On the relation between quantum mechanics with a magnetic field on $\mathbb{R}^n$ and a torus $\mathbb{T}^n$

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Introduction

Tori are simplest manifolds with nontrivial topology, quotients $\mathbb{T}^n = \mathbb{R}^n/\Lambda$, with $\Lambda \subset \mathbb{R}^n$ a lattice of rank n. Here:

$U(1)$-gauge theory of a scalar charged particle on $\mathbb{T}^n$. $\equiv Th_1$	$\Leftrightarrow$	$U(1)$-gauge theory of a scalar ch. particle on $\mathbb{R}^n$ *quasiperiodic* under translations $\lambda \in \Lambda$. $\equiv Th_2$

Th_2 is: a) much simpler: global description complex line bundle;
b) physically relevant in itself: quantum mechanics of a Bloch electron in periodic magnetic field and scalar potential $\mathbf{V}$ ($\hbar = 1$)

$$\mathsf{H}\psi = i\partial_t\psi, \qquad \mathsf{H} = \frac{1}{2m}\nabla_a\nabla_a + \mathbf{V}, \qquad \nabla_a = -i\partial_a + qA_a. \quad (1)$$

Topic not new. Many results = 'smooth' analog of 'holomorphic' ones on *complex* tori (only $n=2m$) see e.g. [BirLan04,Mum83]. Here $\forall n \geq 1$, all gauges, no additional structure needed.

But our treatment is based only on the basic definition of a real torus, thus showing that additional structures such as holomorphic ones on *complex* tori (which can arise only for $n = 2k$), complex abelian varieties, etc., are not necessary. Moreover, it is valid for all gauges and all n (this is not so e.g. in [**?**, **?**]). Also the physical identification of the central generator of $\mathbf{g}_Q$ as the electric charge operator seems new.

Translation symmetry becomes ‘projective’, w.r.t. a Lie group $G_Q \simeq \mathbb{R}^k \times$Heisenberg group = translation group G_0 centrally extended with electric charge operator Q.
$\exists g \in GL(n)$ such that $g\Lambda = 2\pi\mathbb{Z}^n$, so can choose $\Lambda = 2\pi\mathbb{Z}^n$ without loss of generality ($\Rightarrow$ reciprocal lattice = $\mathbb{Z}^n$).

PLAN

1. Introduction
2. Th_2: quasiperiodicity factors V, $\psi \in \mathcal{X}^V$, ∇_a, G_Q ,...
3. $\mathcal{X}^V \xrightarrow{\sim} \Gamma(\mathbb{T}^n,E)$:=space of sections of a hermitean line bundle $E \overset{\pi}{\mapsto} \mathbb{T}^n$ (Th_1). $V \leftrightarrow$ transition functions on E
4. Decomposition and irreducible representations (for $q \in \mathbb{Z}$) of the algebra $\mathcal{O}$ and of the group Y of observables, *explicitly* in configuration space.
5. Conclusions and perspectives. (Henceforth: **LENSES!!**)

2. Quasiperiodic wavefunctions and related ∇ on $\mathbb{R}^n$

$$\mathcal{X}^V := \{\psi \in C^\infty(\mathbb{R}^n) \quad | \quad \psi(x{+}2\pi l) = V(l,x)\,\psi(x) \qquad \forall\, x \in \mathbb{R}^n,\ l \in \mathbb{Z}^n\}, \qquad (2)$$

$\psi, \mathcal{X}^V$ well-defined $\Leftrightarrow$

$$\begin{tikzcd} x \arrow[rr, "l+l'"] \arrow[dr, "l"'] & & x{+}2\pi(l{+}l') \\ & x{+}2\pi l \arrow[ur, "l'"'] & \end{tikzcd}$$

is commutative $\Leftrightarrow$ $V(l{+}l',x) = V(l,x{+}2\pi l')V(l,x).$ (3)

A connection $\nabla = -id + A(x)Q$ **is compatible with $\mathcal{X}^V$ iff** $\nabla_a : \mathcal{X}^V \mapsto \mathcal{X}^V$**.**

$$\Rightarrow \qquad -2iQB_{ab}\cdot = [\nabla_a, \nabla_b] \ :\ \mathcal{X}^V \mapsto \mathcal{X}^V \qquad \Rightarrow \qquad B_{ab}(x) \ \text{periodic} \qquad \Rightarrow$$

$$B_{ab}(x) = \tfrac{1}{2}\beta^A_{ab} + \underbrace{\sum_{l\neq 0} \beta^l_{ab} e^{il\cdot x}}_{B'_{ab}(x)} \qquad \Rightarrow \qquad A_a(x) = \tfrac{1}{2}x^b\beta^A_{ba} + \underbrace{A'_a(x)}_{\text{periodic, } B'=dA'} \text{ (+ gauge tr.)}$$

and $\nabla_a\psi \in \mathcal{X}^V$ if $V(l,x) = e^{-iq\pi l^t\beta^A x}$ (mod gauge tr); $\overset{(3)}{\Rightarrow}$ $\nu_{ab} = \pi\beta^A_{ab} \in \mathbb{Z}$ (4)

On $\mathbb{T}^n$ $\beta^A_{ab}dx^a dx^b$ identifies 1st Chern class; flux ϕ_{ab} of B through the (a,b) 2-cycle is $= 2\pi\nu_{ab}$

$$\nabla_a = p_a + QA'_a(x), \qquad p_a := -i\partial_a + \frac{Q}{2} x^b \beta^A_{ba} + Q\alpha_a \ \text{(+ gauge tr.)} \tag{5}$$

$Q, p_a, f\cdot, \nabla_a, \mathsf{H} \ : \ \mathcal{X}^V \mapsto \mathcal{X}^V$; they belong to the $*$-algebra of observables $\mathcal{O} \equiv$ algebra of polynomials in $Q, p_1, ..., p_m$ with coefficients f in $\mathcal{X} := \mathcal{X}^1$, constrained by

$$[p_a, p_b] = -i\beta^A_{ab} Q, \qquad [Q, \cdot] = 0, \qquad [p_a, f] = -i(\partial_a f). \tag{6}$$

Note that (6) depend on A only through β^A, in particular are gauge-independent.
Q, p_a generate the real Lie algebra $\mathbf{g}_Q$ of a Lie group G_Q. We define groups G_Q, L, Y_Q by

$$\begin{aligned}
&G_Q := \left\{ g_{(z^0,z)} := e^{i[p\cdot z + \frac{Q}{2} z^0]} \mid (z^0, z) \in \mathbb{R}^{n+1} \right\}, \qquad \text{“projective translation group”} \\
&L := \{ e^{il\cdot x + h} \mid l \in \mathbb{Z}^n, \ h \in \mathbb{R} \}, \\
&Y_Q := \{ e^{i[l\cdot x + h + \frac{Q}{2} z^0 + p\cdot z]} \mid l \in \mathbb{Z}^n, \ (h, z^0, z) \in \mathbb{R}^{n+2} \} \qquad \text{“ observables’ group”};
\end{aligned} \tag{7}$$

the group law from Weyl forms of (6). Actually $Y_Q = G_Q \ltimes L$. $y^* = y^{-1}$,
$y : \mathcal{X}^V \mapsto \mathcal{X}^V$ for all $y \in Y_Q$: $t \in L$ act by multiplication, whereas $g \in G_Q$ act by

$$\left[g_{(z^0,z)} \psi \right](x) = \underbrace{e^{iq\left[z^0 + x^t \frac{\beta^A}{2} z \right]}}_{\text{phase factor}} \underbrace{\psi(x+z)}_{\text{translation}} \quad \text{(up to gauge tr.)}. \tag{8}$$

$$\frac{i}{4\pi}\log\left[\frac{V(l,x+2\pi l')V(l',x)}{V(l',x+2\pi l)V(l,x)}\right] = ql^t\nu l' =: \tilde{\nu}(l,l') \tag{9}$$

is x-independent (and gauge-independent) and defines an antisymmetric bilinear $\mathbb{Z}$-valued form $\tilde{\nu}$ on $\Lambda = \mathbb{Z}^n$: $2\pi\beta^A = \nu$ can be recovered from V.

Let $r := \frac{1}{2}\text{rank}(\beta^A)$; it is $r \in \mathbb{N}_0$. By the Frobenius theorem $\exists$ a matrix S with $S_{ab} \in \mathbb{Z}$, $\det S = \pm 1$ such that after the change of generators

$$p_a \mapsto (S^t p)_a, \qquad\qquad x^a \mapsto (S^{-1}x)^a, \tag{10}$$

$[x^a, p_b] = i\delta^a_b$ remain true, while $(6)_1$ become

$$[p_j, p_{r+j}] = ib_jQ \qquad j = 1, ..., r, \qquad\qquad [p_a, p_b] = 0 \qquad \text{otherwise}, \tag{11}$$

where $\nu_j := \pi b_j \in \mathbb{N}$. This shows that

$$\mathbf{g}_Q \simeq \mathbf{h}_{Q\,2r+1} \oplus \mathbb{R}^{n-2r}, \qquad\qquad G_Q \simeq H_{Q\,2r+1} \times \mathbb{R}^{n-2r}; \tag{12}$$

$\mathbf{h}_{Q\,k}$,$H_{Q\,k}$ denote the Heisenberg Lie algebra, group of dimension k and central generator Q.

$$(\psi', \psi) := \int_0^{2\pi} dx^1 ... \int_0^{2\pi} dx^n \ \overline{\psi'(x)}\, \psi(x) \tag{13}$$

defines an Hermitean structure on $\mathcal{X}^V$, as $\overline{\psi'}\psi$ is periodic; p_a, ∇_a are essentially self-adjoint.
We shall call $\mathcal{H}^V$ the Hilbert space completion of $\mathcal{X}^V$.
Y_Q extends as a group of unitary transformations of $\mathcal{H}^V$, acting in the same way.

Gauge transformation $U(x)$ = unitary transformation $\big(\rho(Y_Q), \mathcal{H}^V\big) \mapsto \big(\rho^U(Y_Q), \mathcal{H}^{V^U}\big)$:

$$\psi \mapsto \psi^U = U\psi, \qquad V \mapsto V^U, \qquad p_a \mapsto p_a^U = Up_aU^*, \qquad f \mapsto f. \tag{14}$$

This definition works also for the associated representation of Y_Q as a group of unitary operators on a Hilbert space.
All the previous relations (2-13) remain valid.

After the change (10) and the gauge transformation $U(x) = e^{i\frac{q}{4}x^t\beta^S x}$, with β^S such that $\beta := \beta^A + \beta^S$ is lower-triangular, quasiperiodicity (2) implies that each $\psi \in \mathcal{X}^\beta := \mathcal{X}^{V^U}$ is periodic in $x^1, ..., x^r, x^{2r+1}, ..., x^n$ and can be expressed as

$$\psi(x) = \sum_{h\in H}\sum_{l\in\mathbb{Z}^r} e^{i\sum\limits_{j=1}^{r}(h_j+2q\nu_j l_j)x^j} \psi_h(x^{r+1}+2\pi l_1, ..., x^{2r}+2\pi l_r, x^{2r+1}, ..., x^n) \tag{15}$$

in terms of finitely many functions $\psi_h(x^{r+1}, ..., x^n) \in \mathcal{L}^2\left(\mathbb{R}^r \times \mathbb{T}^{n-2r}\right)$, with

$$H := \{0, 1, ..., |2q\nu_1|-1\} \times ... \times \{0, 1, ..., |2q\nu_r|-1\} \subset \mathbb{Z}^r. \tag{16}$$

Replacing in (13) we find

$$(\psi', \psi) \; = \; (2\pi)^r \sum_{h\in H} \int_{\mathbb{R}} dx^{r+1} ... \int_{\mathbb{R}} dx^{2r} \int_0^{2\pi} dx^{2r+1} ... \int_0^{2\pi} dx^n \; \overline{\psi'_h}\psi_h. \tag{17}$$

3. Mapping $\mathcal{X}^V \xrightarrow{\sim} \Gamma(\mathbb{T}^n, E)$

$T_l : x \in \mathbb{R}^n \mapsto x{+}2\pi l \in \mathbb{R}^n \quad (l \in \mathbb{Z}^n)$ defines a free action of the abelian group $\mathbb{Z}^n$ on $\mathbb{R}^n$. Let $x \sim x'$ iff $x' = T_l(x)$ for some $l \in \mathbb{Z}^n$. The definition $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ amounts to

$$\mathbb{T}^n = \mathbb{R}^n/\!\sim\; \ni [x] = \{T_l(x),\ l \in \mathbb{Z}^n\} \tag{18}$$

Universal cover map: $P : x \in \mathbb{R}^n \mapsto [x] \in \mathbb{T}^n$. Similarly, if $V : \mathbb{Z}^n \times \mathbb{R}^n \mapsto U(1)$ fulfills (3),

$$\chi_l^V : (x, c) \in \mathbb{R}^n \times \mathbb{C} \mapsto \Big(x{+}2\pi l\,,\, V(l,x)\, c\Big), \qquad\qquad l \in \mathbb{Z}^n \tag{19}$$

defines a free action of $\mathbb{Z}^n$ on $\mathbb{R}^n \times \mathbb{C}$. Let $(x, c) \sim_V (x', c')$ iff $(x', c') = \chi_l^V[(x, c)]$ for some $l \in \mathbb{Z}^n$; define

$$E = (\mathbb{R}^n \times \mathbb{C})/\sim_V\; \ni [(x, c)] = \{\chi_l^V\big((x, c)\big),\ l \in \mathbb{Z}^n\}. \tag{20}$$

$\pi\big([(x, c)]\big) := [x]$ defines the projection $\pi : E \mapsto \mathbb{T}^n$. E is trivial (i.e. $E = \mathbb{T}^n \times \mathbb{C}$) if $V(l, x) \equiv 1$. Given a smooth function $\psi : \mathbb{R}^n \mapsto \mathbb{C}$ fulfilling (2) we can define a smooth map

$$\boldsymbol{\psi} : [x] \in \mathbb{T}^n \mapsto \Big[\Big(x, \psi(x)\Big)\Big] = \Big\{\chi_l^V\Big[\Big(x, \psi(x)\Big)\Big],\ l{\in}\mathbb{Z}^n\Big\} \in E,$$

i.e. a (global) section $\boldsymbol{\psi} \in \Gamma(\mathbb{T}^n, E)$.

The correspondence $\psi \in \mathcal{X}^V \mapsto \boldsymbol{\psi} \in \Gamma(\mathbb{T}^n, E)$ is one-to-one.
The dx^a generate the exterior algebra $\bigwedge^*$ not only on $\mathbb{R}^n$ but also on $\mathbb{T}^n$, and by multiplication by $f \in \mathcal{X}$ and linearity the algebra $\Omega^*(\mathbb{T}^n)$ of differential forms on $\mathbb{T}^n$. We can consider $\Omega^*(\mathbb{T}^n)$ as a subspace of $\Omega^*(\mathbb{R}^n)$, and d on $\Omega^*(\mathbb{T}^n)$ as the restriction of d on $\Omega^*(\mathbb{R}^n)$.
Given $\boldsymbol{\psi}, \boldsymbol{\psi}' \in \Gamma(\mathbb{T}^n, E)$, then $\overline{\boldsymbol{\psi}'}\boldsymbol{\psi}([x]) := \overline{\psi'(x)}\psi(x)$ defines a function $\overline{\boldsymbol{\psi}'}\boldsymbol{\psi} \in \mathcal{X}$, and

$$(\boldsymbol{\psi}', \boldsymbol{\psi})_\Gamma := \int_{\mathbb{T}^n} d^n x \; \overline{\boldsymbol{\psi}'}\, \boldsymbol{\psi} = \int_{C^y_{1\ldots n}} d^n x \; \overline{\psi'(x)}\, \psi(x) =: (\psi', \psi) \tag{21}$$

a hermitean structure $(\;,\;)_\Gamma$ equal to that of $\mathcal{X}^V$ and making $\Gamma(\mathbb{T}^n, E)$ a hermitean line bundle.

Therefore we can and shall identify ψ with $\boldsymbol{\psi}$, $\mathcal{X}^V$ with $\Gamma(\mathbb{T}^n, E)$, etc.
The actions of $\mathcal{O}$, $\mathbf{g}$, Y, G are lifted from $\mathcal{X}^V, \mathcal{H}^V$ to $\Gamma(\mathbb{T}^n, E)$ and its completion, e.g. we find

$$[g_{\tilde{z}}\boldsymbol{\psi}]([x]) := \Big[\Big(x, [g_{\tilde{z}}\psi](x)\Big)\Big]\,. \tag{22}$$

Given a smooth gauge transformation $U(x)$ on $\mathbb{R}^n$, replacing $\big(V, \psi, \nabla\big) \to \big(V^U, \psi^U, U\nabla U^{-1}\big)$ [see (14)] yields a new hermitean line bundle $E^U = (\mathbb{R}^n \times \mathbb{C})/\sim_{V^U}$ isomorphic to E, a section $\boldsymbol{\psi}^U \in \Gamma(\mathbb{T}^n, E^U)$ isomorphic to $\boldsymbol{\psi} \in \Gamma(\mathbb{T}^n, E)$ and a covariant derivative ∇^U on $\Gamma(\mathbb{T}^n, E^U)$ isomorphic to ∇ on $\Gamma(\mathbb{T}^n, E)$; in other words, they result into a gauge transformation on $\mathbb{T}^n$.

The above data determine also trivializations of $E, \Gamma(\mathbb{T}^n, E), \nabla$ in a canonical way.
Let $\{X_i\}$ be a finite open cover of $\mathbb{T}^n$, and $\forall i \;\; W_i \subset \mathbb{R}^n$ s. t. $P_i \equiv P|_{W_i} : W_i \mapsto X_i$ is invertible. Let

$$c_{ij}(u) := V\left\{\frac{1}{2\pi}\left[P_i^{-1}(u) - P_j^{-1}(u)\right], P_j^{-1}(u)\right\}, \qquad u \in X_i \cap X_j,$$

$$\psi_i(u) := \psi[P_i^{-1}(u)], \qquad A_{ia}(u) := A_a[P_i^{-1}(u)], \qquad U_i(u) := U[P_i^{-1}(u)], \qquad u \in X_i$$

$$\overset{(2),(3)}{\Rightarrow} \qquad \psi_i = c_{ij}\psi_j \quad \text{in } X_i \cap X_j, \qquad c_{ik} = c_{ij}c_{jk} \quad \text{in } X_i \cap X_j \cap X_k \tag{23}$$

Therefore $\{(X_i, \psi_i, A_i)\}$ defines a trivialization of a section $\boldsymbol{\psi}$ of a line bundle $E \overset{\pi}{\mapsto} \mathbb{T}^n$ with connection A of Chern numbers ν_{ab}. Given a partition $\{X'_\tau\}$ of $\mathbb{T}^n$ s. t. $X'_\tau \subset X_{i(\tau)}$, then

$$(\boldsymbol{\psi}', \boldsymbol{\psi}) := \sum_\tau \int_{X'_\tau} d^n x \, \overline{\psi'_{i(\tau)}} \, \psi_{i(\tau)} \overset{(18)}{=} \int_{[0,2\pi]^n} d^n x \, \overline{\psi'} \, \psi = (\psi', \psi) \tag{24}$$

Similarly, ϕ_{ab} = integral of B over the (a, b)-plaquette, etc.

$\{(X_i, U_i)\}$ defines a gauge transformation; if $U(x)$ is periodic (global gauge tr.) $c_{ij}^U = c_{ij}$.
Changing W_i for fixed $\{X_i\}$, or changing $\{X_i\}$, amounts to gauge transformations.
Up to gauge transformations the map $\mathcal{X}^V \mapsto \Gamma(\mathbb{T}^n, E)$ is one-to-one, and we can identify $\mathcal{X}^V \simeq \Gamma(\mathbb{T}^n, E)$.

4. Physical representations of Y_Q , $\mathcal{O}_Q$

Replace $Q \mapsto q \in \mathbb{Z}$ and drop $_Q$: $Y, G, g, \mathcal{O}, \mathbf{h}_k, H_k$. $\mathcal{Y} \equiv$group C^*-algebra of Y.
Abbreviate $\tilde{\alpha} := q\alpha$, $\tilde{\beta^A} := q\beta^A$, $\tilde{\nu}_j := q\nu_j \in \mathbb{N}$, etc. All commutation rel. depend only on $\tilde{\beta^A}$.
Do the Frobenius transformation $(x, p) \mapsto (S^{-1}x, S^t p)$, and let

$$m_{r+j} := \exp[i(x^j + \pi p_{r+j}/\tilde{\nu}_j)], \qquad m_j := \exp[i(x^{r+j} - \pi p_j/\tilde{\nu}_j)].$$

Proposition 1. *Y decomposes into a product of commuting subgroups as follows*

$$Y = M^1 \ldots M^r \; H_3^1 \ldots H_3^r \; Y_{2r+1} \ldots Y_n, \tag{25}$$

- *M^j is discrete, generated by* $m_j, m_{r+j}, e^{\frac{i\pi}{\tilde{\nu}_j}}$, *that fulfill* $m_j m_{r+j} = m_{r+j} m_j e^{\frac{i\pi}{\tilde{\nu}_j}}$, *and by their inverses;*
- $H_3^j := \{e^{i(h + w p_j + z p_{r+j})} \mid (h, w, z) \in \mathbb{R}^3\} \simeq$ *3-dim Heisenberg Lie group*
- $Y_a := \{e^{i(l x^a + h + z p_a)} \mid l \in \mathbb{Z},\ (h, z) \in \mathbb{R}^2\} \simeq$ *observables' group on a circle.*

$\zeta_j := (m_j)^{2\tilde{\nu}_j}$, $\zeta_{r+j} := (m_{r+j})^{2\tilde{\nu}_j}$ *are central* $(j = 1, \ldots, r)$.

$\mathcal{Z}(Y) = \mathcal{Z}^{(n)} U(1)$, *where* $\mathcal{Z}^{(n)}$ *is the group generated by* ζ_j, ζ_{r+j} *and their inverses.*

Corollary: M^j commute with $\mathsf{H}_0 := \sum_{a=1}^n (p_a)^2$ so are 'magnetic translation groups' [Zak].

$\mathbf{n=1}$ (quantum mechanics on a circle S^1). The Casimir eigenvalue $\zeta = e^{i2\pi\tilde\alpha}$ ($\tilde\alpha \in \mathbb{R}/\mathbb{Z}$) identifies the inequivalent irreps of $Y, \mathcal{Y}$: $(\rho_{\tilde\alpha}, \mathcal{L}^2(S^1))$ with

$$\rho_{\tilde\alpha}\left[e^{ilx}\right]\psi(x) = e^{ilx}\psi(x), \qquad \rho_{\tilde\alpha}(e^{izp})\psi(x) = e^{i\tilde\alpha z}\psi(x+z). \tag{26}$$

Replacing $(20)_2$ by $\rho_{\tilde\alpha}(p)\psi = (\tilde\alpha - i\partial_x)\psi$ defines the associated irrep $(\rho_{\tilde\alpha}, \mathcal{C}^\infty(S^1))$ of $\mathcal{O}$. $\{\frac{e^{ilx}}{\sqrt{2\pi}}\}_{l\in\mathbb{Z}}$ is an orthonormal basis consisting of eigenvectors of p: $\rho_{\tilde\alpha}(p)e^{ilx} = (l+\tilde\alpha)e^{ilx}$.

$\mathbf{n=2=2r}$, implying $Y = M\,\mathbf{H}_3$. The Casimir eigenvalues $\zeta_a = e^{i2\pi\tilde\alpha_a}$ ($\tilde\alpha \in \mathbb{R}^2/\mathbb{Z}^2$) identify the inequivalent irreps of $Y, \mathcal{Y}$: $(\rho_{\tilde\alpha}, \mathcal{H}_{\tilde\alpha})$ with

$$\mathcal{H}_{\tilde\alpha} = \bigoplus_{h=0}^{2\tilde\nu-1} \mathcal{H}_{\tilde\alpha,h}, \qquad \begin{array}{l} \rho_{\tilde\alpha}(m_2)\,\mathcal{H}_{\tilde\alpha,h} = e^{i\frac{\pi}{\tilde\nu}(\tilde\alpha_2-h)}\mathcal{H}_{\tilde\alpha,h}, \qquad \rho_{\tilde\alpha}(m_1)\mathcal{H}_{\tilde\alpha,h} = \mathcal{H}_{\tilde\alpha,h+1} \\ \rho_{\tilde\alpha}(H_3) \text{ as Schrödinger representation on } \mathcal{H}_{\tilde\alpha,h} \simeq \mathcal{L}^2(\mathbb{R}) \end{array} \tag{27}$$

Setting $a := \frac{p_1+ip_2}{\sqrt{2\tilde b}}$, $a^* := \frac{p_1-ip_2}{\sqrt{2\tilde b}}$, $\mathbf{n} := a^*a$, we find $[a, a^*] = \mathbf{1}$.
$\{\psi_{\mathrm{n},h}\}_{\mathrm{n}\in\mathbb{N}_0} \equiv$ orthonormal basis of $\mathcal{H}_{\tilde\alpha,h}$, $\quad \mathbf{n}\,\psi_{\mathrm{n},h} = \mathrm{n}\,\psi_{\mathrm{n},h}$. $\quad a\,\psi_{0,h} = 0$.

$$\begin{array}{l} \psi_{0,0}(x;\tilde\alpha) = N \sum\limits_{k\in\mathbb{Z}} e^{ikx^1 - \frac{1}{2\tilde b}\left(\tilde b x^2 + k + \tilde\alpha_1 + i\tilde\alpha_2\right)^2}, \qquad \psi_{\mathrm{n},h} = \rho_{\tilde\alpha}\left[\frac{(a^*)^{\mathrm{n}}}{\sqrt{\mathrm{n}!}}(m_1)^h\right]\psi_{0,0}, \\ \rho_{\tilde\alpha}(a^*) = \frac{-\partial_2 - i\partial_1 + \tilde b x^2 + \tilde\alpha_1 - i\tilde\alpha_2}{\sqrt{2\tilde b}}, \qquad m_1 = e^{\frac{1}{b}(i\tilde\alpha_1+\partial_1)}, \qquad m_2 = e^{ix^1 + \frac{1}{b}(i\tilde\alpha_2+\partial_2)}. \end{array} \tag{28}$$

The $\psi_{\mathrm{n},h}$ are related to Jacobi Theta function $\vartheta(z,\tau) := \sum_{k\in\mathbb{Z}} e^{i\pi k(2z+k\tau)}$ by

$$\psi_{\mathrm{n},h}\big(x;\tilde{\alpha}\big) = g(x)\,\frac{N}{\sqrt{\mathrm{n}!}}\left[\frac{\partial_z+2\pi\tilde{\alpha}_2}{i\sqrt{2\pi\tilde{\nu}}}\right]^{\mathrm{n}}\vartheta\Big[z+\frac{h-\tilde{\alpha}_2+i\tilde{\alpha}_1}{2\tilde{\nu}},\,\frac{i}{2\tilde{\nu}}\Big], \qquad z=\frac{x^1+ix^2}{2\pi}, \tag{29}$$

$g(x) := e^{\frac{-(\tilde{b}x^2+\tilde{\alpha}_1+i\tilde{\alpha}_2)^2}{2\tilde{b}}}$. Hence, up to the gaussian factor $g(x)$ the $\psi_{\mathrm{n},h}$ are *analytic* (actually *entire*) functions of z

By Proposition 1, irreps for $n\geq 3$ are Tensor Products of irreps for $n=1,2$.

$\mathcal{H}_{\tilde{\alpha}}$ carries also the associated representation of the group algebra $\mathcal{Y}$ of Y (a C^*-algebra).

Conclusions and perspectives

The basic notion of *quasiperiodicity factors* $V : \mathbb{Z}^n \times \mathbb{R}^n \mapsto U(1)$ allows a clean, gauge-independent formulation of quantum mechanics of a charged scalar particle on $\mathbb{R}^n$ in a magnetic field and a scalar potential periodic under translations in Λ.

This formulation can be reinterpreted and re-used for the same theory on the torus $\mathbb{T}^n = \mathbb{R}^n/\Lambda$, using the one-to-one correspondences between smooth quasiperiodicity factors V and hermitean line bundles $E \stackrel{\pi}{\mapsto} \mathbb{T}^n$, pre-Hilbert spaces $\mathcal{X}^V$ of the type (2) & (3) and pre-Hilbert spaces $\Gamma(\mathbb{T}^n,E)$ of smooth sections of E, and between the covariant derivatives, algebras/groups of observables acting on $\mathcal{X}^V, \mathcal{H}^V$ and those acting on $\Gamma(\mathbb{T}^n,E)$ and the Hilbert space completion of the latter. Working on the former is an elegant and convenient way to avoid the bothering work with local trivializations.

Generalization to gauge theories with more complicated gauge groups, particles with spin, etc., seem at hand.

These formulations seem well-suited for applying noncommutative deformations (gauge theories on noncommutative tori).