitary transform $\mathcal{F}_\infty : \mathcal{H}_\infty \to L^2(\mathscr{E}_\infty)$. The vector bundle $\mathscr{E}_\infty$ is non trivial with (first) Chern number $C_1(\mathscr{E}_\infty) = 1$.*

The twofold way

We obtain the TKNN formula as a corollary of the duality between vector bundles emerging in two inequivalent representations of the same C^*-algebra.

$$
\begin{tikzcd}
 & P_0 \arrow[r, "\mathcal{U}_0 \ldots \mathcal{U}_0^{-1}"] & P_0(\cdot) \arrow[r, "\mathrm{Ran}"] & \mathscr{L}_0(\mathfrak{p}) \arrow[r, "C"] & C_0(\mathfrak{p}) \\
\mathfrak{p} \in \mathrm{Proj}(\mathfrak{A}_\theta) \arrow[ur, "\Pi_0"] \arrow[dr, "\Pi_\infty"'] & \mathscr{B}(\mathcal{H}) & \Gamma(\mathrm{End}(\mathcal{E})) & \mathcal{E} \to \mathbb{T}^2 & \mathbb{Z} \\
 & P_\infty \arrow[r, "\mathcal{U}_\infty \ldots \mathcal{U}_\infty^{-1}"'] & P_\infty(\cdot) \arrow[r, "\mathrm{Ran}"'] & \mathscr{L}_\infty(\mathfrak{p}) \arrow[r, "C"'] & C_\infty(\mathfrak{p})
\end{tikzcd}
$$

$C :=$ first Chern number.

What is the relation between $C_0(\mathfrak{p})$ and $C_\infty(\mathfrak{p})$ for a given $\mathfrak{p} \in \mathrm{Proj}(\mathfrak{A}_\theta)$ for $\theta = M/N$?

Outline

Let $\mathfrak{p} \in \mathrm{Proj}(\mathfrak{A}_{M/N})$ and $r \in \{0, \infty\}$. The orthogonal projector $\pi_r(\mathfrak{p}) \in \mathscr{B}(\mathscr{H}_r)$ defines a vector subbundle $\mathscr{L}(\pi_r(\mathfrak{p}))$ of $\mathscr{E}_r \to \mathbb{T}^2$, which we denote shortly by $\mathscr{L}_r(\mathfrak{p})$. We write $C_r(\mathfrak{p}) := C_1(\mathscr{L}_r(\mathfrak{p}))$.

THEOREM (Geometric duality (De Nittis, Faure, P))

Let $\mathfrak{A}_{M/N}$ be the rational NCT-algebra and $\mathfrak{p} \in \mathrm{Proj}(\mathfrak{A}_{M/N})$. Then:

(i) *The vector bundles $\mathscr{L}_0(\mathfrak{p})$ and $\mathscr{L}_\infty(\mathfrak{p})$ have rank equal to $\mathrm{Rk}(\mathfrak{p})$.*

(ii) *Consider the continuous functions $\mathbb{T}^2 \to \mathbb{T}^2$ defined by*

$$\alpha(z_1, z_2) = (z_1^N, z_2) \qquad \beta(z_1, z_2) = (z_1^{-M}, z_2). \tag{2}$$

Then the following isomorphism of Hermitian vector bundles holds true,

$$\alpha^* \mathscr{L}_\infty(\mathfrak{p}) \simeq \beta^* \mathscr{L}_0(\mathfrak{p}) \otimes \mathscr{I} \tag{3}$$

where $\mathscr{I}$ is the determinant bundle of $\mathscr{E}_\infty$.

Derivation of the TKNN equations

$$\alpha^* \mathcal{L}^\infty_{\mathfrak{p}} \simeq \beta^* \mathcal{L}^0_{\mathfrak{p}} \otimes \mathfrak{I}$$

$\downarrow$ c_1 first Chern class of a product VB

$$c_1(\alpha^* \mathcal{L}^\infty_{\mathfrak{p}}) = c_1(\beta^* \mathcal{L}^0_{\mathfrak{p}} \otimes \mathfrak{I}) = \mathrm{Rk}(\beta^* \mathcal{L}^0_{\mathfrak{p}})\, c_1(\mathfrak{I}) + \mathrm{Rk}(\mathfrak{I})\, c_1(\beta^* \mathcal{L}^0_{\mathfrak{p}})$$

$\downarrow$ functoriality of c_1, $\mathrm{Rk}(\mathfrak{I}) = 1$, $\mathrm{Rk}(\beta^* \mathcal{L}^0_{\mathfrak{p}}) = \mathrm{Rk}(P_0)$

$$\alpha^* c_1(\mathcal{L}^\infty_{\mathfrak{p}}) = \mathrm{Rk}(P_0)\, c_1(\mathfrak{I}) + \beta^* c_1(\mathcal{L}^0_{\mathfrak{p}})$$

$\downarrow$ $C(\cdot) := \int_{\mathbb{T}^2} c_1(\cdot)$ (first) Chern number, $C(\mathfrak{I}) = 1$

$$NC(\mathcal{L}^\infty_{\mathfrak{p}}) = \mathrm{Rk}(P_0) - MC(\mathcal{L}^0_{\mathfrak{p}})$$

$C_\natural(\mathfrak{p}) := C(\mathcal{L}^\natural_{\mathfrak{p}}), \quad \natural \in \{0, \infty\}.$

COROLLARY (TKNN equations)

Let $\mathfrak{p} \in Proj(\mathfrak{A}_\theta)$, $\theta = \frac{M}{N}$. The Harper and Hofstadter Chern numbers of $\mathfrak{p}$ are related by the following (diophantine) equation

$$NC_\infty(\mathfrak{p}) + MC_0(\mathfrak{p}) = Rk(\mathfrak{p}).$$

In particular if $\mathfrak{h}$ has N disjoint bands (as in the case of $\mathfrak{h} = \mathfrak{u} + \mathfrak{u}^{-1} + \mathfrak{v} + \mathfrak{v}^{-1}$ for N odd), then $\mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_N = \mathbb{1}$.

$$NC_\infty(\mathfrak{p}_k) + MC_0(\mathfrak{p}_k) = 1 \quad \forall\ k = 1, \ldots, N.$$

Let $\mathfrak{P}_k := \mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_k$ (k^{th} gap projection)

$$NC_\infty(\mathfrak{P}_k) + MC_0(\mathfrak{P}_k) = k \quad \forall\ k = 1, \ldots, N.$$

Remark!! if $k = N$ then $\mathfrak{P}_N = \mathbb{1}$. $C_\infty(\mathbb{1}) = 1$ (Chern number of the total Harper bundle), $C_0(\mathbb{1}) = 0$ (Chern number of the total trivial Hofstadter bundle).

Chern numbers: "abstract" and "concrete"

$$N C_\infty(\mathfrak{p}) + M C_0(\mathfrak{p}) = \mathrm{Rk}(\mathfrak{p})$$

$$\downarrow$$

$$C_\infty(\mathfrak{p}) = \frac{1}{N}\mathrm{Rk}(\mathfrak{p}) - \frac{M}{N} C_0(\mathfrak{p})$$

$$\downarrow \qquad (\mathfrak{Ch} = C_0, \quad \mathfrak{f} = \frac{1}{N}\mathrm{Rk})$$

$$\boxed{C_\infty(\mathfrak{p}) = \mathfrak{f}(\mathfrak{p}) - \theta\,\mathfrak{Ch}(\mathfrak{p}) \qquad (\star)}$$

- The "concrete" Chern number of an "abstract" projection $\mathfrak{p}$ in the Harper representation is a linear combination of the "astract" operations $\mathfrak{Ch}$ and $\mathfrak{f}$.
- The differential-geometric Chern number depends on the representation considered.

Thank you for your attention!!