Geometric derivation of the TKNN equations

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based on joint work with:
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In a milestone paper [TKNN82] Thouless et al. contributed to the understanding of the Quantum Hall effect (QHE) with two important discoveries.

1. For rational magnetic flux the Hall conductance, defined as a bulk observable by linear response theory (Kubo formula), has a topological meaning, which explains its integer value. This topological structure emerges since the Hamiltonian operator commutes with the magnetic translations.

2. The integers corresponding to the Hall conductance, in the limit of weak and strong magnetic field respectively, satisfy a Diophantine equation, nowadays known as TKNN equation (an equation for each spectral gap).
The first intuition has been later dramatically pushed forward by realizing that the notion of topological invariant can be extended to the case of irrational flux, or of a random perturbation, by replacing the algebra generated by magnetic translations with the rotation C*-algebra, alias the algebra of the non-commutative torus. The relation with alternative approaches to the QHE, based either on a quantum pumping mechanism or on the edge currents, has been clarified.

On the contrary, the status and the meaning of the TKNN equations have not been completely clarified. No one of the existing derivations took into account the fact that the integers are Chern numbers of suitable vector bundles over $\mathbb{T}^2$. 
1 Introduction
   ■ The model
      ■ Effective models: Hofstadter regime
      ■ Effective models: Landau regime
      ■ Chern numbers in the effective models
      ■ The TKNN equations

2 Common algebraic structure & NCT
   ■ Abstract algebraic structure & isospectrality
   ■ Measure and differentiable structure on the NCT
   ■ The band spectrum

3 Main results
   ■ Bundle decomposition
   ■ Duality of vector bundles
Theoretical model (without randomness)

- Independent electrons, effectively 2-dimensional sample
- periodic potential $V_\Gamma : \mathbb{R}^2 \to \mathbb{R}$, $\Gamma \simeq \mathbb{Z}^2$
- uniform orthogonal magnetic field (symmetric gauge)

The dynamics is described by the **Bloch-Landau Hamiltonian**

$$H_{BL} = \frac{1}{2} \left( -i \frac{\partial}{\partial x} - \frac{\varphi_B}{2} y \right)^2 + \frac{1}{2} \left( -i \frac{\partial}{\partial y} + \frac{\varphi_B}{2} x \right)^2 + V_\Gamma(x, y)$$

acting in $L^2(\mathbb{R}^2)$. 

**Landau Hamiltonian**
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acting in $L^2(\mathbb{R}^2)$.

The competition between the periodic and the magnetic length scales makes a direct study of $H_{\text{BL}}$ a formidable task. Thus the need of simpler effective models which capture the physics in suitable asymptotic regimes.
Multiscale structure: separation of scales

A. The Hofstadter regime ($\varphi_B \to 0$)

B. The Landau regime ($\varphi_B \to \infty$)

The ratio of the energy scales considered in [TKNN] is $\hbar \omega_c / |V_\Gamma| \propto \varphi_B$.

The ratio of the typical time-scales is

$$\frac{\text{Crystal period}}{\text{Cyclotron period}} = \frac{ma^2/2\pi\hbar}{mc/eB} = \frac{BS_\Gamma}{hc/e} = \frac{\Phi}{\Phi_0} = \varphi_B$$

Here crystal period is estimated as the time needed to travel through a distance $a$ ($a = \sqrt{S_\Gamma}$ = lattice constant) when the momentum is

$$k = 2\pi \hbar / a$$
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Effective models: Hofstadter regime $\varphi_B \to 0$

In the **Hofstadter regime**, $\varphi_B \to 0$, one considers firstly the unperturbed periodic Hamiltonian $H_{\text{per}} = -\Delta + V_\Gamma$.

The periodicity leads to consider the **Bloch-Floquet transform**

$$
\mathcal{U} : L^2(\mathbb{R}^2) \longrightarrow L^2(\mathbb{B}) \otimes L^2(\mathbb{T}_Y^2) \cong L^2(\mathbb{T}^2, \mathcal{H}_f) \\
(\mathcal{U}\psi)(k, y) = \sum_{\gamma \in \Gamma} e^{-i(y+\gamma) \cdot k} \psi(y + \gamma).
$$
Effective models: Hofstadter regime $\varphi_B \to 0$

By using the Bloch-Floquet transform $\mathcal{U}$, one obtains a fibered operator

$$\mathcal{U} H_{\text{per}} \mathcal{U}^{-1} = \int_{\mathbb{B}} \otimes H_{\text{per}}(k) \, dk \quad \text{in} \quad L^2(\mathbb{B}, \mathcal{H}_f),$$

$$H_{\text{per}}(k) = \frac{1}{2} (-i \nabla_y + k)^2 + V_\Gamma(y) \quad \text{acting on} \quad \mathcal{D} \subseteq L^2(\mathbb{T}_Y^d, dy) =: \mathcal{H}_f.$$

We assume to know the solution of the eigenvalue problem:

$$H_{\text{per}}(k) \chi_n(k, y) = E_n(k) \chi_n(k, y)$$
Effective models: Hofstadter regime $\varphi_B \to 0$
Effective models: Hofstadter regime $\varphi_B \to 0$

To any isolated Bloch band $E_*(k_1, k_2) := \sum h_{n,m} e^{i(nk_1+mk_2)}$ corresponds an effective Hamiltonian acting in $L^2(\mathbb{T}^2, dk_1 dk_2)$ as (formally)

$$E_*(K_1, K_2) \quad \text{with} \quad \begin{cases} K_1 = k_1 + \frac{i}{2} \pi \varphi_B \frac{\partial}{\partial k_2} \\ K_2 = k_2 - \frac{i}{2} \pi \varphi_B \frac{\partial}{\partial k_1} \end{cases}$$

$$[K_1, K_2] = i\pi \varphi_B \mathbb{1}$$

This procedure corresponds to Peierl’s substitution [Peierls 33] and it has been mathematically established in [Bellissard 88] [Hellfer Sjöstrand 89] [Nenciu91] [P Spohn Teufel 03] [DeNittis P 10].
Effective models: Hofstadter regime $\varphi_B \to 0$

To any isolated Bloch band $E_*(k_1, k_2) := \sum h_{n,m} e^{i(nk_1 + mk_2)}$ corresponds an effective Hamiltonian acting in $L^2(T^2, dk_1dk_2)$ as

$$E_*(K_1, K_2) = \sum h_{n,m} e^{i(nK_1 + mK_2)} = \sum h_{n,m} e^{i\pi nm \varphi_B} U_0^m V_0^n$$

$$U_0 = e^{iK_2}, \quad V_0 = e^{iK_1}, \quad U_0 V_0 = e^{i2\pi \varphi_B} V_0 U_0$$

Hofstadter model: $E_*(k_1, k_2) = 2[\cos(k_1) + \cos(k_2)].$

$$\hat{h}_* = U_0 + U_0^{-1} + V_0 + V_0^{-1} =: H_{\text{Hof}}(\varphi_B).$$
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Effective models: Landau regime $\varphi_B \to +\infty$

In the **Landau regime**, $\varphi_B \to \infty$, one considers a perturbation of the Landau Hamiltonian

$$H_L = \frac{1}{2} \left[ \left( -i \frac{\partial}{\partial x} - \frac{\varphi_B}{2} y \right)^2 + \left( -i \frac{\partial}{\partial y} + \frac{\varphi_B}{2} x \right)^2 \right] = \frac{1}{2} [\tilde{K}_1^2 + \tilde{K}_2^2].$$

For $V_\Gamma = 0$ the Hamiltonian is quadratic (**Landau levels**). The operators $G_1$ and $G_2$, corresponding to the classical coordinates of the center of the cyclotron orbit, commute with the Hamiltonian.

**Classical Mechanics**

\[
\begin{align*}
\tilde{x}_1 &= x_1(t) + \frac{1}{\varphi_B} v_2(t) \\
\tilde{x}_2 &= x_2(t) - \frac{1}{\varphi_B} v_1(t)
\end{align*}
\]

**Quantum Mechanics**

\[
\begin{align*}
G_1 &= X_1 + \frac{1}{\varphi_B} \tilde{K}_2 = \frac{1}{\varphi_B} \left( P_2 + \frac{\varphi_B}{2} X_1 \right) \\
G_2 &= X_2 - \frac{1}{\varphi_B} \tilde{K}_1 = \frac{1}{\varphi_B} \left( -P_1 + \frac{\varphi_B}{2} X_2 \right)
\end{align*}
\]

\[ [\tilde{K}_1, \tilde{K}_2] = i \varphi_B \mathbb{1} \]

\[ [G_j, \tilde{K}_i] = 0 \]

\[ [G_1, G_2] = i \frac{1}{\varphi_B} \mathbb{1} \]
Effective models: Landau regime $\varphi_B \to +\infty$

For $V_{\Gamma} \neq 0$ to each Landau level corresponds an effective Hamiltonian acting in $L^2(\mathbb{R}, ds)$.

In the representation which diagonalizes $G_1$ the effective Hamiltonian is given (assuming $V_{\Gamma} \in C^\infty$) by

$$V_{\Gamma}(G_1, G_2) \quad \text{with} \quad \begin{cases} G_1 = s \\ G_2 = -i \frac{1}{\varphi_B} \frac{d}{ds} \end{cases}$$

Here $V_{\Gamma}(G_1, G_2)$ can be interpreted by means of ordinary $\epsilon$-Weyl quantization, with $\epsilon = \epsilon_\infty = \frac{1}{\varphi_B}$. Since the choice

$$V_{\Gamma}(p, q) = 2 \cos(2\pi p) + 2 \cos(2\pi q)$$

yields the Harper operator, we say that $V_{\Gamma}(G_1, G_2)$ is a Harper-like Hamiltonian.
Effective models: Landau regime $\varphi_B \to +\infty$

For $V_\Gamma \neq 0$ to each Landau level corresponds an effective Hamiltonian acting in $L^2(\mathbb{R}, ds)$.

Explicitly, for $V_\Gamma(q, p) = \sum_{n,m \in \mathbb{Z}} v_{n,m} e^{i2\pi(np+mq)}$,

$$V_\Gamma(G_1, G_2) = \sum_{n,m} v_{n,m} e^{i2\pi(nG_1+mG_2)} = \sum_{n,m} v_{n,m} e^{i\pi nm\varepsilon_\infty} U_\infty^m V_\infty^{-n}$$

$$U_\infty := e^{i2\pi G_1}, \quad V_\infty := e^{-i2\pi G_2}, \quad U_\infty V_\infty := e^{i2\pi \varepsilon_\infty} V_\infty U_\infty.$$  

The validity of $V_\Gamma$ as effective Hamiltonian for $\varphi_B \to \infty$ is proved in [Bellissard 88] [Hellfer Sjöstrand 89] [DeNittis P 10].
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Conductance in the effective models

How can the value of the **conductance** be recovered? The key idea is that it corresponds to a TQN related to the symmetries of the system [TKNN].

Each of the models has a $\mathbb{Z}^2$-symmetry, provided $\varphi_B = \frac{M}{N} \in \mathbb{Q}$. For any $n = (n_1, n_2) \in \mathbb{Z}^2$ the symmetry is provided by

<table>
<thead>
<tr>
<th>$L^2(\mathbb{R}^2)$</th>
<th>$H_{BL}$</th>
<th>magnetic translations $T_1^N, T_2$ ($\varphi_B \in \mathbb{Q}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L^2(\mathbb{T}^2) \cong \ell^2(\mathbb{Z}^2)$</td>
<td>$E_*(K_1, K_2)$</td>
<td>$(T_1^N \psi)<em>{n,m} = e^{i\pi m N \varphi_B} \psi</em>{n-1,m}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(T_2 \psi)<em>{n,m} = e^{i\pi n \varphi_B} \psi</em>{n,m+1}$</td>
</tr>
<tr>
<td>$L^2(\mathbb{R})$</td>
<td>$V_\Gamma(G_1, G_2)$</td>
<td>$(T_1^M \psi)(x) = \psi(x - M)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(T_2 \psi)(x) = e^{i\frac{2\pi}{\varphi_B} x} \psi(x)$</td>
</tr>
</tbody>
</table>

One can use a **generalized Bloch-Floquet transform** to decompose the Hilbert space with respect to the $\mathbb{Z}^2$-symmetry.
The generalized **Bloch-Floquet transform** is defined as

\[ \mathcal{U} : \mathcal{H} \longrightarrow \int_{\mathbb{T}^2}^\oplus \mathcal{H}(k) \, dk \]

\[ (\mathcal{U}\psi)(k) := \sum_{n \in \mathbb{Z}^2} e^{-in \cdot k} \tilde{T}_1^{n_1} \tilde{T}_2^{n_2} \psi, \quad k \in \mathbb{T}^2. \]

where \( \tilde{T}_1, \tilde{T}_2 \in \mathcal{B}(\mathcal{H}) \) are the generators of the \( \mathbb{Z}^2 \)-symmetry mentioned above. Hereafter we assume the rationality condition \( \varphi_B = \frac{M}{N} \).

The relevant Hamiltonian \( H \) (resp. \( H_{BL} \), \( H_{Hof} \) or \( H_{Har} \)), after BF transform, becomes a **fibered operator** i.e.

\[ \mathcal{U} H \mathcal{U}^{-1} = \int_{\mathbb{T}^2}^\oplus H(k) \, dk \quad \text{in} \quad \int_{\mathbb{T}^2}^\oplus \mathcal{H}(k) \, dk. \]
The direct integral defines a relevant collection of vector spaces. Fix $\mu \in \mathbb{R}$ (chemical potential) so that $\mu \notin \text{Spec } H(k)$ for all $k \in \mathbb{T}^2$ (physical condition). Denote by

$$P_\mu(k) := \chi_{(-\infty,\mu]}(H(k))$$

the spectral projector up to the energy $\mu$ and pose

$$F(k) := \text{Ran } P_\mu(k) \subset \mathcal{H}(k)$$
The direct integral defines a relevant collection of vector spaces. Fix \( \mu \in \mathbb{R} \) (chemical potential) so that \( \mu \notin \text{Spec} \, H(k) \) for all \( k \in \mathbb{T}^2 \) (physical condition). Denote by

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The collection of such vector spaces is a vector bundle over \( \mathbb{T}^2 \). The Chern number of this vector bundle is the Topological Quantum Number we are interested in.
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Color-coded quantum butterflies

\[ E_*(k_1, k_2) = 2(\cos k_1 + \cos k_2) \]

Hofstadter Hamiltonian

\[ V_\Gamma(p, x) = 2(\cos(2\pi p) + \cos(2\pi x)) \]

Harper Hamiltonian
Color-coded quantum butterflies

\[ \varphi_B = \theta = \frac{1}{\varphi_B'} \]

\[ E_*(k_1, k_2) = 2(\cos k_1 + \cos k_2) \quad \text{Hofstadter Hamiltonian} \]

\[ V_\Gamma(p, x) = 2(\cos(2\pi p) + \cos(2\pi x)) \quad \text{Harper Hamiltonian} \]
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Color-coded quantum butterflies

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Hofstadter Hamiltonian

\[ V_G(p, x) = 2(\cos(2\pi p) + \cos(2\pi x)) \]
Harper Hamiltonian

Conductance in the gap (Chern number)
Observation: the colors (Chern numbers) in the limiting models are related by the TKNN-equation

\[ NC_{\infty}(g) + MC_0(g) = d_g \quad \text{for} \quad g \in \{0, 1, \ldots, g_{\text{max}}\} \]
Here the integer $d_g$ is a gap label: it is equal to $g$ for $N$ odd, while for $N$ even is defined by

$$d_g = \begin{cases} 
  g & \text{for } 0 \leq g \leq N/2 - 1 \\
  g + 1 & \text{for } N/2 \leq g \leq g_{\text{max}}.
\end{cases}$$

Moreover, one has the constraint

$$2|C_0(g)| \leq N.$$ 

For any $g \in \{0, 1, \ldots, g_{\text{max}}\}$, the TKNN equation is solved by infinite pairs $(t_g, s_g) \in \mathbb{Z}^2$. If the constraint is imposed, the solution is unique (provided $M, N$ are coprime, $M \in \mathbb{Z}, N \in \mathbb{N} \setminus \{0\}$).
The twofold way

We obtain the TKNN formula as a corollary of the duality between vector bundles emerging in two inequivalent representations of the same $C^*$-algebra.

\[ P_0 \xrightarrow{U_0 \ldots U_0^{-1}} P_0(\cdot) \xrightarrow{\text{Ran}} \mathcal{L}_p^0 \xrightarrow{C} C_0(\mathfrak{p}) \]

\[ P_\infty \xrightarrow{U_\infty \ldots U_\infty^{-1}} P_\infty(\cdot) \xrightarrow{\text{Ran}} \mathcal{L}_p^\infty \xrightarrow{C} C_\infty(\mathfrak{p}) \]

$C :=$ first Chern number.

What is the relation between $C_0(\mathfrak{p})$ and $C_\infty(\mathfrak{p})$ for a given $\mathfrak{p} \in \text{Proj}(\mathfrak{A}_\theta)$?
Hilbert space: $\mathcal{H}_0 := L^2(\mathbb{T}^2, dk_1 \, dk_2)$.

Hofstadter unitaries:

$$U_0 = e^{iK_2}, \quad V_0 = e^{iK_1}, \quad U_0V_0 = e^{i2\pi\varepsilon_0}V_0U_0.$$  

where

$$K_1 = k_1 + i\pi\varepsilon_0 \frac{\partial}{\partial k_2}, \quad K_2 = k_2 - i\pi\varepsilon_0 \frac{\partial}{\partial k_1}.$$  

Hofstadter-like Hamiltonian:

$$h(k_1, k_2) = \sum h_{n,m} e^{i(nk_1 + mk_2)} \in C^1(\mathbb{T}^2)$$

\[\downarrow\]  

(\text{Peierls substitution})

$$\hat{h}(\varepsilon_0) := \sum h_{n,m} e^{i(nK_1 + mK_2)} = \sum h_{n,m} e^{i\pi nm\varepsilon_0} U_0^m V_0^n \in \mathcal{B}(\mathcal{H}_0)$$

Hofstadter model: $h_{\star}(k_1, k_2) = 2\cos(k_1) + 2\cos(k_2)$. 

$$\hat{h}_{\star} = U_0 + U_0^{-1} + V_0 + V_0^{-1} =: H_{\text{Hof}}(\varepsilon_0).$$
Hilbert space: $\mathcal{H}_\infty := L^2(\mathbb{R}, dq)$.

Harper unitaries:

$$U_\infty := e^{i2\pi Q}, \quad V_\infty := e^{-i2\pi P}, \quad U_\infty V_\infty = e^{i2\pi \varepsilon_\infty} V_\infty U_\infty.$$  

where

$$Q = \text{multiplication by } q, \quad P = -i \frac{\varepsilon_\infty}{2\pi} \frac{\partial}{\partial q}.$$  

Harper-like Hamiltonian:

$$h(p, q) = \sum h_{n,m} e^{i2\pi(np+mq)} \quad \in C^1_{\mathbb{Z}^2}(\mathbb{R}^2)$$

$\downarrow$ (Weyl quantization)

$$\hat{h}(\varepsilon_\infty) := \sum h_{n,m} e^{i2\pi(np+mq)} = \sum h_{n,m} e^{i\pi nm\varepsilon_\infty} U_\infty^m V_\infty^{-n} \quad \in \mathcal{B}(\mathcal{H}_\infty)$$

Harper model: $h_\star(p, q) = 2 \cos(2\pi p) + 2 \cos(2\pi q)$.

$$\hat{h}_\star = U_\infty + U_\infty^{-1} + V_\infty + V_\infty^{-1} =: H_{\text{Har}}(\varepsilon_\infty).$$
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Non-Commutative Torus (NCT) “abstract” C*-algebra $\mathcal{A}_\theta$ ($\theta \in \mathbb{R}$) deformation parameter) generated by:

$$u^* = u^{-1}, \quad v^* = v^{-1}, \quad uv = e^{i2\pi \theta} vu$$

with universal norm $\|a\| := \sup \{ \|\pi(a)\|_\mathcal{H} \mid \pi \text{ rep. of } \mathcal{A}_\theta \text{ on } \mathcal{H} \}$. 

Proposition (isospectrality) $\Pi_0$ and $\Pi_\infty$ are injective. Let $\epsilon_0 = \epsilon_\infty =: \theta$ (duality condition). Then for all $a \in \mathcal{A}_\theta$ $\sigma(\Pi_0(a)) = \sigma(a) = \sigma(\Pi_\infty(a))$. 
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Hofstadter representation:

$$\Pi_{0} : \mathcal{A}_{\varepsilon_{0}} \rightarrow \mathcal{B}(\mathcal{H}_{0}), \quad \Pi_{0}(u) := U_{0}, \quad \Pi_{0}(v) := V_{0}.$$
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Harper representation:
$$\Pi_\infty : \mathcal{A}_{\epsilon_\infty} \rightarrow \mathcal{B}(\mathcal{H}_\infty), \quad \Pi_\infty(u) := U_\infty, \quad \Pi_\infty(v) := V_\infty.$$
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Harper representation:

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**PROPOSITION (isospectrality)**

$\Pi_0$ and $\Pi_\infty$ are injective. Let $\varepsilon_0 = \varepsilon_\infty =: \theta$ (duality condition). Then for all $a \in \mathcal{A}_\theta$

$$\sigma(\Pi_0(a)) = \sigma(a) = \sigma(\Pi_\infty(a)).$$
Let $\mathfrak{h} := u + u^{-1} + v + v^{-1} \in \mathcal{A}_\theta$ and $\theta \in \mathbb{Q}$, [Hof76]:

$$\varepsilon_0(B_1) = \varepsilon_\infty(B_2) \text{ implies } B_1 B_2 \propto 1 \text{ with } B_1 \ll 1 \ll B_2 \text{ (duality)}$$

$$\sigma(H_{Hof}) = \sigma(\mathfrak{h}) = \sigma(H_{Har})$$
Let $\mathfrak{h} := u + u^{-1} + v + v^{-1} \in \mathcal{A}_\theta$ and $\theta \in \mathbb{Q}$, [Hof76]:

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$$\sigma(H_{\text{Hof}}) = \sigma(\mathfrak{h}) = \sigma(H_{\text{Har}})$$
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Smooth algebra:

\[ \mathcal{A}_\theta^\infty := \left\{ \sum a_{n,m} u^n v^m \mid \{a_{n,m}\} \in S(\mathbb{Z}^2) \right\} \subset \mathcal{A}_\theta \]

It is a Fréchet algebra.

Canonical derivations (differentiable structure):

\[ \partial_j : \mathcal{A}_\theta^\infty \rightarrow \mathcal{A}_\theta^\infty \quad j = 1, 2 \]
\[ \partial_1 (u^n v^m) = i2\pi n (u^n v^m), \quad \partial_2 (u^n v^m) = i2\pi m (u^n v^m). \]

Symmetric (\( \partial_j (a^*) = \partial_j (a)^* \)), commuting (\( \partial_1 \circ \partial_2 = \partial_2 \circ \partial_1 \)), trace-compatibility (\( f \circ \partial_j = 0 \)).

Canonical trace (measure structure):

\[ f : \mathcal{A}_\theta \rightarrow \mathbb{C} \quad f(u^n v^m) = \delta_{n,0} \delta_{m,0}. \]

state (linear, positive, normalized), faithful (\( f (a^* a) = 0 \Leftrightarrow a = 0 \)), tracial property (\( f (ab) = f (ba) \)), unique if \( \theta \not\in \mathbb{Q} \).
Smooth algebra:

$$\mathcal{A}_\infty := \left\{ \sum a_{n,m} u^n v^m \mid \{a_{n,m}\} \in S(\mathbb{Z}^2) \right\} \subset \mathcal{A}_\theta$$

It is a Fréchet algebra.

Canonical derivations (differentiable structure):

$$\bar{\mathcal{F}}_j : \mathcal{A}_\infty \to \mathcal{A}_\infty \quad j = 1, 2$$

$$\bar{\mathcal{F}}_1(u^n v^m) = i2\pi n(u^n v^m), \quad \bar{\mathcal{F}}_2(u^n v^m) = i2\pi m(u^n v^m).$$

Symmetric ($\bar{\mathcal{F}}_j (a^*) = \bar{\mathcal{F}}_j (a)^*$), commuting ($\bar{\mathcal{F}}_1 \circ \bar{\mathcal{F}}_2 = \bar{\mathcal{F}}_2 \circ \bar{\mathcal{F}}_1$), trace-compatibility ($f \circ \bar{\mathcal{F}}_j = 0$).

Canonical trace (measure structure):

$$f : \mathcal{A}_\theta \to \mathbb{C} \quad f(u^n v^m) = \delta_{n,0} \delta_{m,0}.$$  

state (linear, positive, normalized), faithful ($f(a^* a) = 0 \iff a = 0$), tracial property ($f(ab) = f(ba)$), unique if $\theta \notin \mathbb{Q}$. 
Projections:

\[ \text{Proj}(\mathcal{A}_\theta) := \{ p \in \mathcal{A}_\theta \mid p = p^* = p^2 \} \]

If \( \theta = \frac{M}{N} \) (with \( M \in \mathbb{Z} \), \( N \in \mathbb{N}^* \) and \( \text{g.c.d}(M, N) = 1 \)), then

\[ f: \text{Proj}(\mathcal{A}_{M/N}) \rightarrow \left\{ 0, \frac{1}{N}, \ldots, \frac{N-1}{N}, 1 \right\} \]
Projections:

$$\text{Proj}(\mathcal{A}_\theta) := \{ p \in \mathcal{A}_\theta | p = p^* = p^2 \}.$$ 

If $\theta = \frac{M}{N}$ (with $M \in \mathbb{Z}$, $N \in \mathbb{N}^*$ and $\text{g.c.d}(M, N) = 1$), then

$$f: \text{Proj}(\mathcal{A}_{M/N}) \rightarrow \left\{0, \frac{1}{N}, \ldots, \frac{N-1}{N}, 1\right\}.$$ 

“Abstract” (first) Chern number:

$$\mathcal{C}_\theta : \text{Proj}(\mathcal{A}_\theta) \cap \mathcal{A}_\theta^\infty \rightarrow \mathbb{Z}$$

$$\mathcal{C}_\theta(p) := \frac{i}{2\pi} \ f \ (p[\mathcal{F}_1(p); \mathcal{F}_2(p)]).$$
Projections:

\[ \text{Proj}(\mathcal{A}_\theta) := \{ p \in \mathcal{A}_\theta \mid p = p^* = p^2 \}. \]

If \( \theta = \frac{M}{N} \) (with \( M \in \mathbb{Z}, N \in \mathbb{N}^* \) and \( \text{g.c.d}(M, N) = 1 \)), then

\[ f: \text{Proj}(\mathcal{A}_{M/N}) \to \left\{ 0, \frac{1}{N}, \ldots, \frac{N-1}{N}, 1 \right\}. \]

“Abstract” (first) Chern number:

\[ \mathcal{C}_\theta : \text{Proj}(\mathcal{A}_\theta) \cap \mathcal{A}_\infty \to \mathbb{Z} \]

\[ \mathcal{C}_\theta(p) := \frac{i}{2\pi} \int (p[\mathcal{F}_1(p); \mathcal{F}_2(p)]). \]
Outline

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   ■ Effective models: Hofstadter regime
   ■ Effective models: Landau regime
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Band projections and gap band projections

\( h \in \mathcal{A}_{M/N} \) selfadjoint (\( h = h^* \)). Band spectrum \( \sigma(h) = \bigcup_{j \in J} I_j \) with \( I_j \subset \mathbb{R} \) closed intervals. **Gap**, open interval between two bands.
Band projections and gap band projections

$h \in \mathcal{A}_{M/N}$ selfadjoint ($h = h^*$). Band spectrum $\sigma(h) = \bigcup_{j \in J} I_j$ with $I_j \subset \mathbb{R}$ closed intervals. Gap, open interval between two bands.

**Lemma (band projection)**

Any band $I_j \subset \sigma(h)$ defines a $p_j \in \text{Proj}(\mathcal{A}_\theta) \cap \mathcal{A}_\theta^\infty$. 
Band projections and gap band projections

\( \mathfrak{h} \in \mathfrak{A}_{M/N} \) selfadjoint \((\mathfrak{h} = \mathfrak{h}^*)\). Band spectrum \( \sigma(\mathfrak{h}) = \bigcup_{j \in J} I_j \) with \( I_j \subset \mathbb{R} \) closed intervals. Gap, open interval between two bands.

**LEMMA (band projection)**

*Any band \( I_j \subset \sigma(\mathfrak{h}) \) defines a \( p_j \in \text{Proj}(\mathfrak{A}_\theta) \cap \mathfrak{A}_\theta^\infty \).*

Gap projection \( \mathfrak{P}_j := \bigoplus_{k=1}^j p_k \). If \( \theta = M/N \) then \( 1 \leq j \leq N \).
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3 Main results
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Let $\Gamma(\mathcal{E})$ be the space of continuous sections of the Hermitian vector bundle $\mathcal{E} \to \mathbb{T}^2$. Any $A \in \Gamma(\text{End}(\mathcal{E}))$ defines a multiplication operator $\tilde{A}$ acting on $s \in L^2(\mathcal{E})$ by $(\tilde{A}s)_z = A_z s_z$, thus $\Gamma(\text{End}(\mathcal{E}))$ can be regarded as a subalgebra of $\mathcal{B}(L^2(\mathcal{E}))$.

**Definition (Bundle decomposition)**

Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a C*-algebra. We say that $\mathcal{A}$ admits a bundle decomposition over $\mathbb{T}^2$ if there exist a Hermitian vector bundle $\mathcal{E} \to \mathbb{T}^2$ and a unitary map $\mathcal{F} : \mathcal{H} \to L^2(\mathcal{E})$ such that $\mathcal{F} \mathcal{A} \mathcal{F}^{-1} \subset \Gamma(\text{End}(\mathcal{E}))$.

**Lemma**

Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a C*-algebra admitting a bundle decomposition over $\mathbb{T}^2$ with respect to the vector bundle $\iota : \mathcal{E} \to \mathbb{T}^2$. Then any projection $p \in \mathcal{A}$ defines a vector subbundle $\mathcal{L}(p) \subset \mathcal{E}$.
**THEOREM** (Bundle decomposition in Hofstadter and Harper representations)

Let $\mathcal{A}_{M/N}$ be the the rational NCT-algebra, $\pi_0 : \mathcal{A}_{M/N} \rightarrow \mathcal{B}(\mathcal{H}_0)$ the Hofstadter representation and $\pi_\infty : \mathcal{A}_{M/N} \rightarrow \mathcal{B}(\mathcal{H}_\infty)$ the Harper representation. Then:

(i) The operator algebra $\pi_0(\mathcal{A}_{M/N}) \subset \mathcal{B}(\mathcal{H}_0)$ admits a bundle decomposition over $\mathbb{T}^2$ with respect to a (rank $N$) Hermitian vector bundle $\mathcal{E}_0 \rightarrow \mathbb{T}^2$ and a unitary operator $F_0 : \mathcal{H}_0 \rightarrow L^2(\mathcal{E}_0)$. The vector bundle $\mathcal{E}_0$ is trivial. The bundle representation $a \mapsto F_0 \pi_0(a) F_0^{-1}$ is generated by the endomorphism sections $U_0(\cdot) := F_0 \pi_0(u) F_0^{-1}$ and $V_0(\cdot) := F_0 \pi_0(v) F_0^{-1}$, which are explicitly given in the proof.

(ii) The operator algebra $\pi_\infty(\mathcal{A}_{M/N}) \subset \mathcal{B}(\mathcal{H}_\infty)$ admits a bundle decomposition over $\mathbb{T}^2$ with respect to a (rank $N$) Hermitian vector bundle $\mathcal{E}_\infty \rightarrow \mathbb{T}^2$ and unitary transform $F_\infty : \mathcal{H}_\infty \rightarrow L^2(\mathcal{E}_\infty)$. The vector bundle $\mathcal{E}_\infty$ is non trivial with (first) Chern number $C_1(\mathcal{E}_\infty) = 1$. 
The twofold way

We obtain the TKNN formula as a corollary of the duality between vector bundles emerging in two inequivalent representations of the same $C^*$-algebra.

\[
\begin{array}{cccc}
P_0 & \overset{U_0 \ldots U_0^{-1}}{\longrightarrow} & P_0(\cdot) & \overset{\text{Ran}}{\longrightarrow} & \mathcal{L}_0(p) & \overset{C}{\longrightarrow} & C_0(p) \\
\Pi_0 & \downarrow & & & & & \\
p \in \text{Proj} (\mathcal{A}_\theta) & & & & & & \\
\mathcal{B}(\mathcal{H}) & \Gamma(\text{End}(\mathcal{E})) & \mathcal{E} & \rightarrow & \mathbb{T}^2 & \mathbb{Z} \\
\Pi_\infty & \downarrow & & & & & \\
P_\infty & \overset{U_\infty \ldots U_\infty^{-1}}{\longrightarrow} & P_\infty(\cdot) & \overset{\text{Ran}}{\longrightarrow} & \mathcal{L}_\infty(p) & \overset{C}{\longrightarrow} & C_\infty(p) \\
\end{array}
\]

$C :=$ first Chern number.

What is the relation between $C_0(p)$ and $C_\infty(p)$ for a given $p \in \text{Proj} (\mathcal{A}_\theta)$ for $\theta = M/N$?
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Let $p \in \text{Proj}(\mathcal{A}_{M/N})$ and $r \in \{0, \infty\}$. The orthogonal projector $\pi_r(p) \in \mathcal{B}(\mathcal{H}_r)$ defines a vector subbundle $\mathcal{L}(\pi_r(p))$ of $\mathcal{E}_r \to \mathbb{T}^2$, which we denote shortly by $\mathcal{L}_r(p)$. We write $C_r(p) := C_1(\mathcal{L}_r(p))$.

**THEOREM (Geometric duality (De Nittis, Faure, P))**

Let $\mathcal{A}_{M/N}$ be the rational NCT-algebra and $p \in \text{Proj}(\mathcal{A}_{M/N})$. Then:

(i) The vector bundles $\mathcal{L}_0(p)$ and $\mathcal{L}_\infty(p)$ have rank equal to $\text{Rk}(p)$.

(ii) Consider the continuous functions $\mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$
\alpha(z_1, z_2) = (z_1^N, z_2) \quad \beta(z_1, z_2) = (z_1^{-M}, z_2).
$$

Then the following isomorphism of Hermitian vector bundles holds true,

$$
\alpha^* \mathcal{L}_\infty(p) \cong \beta^* \mathcal{L}_0(p) \otimes \mathcal{I}
$$

where $\mathcal{I}$ is the determinant bundle of $\mathcal{E}_\infty$. 

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Derivation of the TKNN equations

\[ \alpha^* \mathcal{L}_p^\infty \simeq \beta^* \mathcal{L}_p^0 \otimes \mathcal{I} \]

\[ \downarrow \quad c_1 \text{ first Chern class of a product VB} \]

\[ c_1(\alpha^* \mathcal{L}_p^\infty) = c_1(\beta^* \mathcal{L}_p^0 \otimes \mathcal{I}) = \text{Rk}(\beta^* \mathcal{L}_p^0) \cdot c_1(\mathcal{I}) + \text{Rk}(\mathcal{I}) \cdot c_1(\beta^* \mathcal{L}_p^0) \]

\[ \downarrow \quad \text{functoriality of } c_1, \quad \text{Rk}(\mathcal{I}) = 1, \quad \text{Rk}(\beta^* \mathcal{L}_p^0) = \text{Rk}(P_0) \]

\[ \alpha^* c_1(\mathcal{L}_p^\infty) = \text{Rk}(P_0) \cdot c_1(\mathcal{I}) + \beta^* c_1(\mathcal{L}_p^0) \]

\[ \downarrow \quad C(\cdot) := \int_{\mathbb{T}^2} c_1(\cdot) \quad \text{(first) Chern number,} \quad C(\mathcal{I}) = 1 \]

\[ NC(\mathcal{L}_p^\infty) = \text{Rk}(P_0) - MC(\mathcal{L}_p^0) \]
\[ C_{\sharp}(p) := C(\mathcal{L}_{\sharp}^p), \quad \sharp \in \{0, \infty\}. \]

**Corollary (TKNN equations)**

Let \( p \in \text{Proj}(\mathcal{A}_\theta), \quad \theta = \frac{M}{N}. \) The Harper and Hofstadter Chern numbers of \( p \) are related by the following (diophantine) equation

\[ NC_\infty(p) + MC_0(p) = Rk(p). \]

In particular if \( \mathfrak{h} \) has \( N \) disjoint bands (as in the case of \( \mathfrak{h} = u + u^{-1} + v + v^{-1} \) for \( N \) odd), then \( p_1 \oplus \ldots \oplus p_N = 1. \)

\[ NC_\infty(p_k) + MC_0(p_k) = 1 \quad \forall \ k = 1, \ldots, N. \]

Let \( \mathfrak{p}_k := p_1 \oplus \ldots \oplus p_k \) \((k^{th} \text{ gap projection})\)

\[ NC_\infty(\mathfrak{p}_k) + MC_0(\mathfrak{p}_k) = k \quad \forall \ k = 1, \ldots, N. \]

Remark!! if \( k = N \) then \( \mathfrak{p}_N = 1. \) \( C_\infty(1) = 1 \) (Chern number of the total Harper bundle), \( C_0(1) = 0 \) (Chern number of the total trivial Hofstadter bundle).
Chern numbers: “abstract” and “concrete”

\[ NC_\infty(p) + MC_0(p) = Rk(p) \]
\[ \downarrow \]
\[ C_\infty(p) = \frac{1}{N} Rk(p) - \frac{M}{N} C_0(p) \]
\[ \downarrow \]
\[ (\mathcal{C}_\hbar = C_0, \ f = \frac{1}{N} Rk) \]

\[ C_\infty(p) = f(p) - \theta \mathcal{C}_\hbar(p) \quad (\star) \]

- The “concrete” Chern number of an “abstract” projection \( p \) in the Harper representation is a linear combination of the “abstract” operations \( \mathcal{C}_\hbar \) and \( f \).
- The differential-geometric Chern number depends on the representation considered.
Thank you for your attention!!