Gauge fields over noncommutative manifolds

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Why we are doing this:

put together some Yang–Mills-Higgs systems

with noncommutative spaces

hoping to get interesting stuff
Vortices and gauge fields; Taubes, ....

The Ginzburg–Landau equations for vortices is related to the four dimensional Yang–Mills equations via reduction:

any $SO(3)$ symmetric solution to the $SU(2)$ Y–M eqs on $\mathbb{R}^2 \times S^2$

determines a solution to the G–L eqs on $\mathbb{R}^2$ and vice versa.
\( M \) a compact oriented Riemann surface
(or maybe \( \mathbb{R}^2 \) with suitable boundary conditions,
e.g. locally bounded and globally square integrable curvatures);

holomorphic vector bundles \( \mathcal{E}_0, \mathcal{E}_1 \) over \( M \) and a holomorphic

\[
\mathcal{E}_0 \xrightarrow{\phi} \mathcal{E}_1
\]

self-duality equations

\[
\star F_\nabla = - F_\nabla
\]
on \( M \times S^2 \) are vortex equations

\[
\star F_\nabla_0 = \text{id}_{\mathcal{E}_0} - \phi \circ \phi^* \quad \text{and} \quad \star F_\nabla_1 = - \text{id}_{\mathcal{E}_1} + \phi^* \circ \phi
\]
on \( M \)
Equivariant dimensional reduction:

a systematic procedure for including internal fluxes on $S/R$ (instantons and/or monopoles of $R$-fields)

which are ‘symmetric’ (equivariant) under $S$

Monopoles; relevant for QHE
F.D. Haldane,

Instantons; relevant for Spin HE
S.-C. Zhang, J.-P. Hu
A four-dimensional generalization of the quantum Hall effect, Science (2001)
Physics: A brief history of dimensional reduction

Kaluza (1921), Klein (1926): the observed fundamental forces in 4-dimensions can be understood in terms of the dynamics of a simpler higher dimensional theory

Starting from a 5-D theory on $M_5 = M_4 \times S^1$, the product of a curved 4-D space-time $M_4$ and a circle with radius $r$ (and coordinate $0 \leq y < 2\pi$). Take the line element

$$ds^2_{(5)} = ds^2_{(4)} + (rdy + A(x))^2,$$

with $A(x) = A_\mu(x)dx^\mu$ a 4-dimensional vector potential
The 5-dimensional Einstein action reduces to

\[
\frac{1}{2\pi r} \int_{M_5} \sqrt{-g(5)} R(5) \, d^4x \, dy = \int_{M_4} \sqrt{-g(4)} \left( R(4) - \frac{1}{4} F^2 \right) \, d^4x,
\]

\(F = dA\) is a \(U(1)\) field strength in 4-dimensions.

Matter, e.g. a scalar field \(\Phi\), harmonically expanded on \(S^1\),

\[
\Phi(x, y) = \sum_{n=-\infty}^{\infty} \phi_n(x) e^{\frac{iny}{r}},
\]

then the 5-dimensional kinetic term for \(\Phi\) gives rise to an infinite tower of massive fields \(\phi_n(x)\) in \(M_4\), with masses \(m_n = \frac{n}{r}\).
A non-abelian generalisation of the K-K idea,

Start from $d$-dimensional Y–M theory on $M_4 \times S/R$ with gauge group $G$

If $R \subset G$, integrating over $S/R$ gives a Y–M–H system on $M_4$, with gauge group $K$, the centraliser, i.e. $[R, K] = 0$, of $R$ in $G$

Upon dimensional reduction the internal components of the $d$-dimensional gauge field $A$ play the rôle of Higgs fields in 4-dim:

$$A(x, y) \longrightarrow \begin{cases} A_\mu(x) \text{ (4-dim gauge fields)} \\ \Phi_a(x) \text{ (4-dim Higgs fields)} \end{cases}$$
A Higgs potential is generated from the $d$-dimensional Y–M action. Indeed, the full $d$-dim Y–M action reduces as

$$-\frac{1}{4} \int_{M_d} \sqrt{-g(d)} \text{Tr}(F^2) d^4x d^{d-4}y =$$

$$= \text{vol}(S/R) \int_{M_4} \sqrt{-g(4)} \text{tr} \left( -\frac{1}{4} F^2 + (D\Phi)^\dagger D\Phi - V(\Phi) \right) d^4x$$

The Higgs potential breaks $K$ dynamically: if $S \subset G$, $V(\Phi)$ breaks $K$ spontaneously to $K'$, the centraliser, $[S, K'] = 0$, of $S$ in $G$. 
The Ginsburg–Landu action functional

\[ GL(A, \Phi) = \int_{\mathbb{R}^2} \text{tr} \left( -\frac{1}{4} F^2 + D\Phi^\dagger D\Phi + \lambda(\Phi^\dagger \Phi - 1)^2 \right) \]

with critical points as mentioned self-duality equation are vortex equations:

\[ *F = \text{id}_{\mathbb{R}^2} - \Phi \circ \Phi^* \quad \text{and} \quad D\Phi = 0 \]
a caveat:

In the co-set space dimensional reduction programme, spinors on $M_4 \times S/R$ cannot give a chiral theory on $M_4$

rather anti-climax, one should admit !!!
Equivariant dimensional reduction

Equivariant dimensional reduction is a systematic procedure for including internal fluxes on $S/R$ (instantons and/or monopoles of $R$-fields) which are ‘symmetric’ (equivariant) under $S$

In general, a one-to-one correspondence between $S$-equivariant complex vector bundles over $M_d$

$$B \rightarrow M_d = M_4 \times S/R,$$

and $R$-equivariant bundles over $M_4$,

$$E \rightarrow M_4,$$

where $S$ acts on the space $M_d$ via the trivial action on $M_4$ and by the standard left translation action on $S/R$
In general the reduction yields rise quiver gauge theories on $M_4$.

Including spinor fields, coupling to background equivariant fluxes, can give rise to chiral theories on $M_4$.

Yukawa couplings are induced and the dimensional reduction can give masses to some zero modes of the Dirac operator on $S/R$. 
A simple example: Complex projective line

\( S = SU(2) \) and \( R = U(1) \), giving a 2-dim sphere \( S^2 \simeq SU(2)/U(1) \) (or projective line \( \mathbb{CP}^1 \)), and with \( G = U(k) \).

An embedding \( S \hookrightarrow G \) results into a decomposition

\[
U(k) \rightarrow \prod_{i=0}^{m} U(k_i),
\]

\( k = \sum_{i=0}^{m} k_i \), associated with the \( m + 1 \)-dim I.R. of \( SU(2) \)

\( g \in G \) decomposes as:

\[
g = (g_{k_i} \times k_i), \quad g_{k_i} \times k_i \quad \text{on} \quad \mathbb{C}^{k_i}
\]

each \( \mathbb{C}^{k_i} \) transforms under \( U(k_i) \subset U(k) \)

carries a \( U(1) \) charge \( p_j = m - 2j \), for \( -m \leq p_j \leq m \).
The $U(k)$ gauge potential, $A$ on $M_d$, splits into $k_i \times k_j$ blocks

$$A(x, y) = A(x) + a(y) + \Phi(x)\overline{\beta}(y) + \Phi^\dagger(x)\beta(y),$$

$$a = \bigoplus_{i=0}^{m}a_{m-2i}, \quad a_{m-2i} \text{ charge } m-2i \text{ monopole connection}$$

$A(x) = \bigoplus_{i=0}^{m}A^i(x)$, $A^i(x)$ is a $U(k_i)$ gauge connection on $M_4$,

and $\Phi(x)$ is a collection of Higgs fields.

$$A(x, y) = \begin{pmatrix}
A^0 + a_m \mathbf{1}_{k_0} & \phi_1\overline{\beta} & 0 & \cdots & 0 \\
\phi_1^\dagger\beta & A^1 + a_{m-2} \mathbf{1}_{k_1} & \phi_2\overline{\beta} & \cdots & 0 \\
0 & \phi_2^\dagger\beta & \vdots & \ddots & \vdots \\
\vdots & \vdots & 0 & \ddots & \phi_m\overline{\beta} \\
0 & 0 & 0 & \cdots & A^m + a_{-m} \mathbf{1}_{k_m}
\end{pmatrix}$$
Dimensional reduction generates a 4-dim Higgs potential,

\[ V(\Phi) = \frac{g^2}{2} \text{tr}_k \left( \frac{1}{4g^2 r^2} \begin{pmatrix} m^1_{k0} & 0 & \cdots & 0 \\ 0 & (m-2)^1_{k1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -m^1_{km} \end{pmatrix} - [\Phi, \Phi^\dagger] \right)^2, \]

whose minimization gives a vacuum structure depending on the monopole charges \( p_j = m - 2j \)
all of the above is related to stability of bundles

a form of Hitchin-Kobayashi correspondence

but
let us

move to noncommutative spaces

interesting consequences: e.g. de-singularization of moduli spaces
A dictionary:

<table>
<thead>
<tr>
<th>Classical</th>
<th>Noncommutative</th>
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<tbody>
<tr>
<td>(locally) compact space</td>
<td>(unital) C*-algebra</td>
</tr>
<tr>
<td>smooth manifold</td>
<td>C*-algebra with ‘smooth’ subalgebra</td>
</tr>
<tr>
<td>vector bundle</td>
<td>finite projective module</td>
</tr>
<tr>
<td>spin structure</td>
<td>spectral triple</td>
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....
Symmetries

\[ U = (U, \Delta, S, \varepsilon) \text{ a Hopf } \ast\text{-algebra} \]

\[ \Delta : U \to U \otimes U \quad \text{the coproduct} \]

\[ S : U \to U \quad \text{the antipode} \]

\[ \varepsilon : U \to \mathbb{C} \quad \text{counit} \]

\( \mathcal{A} \text{ a left } U\text{-module } \ast\text{-algebra: there is a left action } \triangleright \text{ of } U \text{ on } \mathcal{A}, \)

\[ h \triangleright xy = (h_{(1)} \triangleright x)(h_{(2)} \triangleright y), \]

\[ h \triangleright 1 = \varepsilon(h)1, \quad (h \triangleright x)^* = S(h)^* \triangleright x^*, \]

notation \( \Delta : U \to U \otimes U, \quad \Delta(h) = h_{(1)} \otimes h_{(2)}. \)
Connections on bundles

a space traded with a noncommutative algebra $\mathcal{A}$ (and a calculus $(\Omega(\mathcal{A}), d)$)

a vector bundle traded with a finitely generated projective (right) $\mathcal{A}$-module $\mathcal{E}$

a connection:

$$\nabla: \mathcal{E} \to \mathcal{E} \otimes \Omega^1(\mathcal{A}), \quad \nabla(\eta a) = \nabla(\eta)a + \eta da$$
$M$ a smooth manifold; $\mathbb{C}P^1_q$ the quantum projective line

Characterize vector bundles over the quantum space

$$\underline{M} := \mathbb{C}P^1_q \times M$$

equivariant under an action of the quantum group $SU_q(2)$

These are finitely-generated and projective $SU_q(2)$-equivariant modules over the algebra of functions

$$\mathcal{A}(\underline{M}) = \mathcal{A}(\mathbb{C}P^1_q) \otimes \mathcal{A}(M)$$
Describe the dimensional reduction of invariant connections

In particular, Yang–Mills gauge theory on $\mathcal{A}(M)$ is reduced to a type of Yang–Mills–Higgs theory on the manifold $M$

The equations of motion give $q$-deformations of known vortex equations, whose solutions possess remarkable properties

In particular desingularization of moduli spaces
$q \in \mathbb{R}_{>0}$ \hspace{1cm} $q \simeq q^{-1}$

$\mathcal{A}(SU_q(2)) :=$ $\ast$-algebra generated by $a$ and $c$, with relations

$UU^* = U^*U = 1$ \hspace{1cm} $U = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}$

$ac = qca, \hspace{1cm} ac^* = qc^*a, \hspace{1cm} cc^* = c^*c,$

$a^*a + c^*c = aa^* + q^2cc^* = 1$

Hopf $\ast$-algebra structure on $\mathcal{A}(SU_q(2))$:

$\Delta U = U \otimes U$ \hspace{1cm} $S(U) = U^*$ \hspace{1cm} $\varepsilon(U) = 1$
These dualize classical operations

\( \mathcal{A}_1 = \mathcal{A}(\text{SU}(2)), \)  polynomial functions on \( \text{SU}(2) \)

\[ \Delta : \mathcal{A}_1 \to \mathcal{A}_1 \otimes \mathcal{A}_1; \quad (\Delta f)(x \otimes y) = f(xy) \]

\[ S : \mathcal{A}_1 \to \mathcal{A}_1; \quad (Sf)(x) = f(x^{-1}) \]

\[ \varepsilon : \mathcal{A}_1 \to \mathbb{C}; \quad (\varepsilon f) = f(e) \]
A (right) action: \( \alpha : U(1) \rightarrow \text{Aut}(\mathcal{A}(\text{SU}_q(2))) \)

\[
\alpha_u \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad \text{for} \quad u \in U(1).
\]

The invariant elements form a subalgebra of \( \mathcal{A}(\text{SU}_q(2)) \), the co-ordinate algebra \( \mathcal{A}(S^2_q) \) of the standard Podleś sphere \( S^2_q \)

\[
\mathcal{A}(S^2_q) = \mathcal{A}(\text{SU}_q(2))^U(1)
\]

the algebra inclusion

\[
\mathcal{A}(S^2_q) \hookrightarrow \mathcal{A}(\text{SU}_q(2))
\]

is a noncommutative principal bundle
As a set of generators for $\mathcal{A}(S^2_q)$ we may take

$$B_- := ac^*, \quad B_+ := ca^*, \quad B_0 := cc^*.$$  

A natural complex structure on the 2-sphere $S^2_q$ for the unique 2-dimensional $\text{SU}_q(2)$-covariant calculus;

$S^2_q$ becomes a quantum Riemannian sphere or quantum projective line $\mathbb{CP}^1_q$. 

noncommutative manifolds

SU_q(2) and S^2_q

admit equivariant spectral triples \((\mathcal{A}, \mathcal{H}, D)\)
A vector space decomposition

\[ \mathcal{A}(\text{SU}_q(2)) = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n, \]

\[ \mathcal{L}_n := \mathcal{A}(\text{SU}_q(2)) \boxtimes_{\rho_n} \mathbb{C} \cong \left\{ x \in \mathcal{A}(\text{SU}_q(2)) \mid \alpha_u(x) = x (u^*)^n \right\} \]

for \( u \in \text{U}(1) \)

Each \( \mathcal{L}_n \) is a finitely-generated projective (right, say) \( \mathcal{A}(\mathbb{C}P^1_q) \)-module of rank one

module of \( \text{SU}_q(2) \)-equivariant sections of a line bundles over the quantum projective line \( \mathbb{C}P^1_q \) with degree (monopole charge) \(-n\)
Left covariant calculus

\[ \Omega^i = \mathcal{A}(SU_q(2)) \otimes \bigwedge^i \{\beta_+, \beta_-, \beta_z\} \quad 0 \leq i \leq 3 \]

\[ \Delta^{(1)}_{L}(\beta_s) = 1 \otimes \beta_s, \quad s = z, \pm, \]

\[ \Delta^{(1)}_{L} \] the left-coaction extended to 1-forms

\[ \bigoplus \bigwedge^i \{\beta_+, \beta_-, \beta_z\} = \text{the } q\text{-Grassmann algebra:} \]

\[ \beta_+ \wedge \beta_+ = \beta_- \wedge \beta_- = \beta_z \wedge \beta_z = 0 \]
\[ \beta_- \wedge \beta_+ + q^{-2} \beta_+ \wedge \beta_- = 0 \]
\[ \beta_z \wedge \beta_- + q^4 \beta_- \wedge \beta_z = 0, \]
\[ \beta_z \wedge \beta_+ + q^{-4} \beta_+ \wedge \beta_z = 0. \]

unique top form: \[ \beta_- \wedge \beta_+ \wedge \beta_z \]
Differential $d : \mathcal{A}(\text{SU}_{q}(2)) \to \Omega^{1}(\text{SU}_{q}(2))$:

$$df = (X_+ \triangleright f) \beta_+ + (X_- \triangleright f) \beta_- + (X_z \triangleright f) \beta_z,$$

$$X_z = \frac{1 - K^4}{1 - q^{-2}}, \quad X_- = q^{-1/2}FK, \quad X_+ = q^{1/2}EK$$

$E, F, K, K^{-1}$ generates the q.u.e.a. $\mathcal{U}_{q}(\text{su}(2))$:

$\mathcal{U}_{q}(\text{su}(2))$ is a Hopf $*$-algebra of twisted derivation on $\text{SU}_{q}(2)$:

$$h \triangleright x := x_{(1)} \langle h, x_{(2)} \rangle, \quad x \triangleleft h := \langle h, x_{(1)} \rangle x_{(2)}$$

with notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$

$\langle \ , \ \rangle$ a natural pairing between $\mathcal{U}_{q}(\text{su}(2))$ and $\mathcal{A}(\text{SU}_{q}(2))$
The holomorphic calculus on $\mathbb{C}P^1_q$

By restriction

$$\Omega(\mathcal{A}(S^2_q)) \simeq \mathcal{A}(S^2_q) \oplus (\mathcal{L}_{-2}\beta_- \oplus \mathcal{L}_2\beta_+) \oplus \mathcal{A}(S^2_q)$$

In particular

$$\Omega^1(\mathcal{A}(S^2_q)) = \Omega^-(\mathcal{A}(S^2_q)) \oplus \Omega^+(\mathcal{A}(S^2_q)) \simeq \mathcal{L}_{-2}\beta_- \oplus \mathcal{L}_2\beta_+$$

a complex structure

$$d = \partial + \overline{\partial}, \quad df = (X_+ \triangleright x)\beta_+ + (X_- \triangleright x)\beta_-$$

$$\partial f = (X_+ \triangleright x)\beta_+, \quad \overline{\partial} f = (X_- \triangleright x)\beta_-$$

Also,

$$\Omega^2(\mathcal{A}(S^2_q)) = \mathcal{A}(S^2_q)(\beta_+ \wedge \beta_-) = (\beta_+ \wedge \beta_-) \mathcal{A}(S^2_q)$$
Enlarging the space

For a smooth manifold $M$, consider $\overline{M} := \mathbb{C}P^1_q \times M$ with ‘coordinate’ algebra,

$$\mathcal{A}(\overline{M}) := \mathcal{A}(\mathbb{C}P^1_q) \otimes \mathcal{A}(M).$$

A coaction of $\text{SU}_q(2)$ on $\mathcal{A}(\overline{M})$; trivially on $\mathcal{A}(M)$ and with canonical coaction $\Delta_L$ on $\mathcal{A}(\mathbb{C}P^1_q)$:

$$\Delta : \mathcal{A}(\overline{M}) \longrightarrow \mathcal{A}(\text{SU}_q(2)) \otimes \mathcal{A}(\overline{M}),$$

$$b \otimes f \mapsto m_{13}\left(\Delta_L(b) \otimes (1 \otimes f)\right) = b_{(-1)} \otimes b_{(0)} \otimes f$$

for $b \in \mathcal{A}(\mathbb{C}P^1_q), f \in \mathcal{A}(M)$. Here $\Delta_L(b) = b_{(-1)} \otimes b_{(0)}$

with $b_{(-1)} \in \mathcal{A}(\text{SU}_q(2)), b_{(0)} \in \mathcal{A}(\mathbb{C}P^1_q)$.
A $\text{SU}_q(2)$-equivariant right $\mathcal{A}(\overset{\sim}{M})$-module $\mathcal{E}$ carries a coaction

$$\delta : \mathcal{E} \longrightarrow \mathcal{A}(\text{SU}_q(2)) \otimes \mathcal{E}$$

compatible with the coaction $\Delta$ of $\mathcal{A}(\text{SU}_q(2))$ on $\mathcal{A}(\overset{\sim}{M})$,

$$\delta(\varphi \cdot f) = \delta(\varphi) \cdot \Delta(f) \quad \text{for all} \quad \varphi \in \mathcal{E}, \ f \in \mathcal{A}(\overset{\sim}{M})$$

Relate $\mathcal{A}(\text{SU}_q(2))$-equivariant bundles $\mathcal{E}$ on the quantum space $\overset{\sim}{M}$ to $\text{U}(1)$-equivariant bundles $E$ over the manifold $M$. 
Proposition 1. Every finitely-generated $\text{SU}_q(2)$-equivariant projective module $\mathcal{E}$ over $\mathcal{A}(\underline{M})$ equivariantly decomposes as

$$\mathcal{E} = \bigoplus_{i=0}^{m} \mathcal{E}_i = \bigoplus_{i=0}^{m} \mathcal{L}_{m-2i} \otimes \mathcal{E}_i$$

( and uniquely up to isomorphism ), for some $m \in \mathbb{N}_0$;

$\mathcal{E}_i$ are modules of sections of (usual) vector bundles $E_i$ over $M$ with trivial $\text{SU}_q(2)$ coactions;

$\mathcal{L}_n$ are the above modules of sections of $\text{SU}_q(2)$-equivariant line bundles over $\mathbb{C}P^1_q$.

( there are also morphisms $\Phi_i \in \text{Hom}_{\mathcal{A}(\underline{M})}(\mathcal{E}_{i-1}, \mathcal{E}_i)$, of $\mathcal{A}(\underline{M})$-modules, coming from the $\text{SU}_q(2)$-coaction ).
On each $\mathcal{A}(M)$-module $\mathcal{E}_i$ in this decomposition fix an $\mathcal{A}(M)$-valued hermitian structure

$$h_i : \mathcal{E}_i \times \mathcal{E}_i \to \mathcal{A}(M).$$

Combined with the hermitian structure on the line bundles $\mathcal{L}_n$ this gives an $\mathcal{A}(M)$-valued hermitian structure on $\mathcal{E}_i$ defined by

$$h_i = \hat{h}_{m-2i} \otimes h_i : \mathcal{E}_i \times \mathcal{E}_i \to \mathcal{A}(\mathbb{C}P^1_q) \otimes \mathcal{A}(M),$$

and in turn a left $\text{SU}_q(2)$-covariant hermitian structure on $\mathcal{E}$ by

$$h = \bigoplus_{i=0}^{m} h_i : \mathcal{E} \times \mathcal{E} \to \mathcal{A}(M).$$
**Lemma 2.** A unitary connection $\nabla$ on $(\mathcal{E}, \mathcal{H})$ decomposes as

$$\nabla = \sum_{i=0}^{m} \left( \nabla_i + \sum_{j<i} \left( \beta_{ji} - \beta_{ji}^* \right) \right),$$

where:

1. Each $\nabla_i$ is a unitary connection on $(\mathcal{E}_i, \mathcal{H}_i)$, i.e.
   $$\mathcal{H}_i(\nabla_i \varphi, \psi) + \mathcal{H}_i(\varphi, \nabla_i \psi) = d \left( \mathcal{H}_i(\varphi, \psi) \right) \text{ for } \varphi, \psi \in \mathcal{E}_i.$$

2. For $j \neq i$,
   $$\beta_{ji} \in \text{Hom}_{\mathcal{A}(\mathcal{M})}(\mathcal{E}_i, \Omega^1(\mathcal{E}_j)) \text{ is the adjoint of } -\beta_{ij}, \text{ i.e.}$$
   $$\mathcal{H}(\beta_{ji} \varphi, \psi) + \mathcal{H}(\varphi, \beta_{ij} \psi) = 0 \text{ for } \varphi \in \mathcal{E}_i, \psi \in \mathcal{E}_j.$$
In an analogous way, any element $A$ of the space (of anti-hermitian elements) $\text{Hom}^a_{A(\mathcal{M})}(\mathcal{E}, \Omega^1(\mathcal{E}))$ decomposes as

$$A = \sum_{i=0}^{m} \left( A_i + \sum_{j<i} \left( A_{ji} - A_{ji}^* \right) \right),$$

with

$$A_i \in \text{Hom}^a_{A(\mathcal{M})}(\mathcal{E}_i, \Omega^1(\mathcal{E}_i)), \quad A_{ji} \in \text{Hom}_{A(\mathcal{M})}(\mathcal{E}_i, \Omega^1(\mathcal{E}_j))$$

leading to a decomposition

$$\text{Hom}^a_{A(\mathcal{M})}(\mathcal{E}, \Omega^1(\mathcal{E})) \cong \bigoplus_{i=0}^{m} \left( \text{Hom}^a_{A(\mathcal{M})}(\mathcal{E}_i, \Omega^1(\mathcal{E}_i)) \right)$$

$$\oplus \bigoplus_{j<i} \text{Hom}_{A(\mathcal{M})}(\mathcal{E}_i, \Omega^1(\mathcal{E}_j)),$$
SU\(_q\)(2)-invariant connections and gauge transformations

On \(\mathcal{E} = \bigoplus_i \mathcal{E}_i = \bigoplus_i \mathcal{L}_{m-2i} \otimes \mathcal{E}_i\) a coaction \(\Delta_\mathcal{E}\) of \(\mathcal{A}(SU_q(2))\) which combines the natural coaction of \(\mathcal{A}(SU_q(2))\) on the modules \(\mathcal{L}_{m-2i}\) and the trivial coaction on the modules \(\mathcal{E}_i\):

\[
\Delta_\mathcal{E} : \mathcal{E} \rightarrow \mathcal{A}(SU_q(2)) \otimes \mathcal{E}.
\]

‘Adjoint’ coactions of \(\mathcal{A}(SU_q(2))\) on

the space \(\mathcal{C}(\mathcal{E})\) of unitary connections,

the group \(\mathcal{U}(\mathcal{E})\) of gauge transformations,

the spaces \(\text{Hom}_{\mathcal{A}(\underline{M})}(\mathcal{E}_i, \mathcal{E}_j)\) and \(\text{Hom}_{\mathcal{A}(\underline{M})}(\mathcal{E}_i, \Omega^1(\mathcal{E}_j))\).
The corresponding space of coinvariant elements, i.e.

\[ \mathcal{C}(\mathcal{E})^{SU_q(2)} = \{ \nabla \in \mathcal{C}(\mathcal{E}) \mid \Delta^C(\nabla) = 1 \otimes \nabla \} , \]

and similarly for the other spaces and coactions

These spaces of invariants are given in terms of objects defined on \( M \) and of canonical (and unique) objects defined on \( \mathbb{C}P^1_q \).
Proposition 3. There is a bijection

\[ \mathcal{C}(\mathcal{E})^{\text{SU}_q(2)} \simeq \mathcal{C}(\mathcal{E}) := \prod_{i=0}^{m} \left( \mathcal{C}(\mathcal{E}_i) \times \text{Hom}_{A(M)}(\mathcal{E}_i, \mathcal{E}_{i+1}) \right) \]

associate, to any element \((\nabla, \phi)\) of \(\mathcal{C}(\mathcal{E})\), given by:

- connections \(\nabla_i \in \mathcal{C}(\mathcal{E}_i)\)
- Higgs fields \(\phi_{i+1} \in \text{Hom}_{A(M)}(\mathcal{E}_i, \mathcal{E}_{i+1})\),

the \(\text{SU}_q(2)\)-invariant unitary connection \(\nabla \in \mathcal{C}(\mathcal{E})^{\text{SU}_q(2)}\):

\[ \nabla = \sum_{i=0}^{m} \left( \nabla_i + \beta_+ \otimes \phi_{i+1} + \beta_- \otimes \phi_{i+1}^* \right). \]
Here $\nabla_i$ is the unitary connection on $(\mathcal{E}_i, h_i)$ given by
\[
\nabla_i = \widehat{\nabla}_{m-2i} \otimes \text{id} + \text{id} \otimes \nabla_i,
\]
with
\[
\widehat{\nabla}_{m-2i} \text{ the unique } SU_q(2)\text{-invariant unitary connection on the hermitian line bundle } (\mathcal{L}_{m-2i}, \hat{h}_{m-2i}) \text{ over } \mathbb{C}P^1_q
\]
$\beta_+ , \beta_- \text{ the basis holomorphic, anti-holomorphic, 1-forms on } \mathbb{C}P^1_q$
Proposition 4. There is a bijection

\[ \mathcal{U}(\mathcal{E})^{SU_q(2)} \simeq \mathcal{U}(\mathcal{E}) := \prod_{i=0}^{m} \mathcal{U}(\mathcal{E}_i) , \]

which associates to

any element \( u = (u_0, u_1, \ldots, u_m) \in \mathcal{U}(\mathcal{E}) \)

the \( SU_q(2) \)-invariant gauge transformation of \((\mathcal{E}, h)\)

\[ u = \sum_{i=0}^{m} u_i \quad \text{with} \quad u_i = 1 \otimes u_i \in \mathcal{U}(\mathcal{E}_i)^{SU_q(2)} \simeq \mathbb{C} \otimes \mathcal{U}(\mathcal{E}_i) . \]
Integrable connections

$M$ be a complex manifold, with standard complex structure. Combined with the complex structure for the differential calculus on $\mathcal{A}(\mathbb{C}P^q_1)$ we get a natural complex structure for the calculus on the algebra $\mathcal{A}(M) = \mathcal{A}(\mathbb{C}P^q_1) \otimes \mathcal{A}(M)$.

If $\nabla$ is a connection on the $\mathcal{A}(M)$-module $\mathcal{E}$ with equivariant decomposition as before, then the $(0,2)$-component of its curvature $F^{0,2}_{\nabla}$ is an element of $\text{Hom}_{\mathcal{A}(M)}(\mathcal{E}, \Omega^{0,2}(\mathcal{E}))$.

The connection $\nabla$ is then integrable if $F^{0,2}_{\nabla} = 0$. In this case the pair $(\mathcal{E}, \nabla)$ is a holomorphic vector bundle.
Let $\mathcal{C}(\mathcal{E}_i)^{1,1}$ be the integrable unitary connections on $(\mathcal{E}_i, h_i)$.

Let $\mathcal{C}(\mathcal{E})^{1,1} \subset \mathcal{C}(\mathcal{E})$ made of

- integrable connections $\nabla^\partial_i \in \mathcal{C}(\mathcal{E}_i)^{1,1}$

- Higgs fields $\phi^*_{i+1} \in \text{Hom}_{\mathcal{A}(M)}(\mathcal{E}_{i+1}, \mathcal{E}_i)$ s. t.

  $\nabla^\partial_{i+1,i}(\phi^*_{i+1}) := \phi^*_{i+1} \circ \nabla^\partial_{i+1} - \nabla^\partial_i \circ \phi^*_{i+1} = 0$ .

Then, Proposition 3 yields a bijection

$$\mathcal{C}(\mathcal{E})^{1,1} \simeq (\mathcal{C}(\mathcal{E})^{1,1})^{\text{SU}_q(2)}$$

with the space of $\text{SU}_q(2)$-invariant integrable connections.
Yet another ingredient: an integral on $\mathbb{CP}^1_q$

The Haar state $H$ on $\mathcal{A}(\text{SU}_q(2))$ when restricted to $\mathcal{A}(\mathbb{CP}^1_q)$ yields a faithful, invariant state, $H(a \triangleleft X) = H(a) \epsilon(X)$ for $a \in \mathcal{A}(\mathbb{CP}^1_q)$ and $X \in \mathcal{U}_q(\text{su}(2))$, with modular automorphism

$$\vartheta(a) = a \triangleleft K^2$$

such that

$$H(ab) = H(\vartheta(b) a)$$

With $\beta_- \wedge \beta_+$ the generator of $\Omega^2(\mathbb{CP}^1_q)$, the linear functional

$$\int_{\mathbb{CP}^1_q} : \Omega^2(\mathbb{CP}^1_q) \rightarrow \mathbb{C} , \quad \int_{\mathbb{CP}^1_q} a \beta_- \wedge \beta_+ := H(a)$$

defines a non-trivial $\vartheta$-twisted cyclic two-cocycle $\tau$ on $\mathcal{A}(\mathbb{CP}^1_q)$

$$\tau(a_0, a_1, a_2) := \frac{1}{2} \int_{\mathbb{CP}^1_q} a_0 \text{ da}_1 \wedge \text{ da}_2 .$$

i.e.

$$b_\vartheta \tau = 0 \quad \text{and} \quad \lambda_\vartheta \tau = \tau$$
$b_\vartheta$ is the $\vartheta$-twisted coboundary operator

$$(b_\vartheta \tau)(f_0, f_1, f_2, f_3) := \tau(f_0 f_1, f_2, f_3) - \tau(f_0, f_1 f_2, f_3) + \tau(f_0, f_1, f_2 f_3) - \tau(\vartheta(f_3) f_0, f_1, f_2)$$

$\lambda_\vartheta$ is the $\vartheta$-twisted cyclicity operator

$$(\lambda_\vartheta \tau)(f_0, f_1, f_2) := \tau(\vartheta(f_2), f_0, f_1)$$

With $M$ a connected Kähler manifold of complex dimension $d$, using this we get an integral

$$\int_M := \int_{\mathbb{CP}^1_q} \otimes \int_M : \Omega^2(\mathbb{CP}^1_q) \otimes \Omega^{2d}(M) \rightarrow \mathbb{C}$$

We set $\int_M \alpha := 0$ whenever $\alpha \notin \Omega^2(\mathbb{CP}^1_q) \otimes \Omega^{2d}(M)$.
There is also a Hodge operator (as a bimodule map)

\[ \star := \hat{\star} \otimes \star : \Omega^p(M) \longrightarrow \Omega^{2(d+1)-p}(M) \]

Let \( \mathcal{C}(\mathcal{E}) \) be the space of unitary connections on an SU\(_q(2)\)-equivariant hermitian \( A(M) \)-module \( (\mathcal{E}, h) \).

The Y–M action functional \( \text{YM} : \mathcal{C}(\mathcal{E}) \rightarrow [0, \infty) \) is as usual

\[ \text{YM}(\nabla) = \left\| F_\nabla \right\|^2_h \] (5)

from a suitable \( L^2 \)-norm \( \| - \|_h \) on the space \( \text{Hom}_{A(M)}(\mathcal{E}, \Omega^p(\mathcal{E})) \)
Dimensional reduction of the Yang–Mills action functional
Proposition 6.

The functional $\text{YM}_{C(\mathcal{E})^\text{SU}_q(2)}$ on the quantum space $\mathcal{M}$, when restricted to $\text{SU}_q(2)$-invariant unitary connections coincides with the Y–M–H functional $\text{YMH}_{q,m}$ on $\mathcal{M}$:

$$\text{YMH}_{q,m}(\nabla, \phi) = \sum_{i=0}^{m} \left( \| F_{\nabla_i} \|_{h_i}^2 + (q^2 + 1) \| \nabla_{i-1,i}(\phi_i) \|_{h_{i-1,i}}^2 ight. 
+ \left. \| \phi_i^{*} \circ \phi_i + q^2 \phi_i \circ \phi_i^{*} - q^{m-2i+1}(m-2i)_{q} \text{id}_{\mathcal{E}_i} \|_{h_i}^2 \right),$$

with

- $F_{\nabla_i} = \nabla_i^2$, the curvature of the connection $\nabla_i \in C(\mathcal{E}_i)$ on $\mathcal{M}$

- $\nabla_{i-1,i}$ the connection on $\text{Hom}_{A(M)}(\mathcal{E}_{i-1}, \mathcal{E}_i)$ induced by $\nabla_{i-1}$ on $\mathcal{E}_{i-1}$ and $\nabla_i$ on $\mathcal{E}_i$ and given by

  $$\nabla_{i-1,i}(\phi_i) = \phi_i \circ \nabla_{i-1} - \nabla_i \circ \phi_i.$$
Symbol

\[ [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \]

Also \( \phi_0 := 0 =: \phi_0^* \) and \( \phi_{m+1} := 0 =: \phi_{m+1}^* \)

This functional restricts to a map on gauge orbits

\[ \text{YMH}_{q,m} : \mathcal{C}(E) / \mathcal{U}(E) \rightarrow [0, \infty) \]
Characterize stable critical points of the Y–M functional (5) on \( M \), and study their reduction to configurations on \( M \).

generalized instantons

**Lemma 7.** Let \( \nabla \in \mathcal{C}(E) \) be a unitary connection such that

\[
\star F_{\nabla} = -F_{\nabla} \wedge \Xi
\]

for \( \Xi \in \Omega^{2d-2}(M) \) a closed form of degree \( 2d-2 \).

Then \( \nabla \) is a critical point of the Y–M functional and

\[
YM(\nabla) = Top_2(E, \Xi) := -\left( F_{\nabla}, \star (F_{\nabla} \wedge \Xi) \right)_h
\]
The functional $\text{Top}_2(\mathcal{E}, \Xi)$ does not depend on the choice of $\nabla$

It defines a ‘topological action’ depending only on the $\mathcal{A}(\mathcal{M})$-module $\mathcal{E}$ and the closed form $\Xi$

Provides an $a\ priori$ lower bound on the Y–M functional

The gauge invariant equation (8) is the $\Xi$-anti-selfduality eqn

The gauge equivalence classes in $\mathcal{C}(\mathcal{E})/\mathcal{U}(\mathcal{E})$ of solutions are

generalized instantons or $\Xi$-instantons
Holomorphic chain $q$-vortex equations

$(M, \omega)$ a Kähler manifold

A natural closed $(1, 1)$-form on $\mathcal{A}(M)$

$$\omega = (\beta_- \wedge \beta_+) \otimes 1 + 1 \otimes \omega$$

Set

$$\Xi = \frac{\omega^{d-1}}{(d-1)!} = 1 \otimes \frac{\omega^{d-1}}{(d-1)!} + (\beta_- \wedge \beta_+) \otimes \frac{\omega^{d-2}}{(d-2)!}$$

Recall the generalized instanton equation

$$\star F_\nabla = -F_\nabla \wedge \Xi$$
Proposition 9. The subspace of $SU_q(2)$ invariant connections

$$\nabla \bar{\partial} \in (\mathcal{C}(\mathcal{E})^{1,1})^{SU_q(2)}$$

solving the generalized instanton equation on $M$ corresponds bijectively to the subspace of $\mathcal{C}(\mathcal{E})^{1,1}$ of elements $(\nabla \bar{\partial}, \phi^*)$ satisfying the holomorphic chain $q$-vortex equations on $M$

$$F_{\nabla_i} = q^2 \phi_i \circ \phi_i^* - \phi_{i+1} \circ \phi_{i+1} + q^{m-2i+1} [m - 2i]_q \text{id}_{\mathcal{E}_i} \quad (10)$$

for $i = 0, 1, \ldots, m$. Here

$$F_{\nabla_i}^\omega = * \left( (F_{\nabla_i}^{1,1})^* \wedge *\omega \right) \in \text{End}_{A(M)}(\mathcal{E}_i),$$

the component of the curvature of $\nabla_i$ along the Kähler form $\omega$
Stability conditions

A hermitian finitely-generated projective $\mathcal{A}(M)$-module $(\mathcal{E}, h)$ has degree and slope given by

$$\deg(\mathcal{E}) = \frac{\text{Top}_1(\mathcal{E}, \omega)}{\text{vol}_\omega(M)} \quad \text{and} \quad \mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\text{rank}(\mathcal{E})},$$

with $\text{rank}(\mathcal{E}) = \text{tr}(\text{id}_\mathcal{E})$.

Analogously, the $(q, m)$-degree of a finitely-generated $\text{SU}_q(2)$-equivariant projective module $\mathcal{E}$ over the algebra $\mathcal{A}(M)$, with equivariant decomposition $\mathcal{E} = \bigoplus_{i=0}^m \mathcal{E}_i = \bigoplus_{i=0}^m \mathcal{L}_{m-2i} \otimes \mathcal{E}_i$ is

$$\deg_{q,m}(\mathcal{E}) = \sum_{i=0}^m \left( \deg(\mathcal{E}_i) - q^{m-2i+1} [m-2i]_q \text{rank}(\mathcal{E}_i) \right).$$
and, its \((q,m)\)-slope is

\[
\mu_{q,m}(E) = \frac{\deg_{q,m}(E)}{\text{rank}(E)}
\]

with \(\text{rank}(E) = \sum_i \text{rank}(E_i)\).

Natural topological meaning: the \(q\)-integers \([m-2i]_q\) label classes in the \(SU_q(2)\)-equivariant K-theory \(K_0^{U_q(su(2))}(\mathbb{C}P^1_q)\).

\(SU_q(2)\) acting trivially on \(M\), the \((q,m)\)-degree labels classes in the \(SU_q(2)\)-equivariant K-theory of \(M = \mathbb{C}P^1_q \times M\).

Thus the usual assignment of D-brane charges in equivariant K-theory to quiver vortices extends to our \(q\)-vortices as well.

The parameters \(q, m\) and the topology of the bundles \(E_i\) over \(M\) are constrained by the following (semi-)stability criteria.
Proposition 11. A stable $q$-quiver bundle on $M$ has slopes constrained by the inequalities:

(a) $\mu(\mathcal{E}_0) \leq q^{m+1}[m]_q$, with equality iff $\mathcal{E}_0$ admits a holomorphic connection $\nabla_0$ solving the hermitian Yang–Mills eqn
$$F_{\nabla_0}^\omega = q^{m+1}[m]_q \text{id}_{\mathcal{E}_0}.$$ 

(b) $\mu(\mathcal{E}_m) \geq -q^{-m+1}[m]_q$, with equality iff $\mathcal{E}_m$ admits a holomorphic connection $\nabla_m$ solving the hermitian Yang–Mills eqn
$$F_{\nabla_m}^\omega = -q^{-m+1}[m]_q \text{id}_{\mathcal{E}_m}.$$ 

(c) $\mu_{q,m}(\mathcal{E}) \leq 0$, with equality iff $\mathcal{E}_i$ admits a holomorphic connection $\nabla_i$ solving the hermitian Yang–Mills eqn
$$F_{\nabla_i}^\omega = q^{m-2i+1}[m-2i]_q \text{id}_{\mathcal{E}_i} \quad \text{for each} \quad i = 0, 1, \ldots, m.$$
Examples

Some explicit examples of the $q$-vortex equations

$$F_{\nabla_i} = q^2 \phi_i \circ \phi_i^* - \phi_{i+1} \circ \phi_{i+1} + q^{m-2i+1} [m - 2i]_q \text{id}_{\mathcal{E}_i}$$

$i = 0, 1, \ldots, m$.

The $q$-deformations affect stability conditions for the existence of solutions and the structure of the corresponding moduli spaces.
1. Deformations of holomorphic triples and stable pairs

A holomorphic triple \((E_0, E_1, \phi)\) on a compact Kähler manifold \((M, \omega)\) is a pair of holomorphic vector bundles \(E_0, E_1\) over \(M\) and a holomorphic morphism

\[ E_0 \xrightarrow{\phi} E_1 \]

With \(\phi := \phi_1\), we get

\[
F^\omega_{\nabla 0} = q^2 \left( \text{id}_{E_0} - q^{-2} \phi \circ \phi^* \right) \quad \text{and} \quad F^\omega_{\nabla 1} = - \left( \text{id}_{E_1} - q^2 \phi^* \circ \phi \right) \tag{\play}
\]

The degrees of the bundles are related by

\[
\deg(E_0) + q^{-2} \deg(E_1) = q^2 \text{rank}(E_0) - q^{-2} \text{rank}(E_1)
\]

Much more stringent than the undeformed stability condition
2. $q$-vortices on Riemann surfaces

$M$ a compact oriented Riemann surface of genus $g$ and scalar curvature $\kappa$. Eq. (♦) describe $q$-non-abelian vortices on $M$.

A particular case:

$\mathcal{E} := \mathcal{E}_0, \quad \nabla := \nabla_0 \quad ; \quad \mathcal{E}_1 \simeq \mathbb{C}^r \otimes \mathcal{A}(M) \quad r = \text{rank}(\mathcal{E}),$

the Higgs field $\phi = q^{-1}\varphi$ can be regarded as an element of $\mathbb{C}^r \otimes \mathcal{E}$

Also $\frac{1}{2\pi} \text{Top}_1(M, \omega) = c_1(\mathcal{E})$.

A non-empty moduli space of solutions to the $q$-vortex equations (♦) is ensured by the stability condition

$$c_1(\mathcal{E}) = \frac{4r}{\kappa} \left( q^2 - q^{-2} \right) (1 - g) \quad \text{for} \quad g \neq 1$$
Since $0 < q < 1$, this degree satisfies the bound

$$c_1(\mathcal{E}) < \frac{4q^2}{\kappa} (1 - g).$$

Hence the pair $(\mathcal{E}, \varphi)$ is $\tau$-stable and by the Hitchin–Kobayashi correspondence it is gauge equivalent to a solution of the non-abelian $q$-vortex equations.

For abelian vortices, $r = 1$, this moduli space is the $|n|$-th symmetric product orbifold of $M$, i.e. the space of effective divisors on $M$ of degree $n = c_1(\mathcal{E})$. 
3. $q$-instantons

Let $(M, \omega)$ be a Kähler surface. Set $\mathcal{E}_0 \simeq \mathcal{E}_1 =: \mathcal{E}$.

Since $\phi$ is a holomorphic section, $\nabla_{\bar{\partial}}^{0,1}(\phi) = 0$; we have $\nabla_0 = \nabla_1 =: \nabla$ and both equations in (♦) simplify to

$$F^\omega_\nabla = (q^2 - 1) \text{id}_\mathcal{E}$$

a deformation of the hermitian Yang–Mills equation on $M$, and hence of the standard anti-selfduality equations $\star F_\nabla = -F_\nabla$. Its gauge equivalence classes of solutions are thus called $q$-instantons.
The natural $\mathcal{U}(\mathcal{E})$-invariant symplectic form $\omega_C$ on $\mathcal{C}(\mathcal{E})$:

$$\omega_C(\alpha, \alpha') = \frac{1}{2} \int_M \text{tr} (\alpha^* \wedge \alpha') \omega \omega^2$$

for $\alpha, \alpha' \in \text{Hom}^a_{\mathcal{A}(M)}(\mathcal{E}, \Omega^1(\mathcal{E}))$.

Then corresponding moment map $\mu_C : \mathcal{C}(\mathcal{E}) \to (\text{Lie } \mathcal{U}(\mathcal{E}))^*$ is

$$\mu_C(\nabla) = F^\omega_\nabla.$$

The moduli space of $q$-instantons on $M$ is the symplectic quotient

$$\mu_C^{-1}((q^2 - 1) \text{id}_\mathcal{E}) / \mathcal{U}(\mathcal{E}),$$

and $q$-vortices correspond to points of $\mu_C^{-1}((q^2 - 1) \text{id}_\mathcal{E})$ which lie inside the Kähler submanifold $\mathcal{C}(\mathcal{E})^{1,1}$ (outside the singularities).
When $M = \mathbb{C}^2$, the constant shift in the moment map condition

\[ \mu_C = 0 \quad \text{to} \quad \mu_C = (q^2 - 1) \operatorname{id}_\mathcal{E} \]

induces a shift in the corresponding real ADHM equation.

**NS:** this modification arises in the equations which determine
instantons on a certain noncommutative deformation of $\mathbb{R}^4$.

Here we have the same sort of resolution of instanton moduli
space via our $q$-deformed dimensional reduction procedure over
the quantum projective line $\mathbb{CP}_q^1$. 
Summing up:

Characterized vector bundles over the quantum space

\[ \mathcal{M} := \mathbb{CP}_q^1 \times M \]

equivariant under an action of the quantum group \( SU_q(2) \)

Described the dimensional reduction of invariant connections

In particular, Yang–Mills gauge theory on \( A(\mathcal{M}) \) is reduced to a type of Yang–Mills–Higgs theory on the manifold \( \mathcal{M} \)

The equations of motion give \( q \)-deformations of known vortex equations, whose solutions possess remarkable properties.
Thank you