Propagation, Observation, Control and Numerical Approximation of Waves

July 23, 2003

Enrique Zuazua[1]
Departmento de Matemáticas
Universidad Autónoma
28049 Madrid, Spain
enrique.zuazua@uam.es

Abstract

We survey existing work on some topics related with the propagation of waves and its discrete numerical approximation. We focus on the problem of *observability* arising in control theory but relevant also in other contexts like optimal design and inverse problems. The problem of observability consists in analyzing whether the total energy of solutions can be estimated by means of partial measurements on a subset of the domain or of the boundary.

Most of the paper is devoted to analyzing the constant-coefficient, scalar, linear wave equation. We show that for most numerical schemes, due to high frequency spurious solutions for which the group velocity is not that of the continuous equation, observability may be lost under numerical discretizations when the mesh size tends to zero. We then discuss some possible remedies: Tychonoff regularization, multigrid methods, mixed finite elements, numerical viscosity and filtering of high frequencies. As a consequence of the analysis, we conclude that controlling a numerical discretized model is not a guarantee of computing a good numerical approximation of the control of the continuous model. In other words, the classical convergence (consistency+stability) property of a numerical scheme does not suffice to guarantee its suitability for providing good approximations of the controls. We also briefly discuss the same topics for the heat and Schrödinger equations and we show that both diffusive and dispersive effects

[1]Supported by grant BFM2002-03345 of the MCYT, Spain and the Networks "Homogenization and Multiple Scales" and "New materials, adaptive systems and their nonlinearities: modelling, control and numerical simulation (HPRN-CT-2002-00284)" of the EU.

may help to retain the observability properties at the discrete level. We conclude with a list of open problems and future subjects of research.

Contents

1 Introduction

In recent years important progress has been made on problems of observation and control of wave phenomena. There is now a well established theory for wave equations with sufficiently smooth coefficients for a number of systems and variants: Lamé and Maxwell systems, Schrödinger and plate equations, etc. However, when waves propagate in highly heterogeneous or discrete media much less is known.

These problems of observability and controllability can be stated as follows:

- *Observability.* Assuming that waves propagate according to a given wave equation and with suitable boundary conditions, can one guarantee that their whole energy can be estimated (independently of the solution) in terms of the energy concentrated on a given subregion of the domain (or its boundary) where propagation occurs in a given time interval ?

- *Controllability.* Can solutions be driven to a given state at a given final time by means of a control acting on the system on that subregion?

It is well known that the two problems are equivalent provided one chooses an appropriate functional setting, which depends on the equation (see for, instance, [54],[93]). It is also well known that in order for the observation and/or control property to hold, a *Geometric Control Condition (GCC)* should be satisfied [5]. According to the GCC all rays of Geometric Optics should intersect the observation/control region in the time given to observe/control.

In this work we shall mainly focus on the issue of how these two properties behave under numerical approximation schemes. More precisely, we shall discuss the problem of whether, when the continuous wave model under consideration has an observation and/or control property, it is preserved for numerical approximations, and whether this holds uniformly with respect to the mesh size so that, in the limit as the mesh size tends to zero, one recovers the observation/control property of the continuous model.

But, before getting into the matter, let us briefly indicate some of the industrial and/or applied contexts in which this kind of problems arises. The interested reader on learning more on this matter is refered to the SIAM Report [79], or, for more historical and engineering oriented applications, to [53]. The problem of controllability is classical in the field of Mathematical Control Theory and Control Engineering. We refer to the books by Lee and Marcus [51] and Sontag [80] for a classical and more modern, respectively, account of the theory for finite-dimensional systems with applications. The book by Fattorini [27] provides an updated account of theory in the context of semigroups which is therefore more adapted to the models governed by partial differential equations (PDE) and provides also some interesting examples of applications. Being more especific, the problems of controllability and/or observability are in fact only some of those arising in the applications of control theory nowadays. In fact, an important part of the modelling effort needs to be devoted to defining the appropriate control problem to be addressed. But, whatever the control question we address is, the deep mathematical issues that arise when facing these problems of observability and controllability end up entering in one way or another. Indeed, understanding the properties of observation and controllability for a given system requires understanding, first, the fine dynamical properties of the system and, second, the effect of the controllers on its dynamics. In the context of control for PDE one needs to distinguish, necessarily, the elliptic, parabolic and hyperbolic ones since their distinguished qualitative properties make them to behave also very differently from a control point of view. The issue of controllability being typically of dynamic nature (although it also makes senses for elliptic or stationary problems) it is natural to address parabolic and hyperbolic equations, and, in particular, the heat and the wave equation. Most of this article is devoted to the wave equation (although we shall also discuss briefly the beam equation, the Schrödinger equation and the heat equation). The wave equation is a simplified hyperbolic problem arising in many areas of Mechanics, Engineering and Technology. It is indeed, a model for describing the vibrations of structures, the propagation of acoustic or seismic waves, etc. Therefore, the control of the wave equation enters in a way or another in problems related with control mechanisms for structures, buildings in the presence of earthquakes, for noise reduction in cavities and vehicles, etc. We refer to [2], and [75] for insteresting practical applications in these areas. But the wave equation, as we said, is also a prototype of infinite-dimensional, purely conservative dynamical system. As we shall see, most of the theory can be adapted to deal also with Schrödinger equation which opens the frame of applications to the challenging areas of Quantum computing and control (see [8]).

It is well known that the interaction of waves with a numerical mesh produces dispersion phenomena and spurious[2] high frequency oscillations ([85], [82]). In particular, because of this nonphysical interaction of waves with the discrete medium, the velocity of propagation of numerical waves, the so called *group velocity*, may converge to zero when the wavelength of solutions is of the order of the size of the mesh and the latter tends to zero. As a consequence of this fact, the time needed to uniformly (with respect to the mesh size) observe (or control) the numerical waves from the boundary or from a subset of the medium in which

[2]The adjective spurious will be used to designate any component of the numerical solution that does not correspond to a solution of the underlying PDE. In the context of the wave equation, this happens at the high frequencies and, consequently, these spurious solutions weakly converge to zero as the mesh size tends to zero. Thus, they are present for any value of the mesh size parameter but vanish in the limit as it tends to zero. Consequently, the existence of these spurious oscillations is compatible with the convergence (in the classical sense) of the numerical scheme.

they propagate may tend to infinity as the mesh becomes finer. Thus, the observation and control properties of the discrete model may eventually disappear.

This effect is compatible and not in contradiction (as one's first intuition might suggest) with the convergence of the numerical scheme in the classical sense and with the fact that the observation and control properties of the continuous model do hold.

The main objectives of this paper are:

- To explain how numerical dispersion and spurious high frequency oscillations occur;
- To describe their consequences for observation/control problems;
- To describe possible remedies for these pathologies;
- To consider to what extent these phenomena occur for other models like plate or heat-like equations.

The previous discussion can be summarized by saying that discretization and observation or control do not commute:

$$Continuous\ Model + Observation/Control + Numerics$$

$$\neq$$

$$Continuous\ Model + Numerics + Observation/Control.$$

This fact has important impact when computing the control for the continuous model. Of course one may approximate the control of the underlying PDE through its corresponding optimality system or Euler-Lagrange equations, which, depending on the complexity of the model under consideration can be difficult to compute. In other words, one may apply numerical algorithms to compute the controls we have obtained for a continuous wave model by some rigorous analytical derivation. This will certainly provide a convergent algorithm that will produce good numerical approximations of the true control of the continuous wave model. But we may also first discretize the continuous model, then compute the control of the discrete system and use it as a numerical approximation of the continuous one. However, this second procedure, which is often used in the literature without comment, may diverge. We shall describe how this divergence may occur and develop some numerical remedies.

Thus, *controlling a discrete version of a continuous wave model is often a bad way of controlling the continuous wave model itself.*

It is important to emphasize that this lack of convergence has nothing to do with the classical notion of convergence (consistency+stability) for numerical schemes. In fact, we shall only discuss classical and well known convergent semi-discrete and fully discrete approximations of the wave equation, but we shall see that this failure of convergence occurs at the level of observation and control properties. As we said above, this is due to the fact that most numerical schemes exhibit dispersion dyagrams (we shall give a few examples below) showing that the group velocity of high frequency numerical solutions tends to zero.

As a consequence of the results we shall present in this paper, it will become clear that one has to distinguish between the following two problems, which are often confused in the literature. Each is relevant in its own context, but the conclusions should not be transferred automatically from one to another since their behavior is often of a very different nature from the point of view of numerical simulation, and they correspond to two completely different mathematical questions.

- *Computing the control of the continuous model*, previously characterized by some minimality or optimality criteria.

- *Controlling discrete equations resulting from numerical approximations of the continuous models* or directly derived by means of discrete or semi-discrete modelling.

Indeed, as mentioned above, our analysis shows that, except for very particular situations corresponding to carefully chosen numerical schemes, the controls of the discretized models do not necessarily converge to the control of the continuous one. Note however that the control of the discrete model may be of independent interest and very efficient in practice if the discrete model is accurate enough for the application we have in mind. But this, of course, requires adopting the discrete model itself as a valuable one for the application under consideration and avoiding passing to the limit in the control as the mesh size tends to zero.

On the contrary, if one considers the continuous model as the appropriate one, one has then to compute the corresponding continuous control. When doing this, surprisingly, relaxing the control requirement on the discrete system by taking controls that fulfill weaker controllability properties at the discrete level, may improve the convergence properties of the controls. This is due to a simple fact: *If the discrete dynamics generates spurious solutions that do not exist at the continuous level, then in order to control them, the controller will necessarily need to differ significantly from the control of the continuous model.* Thus, weakening the control requirement at the discrete level allows one to ignore the spurious oscillations, and this facilitates the convergence of the discrete control to the continuous one.

Up to now, we have discussed control problems in quite a vague way. In fact, rigorously speaking, the discussion above concerns the problem of *exact controllability* in which the goal is to drive the solution of an evolution problem to a given final state exactly in a given time. It is in this setting where the pathological numerical high frequency waves may lead to lack of convergence. But this lack of convergence does not occur if the control problem is relaxed to an approximate or optimal control problem. In this paper we shall illustrate this fact by showing that, although controls may diverge when we impose an exact control condition requiring the whole solution to vanish at the final time, when relaxing this condition (by simply requiring the solution to become smaller in norm than a given arbitrarily small number ε (approximate control) or to minimize the norm of the solution within a class of bounded controls (optimal control)) then the controls are bounded and converge as $h \to 0$ to the corresponding control of the continuous wave equation.

However, even if one is interested on those weakened versions of the control problem, taking into account that the exact controllability one can be obtained as a suitable limit of them, *the previous discussion indicates the instability and extreme sensitivity of all control problems for waves under numerical discretizations.*

As a consequence of this, computing efficiently the control of the continuous wave model may be a difficult task, which has has been undertaken in a number of works by Glowinski, Li, and Lions [36], Glowinski [34], and Asch and Lebeau [1], among others. The effort that has been carried out in these papers is closely related to the existing work on developing numerical schemes with suitable dispersion properties ([85], [82]), based on the classical notion of *group velocity*. But a full understanding of these connections in the context of control and observation of numerical waves requires an additional effort to which this paper is devoted.

In this paper, avoiding unnecessary technical difficulties, we shall present the main advances in this field, explaining the problems under consideration, the existing results and methods and also some open problems that, in our opinion, are particularly important. We shall describe some possible alternatives for avoiding these high frequency spurious oscillations, including Tychonoff regularization, multigrid methods, mixed finite elements, numerical viscosity, and filtering of high frequencies. All these methods, although they may look very different one from another in a first view, are in fact different ways of taking care of the spurious high frequency oscillations that common numerical approximation methods introduce. Despite the fact that the proofs of convergence may be lengthy and technically difficult (and often constitute open problems), once the high frequency numerical pathologies under consideration are well understood, it is easy to believe that they are indeed appropriate methods for computing the controls.

Our analysis is mainly based on the Fourier decomposition of solutions and classical results on the theory of non-harmonic Fourier series. In recent works by F. Macià [59], [60] tools of discrete Wigner measures (in the spirit of Gérard [32] and Lions and Paul [57]) have been developed to show that, as in the continuous wave equation, in the absence of boundary effects, one can characterize the observability property in terms of geometric properties related to the propagation of bicharacteristic rays. In this respect it is important to observe that the bicharacteristic rays associated with the discrete model do not obey the same

Hamiltonian system as the continuous ones but have their own dynamics (as was pointed out before in [82]). As a consequence, numerical solutions develop quite different dynamical properties at high frequencies since both velocity and direction of propagation of energy may differ from those of the continuous model. In this way one can for instance check the existence of rays for the discrete model that do not propagate in space as time evolves. They correspond to *wave packets* with vanishing group velocity. These rays propagate solutions whose energy is concentrated more and more along the ray as the mesh size decreases and make it impossible for the observability property to hold uniformly with respect to the mesh size in the natural geometric setting for the wave equation. This ray analysis allows one to be very precise when filtering the high frequencies and to do this filtering microlocally. In this article we shall briefly comment on this discrete ray theory but shall mainly focus on the Fourier point of view, which is sufficient to understand the main issues under consideration. This ray theory provides a rigorous justification of a fact that can be formally analyzed and understood through the notion of group velocity of propagation of numerical waves [82].

All we have said up to now concerning the wave equation can be applied with minor changes to several other models that are purely conservative. However, many models from physics and mechanics have some damping mechanism built in. In order to illustrate the effect of dissipativity on the observation and control properties of the numerical approximations of continuous models, we shall analyze the 1D heat equation. We shall see that, because of its intrinsic and strong dissipativity properties, the controls of the simplest numerical approximation schemes remain bounded and converge as the mesh size tends to zero to the control of the continuous heat equation, in contrast with the pathological behavior for wave equations. This fact can be easily understood: the dissipative effect of the 1D heat equation acts as a filtering mechanism by itself and is strong enough to exclude high frequency spurious oscillations. However, the situation is more complex in several space dimensions, where the thermal effects are not enough to guarantee the uniform boundedness of the controls. We shall discuss this interesting open problem that, in any case, indicates that viscosity helps to reestablish uniform observation and control properties in numerical approximation schemes. We shall also see that plate and Schrödinger equations behave better than the wave equation. This is due to the fact that the dispersive nature of the continuous model also helps at the discrete level. Indeed, the dispersive nature of the continuous model introduces some dispersion at the level of numerical schemes and this allows the group velocity of high frequency numerical waves not to vanish.

Most of the analysis we shall present here has been also developed in the context of a more difficult problem, related to the behavior of the observation/control properties of the wave equation in the context of homogenization. There, the coefficients of the wave equation oscillate rapidly on a scale ε that tends to zero, so that the equation homogenizes to a constant coefficient one. In that framework it was pointed out that the interaction of high frequency waves with the microstructure produce localized waves at high frequency. These localized waves are an impediment for the uniform observation/control properties to hold. This suggests the need for filtering of high frequencies. It has been proved in a number of situations that this filtering technique suffices to reestablish uniform observation and control properties ([13] and [49]).

The analogies between both problems (homogenization and numerical approximation) are clear: the mesh size h in numerical approximation schemes plays the same role as the ε parameter in homogenization (see [95] and [15] for a discussion of the connection between these problems). Although the analysis of the numerical problem is much easier from a technical point of view, it was only developed after the problem of homogenization was understood. This is probably due to the fact that, from a control theoretical point of view, there was a conceptual difficulty to match the existing finite-dimensional and infinite-dimensional theories. In this article we illustrate how to do this in the context of the wave equation, a model of purely conservative dynamics in infinite dimension.

The rest of this paper is organized as follows. Section 2 is devoted to presenting and discussing the problems of observability and controllability for the constant coefficient wave equation. In section 3 we briefly discuss some aspects related to the multi-dimensional wave equation such as the concentration and the lack of propagation of waves. In section 4 we analyze the variable coefficient 1D case and show that BV-regularity of coefficients is the least regularity we may require on them for the observability inequality to hold. In section 5 we discuss the finite-difference space semi-discretization of the 1D wave equation and present the main results on the lack of controllability and observability. We also comment on how filtering of high frequencies can be used to get uniform controllability results and on the impact of all this on other relaxed versions of the control problem (approximate controllability and optimal control). In Section 6 we discuss some other methods for curing the high frequency pathologies: viscous numerical damping, mixed finite elements, etc. In Section 7 we briefly analyze full space-time discretizations in 1D. Section 8 is devoted to analyzing semi-discretizations for the 2D wave equation in a square. As we shall see, in this case, numerical approximations affect not only the velocity of propagation of energy but also its direction, and further filtering is needed. More precisely, one has to restrict the wavelength of solutions in all space directions to get uniform observability and control properties. Finally, in Section 9 we discuss the finite difference space semi-discretizations for the heat and beam equations, showing that both viscous and dispersive properties of the original continuous models may help in the numerical approximation procedure. We close this paper with some further comments and a list of open problems.

The interested reader is referred to the survey articles [90] and [93] for a more complete discussion of the state of the art in the controllability of partial differential equations.

Acknowledgements. This work is the outcome of many discussions and joint collaborations with other colleagues and experts on various topics. Much of it has been developed under the influence of J.-L. Lions. I must also acknowledge, in particular, R. Glowinski for many fruitful discussions, D. Arnold for pointing out reference [82], M. Asch and J. Rasmussen for a careful reading of a preliminary version of the article, and many other colleagues and collaborators. In particular I should mention C. Castro, R. Dáger, L. León, F. Macià, S. Micu, M. Negreanu, and L. R. Tcheougoué-Tebou, with whom I had the pleasure of developing some of the material included in this paper and who provided also essential help in the preparation of the manuscript. Finally, my thanks go to M. Cea, J. Rasmussen and R. Rodriguez del Rio who helped me with the figures, numerical simulations and some other technical matters.

2 The constant coefficient wave equation

In order to motivate the problems we have in mind let us first consider the constant coefficient 1D wave equation:

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 1,\ 0 < t < T \\ u(0,t) = u(1,t) = 0, & 0 < t < T \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), & 0 < x < 1. \end{cases} \tag{2.1}$$

In (2.1) $u = u(x,t)$ describes the displacement of a vibrating string occupying the interval $(0,1)$.

The energy of solutions of (2.1) is conserved in time, i.e.

$$E(t) = \frac{1}{2}\int_0^1 \left[|u_x(x,t)|^2 + |u_t(x,t)|^2\right] dx = E(0),\ \ \forall 0 \le t \le T. \tag{2.2}$$

The problem of boundary observability of (2.1) can be formulated, roughly, as follows: *To give sufficient conditions on the length of the time interval T such that there exists a constant $C(T) > 0$ so that the following inequality holds for all solutions of (2.1):*

$$E(0) \le C(T)\int_0^T |u_x(1,t)|^2\, dt. \tag{2.3}$$

Inequality (2.3), when it holds, guarantees that the total energy of a solution can be "observed" or estimated from the energy concentrated or measured on the extreme $x = 1$ of the string during the time interval $(0,T)$.

Here and in the sequel, the best constant $C(T)$ in inequality (2.3) will be referred to as the *observability constant*.

Of course, (2.3) is just an example of a variety of similar observability problems. Among its possible variants, the following are worth mentioning: (a) one could observe the energy concentrated on the extreme $x = 0$ or in the two extremes $x = 0$ and 1 simultaneously; (b) the $L^2(0,T)$-norm of $u_x(1,t)$ could be replaced by some other norm, (c) one could also observe the energy concentrated in a subinterval (α, β) of the space interval $(0,1)$ occupied by the string, etc.

The observability problem above is equivalent to a boundary controllability problem. More precisely, the observability inequality (2.3) holds, if and only if, for any $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$ there exists $v \in L^2(0,T)$ such that the solution of the controlled wave equation

$$\begin{cases} y_{tt} - y_{xx} = 0, & 0 < x < 1,\, 0 < t < T \\ y(0,t) = 0;\, y(1,t) = v(t), & 0 < t < T \\ y(x,0) = y^0(x),\, y_t(x,0) = y^1(x), & 0 < x < 1 \end{cases} \tag{2.4}$$

satisfies

$$y(x,T) = y_t(x,T) = 0, \quad 0 < x < 1. \tag{2.5}$$

Needless to say, in this control problem the goal is to drive solutions to equilibrium at the time $t = T$. Once the equilibrium configuration is reached at time $t = T$, , the solution remains at rest for all $t \geq T$, by taking null control for $t \geq T$, i. e. $v \equiv 0$ for $t \geq T$.

The exact controllability property of the controlled state equation (2.4) is completely equivalent [3] *to the observability inequality for the adjoint system (2.1).*

At this respect it is convenient to note that (2.1) is not, strictly speaking, the adjoint of (2.4). The initial data for the adjoint system should be given at time $t = T$. But, in view of the time-irreversibility of the wave equations under consideration this is irrelevant. This is why we prefer to write the initial conditions both for the state equation at its adjoint at time $t = 0$. The same will be done for the time discretizations we shall consider. Obviously, one has to be more careful about this when dealing with time irreversible systems as the heat equation in section 9.1.

Let us check first that observability implies controllability. Furthermore, we describe a constructive method to build the control of minimal norm ($L^2(0,T)$-norm in the present situation) by minimizing a convex, continuous and coercive functional in a Hilbert space. In the present case, given $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$ the control $v \in L^2(0,T)$ of minimal norm for which (2.5) holds is of the form

$$v(t) = u_x^*(1,t), \tag{2.6}$$

[3] We refer to J.L. Lions [54] for a systematic analysis of the equivalence between controllability and observability through the so called Hilbert Uniqueness Method (HUM).

where u^* is the solution of the adjoint system (2.1) corresponding to initial data $(u^{0,*}, u^{1,*}) \in H_0^1(0,1) \times L^2(0,1)$ minimizing the functional

$$J((u^0, u^1)) = \frac{1}{2}\int_0^T |u_x(1,t)|^2 dt + \int_0^1 y^0 u^1 dx - < y^1, u^0 >_{H^{-1}\times H_0^1}, \tag{2.7}$$

in the space $H_0^1(0,1) \times L^2(0,1)$. Here and in the sequel $< \cdot, \cdot >_{H^{-1}\times H_0^1}$ denotes the duality pairing between $H^{-1}(0,1)$ and $H_0^1(0,1)$.

Note that J is convex. The continuity of J in $H_0^1(0,1)\times L^2(0,1)$ is guaranteed by the fact that the solutions of (2.1) satisfy the extra regularity property that $u_x(1,t) \in L^2(0,T)$ (a fact that holds also for the Dirichlet problem for the wave equation in several space dimensions, see [47], [54], [55]). More, precisely, for all $T > 0$ there exists a constant $C_*(T) > 0$ such that

$$\int_0^T \left[|u_x(0,t)|^2 + |u_x(1,t)|^2\right] dt \le C_*(T)E(0), \tag{2.8}$$

for all solution of (2.1).

Thus, in order to guarantee that the functional J achieves its minimum, it is sufficient to prove that it is coercive. This is guaranteed by the observability inequality (2.3).

Once coercivity is known to hold the Direct Method of the Calculus of Variations (DMCV) allows showing that the minimum of J over $H_0^1(0,1) \times L^2(0,1)$ is achieved. By the strict convexity of J the minimum is unique and we denote it, as above, by $(u^{0,*}, u^{1,*}) \in H_0^1(0,1)\times L^2(0,1)$, the corresponding solution of the adjoint system (2.1) being u^*.

The functional J is of class C^1. Consequently, the gradient of J at the minimizer vanishes. This yields the following Euler-Lagrange equations:

$$\begin{aligned} < DJ((u^{0,*}, u^{1,*})), (w^0, w^1) >= \int_0^T u_x^*(1,t) w_x(1,t) dt \\ + \int_0^1 y^0 w^1 dx - < y^1, w^0 >_{H^{-1}\times H_0^1} = 0, \end{aligned} \tag{2.9}$$

for all $(w^0, w^1) \in H_0^1(0,1)\times L^2(0,1)$, where w stands for the solution of the adjoint equation with initial data (w^0, w^1). By choosing the control as in (2.6) this identity yields:

$$\int_0^T v(t) w_x(1,t) dt + \int_0^1 y^0 w^1 dx - < y^1, w^0 >_{H^{-1}\times H_0^1} = 0. \tag{2.10}$$

On the other hand, multiplying in (2.4) by w and integrating by parts we get:

$$\begin{aligned} \int_0^T v(t) w_x(1,t) dt + \int_0^1 y^0 w^1 dx - < y^1, w^0 >_{H^{-1}\times H_0^1} \\ - \int_0^1 y(T) w_t(T) dx + < y_t(T), w(T) >_{H^{-1}\times H_0^1} = 0. \end{aligned} \tag{2.11}$$

Combining these two identities we get:

$$\int_0^1 y(T) w_t(T) dx - < y_t(T), w(T) >_{H^{-1}\times H_0^1} = 0, \tag{2.12}$$

for all $(w^0, w^1) \in H_0^1(0,1) \times L^2(0,1)$, which is equivalent to the exact controllability condition $y(T) \equiv y_t(T) \equiv 0$.

This argument shows that *observability implies controllability*. The reverse is also true. If controllability holds, then the linear map that to all initial data $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$ of the state equation (2.4) associates the control v of minimal $L^2(0,T)$-norm is bounded. Multiplying in the state equation (2.4) with that control by u, solution of the adjoint system, and using that $y(T) \equiv y_t(T) \equiv 0$ we obtain the identity:

$$\int_0^T v(t) u_x(1,t) dt + \int_0^1 y^0 u^1 dx - < y^1, u^0 >_{H^{-1} \times H_0^1} = 0. \tag{2.13}$$

Consequently,

$$\begin{aligned} \left| \int_0^1 y^0 u^1 dx - < y^1, u^0 >_{H^{-1} \times H_0^1} \right| = \left| \int_0^T v(t) u_x(1,t) dt \right| &\leq ||v||_{L^2(0,T)} ||u_x(1,t)||_{L^2(0,T)} \\ &\leq C ||(y^0, y^1)||_{L^2(0,1) \times H^{-1}(0,1)} ||u_x(1,t)||_{L^2(0,T)}, \end{aligned} \tag{2.14}$$

for all $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$, which implies the observability inequality (2.3).

Throughout this paper we shall mainly focus on the problem of observability. However, in view of the equivalence above, all the results we shall present have immediate consequences for controllability. The most important ones will also be stated. Note however that controllability is not the only application of the observability inequalities, which are also of systematic use in the context of Inverse Problems (Isakov, [45]). We shall discuss this issue briefly in Open Problem #10 in Section 10.2.

The first easy fact to be mentioned is that system (2.1) is observable if $T \geq 2$. More precisely, the following can be proved:

Proposition 2.1 *For any $T \geq 2$, system* (2.1) *is observable. In other words, for any $T \geq 2$ there exists $C(T) > 0$ such that* (2.3) *holds for any solution of* (2.1)*. Conversely, if $T < 2$,* (2.1) *is not observable, or, equivalently,*

$$\sup_{u \text{ solution of } (2.1)} \left[\frac{E(0)}{\int_0^T \mid u_x(1,t) \mid^2 dt} \right] = \infty. \tag{2.15}$$

The proof of observability for $T \geq 2$ can be carried out in several ways, including Fourier series, multipliers (Komornik, [47]; Lions, [54]), Carleman inequalities (Zhang, [87]), and microlocal tools (Bardos et al., [5]; Burq and Gérard, [10]). Let us explain how it can be proved using Fourier series. Solutions of (2.1) can be written in the form

$$u = \sum_{k \geq 1} \left(a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right) \sin(k\pi x) \tag{2.16}$$

where a_k, b_k are such that

$$u^0(x) = \sum_{k\geq 1} a_k \sin(k\pi x), \quad u^1(x) = \sum_{k\geq 1} b_k \sin(k\pi x).$$

It follows that

$$E(0) = \frac{1}{4} \sum_{k\geq 1} \left[a_k^2 k^2 \pi^2 + b_k^2 \right].$$

On the other hand,

$$u_x(1,t) = \sum_{k\geq 1} (-1)^k \left[k\pi a_k \sin(k\pi t) + b_k \cos(k\pi t) \right].$$

Using the orthogonality properties of $\sin(k\pi t)$ and $\cos(k\pi t)$ in $L^2(0,2)$, it follows that

$$\int_0^2 |u_x(1,t)|^2 \, dt = \sum_{k\geq 1} \left(\pi^2 k^2 a_k^2 + b_k^2 \right).$$

The two identities above show that the observability inequality holds when $T = 2$ and therefore for any $T > 2$ as well. In fact, in this particular case, we actually have the identity

$$E(0) = \frac{1}{4} \int_0^2 |u_x(1,t)|^2 \, dt. \tag{2.17}$$

On the other hand, for $T < 2$ the observability inequality does not hold. Indeed, suppose that $T \leq 2 - 2\delta$ with $\delta > 0$. We solve the wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 1, \ 0 < t < T \\ u(0,t) = u(1,t) = 0, & 0 < t < T \end{cases} \tag{2.18}$$

with data at time $t = T/2$ with support in the subinterval $(0, \delta)$. Note that, in view of the time reversibility of the wave equation, the solution is determined uniquely for $t \geq T/2$ and $t \leq T/2$.This solution is such that $u_x(1,t) = 0$ for $\delta < t < T - \delta$. This can be seen using the classical fact that the time segment $x = 1$, $t \in (\delta, T - \delta)$ remains outside the domain of influence of the space segment $t = T/2$, $x \in (0, \delta)$ (see Figure 1 below). This is a consequence of the fact that the velocity of propagation in this system is one and shows that the observability inequality fails for any time interval of length less[4]than 2.

Proposition 2.1 states that a necessary and sufficient condition for the observability to hold is that $T \geq 2$. We have just seen that the necessity is a consequence of the finite speed of propagation. The sufficiency, which was proved using Fourier series, is also related to the

[4]This simple construction provides a $1D$ motivation of the Geometric Control Condition (GCC) mentioned in the introduction which is essentially necessary and sufficient for observability to hold in several space dimensions too.

finite speed of propagation. Indeed, when developing solutions of (2.1) in Fourier series the solution is decomposed into the particular solutions

$$u_k = \sin(k\pi t)\sin(k\pi x), \quad \tilde{u}_k = \cos(k\pi t)\sin(k\pi x).$$

Observe that both u_k and $\tilde{u}_k$ can be written in the form

$$u_k = \frac{\cos(k\pi(t-x)) - \cos(k\pi(t+x))}{2}, \quad \tilde{u}_k = \frac{\sin(k\pi(x+t)) - \sin(k\pi(t-x))}{2}$$

and therefore they are linear combinations of functions of the form $f(x+t)$ and $g(x-t)$ for suitable profiles f and g. This shows that, regardless of the frequency of oscillation of the initial data of the equation, solutions propagate with velocity 1 in space and therefore can be observed at the end $x = 1$ of the string, at the latest, at time $T = 2$. Note that the observability time is twice the length of the string. This is due to the fact that an initial disturbance concentrated near $x = 1$ may propagate to the left (in the space variable) as t increases and only reach the extreme $x = 1$ of the interval after bouncing at the left extreme $x = 0$ (as described in Figure 1). A simple computation shows that this requires the time interval to be $T \geq 2$ for observability to hold.

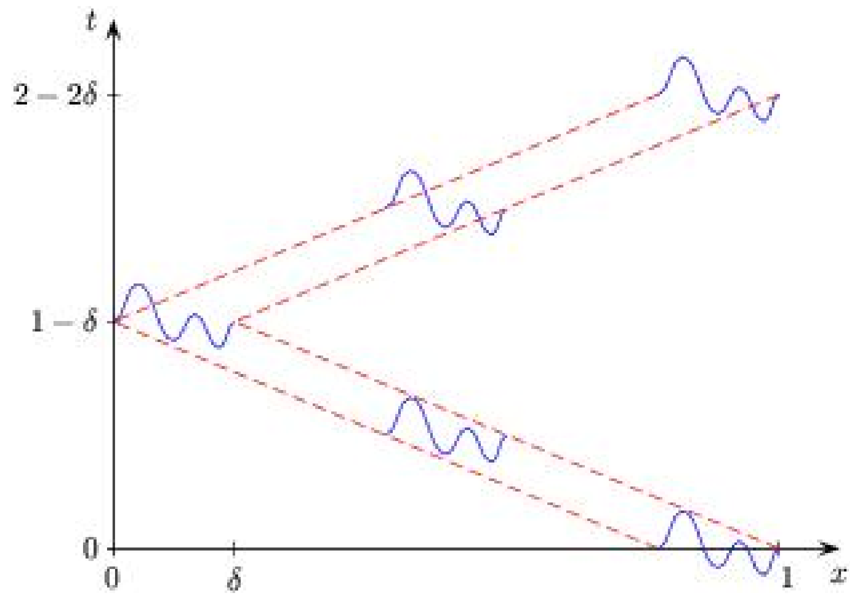

Figure 1: Wave localized at $t = 0$ near the endpoint $x = 1$ that propagates with velocity 1 to the left, bounces at $x = 0$ and reaches $x = 1$ again in a time of the order of 2.

As we have seen, in 1D and with constant coefficients, the observability inequality is easy to understand. The same results are true for sufficiently smooth coefficients (BV-regularity suffices). However, when the coefficients are simply Hölder continuous, these properties may fail, thereby contradicting a first intuition. This issue will be discussed in Section 4 below.

3 The multi-dimensional wave equation

In several space dimensions the observability problem for the wave equation is much more complex and can not be solved using Fourier series. The velocity of propagation is still one for all solutions but energy propagates along bicharacteristic rays. In order to guarantee that the observability inequality holds, it is necessary that all rays reach the observation subset of the boundary in a uniform time and therefore this observation subset has to be selected in an appropriate way and has to be, in general, large enough. As we mentioned above it has to fulfill the Geometric Control Condition (see for instance Bardos et al. [5] and Burq and Gérard, [10]). For instance, when the domain is a ball, the subset of the boundary where the control is being applied needs to contain a point of each diameter. Otherwise, if a diameter skips the control region, it may support solutions that are not observed (see Ralston [70]).

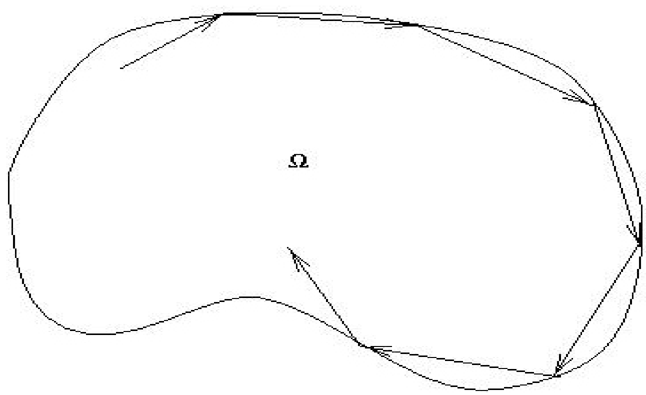

Figure 2: Ray that propagates inside the domain Ω following straight lines that are reflected on the boundary according to the laws of geometrical optics.

Let us formulate the problem and discuss it in some more detail. We shall address here only the case of smooth domains.[5]

Let Ω be a bounded domain of $\mathbf{R}^n, n \geq 1$, with boundary Γ of class C^2. Let ω be an open and non-empty subset of Ω and $T > 0$.

Consider the linear controlled wave equation in the cylinder $Q = \Omega \times (0, T)$:

$$\begin{cases} y_{tt} - \Delta y = f 1_\omega & \text{in} \quad Q \\ y = 0 & \text{on} \quad \Sigma \\ y(x,0) = y^0(x), y_t(x,0) = y^1(x) & \text{in} \quad \Omega. \end{cases} \tag{3.1}$$

[5]We refer to Grisvard [37] for a discussion of these problems in the context of non-smooth domains.

In (3.1) Σ represents the lateral boundary of the cylinder Q, i.e. $\Sigma = \Gamma \times (0,T), 1_\omega$ is the characteristic function of the set ω, $y = y(x,t)$ is the state and $f = f(x,t)$ is the control variable. Since f is multiplied by 1_ω the action of the control is localized in ω.

When $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $f \in L^2(Q)$ the system (3.1) has a unique solution $y \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right)$.

The problem of *controllability*, generally speaking, is as follows: *Given* $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$, *find* $f \in L^2(Q)$ *such that the solution of system* (3.1) *satisfies*

$$y(T) \equiv y_t(T) \equiv 0. \tag{3.2}$$

Several remarks are in order.

Remark 3.1

(a) Since we are dealing with solutions of the wave equation, for any of these properties to hold, the control time T has to be sufficiently large due to the finite speed of propagation, the trivial case $\omega = \Omega$ being excepted. But, as we said above, the time being large enough does not suffice, since the control subdomain ω needs to satisfy the GCC. Figure 3 provides an example of this fact.

(b) The controllability problem may also be formulated in other function spaces in which the wave equation is well posed.

(c) Most of the literature on the controllability of the wave equation has been written in the framework of the *boundary control* problem discussed in the previous section. The control problems formulated above for (3.1) are usually referred to as *internal controllability* problems since the control acts on the subset ω of Ω. The latter is easier to deal with since it avoids considering non-homogeneous boundary conditions, in which case solutions have to be defined in the sense of transposition [54].

■

Using HUM [6] and arguing as in section 2, the exact controllability property can be shown to be equivalent to the following observability inequality:

$$\left|\left|\left(u^0, u^1\right)\right|\right|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq C \int_0^T \int_\omega u^2 dxdt \tag{3.3}$$

for every solution of

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x) & \text{in} \quad \Omega. \end{cases} \tag{3.4}$$

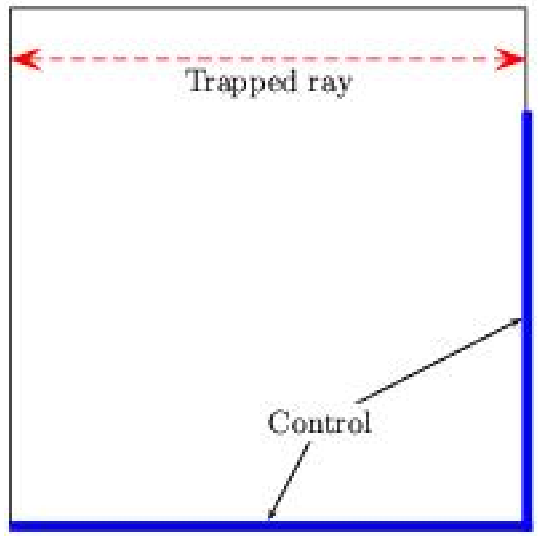

Figure 3: A geometric configuration in which the Geometric Control Condition is not satisfied, whatever $T > 0$ is. The domain where waves evolve is a square. The control is located on a subset of two adjacent sides of the boundary, leaving a small vertical subsegment uncontrolled. There is an horizontal line that constitutes a ray that bounces back and forth for all time perpendicularly on two points of the vertical boundaries where the control does not act.

This inequality makes it possible to estimate the total energy of the solution of (3.4) by means of a measurement in the control region $\omega \times (0, T)$.

When (3.3) holds one can minimize the functional

$$J\left(u^0, u^1\right) = \frac{1}{2}\int_0^T \int_\omega u^2 dx dt + \left\langle \left(u^0, u^1\right), \left(y^1, -y^0\right)\right\rangle \tag{3.5}$$

in the space $L^2(\Omega) \times H^{-1}(\Omega)$. Indeed, the following is easy to prove: *When the observability inequality* (3.3) *holds, the functional J has a unique minimizer $(\widehat{u}^0, \widehat{u}^1)$ in $L^2(\Omega) \times H^{-1}(\Omega)$ for all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$. The control $f = \widehat{u}$ with $\widehat{u}$ a solution of* (3.4) *corresponding to $(\widehat{u}^0, \widehat{u}^1)$ is such that the solution of* (3.1) *satisfies* (3.2).

Consequently, in this way, the exact controllability problem is reduced to the analysis of the observability inequality (3.3).

Let us now discuss what is known about (3.3):

(a) Using multiplier techniques Ho in [40] proved that if one considers subsets of Γ of the form

$$\Gamma(x^0) = \left\{x \in \Gamma : (x - x^0) \cdot n(x) > 0\right\}$$

[6]HUM (Hilbert Uniqueness Method) was introduced by J. L. Lions (see [54] and [55] as a systematic method to address controllability problems for Partial Differential Equations.

for some $x^0 \in \mathbf{R}^n$ (we denote by $n(x)$ the outward unit normal to Ω in $x \in \Gamma$ and by $\cdot$ the scalar product in $\mathbf{R}^n$) and if $T > 0$ is large enough, the following boundary observability inequality holds:

$$\left\| \left(u^0, u^1 \right) \right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq C \int_0^T \int_{\Gamma(x^0)} \left| \frac{\partial u}{\partial n} \right|^2 d\sigma dt \tag{3.6}$$

for all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

This is the observability inequality that is required to solve the boundary controllability problem mentioned above.

Later inequality (3.6) was proved in [54], for any $T > T(x^0) = 2 \parallel x - x^0 \parallel_{L^\infty(\Omega)}$. This is the optimal observability time that one may derive by means of multipliers. More recently Osses in [69] introduced a new multiplier which is basically a rotation of the previous one, making it possible to obtain a larger class of subsets of the boundary for which observability holds.

Proceeding as in [54], one can easily prove that (3.6) implies (3.3) when ω is a neighborhood of $\Gamma(x^0)$ in Ω, i.e. $\omega = \Omega \cap \Theta$ where Θ is a neighborhood of $\Gamma(x^0)$ in $\mathbf{R}^n$, with $T > 2 \parallel x - x^0 \parallel_{L^\infty(\Omega\setminus\omega)}$.

(b) C. Bardos, G. Lebeau and J. Rauch [5] proved that, in the class of C^∞ domains, the observability inequality (3.3) holds if and only if the pair (ω, T) representing the control subdomain and time satisfies the following *geometric control condition (GCC)* in Ω*: Every ray of geometrical optics that propagates in* Ω *and is reflected on its boundary* Γ *intersects* ω *in time less than* T.

In particular, exact controllability holds when ω is a neighborhood of the boundary of Ω.

This result was proved by means of microlocal analysis. Recently the microlocal approach has been greatly simplified by N. Burq [9] by using the microlocal defect measures introduced by P. Gérard [32] in the context of homogenization and kinetic equations. In [9] the GCC was shown to be sufficient for exact controllability for domains Ω of class C^3 and equations with C^2 coefficients. The result for variable coefficients is the same: The observability inequality and, thus, the exact controllability property holds if and only if all rays of geometrical optics intersect the control region before the control time. However, it is important to note that, although in the constant coefficient equation all rays are straight lines, in the variable coefficient case this is no longer the case. In particular, there are smooth coefficients for which there are periodic rays that follow closed trajectories. This may happen in the interior of the domain Ω. Then, for instance, there are variable coefficient wave equations that are not exactly controllable when ω is a neighborhood of the boundary. Note that this is the typical geometrical situation in which the constant coefficient wave equation is exactly controllable. Indeed, in this case, the rays being straight lines, controllability holds when the control time exceeds the diameter of $\Omega \backslash \omega$ since it guarantees that all rays reach the subdomain ω. But, as we said, this property may fail for variable coefficient wave equations. This issue will be further discussed below.

In [61] it was proved that one can build variable coefficients (depending only on the space variable) which are globally Hölder continuous and C^∞ smooth away from a point of the domain so that there are infinitely many concentric rays accumulating in this point. In this case, the only possibility for satisfying the observability inequality is to make observations around that point. Since Hölder continuous coefficients may have this kind of pathological behavior around an infinite number of points, in those cases the control has to be supported everywhere in the domain.

In order to see how this phenomenon occurs it is sufficient to consider the wave equation with a scalar, positive and smooth variable coefficient $a = a(x)$:

$$u_{tt} - \operatorname{div}(a(x)\nabla u) = 0. \tag{3.7}$$

The bicharacteristic rays are then characterized by the Hamiltonian system

$$\begin{cases} x'(s) = -a(x)\xi \\ t'(s) = \ \tau \\ \xi'(s) = \nabla a(x)|\xi|^2 \\ \tau'(s) = 0. \end{cases} \tag{3.8}$$

This system allows one to describe microlocally how the energy of solutions propagates. The projections of the bicharacteristic rays in the (x, t) variables are the rays of geometrical optics that, as we mentioned above, play a fundamental role in the analysis of the observation and control properties through the Geometric Control Condition. As time evolves the rays move in the physical space according to the solutions of (3.8). Moreover, the direction in the Fourier space (ξ, τ) in which the energy of solutions is concentrated as they propagate is given precisely by the projection of the bicharacteristic ray in the (ξ, τ) variables. When the coefficient $a = a(x)$ is constant the ray is a straight line and carries the energy outward, which is always concentrated in the same direction in the Fourier space, as expected. This Hamiltonian system describes the dynamics of rays in the interior of the domain where the equation is satisfied. When rays reach the boundary they are reflected according to the laws of Geometric Optics.[7]

[7]Note however that tangent rays may be diffractive or even enter the boundary. We refer to [5] for a deeper discussion of these issues.

When the coefficient $a = a(x)$ varies in space, the dynamics of this system may be quite complex and can lead to some unexpected behaviour. For instance, let us choose a such that $x \cdot \nabla a(x) - a(x)$ vanishes along the boundary of a ball, the initial point x_0 being on the boundary of that ball with centre at $x = 0$. The initial microlocal direction (ξ_0, τ_0) is chosen such that $\tau_0^2 - a(x_0)|\xi_0|^2 = 0$ and $\xi_0 \cdot x_0 = 0$. It is then easy to see that $\xi(s) \cdot x(s)$ remains identically zero along the whole ray and $|x(s)|^2$ is a constant. That means, in particular, that the ray remains trapped on the surface of the sphere and thus will never reach the boundary of a domain containing that ball.

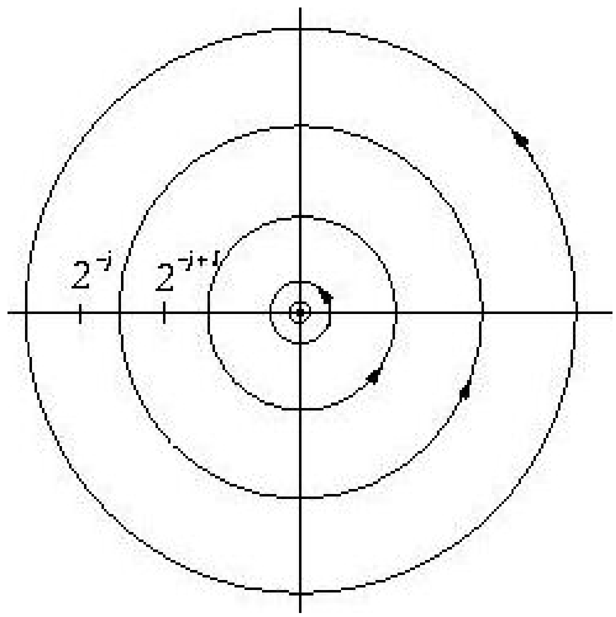

Figure 4: Concentric closed rays concentrating around a centre point (see [61]).

The argument in [61] is based in this observation. Reproducing this construction along surfaces of concentric balls accumulating in its centre, one can indeed show the existence of variable coefficients (which are globally Hölder continuous and C^∞ away from the centre) for which there are solutions that may concentrate the energy as much as one wishes in an arbitrarily small neighborhood of the centre and for an arbitrarily large time. This example shows that our first intuition on how rays propagate may fail completely when the coefficients are not smooth enough.

This example shows that there is a lower limit on the regularity of the coefficients of the wave equation for which the observability inequality holds. But there is still a lot to be done to understand observability inequalities for wave equations with variable $C^{1,\alpha}$ coefficients. Indeed, observability may fail for Hölder continuous coefficients and holds under the GCC for coefficients in $C^{2,\alpha}$ according to [9]. But the issue is to be clarified for $C^{,\alpha}$-coefficients.

4 The 1D wave equation with variable coefficients

Let us now consider the following variable coefficient 1D wave equation:

$$\begin{cases} \rho(x)u_{tt} - u_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\ u(0,t) = u(1,t) = 0, & 0 < t < T \\ u(x,0) = u^0(x), \ u_t(x,0) = u^1(x), & 0 < x < 1. \end{cases} \tag{4.1}$$

We assume that ρ is measurable and that it is bounded above and below by finite, positive constants, i.e.,

$$0 < \rho_0 \le \rho(x) \le \rho_1 < \infty \text{ a.e. } x \in (0,1). \tag{4.2}$$

Under these conditions system (4.1) is well-posed in the sense that for any initial data $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$ there exists a unique solution

$$u \in C\left([0,T];\ H_0^1(0,1)\right) \cap C^1\left([0,T];\ L^2(0,1)\right). \tag{4.3}$$

Moreover, the energy of solutions

$$E(t) = \frac{1}{2}\int_0^1 \left[\rho(x) \mid u_t(x,t) \mid^2 + \mid u_x(x,t) \mid^2\right] dx \tag{4.4}$$

is constant in time. When $\rho \in BV(0,1)$, the following observability properties are well known to hold:

1. **Boundary observability:** If $T > 2\sqrt{\rho_1}$ there exists $C(T) > 0$ such that

$$E(0) \le C \int_0^T |u_x(0,t)|^2 \, dt \tag{4.5}$$

   for every solution of (4.1).

2. **Internal observability:** For any subinterval $(\alpha, \beta) \subset (0,1)$, if

$$T > 2\sqrt{\rho_1}\max(\alpha, 1-\beta),$$

   there exists $C > 0$ such that

$$E(0) \le C \int_0^T \int_\alpha^\beta \left[\rho(x)u_t^2 + u_x^2\right] dx dt, \tag{4.6}$$

   for every solution of (4.1).

These results can be proved using *sidewise energy estimates* for the wave equation in which the role of space and time are interchanged (see [96], [20]). We view the differential equation in (4.1) as

$$u_{xx} - \rho(x)u_{tt} = 0, \tag{4.7}$$

and consider the vertical energy functional

$$F(x) = \frac{1}{2}\int_{\sqrt{\rho_1}x}^{T-\sqrt{\rho_1}x} \Big[\rho(x)|u_t(x,t)|^2 + |u_x(x,t)|^2\Big]\, dt. \tag{4.8}$$

Then, it can be proved that

$$\begin{aligned} \frac{dF(x)}{dx} &\leq \frac{1}{2}\int_{\sqrt{\rho_1}x}^{T-\sqrt{\rho_1}x} \rho'(x)|u_t|^2(x,t)dt \leq \frac{|\rho'(x)|}{2\rho_0}\int_{\sqrt{\rho_1}x}^{T-\sqrt{\rho_1}x} \rho(x)|u_t|^2(x,t)dt \leq \\ &\leq \frac{|\rho'(x)|}{|\rho_0|}F(x). \end{aligned} \tag{4.9}$$

Consequently, when $T > 2\sqrt{\rho_1}$, by Gronwall's inequality

$$\begin{aligned} F(x) &\leq \exp\left(\frac{\int_0^x |\rho'(s)|ds}{|\rho_0|}\right) F(0) = \frac{\exp\left(\int_0^x |\rho'(s)|ds/|\rho_0|\right)}{2}\int_0^T |u_x(0,t)|^2 dt, \\ &\forall x \in [0,1]. \end{aligned} \tag{4.10}$$

Integrating this inequality for x in $[0,1]$ we deduce that

$$\int_0^1 F(x)dx \leq \frac{\exp\left(\int_0^1 |\rho'(s)|ds/|\rho_0|\right)}{2}\int_0^T |u_x(0,t)|^2 dt, \quad \forall x \in [0,1]. \tag{4.11}$$

Taking into account that $T > 2\sqrt{\rho_1}$ and using conservation of energy we have

$$(T - 2\sqrt{\rho_1})E(0) \leq \int_0^1 F(x)dx \leq \frac{\exp\left(\int_0^1 |\rho'(s)|ds/|\rho_0|\right)}{2}\int_0^T |u_x(0,t)|^2. \tag{4.12}$$

These observability inequalities hold provided ρ is in the BV-class.

In [14] it was proved that these inequalities do not hold under the weaker assumption that the density ρ is Hölder continuous. This shows that the BV assumption is sharp. The construction in [14] shows that there exist Hölder continuous densities ρ for which the eigenvalue problem

$$-w'' = \lambda\rho(x)w, \quad -\infty < x < \infty \tag{4.13}$$

admits a sequence of solutions for which λ_k tends to infinity and the corresponding sequence of eigenfunctions w_k is exponentially concentrated in a neighborhood of a given point. By separation of variables this allows the building of solutions of the wave equation in the whole line which are also exponentially concentrated. In particular, the Dirichlet and Neumann traces at the endpoints $x = 0, 1$ of the interval $(0,1)$ are exponentially small compared to the total energy of solutions. This makes it possible to correct the solutions so that the Dirichlet traces vanish but still being incompatible with the observability inequalities in the sense that solutions $\{u_k\}$ of (4.1) obtained in this way satisfy that the ratio between the total energy and the observed one tends to infinity.

This shows that the situation in 1D is quite clear: *Observability holds for BV coefficients but not under the weaker Hölder regularity assumption.* As we said in the previous section, the situation is more incomplete in several space dimensions where the case of $C^{1,\alpha}$ coefficients is open.

5 1D Finite-Difference Semi-Discretizations

In section 2 we showed how the observability problem for the constant coefficient wave equation can be solved by Fourier series expansions. We now address the problem of the continuous dependence of the observability constant $C(T)$ in (2.3) with respect to finite difference space semi-discretizations as the parameter h of the discretization tends to zero. This problem arises naturally in the numerical implementation of the controllability and observability properties of the continuous wave equation but is of independent interest in the analysis of discrete models for vibrations.

There are several important facts and results that deserve to be underlined and that we shall discuss below:

- The observability constant for the semi-discrete model tends to infinity for any T as $h \to 0$. This is related to the fact that the velocity of propagation of solutions tends to zero as $h \to 0$ and the wavelength of solutions is of the same order as the size of the mesh.

- As a consequence of this fact and of Banach-Steinhaus Theorem, there are initial data for the wave equation for which the controls of the semi-discrete models diverge. This proves that one can not simply rely on the classical convergence (consistency + stability) analysis of the underlying numerical schemes to design algorithms for computing the controls.

- The observability constant may be uniform if the high frequencies are filtered in an appropriate manner.

Let us now formulate these problems and state the corresponding results in a more precise way.

5.1 Finite-difference approximations

Given $N \in \mathbf{N}$ we define $h = 1/(N+1) > 0$. We consider the mesh

$$x_0 = 0;\ x_j = jh,\ j = 1, \ldots, N;\ x_{N+1} = 1, \tag{5.1}$$

which divides $[0,1]$ into $N+1$ subintervals $I_j = [x_j, x_{j+1}]$, $j = 0, ..., N$.

Consider the following finite difference approximation of the wave equation (2.1):

$$\begin{cases} u_j'' - \frac{1}{h^2}\left[u_{j+1} + u_{j-1} - 2u_j\right] = 0, & 0 < t < T,\ j = 1, \ldots, N \\ u_j(t) = 0, & j = 0,\ N+1,\ 0 < t < T \\ u_j(0) = u_j^0,\ u_j'(0) = u_j^1, & j = 1, \ldots, N. \end{cases} \tag{5.2}$$

Observe that (5.2) is a coupled system of N linear differential equations of second order. The function $u_j(t)$ provides an approximation of $u(x_j, t)$ for all $j = 1, \ldots, N$, u being the solution of the continuous wave equation (2.1). The conditions $u_0 = u_{N+1} = 0$ take account of the homogeneous Dirichlet boundary conditions, and the second order differentiation with respect to x has been replaced by the three-point finite difference $[u_{j+1} + u_{j-1} - 2u_j]/h^2$.

We shall use a vector notation to simplify the expressions. In particular, the column vector

$$\overrightarrow{u}(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_N(t) \end{pmatrix} \tag{5.3}$$

will represent the whole set of unknowns of the system. Introducing the matrix

$$A_h = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \tag{5.4}$$

the system (5.2) reads as follows

$$\begin{cases} \vec{u}''(t) + A_h \vec{u}(t) = 0, \qquad 0 < t < T \\ \vec{u}(0) = \vec{u}^0, \; \vec{u}'(0) = \vec{u}^1. \end{cases} \tag{5.5}$$

Obviously the solution $\overrightarrow{u}$ of (5.5) depends also on h so that we should actually use the subindex h in its notation. But, for the sake of simplicity, we shall only use it when strictly needed.

The energy of the solutions of (5.2) is as follows:

$$E_h(t) = \frac{h}{2} \sum_{j=0}^{N} \left[\mid u_j' \mid^2 + \left| \frac{u_{j+1} - u_j}{h} \right|^2 \right], \tag{5.6}$$

and it is constant in time. Obviously (5.6) is a natural discretization of the continuous energy (2.2).

The problem of observability of system (5.2) can be formulated as follows: *To find $T > 0$ and $C_h(T) > 0$ such that*

$$E_h(0) \leq C_h(T) \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt \tag{5.7}$$

holds for all solutions of (5.2).

Observe that $\mid u_N/h \mid^2$ is a natural approximation [8] of $\mid u_x(1,t) \mid^2$ for the solution of the

[8] Here and in the sequel u_N refers to the N-th component of the solution of the semidiscrete system $\vec{u}$, which obvioulsy depends also on h. This dependence is not made explicit in the notation not to make it more complex.

continuous system (2.1). Indeed $u_x(1,t) \sim [u_{N+1}(t) - u_N(t)]/h$ and, taking into account that $u_{N+1} = 0$, it follows that $u_x(1,t) \sim -u_N(t)/h$.

System (5.2) is finite-dimensional. Therefore, if observability holds for some $T > 0$, then it holds for all $T > 0$. This is an immediate consequence of the classical result by Kalman (see [51]) characterizing the observability/controllability properties of linear, time-independent, finite-dimensional systems, in algebraic terms, involving the matrix governing the state equation and the controller. This algebraic characterization is independent of time and, consequently, finite-dimensional systems are either observable for all time or non observable whatever the observation time is. For the sake of completeness let us recall the basic ingredients of Kalman's result and its precise statement. Consider a linear, constant-coefficient, finite-dimensional system $x' + \mathcal{A}x = \mathcal{B}v$, of dimension N, where x is the state, v is the control, with $M \leq N$ components, $\mathcal{A} \in \mathcal{M}_{N\times N}$ is the matrix governing the dynamics of the state equation and $\mathcal{B} \in \mathcal{M}_{N\times M}$ stands for the control operator indicating how the control v affects the various components of the state. The Kalman necessary and sufficient condition for controllability is of algebraic nature and reads: $\text{rank}[B, AB, ..., A^{N-1}B] = N$. When this rank condition is satisfied the system is controllable for all T. When this condition fails and the rank is, for instance, $N - m$ with $m \geq 1$, for all T the hyperplane of reachable states at time T from any initial state is of dimension $N - m$ and therefore controllability does not hold. From the point of view of observability the Kalman condition can be interpreted as follows. Consider the adjoint system $-\varphi' + \mathcal{A}^*\varphi = 0$. The observability problem is now as follows: *Does condition $\mathcal{B}^*\varphi(t) \equiv 0$ for all $0 \leq t \leq T$ imply that $\varphi \equiv 0$?* The answer is once again that observability holds if and only if the Kalman algebraic condition is satisfied.

It is easy to see that, for all $h > 0$, (5.7) does indeed hold. But we are interested mainly in the uniformity of the constant $C_h(T)$ as $h \to 0$. If $C_h(T)$ remains bounded as $h \to 0$ we say that system (5.2) is uniformly (with respect to h) observable as $h \to 0$. Taking into account that the observability of the limit system (2.1) only holds for $T \geq 2$, it might seem natural to expect $T \geq 2$ to be a necessary condition for the uniform observability of (5.2). This is indeed the case but, as we shall see, the condition $T \geq 2$ is far from being sufficient. In fact, *uniform observability fails for all $T > 0$*. In order to explain this fact it is convenient to analyze the spectrum of (5.2).

Let us consider the eigenvalue problem

$$\begin{array}{c} -\left[w_{j+1} + w_{j-1} - 2w_j\right]/h^2 = \lambda w_j,\ j = 1, \ldots, N \\ w_0 = w_{N+1} = 0. \end{array} \tag{5.8}$$

The spectrum can be computed explicitly in this case (Isaacson and Keller [44]). The eigenvalues

$$0 < \lambda_1(h) < \lambda_2(h) < \ldots < \lambda_N(h)$$

are

$$\lambda_k^h = \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2}\right) \tag{5.9}$$

and the corresponding eigenvectors are

$$\vec{w}_k^h = (w_{k,1},\ldots,w_{k,N})^T : w_{k,j} = \sin(k\pi jh),\ k,j = 1,\ldots,N, \tag{5.10}$$

Obviously,

$$\lambda_k^h \to \lambda_k = k^2\pi^2,\ \text{as } h \to 0 \tag{5.11}$$

for each $k \geq 1$, $\lambda_k = k^2\pi^2$ being the $k-$th eigenvalue of the continuous wave equation (2.1). On the other hand we see that the eigenvectors $\vec{w}_k^h$ of the discrete system (5.8) coincide with the eigenfunctions $w_k(x) = \sin(k\pi x)$ of the continuous wave equation (2.1). [9]

According to (5.9) we have

$$\sqrt{\lambda_k^h} = \frac{2}{h}\sin\left(\frac{k\pi h}{2}\right),$$

and therefore, in a first approximation, we have

$$\left|\sqrt{\lambda_k^h} - k\pi\right| \sim \frac{k^3\pi^3h^2}{24}. \tag{5.12}$$

This indicates that the convergence in (5.11) is only uniform in the range $k \ll h^{-2/3}$. Thus, one can not expect to solve completely the problem of uniform observability for the semi-discrete system (5.2) as a consequence of the observability property of the continuous wave equation and a perturbation argument with respect to h. A more careful analysis of the behavior of the eigenvalues and eigenvectors at high frequencies is needed.

5.2 Non uniform observability

The following identity holds:

Lemma 5.1 *For any $h > 0$ and any eigenvector of* (5.8) *associated with the eigenvalue λ,*

$$h\sum_{j=0}^{N}\left|\frac{w_{j+1}-w_j}{h}\right|^2 = \frac{2}{4-\lambda h^2}\left|\frac{w_N}{h}\right|^2. \tag{5.13}$$

We now observe that the largest eigenvalue λ_N^h of (5.8) is such that

$$\lambda_N^h h^2 \to 4 \text{ as } h \to 0. \tag{5.14}$$

Indeed

$$\begin{aligned}\lambda_N^h h^2 &= 4\sin^2\left(\frac{\pi N h}{2}\right) = 4\sin^2\left(\frac{\pi(1-h)}{2}\right)\\ &= 4\cos^2(\pi h/2) \to 4 \text{ as } h \to 0.\end{aligned}$$

Combining (5.13) and (5.14) we get the following result on non-uniform observability:

[9] This is a non-generic fact that only occurs for the constant coefficient 1D problem with uniform meshes.

Theorem 5.1 *For any $T > 0$ it follows that*

$$\sup_{u \text{ solution of } (5.2)} \left[\frac{E_h(0)}{\int_0^T \mid u_N/h \mid^2 dt} \right] \to \infty \tag{5.15}$$

as $h \to 0$.

Proof of Theorem 5.1. We consider solutions of (5.2) of the form

$$\vec{u}^h = \cos\left(\sqrt{\lambda_N^h} t\right) \vec{w}_N^h,$$

where λ_N^h and $\vec{w}_N^h$ are the $N-$th eigenvalue and eigenvector of (5.8) respectively. We have

$$E_h(0) = \frac{h}{2} \sum_{j=0}^{N} \left| \frac{w_{N,j+1}^h - w_{N,j}^h}{h} \right|^2 \tag{5.16}$$

and

$$\int_0^T \left| \frac{u_N^h}{h} \right|^2 dt = \left| \frac{w_{N,N}^h}{h} \right|^2 \int_0^T \cos^2\left(\sqrt{\lambda_N^h} t\right) dt. \tag{5.17}$$

Taking into account that $\lambda_N^h \to \infty$ as $h \to 0$ it follows that

$$\int_0^T \cos^2\left(\sqrt{\lambda_N^h} t\right) dt \to T/2 \quad \text{as} \quad h \to 0. \tag{5.18}$$

By combining (5.13), (5.16), (5.17) and (5.18), (5.15) follows immediately.

Remark 5.1 Note that the construction above applies to any sequence of eigenvalues $\lambda_{j(h)}^h$ such that

$$h^2 \lambda_{j(h)}^h \to 4 \ \text{ as } \ h \to 0, \tag{5.19}$$

which is equivalent to

$$\sin^2\left(\frac{\pi j(h) h}{2}\right) \to 1$$

or, in other words, to

$$j(h)h \to 1. \tag{5.20}$$

■

However, the solution we have used in the proof of this theorem is not the only impediment for the uniform observability inequality to hold.

Indeed, let us consider the following solution of the semi-discrete system (5.2), constituted by the last two eigenvectors, which illustrates the relevance of the lack of uniform gap in the spectrum:

$$\vec{u} = \frac{1}{\sqrt{\lambda_N}} \Big[\exp(i\sqrt{\lambda_N} t) \vec{w}_N - \exp(i\sqrt{\lambda_{N-1}} t) \vec{w}_{N-1} \Big]. \tag{5.21}$$

This solution is a simple superposition of two monochromatic semi-discrete waves corresponding to the last two eigenfrequencies of the system. The total energy of this solution is of the order 1 (because each of both components has been normalized in the energy norm and the eigenvectors are orthogonal one to each other). However, the trace of its discrete normal derivative is of the order of h in $L^2(0,T)$. This is due to two facts.

- First, the trace of the discrete normal derivative of each eigenvector is very small compared to its total energy, as Lemma 5.1 shows.
- Second and more important, the gap between $\sqrt{\lambda_N}$ and $\sqrt{\lambda_{N-1}}$ is of the order of h, as it is shown in Figure 5.

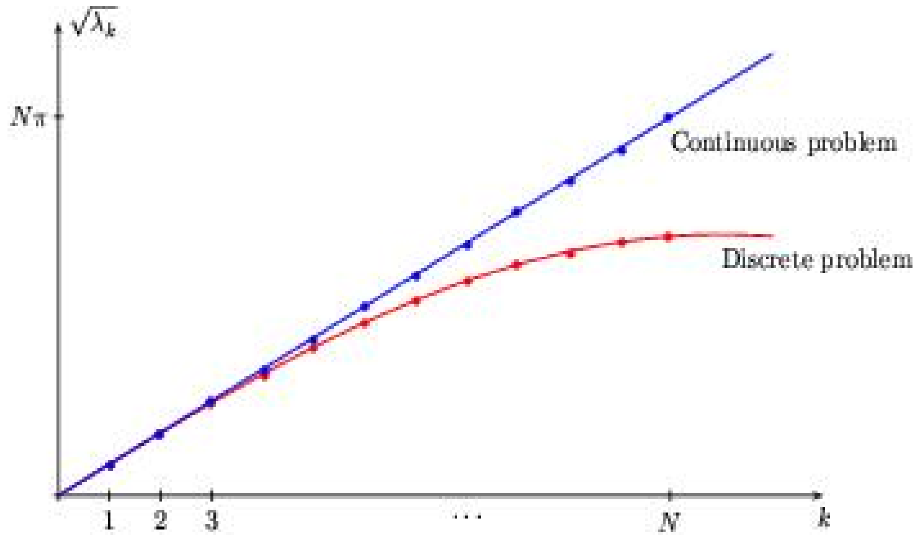

Figure 5: Square roots of the eigenvalues in the continuous and discrete case. The gaps between these numbers are clearly independent of k in the continuous case and of order h for large k in the discrete one.

Thus, by Taylor expansion, the difference between the two time-dependent complex exponentials $\exp(i\sqrt{\lambda_N}t)$ and $\exp(i\sqrt{\lambda_{N-1}}t)$ is of the order Th. Thus, in order for it to be of the order of 1 in $L^2(0,T)$, we need a time T of the order of $1/h$. In fact, by drawing the graph of the function in (5.21) above one can immediately see that it corresponds to a wave propagating at a velocity of the order of h (Figure 6).

This construction makes it possible to show that, whatever the time T is, the observability constant $C_h(T)$ in the semi-discrete system is at least of order $1/h$. In fact, this idea can be used to show that the observability constant has to blow-up at infinite order. To do this it is sufficient to proceed as above but combining an increasing number of eigenfrequencies. This argument allows one to show that the observability constant has to blow-up as an arbitrary negative power of h. Actually, S. Micu in [64] proved that the constant $C_h(T)$ blows up exponentially[10] by means of a careful analysis of the biorthogonal sequences to the family of exponentials $\{\exp(i\sqrt{\lambda_j}t)\}_{j=1,...,N}$ as $h \to 0$.

All these high frequency pathologies are in fact very closely related with the notion of group velocity (see [85], [82] for an in depth analysis of this notion).

On the other hand, according to the fact that the eigenvector w_j is a sinusoidal function (see (5.10)) we see that these functions can also be written as linear combinations of complex exponentials (in space-time):

$$\exp\Big[\pm ij\pi\Big[\frac{\sqrt{\lambda_j}}{j\pi}t - x\Big]\Big].$$

In view of this, we see that each monochromatic wave propagates at a speed

$$\frac{\sqrt{\lambda_j}}{j\pi} = \frac{2\sin(j\pi h/2)}{j\pi h} = \frac{\omega_h(\xi)}{\xi}\Big|_{\{\xi=j\pi h\}} = c_h(\xi)\Big|_{\{\xi=j\pi h\}}, \tag{5.22}$$

with

$$\omega_h(\xi) = 2\sin(\xi/2).$$

This is the so called *phase velocity.* The velocity of propagation of monochromatic semi-discrete waves (5.22) turns out to be bounded above and below by positive constants, independently of h, i. e.

$$0 < \alpha \leq c_h(\xi) \leq \beta < \infty, \quad \forall h > 0, \forall \xi \in [0, \pi].$$

Note that $[0, \pi]$ is the relevant range of frequencies. Indeed, $\xi = j\pi h$ and $j = 1, ..., N$ and $Nh = 1 - h$.

However, it is well known that, even though the velocity of propagation of each eigenmode is bounded above and below, wave packets may travel at a different speed because of the cancellation phenomena we have exhibited above (see (5.21)). The corresponding speed for those semi-discrete wave packets accumulating is given by the derivative of $\omega_h(\cdot)$ (see [82]). At the high frequencies ($j \sim N$) the derivative of $\omega_h(\xi)$ at $\xi = N\pi h = \pi(1-h)$, is of the order of h and therefore the wave packet (5.21) propagates with velocity of the order of h.

Note that the fact that this group velocity is of the order of h is equivalent to the fact that the gap between $\sqrt{\lambda_{N-1}}$ and $\sqrt{\lambda_N}$ is of order h.

Indeed, the definition of group velocity as the derivative of ω_h is a natural consequence of the classical properties of the superposition of linear harmonic oscillators with close but not identical phases (see [19]). The group velocity is thus, simply, the derivative of the curve in

[10] According to [64] we know that the observability constant $C_h(T)$ necessarily blows-up exponentially as $h \to 0$. On the other hand, it is known that the observability inequality is true if $C_h(T)$ is large enough. The problem of obtaining sharp asymptotic (as $h \to 0$) estimates on the observability constant is open.

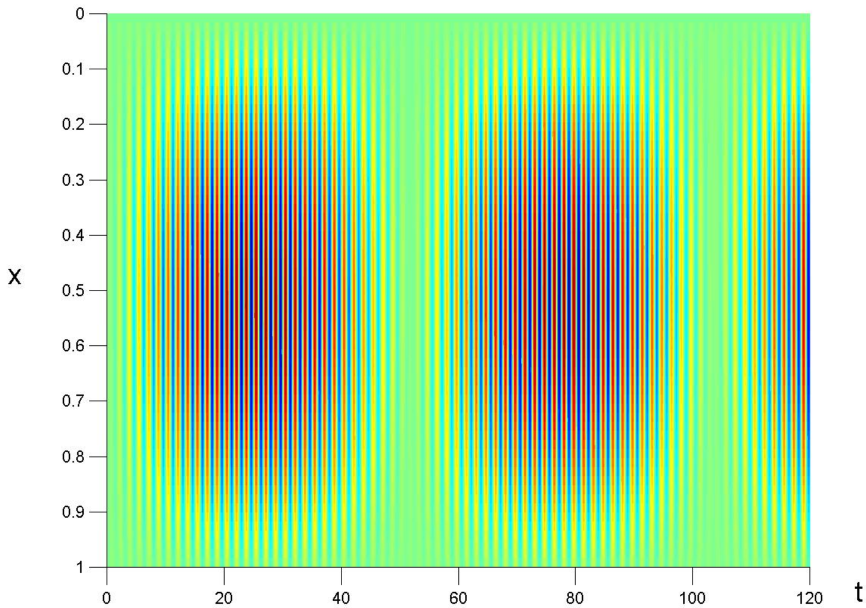

Figure 6: Time evolution of solution (5.21) for $h = 1/61$ ($N = 60$) and $0 \leq t \leq 120$. It is clear that, according to the figure, the solution seems to exhibit a time-periodicity property with period τ of the order of $\tau \sim 50$. Note however that all solutions of the wave equation are time-periodic of period 2. In the figure it also clear that fronts propagate in space at velocity of the order of $1/50$. This is in agreement with the prediction of the theory in the sense that high frequency wave packets travel at a group velocity of the order of h.

the dispersion dyagram of Figure 5 describing the velocity of propagation of monocromatic waves, as a function of frequency. Taking into account that this curve is constituted by the square roots of the eigenvalues of the numerical scheme, we see that there is a one-to-one correspondence between group velocity and spectral gap. In particular, the group velocity decreases when the gap between consecutive eigenvalues does it.

Consequently, we see that the group velocity is indeed related to the separation of the spectrum. Thus, building numerical schemes for which the group velocity remains uniformly bounded below, which is a necessary condition for the observability inequality to hold uniformly with respect to h, turns out to be equivalent to constructing numerical schemes for which the sepectrum is uniformly separated, with respect to the frequency and the mesh-size h.

This well-known fact is relevant when analyzing the adequacy of numerical approximation methods for wave-like equations, since they arise independently of whether the schemes converge or not (see [82]). In fact, the convergence of the numerical scheme only guarantees

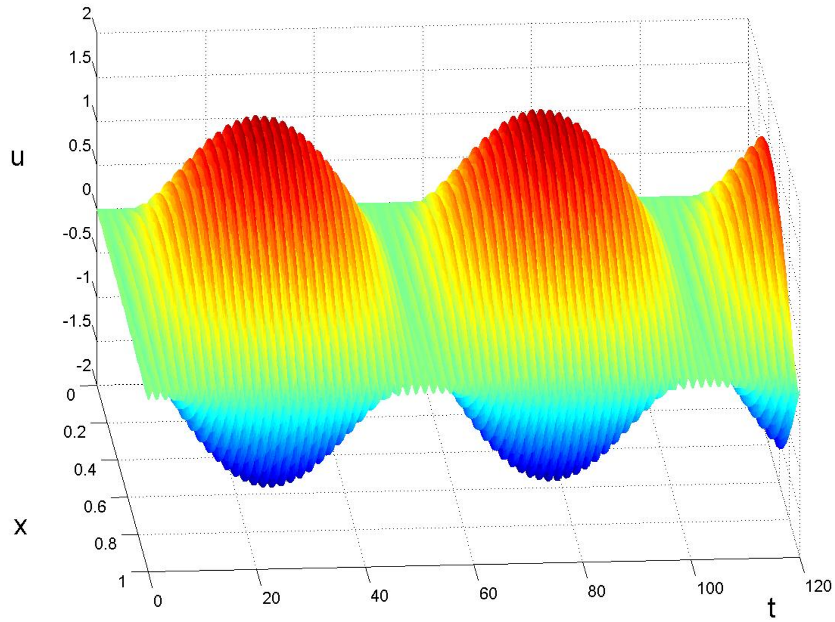

Figure 7: 3D view of the solution of Figure 6.

that the group velocity is correct for low frequency wave packets.[11] Consequently, the negative results we have mentioned above are simply a new reading of well-known pathologies of finite difference scheme for the wave equation.

However, the careful analysis of this negative example is extremely useful when designing possible remedies, i.e., to determine how one could modify the numerical scheme in order to reestablish the uniform observability inequality, since we have only found two obstacles and both happen at high frequencies. The remedy is very natural: To cut off the high frequencies or, in other words, to ignore the high frequency components of the numerical solutions. As we shall see in section 5.3, this method works and, as soon as we deal with solutions where the only Fourier components are those corresponding to the eigenvalues $\lambda \leq \gamma h^{-2}$ with $0 < \gamma < 4$ or with indices $0 < j < \delta h^{-1}$ with $0 < \delta < 1$, the observability inequality becomes uniform. Note that these classes of solutions correspond to taking projections of the complete solutions by cutting off all frequencies with $\gamma h^{-2} < \lambda < 4h^{-2}$.

All this might seem surprising in a first approach to the problem but it is in fact very natural. The numerical scheme, which converges in the classical sense, reproduces, at low frequencies, as $h \to 0$, the whole dynamics of the continuous wave equation. But, it also

[11]Note that in Figure 5 the semidiscrete and continuous curves are tangent. This is in agreement with the convergence property of the numerical algorithm under consideration and with the fact that low frequency wave packets travel essentially with the velocity of the continuous model.

introduces a lot of high frequency spurious solutions. The scheme then becomes more accurate if we ignore that part of the solutions and, at the same time, makes the observability inequality uniform provided the time is taken to be large enough.[12]

It is also important to observe that the high frequency pathologies we have described can not be avoided by simply taking, for instance, a different approximation of the discrete normal derivative as one might think in view of Lemma 5.1. Indeed, the fact that high frequency wave packets propagate at velocity h is due to the scheme itself and, therefore, can not be compensated by suitable boundary measurements. More precisely, even if we have had the right uniform observability inequality for each individual eigenvector, the uniform observability inequality would still be false for the semi-discrete wave equation.

Up to now, no estimate was given on the size of $C_h(T)$. This can be done as follows. One can proceed by the sidewise energy method described in section 4 but this time applied to the semi-discrete system. This provides a very rough estimate. But, in fact, the estimate turns out to be basically sharp since we know from [64] that $C_h(T)$ blows up exponentially as $h \to 0$. To do that we recall that solutions of the semi-discrete system vanish at the boundary point $x = 1$ of the boundary, i.e., $u_{N+1} \equiv 0$. On the other hand, the right hand side of the observability inequality provides an estimate of u_N in $L^2(0,T)$. Now, we can read the semi-discrete equation at the node $j = N$ as follows:

$$u_{N-1} = h^2 u_N'' + 2u_N, \quad 0 < t < T. \tag{5.23}$$

This provides an estimate of u_{N-1} in $L^2(0,T)$. Indeed, in principle, in view of (5.23), one should lose two time derivatives when doing this. However, this can be compensated by the fact that we are dealing with a finite-dimensional model in which two time derivatives are of the order of $A_h u$ where A_h is the matrix in (5.4), which is of norm $4/h^2$. Iterating this argument we can end up getting an estimate in $L^2(0,T)$ for all u_j with $j = 1, ..., N$. But, taking into account that $N \sim 1/h$, the constant in the bound will necessarily be exponential in $1/h$.

In the following section we show how, by a suitable filtering of the high frequencies, the uniform observability inequality can be proved.

5.3 Uniform observability for filtered solutions

In this section we prove the main uniform observability result for system (5.2). In addition to the sharp spectral results of the previous section we shall use a classical result due to Ingham in the theory of non-harmonic Fourier series (see Ingham [43] and Young [86]).

[12] As we will see below, computing the right optimal time for the observability inequality to hold requires taking again into account the notion of group velocity. The minimal time for uniform observability turns out to depend on the cutoff parameter γ.

Ingham's Theorem. *Let* $\{\mu_k\}_{k\in\mathbf{Z}}$ *be a sequence of real numbers such that*

$$\mu_{k+1} - \mu_k \geq \gamma > 0, \forall k \in \mathbf{Z}. \tag{5.24}$$

Then, for any $T > 2\pi/\gamma$ *there exists a positive constant* $C(T,\gamma) > 0$ *such that*

$$\frac{1}{C(T,\gamma)} \sum_{k\in\mathbf{Z}} | a_k |^2 \leq \int_0^T \left| \sum_{k\in\mathbf{Z}} a_k e^{i\mu_k t} \right|^2 dt \leq C(T,\gamma) \sum_{k\in\mathbf{Z}} | a_k |^2 \tag{5.25}$$

for all sequences of complex numbers $\{a_k\} \in \ell^2$.

Remark 5.2 Ingham's inequality can be viewed as a generalization of the orthogonality property of trigonometric functions. Indeed, assume that $\mu_k = k\gamma$, $k \in \mathbf{Z}$ for some $\gamma > 0$. Then (5.24) holds with equality for all k. We set $T = 2\pi/\gamma$. Then

$$\int_0^{2\pi/\gamma} \left| \sum_{k\in\mathbf{Z}} a_k e^{i\gamma k t} \right|^2 dt = \frac{2\pi}{\gamma} \sum_{k\in\mathbf{Z}} | a_k |^2 . \tag{5.26}$$

Note that under the weaker gap condition (5.24) we obtain upper and lower bounds instead of identity (5.26). Observe also that Ingham's inequality does not apply at the minimal time $2\pi/\gamma$. This fact is also sharp [86].

■

In the previous section we have seen that, in the absence of spectral gap (or, when the group velocity vanishes) the uniform observability inequality fails. Ingham's inequality provides the positive counterpart, showing that, as soon as the gap condition is satisfied, there is uniform observability provided the time is large enough. Note that the observability time is inversely proportional to the gap, and this is once more in agreement with the interpretation of the previous section.

All these facts confirm that a suitable cutoff or filtering of the spurious numerical high frequencies may be a cure for these pathologies.

Let us now describe the basic *Fourier filtering mechanism.*

We recall that solutions of (5.2) can be developed in Fourier series as follows:

$$\vec{u} = \sum_{k=1}^{N} \left(a_k \cos\left(\sqrt{\lambda_k^h} t\right) + \frac{b_k}{\sqrt{\lambda_k^h}} \sin\left(\sqrt{\lambda_k^h} t\right) \right) \vec{w}_k^h \tag{5.27}$$

where a_k, b_k are the Fourier coefficients of the initial data, i.e.,

$$\vec{u}^0 = \sum_{k=1}^{N} a_k \vec{w}_k^h, \quad \vec{u}^1 = \sum_{k=1}^{N} b_k \vec{w}_k^h.$$

Given $0 < \delta < 1$, we introduce the following classes of solutions of (5.2):

$$\mathcal{C}_\delta(h) = \left\{ \vec{u} \text{ sol. of (5.2) s.t. } \vec{u} = \sum_{k=1}^{[\delta/h]} \left(a_k \cos\left(\sqrt{\lambda_k^h} t\right) + \frac{b_k}{\sqrt{\lambda_k^h}} \sin\left(\sqrt{\lambda_k^h} t\right)\right) \vec{w}_k^h \right\}. \quad (5.28)$$

Note that in the class $\mathcal{C}_\delta(h)$ the high frequencies corresponding to the indices $j > [\delta(N+1)]$ have been cut off. We have the following result:

Theorem 5.2 *([41], [42]) For any $\delta > 0$ there exists $T(\delta) > 0$ such that for all $T > T(\delta)$ there exists $C = C(T, \delta) > 0$ such that*

$$\frac{1}{C} E_h(0) \leq \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt \leq C E_h(0) \quad (5.29)$$

for every solution u of (5.2) in the class $\mathcal{C}_\delta(h)$, and for all $h > 0$. Moreover, the minimal time $T(\delta)$ for which (5.29) holds is such that $T(\delta) \to 2$ as $\delta \to 0$ and $T(\delta) \to \infty$ as $\delta \to 1$.

Remark 5.3 Theorem 5.2 guarantees the uniform observability in each class $\mathcal{C}_\delta(h)$, for all $0 < \delta < 1$, provided the time T is larger than $T(\delta)$.

The last statement in Theorem 5.2 shows that when the filtering parameter δ tends to zero, i.e. when the solutions under consideration contain fewer and fewer frequencies, the time for uniform observability converges to $T = 2$, which is the corresponding for the continuous equation. This is in agreement with the observation that the group velocity of the low frequency semi-discrete waves coincides with the velocity of propagation in the continuous model.

By contrast, when the filtering parameter increases, i.e. when the solutions under consideration contain more and more frequencies, the time of uniform control tends to infinity. This is in agreement and explains further the negative result in Theorem 5.1 showing that, if the absence of filtering, there is no finite time T for which the uniform observability inequality holds.

The proof of Theorem 5.2 below provides an explicit estimate on the minimal observability time in the class $\mathcal{C}_\delta(h)$: $T(\delta) = 2/\cos(\pi\delta/2)$.

■

Remark 5.4 In the context of the numerical computation of the boundary control for the wave equation the need of an appropriate filtering of the high frequencies was observed by R. Glowinski [34]. This issue was further investigated numerically by M. Asch and G. Lebeau in [1]. There, finite difference schemes were used to test the Geometric Control Condition in various geometrical situations and to analyze the cost of the control as a function of time.

■

Proof of Theorem 5.2. The statement in Theorem 5.2 is a consequence of Ingham's inequality and the gap properties of the semi-discrete spectra. Let us analyze the gap between consecutive eigenvalues. We have

$$\begin{aligned}\sqrt{\lambda_k^h} - \sqrt{\lambda_{k-1}^h} &= \frac{2}{h}\left[\sin\left(\frac{\pi k h}{2}\right) - \sin\left(\frac{\pi(k-1)h}{2}\right)\right] \\ &= \pi\cos\left(\frac{\pi(k-1+\eta)h}{2}\right)\end{aligned}$$

for some $0 < \eta < 1$. Observe that

$$\cos\left(\frac{\pi(k-1+\eta)h}{2}\right) \geq \cos\left(\frac{\pi k h}{2}\right).$$

Therefore

$$\sqrt{\lambda_k^h} - \sqrt{\lambda_{k-1}^h} \geq \pi\cos\left(\frac{\pi k h}{2}\right).$$

It follows that

$$\sqrt{\lambda_k^h} - \sqrt{\lambda_{k-1}^h} \geq \pi\cos\left(\frac{\pi\delta}{2}\right), \text{ for } k \leq \delta h^{-1}. \tag{5.30}$$

We are now in the conditions for applying Ingham's Theorem. We rewrite the solution $\vec{u} \in \mathcal{C}_\delta(h)$ of (5.2) as

$$\vec{u} = \sum_{\substack{|k|\leq\delta/h \\ k\neq 0}} c_k e^{i\mu_k^h t}\vec{w}_k^h \tag{5.31}$$

where

$$\mu_{-k}^h = -\mu_k^h;\ \mu_k^h = \sqrt{\lambda_k^h};\ \vec{w}_{-k} = \vec{w}_k;\ c_k = \frac{a_k - ib_k/\mu_k^h}{2}; c_{-k} = \overline{c_k}.$$

Then,

$$u_N(t) = \sum_{\substack{|k|\leq\delta/h \\ k\neq 0}} c_k e^{i\mu_k^h t} w_{k,N}.$$

Therefore

$$\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt = \int_0^T \left|\sum_{\substack{|k|\leq\delta/h \\ k\neq 0}} c_k e^{i\mu_k^h t}\frac{w_{k,N}}{h}\right|^2 dt. \tag{5.32}$$

In view of the gap property (5.30) and, according to Ingham's inequality, it follows that if $T > T(\delta)$ with

$$T(\delta) = 2/\cos(\pi\delta/2) \tag{5.33}$$

there exists a constant $C = C(T, \delta) > 0$ such that

$$\frac{1}{C}\sum_{\substack{|k|\leq\delta/h \\ k\neq 0}} |\ c_k\ |^2 \left|\frac{w_{k,N}}{h}\right|^2 \leq \int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt \leq C\sum_{\substack{|k|\leq\delta/h \\ k\neq 0}} |\ c_k\ |^2 \left|\frac{w_{k,N}}{h}\right|^2 \tag{5.34}$$

for every solution of (5.2) in the class $\mathcal{C}_\delta(h)$. On the other hand, we observe that

$$\lambda_k^h h^2 = 4\sin^2\left(\frac{\pi k h}{2}\right) \leq 4\sin^2\left(\frac{\pi\delta}{2}\right) \tag{5.35}$$

for all $k \leq \delta/h$. Therefore, according to Lemma 5.1, it follows that

$$\frac{1}{2}\left|\frac{w_N}{h}\right|^2 \leq h\sum_{j=0}^{N}\left|\frac{w_{j+1}-w_j}{h}\right|^2 \leq \frac{1}{2\cos^2(\pi\delta/2)}\left|\frac{w_N}{h}\right|^2 \tag{5.36}$$

for all eigenvalues with index $k \leq \delta/h$.

Combining (5.34) and (5.36) we deduce that for all $T > T(\delta)$ there exists $C > 0$ such that

$$\frac{1}{C}\sum_{\substack{|k|\leq\delta/h\\ k\neq 0}} | \, c_k \, |^2 \leq \int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt \leq C \sum_{\substack{|k|\leq\delta/h\\ k\neq 0}} | \, c_k \, |^2 . \tag{5.37}$$

Finally we observe that

$$\sum_{\substack{|k|\leq\delta/h\\ k\neq 0}} | \, c_k \, |^2 \sim E_h(0).$$

This concludes the proof of Theorem 5.2.

5.4 Conclusion and controllability results

We have shown that the uniform observability property of the finite difference approximations (5.2) fails for any $T > 0$. On the other hand, we have proved that by filtering the high frequencies or, in other words, considering solutions in the classes $\mathcal{C}_\delta(h)$ with $0 < \delta < 1$, the uniform observability holds in a minimal time $T(\delta)$ that satisfies

- $T(\delta) \to \infty$ as $\delta \to 1$;
- $T(\delta) \to 2$ as $\delta \to 0$.

Observe that, as $\delta \to 0$, we recover the minimal observability time $T = 2$ of the continuous wave equation (2.1). This allows us to obtain, for all $T > 2$, the observability property of the continuous wave equation (2.1) as the limit $h \to 0$ of uniform observability inequalities for the semi-discrete systems (5.2). Indeed, given any $T > 2$ there exists $\delta > 0$ such that $T > T(\delta)$ and, consequently, by filtering the high frequencies corresponding to the indices $k > \delta N$, the uniform observability in time T is guaranteed. As we mentioned above, this confirms that the semi-discrete scheme provides a better approximation of the wave equation when the high frequencies are filtered.

On the other hand, we have seen that when $\lambda \sim 4/h^2$ or, equivalently, $k \sim 1/h$, the gap between consecutive eigenvalues vanishes. This allows the construction of high-frequency wave-packets that travel at a group velocity of the order of h and it forces the uniform observability time $T(\delta)$ in the classes $\mathcal{C}_\delta(h)$ to tend to infinity as $\delta \to 1$.

Note that, for any fixed $h > 0$, (5.2) is a linear system of dimension N. Therefore, in view of Lemma 5.1 and the fact that the eigenvalues $\lambda_1^h, \ldots, \lambda_N^h$ are all distinct, it follows that (5.2) is observable for all $T > 0$. But a singular phenomenon arises when letting $h \to 0$ since the limit wave equation (2.1) is only observable when $T > 2$. The analysis we have carried out in this section explains this fact: *To obtain uniform observability inequalities as $h \to 0$, high frequencies have to be filtered out and the time T needs to be larger than $T(\delta)$, which depends on the filtering parameter δ.* The time $T(\delta)$ converges to the optimal time $T = 2$ when the filtering parameter δ tends to zero.

Here we have used Inghman's inequality and spectral analysis. These results may also be obtained using discrete multiplier techniques ([41] and [42]).

In this sub-section we explain the consequences of these results in the context of controllability. Before doing this, it is important to distinguish two notions of the controllability of any evolution system, regardless of whether it is finite or infinite-dimensional:

- To control exactly to zero the whole solution for initial data in a given subspace.
- To control the projection of the solution over a given subspace for all initial data.

Before discussing these issues in detail it is necessary to write down the control problem we are analyzing. The state equation is as follows:

$$\begin{cases} y_j'' - \frac{1}{h^2}\left[y_{j+1} + y_{j-1} - 2y_j\right] = 0, & 0 < t < T,\ j = 1, \ldots, N \\ y_0(0,t) = 0;\ y_{N+1}(1,t) = v(t), & 0 < t < T \\ y_j(0) = y_j^0,\ y_j'(0) = y_j^1, & j = 1, \ldots, N, \end{cases} \tag{5.38}$$

and the question we consider is whether, for a given $T > 0$ and given initial data $(\vec{y}^0, \vec{y}^1)$, there exists a control $v_h \in L^2(0,T)$ such that

$$\vec{y}(T) = \vec{y}'(T) = 0. \tag{5.39}$$

As we shall see below, the answer to this question is positive. It is then natural to address the issue of the convergence of the controls v_h, as $h \to 0$, towards the controls of the continuous wave equation (2.4).

We have the following main results:

- For all $T > 0$ and all $h > 0$ the semi-discrete system (5.38) is controllable. In other words, for all $T > 0$, $h > 0$ and initial data $(\vec{y}^0, \vec{y}^1)$, there exists $v \in L^2(0,T)$ such that the solution $\vec{y}$ of (5.38) satisfies (5.39). Moreover, the control v of minimal $L^2(0,T)$-norm can be built as in section 2. It suffices to minimize the functional

$$J_h((\vec{u}^0, \vec{u}^1)) = \frac{1}{2}\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt + h\sum_{j=1}^N y_j^0 u_j^1 - h\sum_{j=1}^N y_j^1 u_j^0 \tag{5.40}$$

over the space of all initial data $(\vec{u}^0, \vec{u}^1)$ for the adjoint semi-discrete system (5.2).

Of course, this strictly convex and continuous functional is coercive and, consequently, has a unique minimizer. The coercivity of the functional is a consequence of the observability inequality (5.7) that does indeed hold for all $T > 0$ and $h > 0$. To check that (5.7) holds one can argue in two different ways: a) By using the classical Kalman rank condition (see [51]) for the controllability/observability of finite-dimensional linear time-independent systems; b) proceeding as in the previous section by sidewise energy estimates. Propagating the information provided by the right hand side of the observability inequality (5.7) from the last node $j = N$ to the first one $j = 1$ in a recurrent way, one can indeed deduce (5.7).

Once we know that the minimum of J_h is achieved, the control is easy to compute. It suffices to take

$$v_h(t) = \frac{u_N^*(t)}{h}, \; 0 < t < T, \tag{5.41}$$

where $\vec{u}^*$ is the solution of the semi-discrete adjoint system (5.2), corresponding to the initial data $(\vec{u}^{0,*}, \vec{u}^{1,*})$ that minimize the functional J_h, as control to guarantee that (5.39) holds.

The control we obtain in this way is optimal in the sense that it is the one of minimal $L^2(0,T)$-norm. We can also get an upper bound on its size. Indeed, using the fact that $J_h \leq 0$ at the minimum (which is a trivial fact since $J_h((0,0)) \leq 0$), and the observability inequality (5.7), we deduce that

$$||v_h||_{L^2(0,T)} \leq 4C_h(T)||(y^0, y^1)||_{*,h}, \tag{5.42}$$

where $||\cdot||_{*,h}$ denotes the norm

$$||(y^0, y^1)||_{*,h} = \sup_{(u_j^0, u_j^1)_{j=1,\ldots,N}} \left[\left| h\sum_{j=1}^{N} y_j^0 u_j^1 - h\sum_{j=1}^{N} y_j^1 u_j^0 \right| \Big/ E_h^{1/2}(u^0, u^1) \right] \tag{5.43}$$

It is easy to see that this norm converges as $h \to 0$ to the norm in $L^2(0,1) \times H^{-1}(0,1)$. This norm can also be written in terms of the Fourier coefficients. It becomes a weighted euclidean norm whose weights are uniformly (with respect to h) equivalent to those of the continuous $L^2 \times H^{-1}$-norm.

Remark 5.5 In [41] and [42], in one space dimension, similar results were proved for the finite element space semi-discretization of the wave equation (2.1) as well. More precisely, it was proved that the uniform observability and controllability properties fail as the mesh size tends to zero. It was also proved that these properties can be reestablished if the high frequencies are filtered and the control time is taken to be large enough. In Figure 9 below we plot the dispersion dyagram for the piecewise linear finite element space semi-discretization. This time the discrete spectrum and, consequently, the dispersion dyagram lies above the one corresponding to the continuous wave equation. But, the group velocity high frequency numerical solutions vanishes again. This is easily seen on the slope of the discrete dispersion curve.

■

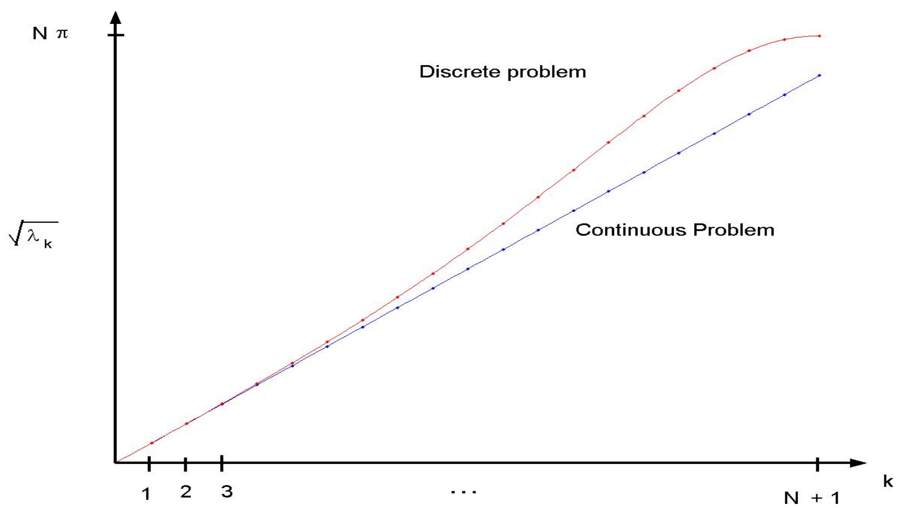

Figure 8: Dispersion dyagram for the piecewise linear finite element space semi-discretization versus the continuous wave equation.

- The estimate (5.42) is sharp. On the other hand, according to Theorem 5.1, for all $T > 0$ the constant $C_h(T)$ diverges as $h \to 0$. This shows that there are initial data for the wave equation in $L^2(0,1) \times H^{-1}(0,1)$ such that the controls of the semi-discrete systems $v_h = v_h(t)$ diverge as $h \to 0$. There are different ways of making this result precise. For instance, given initial data $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$ for the continuous system, we can consider in the semi-discrete control system (5.38) the initial data that take the same Fourier coefficients as (y^0, y^1) for the indices $j = 1, \ldots, N$. It then follows by the Banach-Steinhaus Theorem that, because of the divergence of the observability constant $C_h(T)$, there is necessarily some initial data $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$ for the continuous system such that the corresponding controls v_h for the semi-discrete system diverge in $L^2(0,T)$ as $h \to 0$. Indeed, assume that for any initial data $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$, the controls v_h remain uniformly bounded in $L^2(0,T)$ as $h \to 0$. Then, according to the uniform boundedness principle, we would deduce that the maps that associate the controls v_h to the initial data are also uniformly bounded. But this implies the uniform boundedness of the observability constant.

Consequently, *we should not use the control* v_h *of the semi-discrete system* (5.38) *as an approximation of the control* v *for the continuous wave equation since we know that, in some cases, it diverges.*

This lack of convergence is in fact easy to understand. As we have shown above, the semi-discrete system generates a lot of spurious high frequency oscillations. The control of the semi-discrete system has to take these into account. When doing this it gets further and further away from the true control of the continuous wave equation.

At this respect it is important to note that the following alternative holds:

a) Either the controls v_h of the numerical approximation schemes are not bounded in $L^2(0,T)$ as $h \to 0$;

or

b) If the controls are bounded, they converge in $L^2(0,T)$ to the control v of the continuous wave equation. Indeed, once the controls are bounded a classical argument of weak convergence and passing to the limit on the semi-discrete controlled systems shows that the limit of the controls is a control for the limit system. A Γ-convergence argument allows showing that it is the control of minimal $L^2(0;T)$ obtained by minimizing the functional J in (2.7). Finally, the convergence of the norms of the controls together with their weak convergence yileds the strong convergence result (see [52] for details of the proofs in the case of beam equations.)

- As shown in Theorem 5.2 above, the observability inequality is uniform in the class of filtered solutions $\mathcal{C}_\delta(h)$, for $T > T(\delta)$. As a consequence of this, one can control uniformly the projection of the solutions of the semi-discretized systems over subspaces in which the high frequencies have been filtered. More precisely, if the control requirement (5.39) is weakened to
$$\pi_\delta \vec{y}(T) = \pi_\delta \vec{y}\,'(T) = 0, \tag{5.44}$$
where π_δ denotes the projection of the solution of the semi-discrete system (5.38) over the subspace of the eigenfrequencies involved in the filtered space $\mathcal{C}_\delta(h)$, then the corresponding control remains uniformly bounded as $h \to 0$ provided $T > T(\delta)$. The control that produces (5.44) can be obtained by minimizing the functional J_h in (5.40) over the subspace $\mathcal{C}_\delta(h)$. Note that the uniform (with respect to h) coercivity of this functional and, consequently, the uniform bound on the controls is an immediate consequence of Theorem 5.2.

We underline the fact that Theorem 5.2 guarantees the uniform control of the projections π_δ for all initial data, but does not yield any result on the uniform complete controllability of any particular initial data.

Note however that one may recover the controllability property of the continuous wave equation as a limit of this partial controllability results since, as $h \to 0$, the projections π_δ end up covering the whole range of frequencies.

It is important to underline that the time of control depends on the filtering parameter δ in the projections π_δ. But, as we mentioned above, for any $T > 2$ there is a $\delta \in (0,1)$ for which $T > T(\delta)$ and so that the uniform (with respect to h) results apply.

However, although the divergence of the controls occurs for some data, it is hard to observe in numerical simulations. This fact has been recently explained by an important result by S. Micu [64]. According to [64], if the initial data of the wave equation has only a finite number of non vanishing Fourier components, the controls of the semi-discrete models are bounded as $h \to 0$ and converge to the control of the continuous wave equation.[13] The proof consists in writing the control problem as a moment problem and then getting estimates on its solutions by means of sharp and quite technical estimates of the family of biorthogonals to the family of complex exponentials $\{\exp(\pm i\sqrt{\lambda_j}t)\}_{j=1,\ldots,N}$. The interested reader will find an introduction to these techniques, based on moment problem theory, in the survey paper by D. Russell [76].

The result in [64], in terms of the observability inequality, is equivalent to proving that for $T > 0$ large enough, and for any finite $M > 0$, there exists a constant $C_M > 0$, independent of h, such that

$$E_h(\pi_M(u^0, u^1)) \leq C_M \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt \tag{5.45}$$

for any solution of the semi-discrete adjoint system. Here π_M denotes the projection over the subspace generated by the first M eigenvectors. The inequality (5.45) provides an estimate of $\pi_M(u^0, u^1)$ for any solution, regardless of the number of Fourier components it involves, rather than an estimate for the solutions involving only those M components. We do not know if this type of estimate can be obtained by multiplier methods or Ingham type inequalities. Very likely the celebrated Beurling-Malliavin Theorem can be of some use when doing this, but this issue remains to be clarified.[14]

[13] In fact, the result in [64] is much more precise since it indicates that, as $h \to 0$, one can control uniformly a space of initial data in which the number of Fourier component increases and tends to infinity.

[14] See [39] and [25] for applications of the Beurling-Malliavin Theorem in the control of plates and networks of vibrating strings.

5.5 Numerical experiments

In this section we briefly illustrate by some simple but convincing numerical experiments the theory developed along this section. These experiments have been developed by J. Rasmussen [70] using MatLab.

We consider the wave equation in the space interval $(0, 1)$ with control time $T = 4$.

We address the case of continuous and piecewise constant initial data of the form in Figure 9 below . In Figure 9 we only draw the initial state y^0 to be controlled but not the initial velocity y^1 since it is taken to be identically zero. In this simple situation and when the control time $T = 4$ the control can be computed explicitly. This can be done using Fourier series and the time periodicity of solutions of the adjoint system (with time period $= 2$).

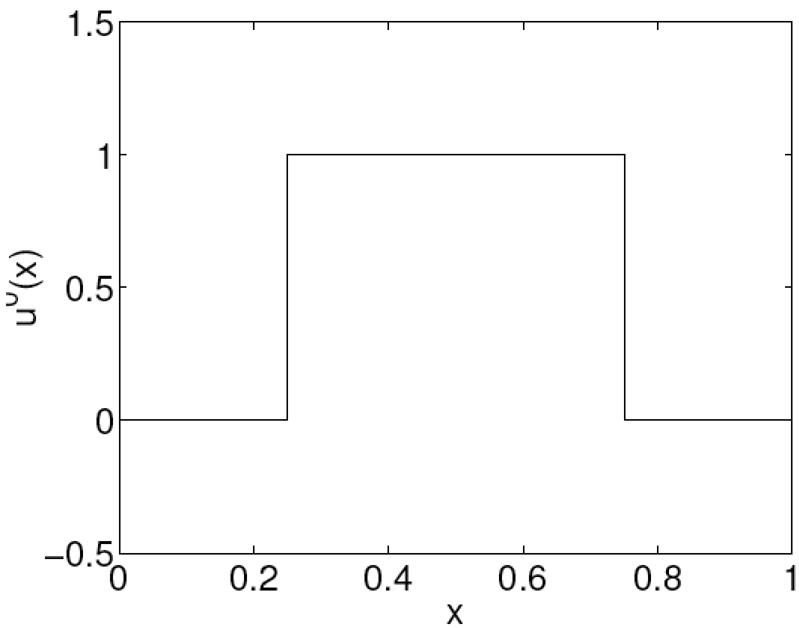

Figure 9: Plot of the initial datum to be controlled for the string occupying the space interval $0 < x < 1$.

Obviously the time $T = 4$ is sufficient for exact controllability to hold, the minimal control time being $T = 2$.

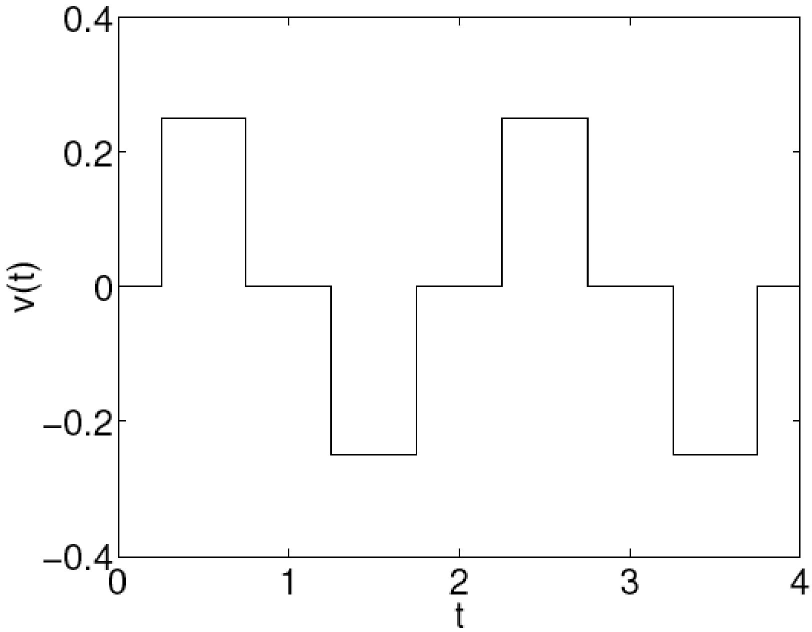

Figure 10: Plot of the time evolution of the exact control for the wave equation in time $T = 4$ with initial data as in Figure 9 above.

We see that the exact explicit control in Figure 10 looks very much like the initial datum to be controlled itslef. This can be easily understood by applying the d'Alembert formula for the explicit representation of solutions of the wave equation in $1D$.

We now consider the finite-difference semi-discrete approximation of the wave equation by finite-differences. First of all, we ignore all the discussion of the present section about the need of filtering. Thus, we merely compute the exact control of the semi-discrete system (5.38), asuming that, as $h \to 0$ that will yield a good approximation of the control of the wave equation (a fact that we know by now it is false). This is done, as described in section 5.4, by minimizing the functional J_h in (5.40) over the space of all solutions of the adjoint equation (2.1). Of course, in practice, we do not deal with the continuous adjoint equation but rather with a fully discrete approximation. We simply take the centered discretization in time with time-step $\Delta t = 0.99\,\Delta x$, $\Delta x = h$, which, of course, guarantees the converegnce of the scheme and the fact that our computations yield results which are very close to the semi-discrete case.

In Figure 11 below we draw the evolution on the error of the control as the number of mesh-points N increases. The solid line describes the evolution of the error when simply doing as indicated above, i. e. ignoring all the previous results about the need of filtering. This solid line diverges very fast as N increasing as a clear evidence of the lack of convergence of the control of the discrete system towards the control of the continuous one as $h \to 0$.

In the dotted line of Figure 11 we describe the evolution of the error when the filtering parameter is taken to be $\gamma = 0.6$. This filtering parameter has been chosen in order to guarantee the uniform observability of the filtered solutions of the adjoint semi-discrete and fully discrete (with $\Delta t = 0.99\Delta x$) schemes in time $T = 4$ and, concequently, the convergence of controls as $h \to 0$ as well. The evolution of the error as the number of mesh-points N increases, or, equivalently, when $h \to 0$, is obvious in the figure.

In the following sequence of figures (Figure 12) we show the evolution of the control of the discrete problem as the number of mesh-points N icreases, or, equivalently, when the mesh-size h tends to zero. We see that ,when $N = 20$, a low number of mesh-points, the control captures essentially the form of the continuous control in Figure 10 but with some extra unwanted oscillations. The situation is very similar when $N = 40$. But when $N = 100$ we see that these oscillations become wild and for $N = 160$ the dynamics of the control is completely chaotic. This is a good example of lack of convergence in the absence of filtering.

The evolution of the control as the number of mesh-points increases confirms the predictions of the theory. The dimension of the state space N increases with the number of mesh-points, N as well. The number of pathological spurious numerical high frequency oscillations increases as well. Of course, to control the finite-dimensional system the control has to take all these unwanted oscillations into account. As a consequence of this, the control gets further and further from the exact control of the continuous wave equation in Figure 10, as $h \to 0$.

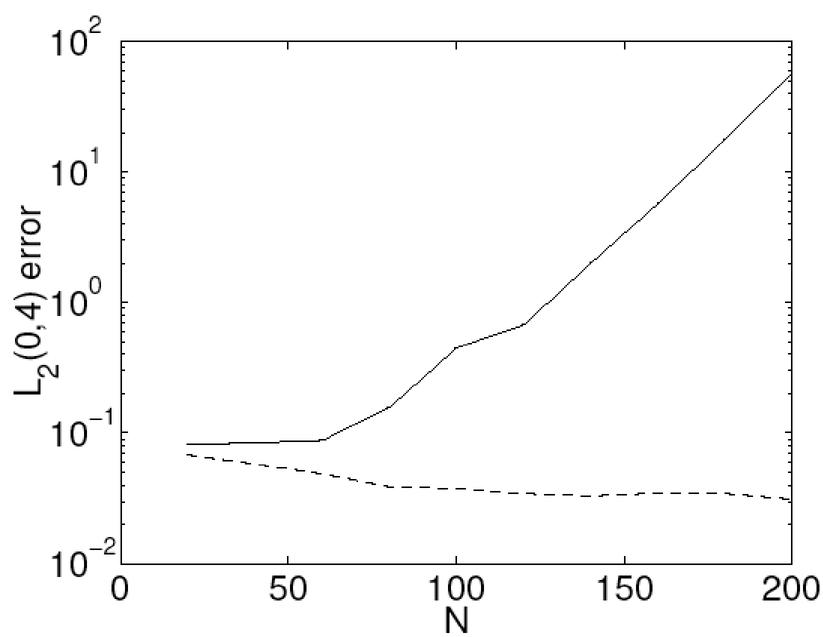

Figure 11: Plot, in solid line, of the evolution of the error in the computation of the control without filtering, versus the error (in dotted line) for the discrete case with filtering parameter 0.6.

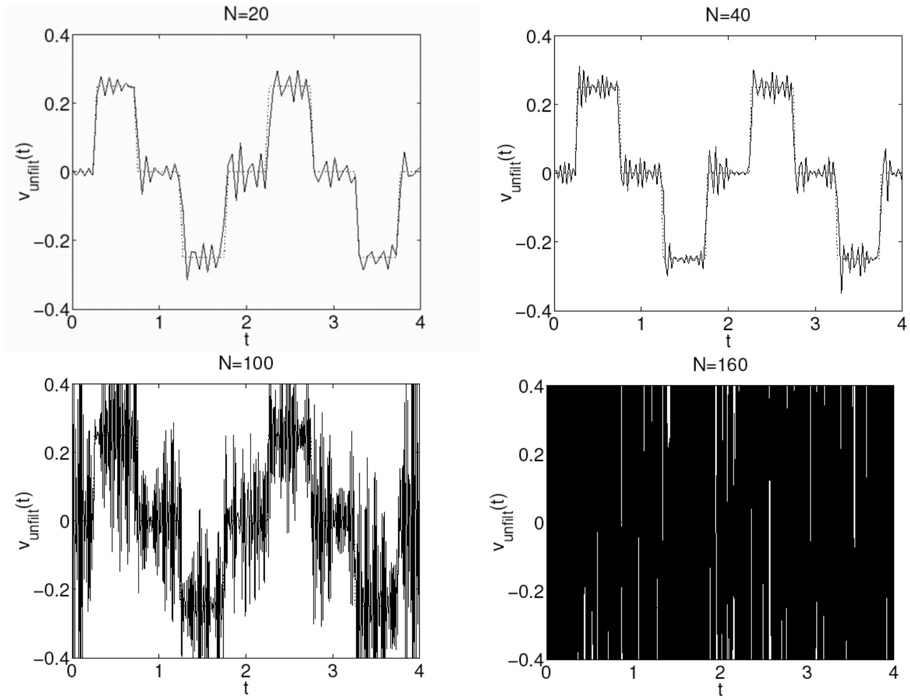

Figure 12: Divergent evolution of the control, in the absence of filtering, when the number N of mesh-points increases.

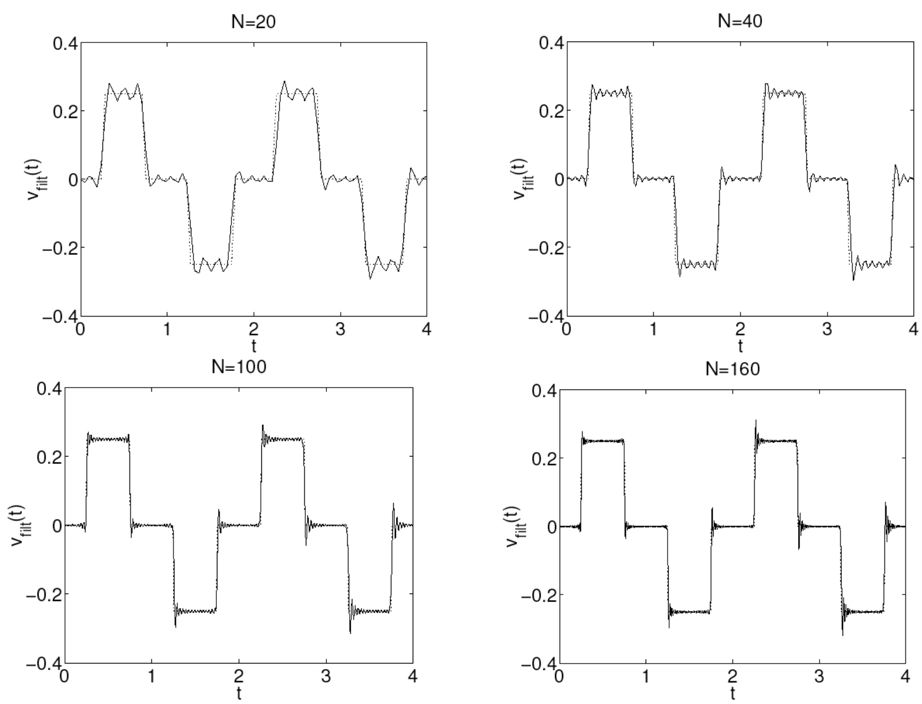

Figure 13: Convergent evolution of the control, with filtering parameter = 0.6, when the number N of mesh-points increases.

We now do the same experiment but now with filtering parameter = 0.6. Theory predicts convergence of controls. The numerical experiments we draw below (Figure 13) confirm this fact.

The evolution of the control as N increases is this time the appropriate one. The filtering parameter acts efficiently and the controller focuses on the low relevant frequencies, ignoring pathological high frequencies. As a consequence of this, the quality of the approximation of the numerical control towards the continuous one in Figure 10 improves as $h \to 0$.

5.6 Robustness of the optimal and approximate control problems

In the previous sections we have shown that the exact controllability property behaves badly under most classical finite difference approximations. It is natural to analyze to what extend the high frequency spurious pathologies do affect other control problems and properties. The following two are worth considering:

- *Approximate controllability.*

  Approximate controllability is a relaxed version of the exact controllability property. The goal this time is to drive the solution of the controlled wave equation (2.4) not exactly to the equilibrium as in (2.5) but rather to an ε-state such that

$$||y(T)||_{L^2(0,1)} + ||y_t(T)||_{H^{-1}(0,1)} \leq \varepsilon. \tag{5.46}$$

  When for all initial data (y^0, y^1) in $L^2(0,1) \times H^{-1}(0,1)$ and for all ε there is a control v such that (5.46) holds, we say that the system (2.5) is approximately controllable. Obviously, approximate controllability is a weaker notion than exact controllability. Whenever the wave equation is exactly controllable, it is approximately controllable too.

  However, the approximate controllability property holds in a much more general geometric setting. For instance, even in several space dimensions, approximate controllability does not require the GCC to hold. In fact, approximate controllability holds for controls acting on any open subset of the domain where the equation holds (or from its boundary) if the time is large enough.

  To be more precise, although exact controllability requires an observability inequality of the form of (2.3) to hold, for approximate controllability one only requires the following uniqueness property: $u \equiv 0$ *whenever the solution* u *of* (2.1) *is such that* $u_x(1,t) \equiv 0$, *in* $(0,T)$. This uniqueness property holds for $T \geq 2$ as well. Its multidimensional version holds as well as an immediate consequence of Holmgren's Uniqueness Theorem (see [54]) in a much more general setting than the observability inequality. Indeed, the uniqueness or unique continuation property holds for general wave equations with analytic coefficients and without geometric conditions, other than the time being large enough. In 1D, because of the trivial geometry, both uniqueness and observability inequality hold simultaneously for $T \geq 2$.

Of course, the approximate controllability property by itself, as stated, does not provide any information of what the cost of controlling to an ε-state as in (5.46) is. Roughly speaking, when exact controllability does not hold (for instance, in several space dimensions, when the GCC is not fulfilled), the cost of controlling blows up exponentially as ε tends to zero (see [74]).[15] But this issue will not be addressed here.

Thus, let us fix some $\varepsilon > 0$ and continue our discussion. Once ε is fixed, we know that when $T \geq 2$, for all initial data (y^0, y^1) in $L^2(0,1) \times H^{-1}(0,1)$, there exists a control $v_\varepsilon \in L^2(0,T)$ such that (5.46) holds.

The question we are interested in is the behavior of this property under numerical discretization.

Thus, let us consider the semi-discrete controlled version of the wave equation (5.38). We also fix the initial data in (5.38) "independently of h" (roughly, by taking a projection over the mesh of a fixed initial data (y^0, y^1) for the continuous wave equation or by truncating the Fourier series of the continuous data).

Of course, (5.38) is also approximately controllable.[16] The question we address is as follows : *Given initial data which are "independent of h", with ε fixed, and given also the control time $T \geq 2$, is the control v_h of the semi-discrete system* (5.38) *(such that the discrete version of* (5.46) *holds) uniformly bounded as $h \to 0$?*

In the previous sections we have shown that the answer to this question in the context of the exact controllability is negative. However, here, in the context of approximate controllability the controls v_h do remain uniformly bounded as $h \to 0$. Moreover, it is not hard to prove that they can be chosen such that they converge to a limit control v for which (5.46) is realized for the continuous wave equation.

This positive result on the uniformity of the approximate controllability property under numerical approximation does not contradict the fact that the controls blow up for exact controllability. These are two complementary facts. For approximate controllability, one is allowed to concentrate an ε amount of energy on the solution at the final time $t = T$. For the semi-discrete problem this is done precisely in the high frequency

[15] This type of result has been also proved in the context of the heat equation in [29]. But there the difficulty does not come from the geometry but rather from the regularizing effect of the heat equation.

[16] In fact, in finite dimensions, exact and approximate controllability are equivalent notions.

components that are badly controllable as $h \to 0$, and this makes it possible to keep the control bounded as $h \to 0$.

The proof of the uniform boundedness of the controls can be carried out easily. We refer the reader to [94] for the details in the context of the homogenization of heat equations.

In the present case, the approximate control of the semi-discrete system can be obtained by minimizing the functional

$$J_h^*(\vec{u}^0, \vec{u}^1) = \frac{1}{2}\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt + \varepsilon||(\vec{u}^0, \vec{u}^1)||_{\mathcal{H}^1\times\ell^2} + h\sum_{j=1}^N y_j^0 u_j^1 - h\sum_{j=1}^N y_j^1 u_j^0 \qquad (5.47)$$

over the space of all initial data $(\vec{u}^0, \vec{u}^1)$ for the adjoint semi-discrete system (5.2). In J_h^*, $||\cdot||_{\mathcal{H}^1\times\ell^2}$ stands for the discrete energy norm, i.e. $||\cdot|| = \sqrt{2E_h}$. Note that there is an extra term $\varepsilon||(\vec{u}^0, \vec{u}^1)||_{\mathcal{H}^1\times\ell^2}$ in this new functional compared with the one we used to obtain the exact control (see (5.40)). Thanks to this term, the functional J_h^* satisfies an extra coercivity property that can be proved to be uniform as $h \to 0$:

$$\underline{\lim}_{||(\vec{u}^0,\vec{u}^1)||_{\mathcal{H}^1\times\ell^2}\to\infty} \frac{J_h^*(\vec{u}^0, \vec{u}^1)}{||(\vec{u}^0, \vec{u}^1)||_{\mathcal{H}^1\times\ell^2}} \geq \varepsilon, \qquad (5.48)$$

uniformly in h, provided $T \geq 2$.

Note that, at this level, the fact that $T \geq 2$ is essential. Indeed, in order to show that the coercivity property above is uniform in $0 < h < 1$ we have to argue by contradiction as in [94]. In particular, we have to consider the case where $h \to 0$ and solutions of the adjoint semi-discrete system (5.2) converge to a solution of the continuous adjoint wave equation (2.18) such that $u_x(1,t) \equiv 0$ in $(0,T)$. Of course, if this happens with $T \geq 2$ we can immediately deduce that $u \equiv 0$, which yields the desired contradiction.

Once the uniform coercivity of the functional is proved, their minimizers are uniformly bounded and in particular, the controls, which are once again given by the formula (2.18), turn out to be uniformly bounded in $L^2(0,T)$. Once this is known it is not hard to prove by a Γ-convergence argument that these controls converge in $L^2(0,T)$ to the control $v \in L^2(0,T)$ for the continuous wave equation that one gets by minimizing the functional

$$\begin{aligned} J^*(u^0, u^1) \quad = \quad & \frac{1}{2}\int_0^T |u_x(1,t)|^2 dt + \varepsilon||(u^0, u^1)||_{H^1(0,1)\times L^2(0,1))} + \\ & + \int_0^1 y^0 u^1 dx - \langle y^1, u^0\rangle_{H^{-1}\times H_0^1} \end{aligned} \qquad (5.49)$$

in the space $H_0^1(0,1)\times L^2(0,1)$ for the solutions of the continuous adjoint wave equation (2.18). This control v is once again obtained as in (2.5) where u^* is the solution of (2.5) with the initial data minimizing the functional J^* and it turns out to be the function of minimal $L^2(0,T)$-norm among all admissible controls satisfying (5.46).

This shows that the approximate controllability property is well-behaved under the semi-discrete finite difference discretization of the wave equation. But the argument is in fact much more general and can be applied in several space dimensions too, and for other numerical approximation schemes.

■

- *Optimal control.*

The optimal control problem can also be viewed as a relaxed version of the exact controllability problem. This time the goal is to drive the solution at time $t = T$ as closely as possible to the desired equilibrium state but penalizing the use of the control. In the continuous context the problem can be simply formulated as that of minimizing the functional

$$L^k(v) = \frac{k}{2}||(y(T), y_t(T))||^2_{L^2(0,1)\times H^{-1}(0,1)} + \frac{1}{2}||v||^2_{L^2(0,T)} \tag{5.50}$$

over $v \in L^2(0,T)$. This functional is continuous, convex and coercive in the Hilbert space $L^2(0,T)$. Thus it admits a unique minimizer that we denote by v_k. The corresponding optimal state is denoted by y_k. The penalization parameter establishes a balance between reaching the distance to the target and the use of the control. As k increases, the need of getting close to the target (the $(0,0)$ state) is emphasized and the penalization on the use of control is relaxed.

When exact controllability holds, i.e. when $T \geq 2$, it is not hard to see that the control one obtains by minimizing L^k converges, as $k \to \infty$, to an exact control for the wave equation.

Of course, once $k > 0$ is fixed, the optimal control v_k does not guarantee that the target is achieved in an exact way. One can then measure the rate of convergence of the optimal solution $(y_k(T), y_{k,t}(T))$ towards $(0,0)$ as $k \to \infty$. When approximate controllability holds but exact controllability does not (a typical situation in several space dimensions when the GCC is not satisfied), the convergence of $(y_k(T), y_{k,t}(T))$ to $(0,0)$ in $L^2(0,1) \times H^{-1}(0,1)$ as $k \to \infty$ is very slow.[17]

But here, once again, we fix any $k > 0$ and we discuss the behavior of the optimal control problem for the semi-discrete equation as $h \to 0$.

[17] We should mention the works of Lebeau [48] and L. Robbiano [74] where the cost of approximate controllability is analyzed for the wave equation when the GCC fails. On the other hand, in [29] a logarithmic convergence rate was proved in the context of the heat equation.

It is easy to write the semi-discrete version of the problem of minimizing the functional L^k. It is sufficient to introduce the corresponding semi-discrete functional L_h^k and for this it suffices to replace the $L^2 \times H^{-1}$-norm in the definition of L^k by the discrete norm introduced in (5.43). It is also easy to prove that, as $h \to 0$, the control v_h^k that minimizes L_h^k in $L^2(0,T)$ converges to the minimizer of the functional L^k. Once this is done, the optimal solutions y_h^k of the semi-discrete system converge to the optimal solution y^k of the continuous wave equation in the appropriate topology[18] too.

This shows that the optimal control problem is also well-behaved with respect to numerical approximation schemes, like the approximate control problem.

The reason for this is basically the same: In the optimal control problem the target is not required to be achieved exactly and, therefore, the pathological high frequency spurious numerical components are not required to be controlled.

■

In view of this discussion it becomes clear that the source of divergence in the limit process as $h \to 0$ in the exact controllability problem is the requirement of driving the high frequency components of the numerical solution exactly to zero. As we mentioned in the introduction, taking into account that optimal and approximate controllability problems are relaxed versions of the exact controllability one, this negative result should be considered as a warning about the limit process as $h \to 0$ in general control problems.

6 Space discretizations of the 2D wave equations

In this section we briefly discuss the results in [91] on the space finite difference semi-discretizations of the 2D wave equation in the square $\Omega = (0,\pi) \times (0,\pi)$ of $\mathbf{R}^2$:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in} \quad Q = \Omega \times (0,T) \\ u = 0 & \text{on} \quad \Sigma = \partial\Omega \times (0,T) \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x) & \text{in} \quad \Omega. \end{cases} \tag{6.1}$$

Given $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, system (6.1) admits a unique solution $u \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right)$. Moreover, the energy

$$E(t) = \frac{1}{2} \int_\Omega \left[\mid u_t(x,t) \mid^2 + \mid \nabla u(x,t) \mid^2 \right] dx \tag{6.2}$$

remains constant, i.e.

$$E(t) = E(0), \quad \forall 0 < t < T. \tag{6.3}$$

[18] Roughly, in $C([0,T]; L^2(0,1)) \cap C^1[0,T]; H^{-1}(0,1))$.

Let Γ_0 denote a subset of the boundary of Ω constituted by two consecutive sides, for instance,

$$\Gamma_0 = \{(x_1, \pi) : x_1 \in (0, \pi)\} \cup \{(\pi, x_2) : x_2 \in (0, \pi)\}. \tag{6.4}$$

It is well known (see [54]) that for $T > 2\sqrt{2}\pi$ there exists $C(T) > 0$ such that

$$E(0) \leq C(T) \int_0^T \int_{\Gamma_0} \left| \frac{\partial u}{\partial n} \right|^2 d\sigma dt \tag{6.5}$$

holds for every finite-energy solution of (6.1). In (6.5), n denotes the outward unit normal to Ω, $\partial \cdot / \partial n$ the normal derivative and $d\sigma$ the surface measure.

Remark 6.1 The lower bound $2\sqrt{2}\pi$ on the minimal observability time is sharp.

On the other hand inequality (6.5) fails if in the right-hand side, instead of Γ_0, we only consider the energy concentrated on a strict subset of Γ_0. Assume for instance that the segment $\{(x_1, \pi) : x_1 \in (0, \alpha)\}$ does not belong to the control subset Γ_0 for some $0 < \alpha < \pi$. In this case, every segment $\{(x_1^0, x_2) : 0 < x_2 < \pi\}$ with $x_1^0 \in (0, \alpha)$ constitutes a ray of geometrical optics that propagates in the domain Ω and bounces on its boundary for all time without ever reaching the control subset. Thus, the Geometric Control Condition is not satisfied and the observability inequality fails. This situation is similar to the one described in Figure 3.

■

Let us now introduce the standard 5-point finite difference space semi-discretization scheme for the 2D wave equation. Given $N \in \mathbf{N}$ we set

$$h = \frac{\pi}{N+1}. \tag{6.6}$$

We denote by $u_{j,k}(t)$ the approximation of the solution u of (6.1) at the point $x_{j,k} = (jh, kh)$. The finite difference semi-discretization of (6.1) is as follows:

$$\begin{cases} u''_{jk} - \frac{1}{h^2}\left[u_{j+1,k} + u_{j-1,k} - 4u_{j,k} + u_{j,k+1} + u_{j,k-1}\right] = 0, \\ \qquad\qquad\qquad\qquad 0 < t < T, j,\ k = 1, \dots, N \\ u_{j,k} = 0, \qquad\qquad 0 < t < T,\ j = 0, N+1; k = 0, N+1 \\ u_{j,k}(0) = u^0_{j,k}, u'_{j,k}(0) = u^1_{j,k}, \qquad\qquad j,\ k = 1, \dots, N. \end{cases} \tag{6.7}$$

This is a coupled system of N^2 linear differential equations of second order.

It is well known that this semi-discrete scheme provides a convergent numerical scheme for the approximation of the wave equation. Let us now introduce the *discrete energy* associated with (6.7):

$$E_h(t) = \frac{h^2}{2} \sum_{j=0}^{N} \sum_{k=0}^{N} \left[|\ u'_{jk}(t)\ |^2 + \left| \frac{u_{j+1,k}(t) - u_{j,k}(t)}{h} \right|^2 + \left| \frac{u_{j,k+1}(t) - u_{j,k}(t)}{h} \right|^2 \right], \tag{6.8}$$

that remains constant in time, i.e.,

$$E_h(t) = E_h(0), \quad \forall 0 < t < T \tag{6.9}$$

for every solution of (6.7).

Note that the discrete version of the energy observed on the boundary is given by

$$\int_0^T \int_{\Gamma_0} \left|\frac{\partial u}{\partial n}\right|^2 d\sigma dt \sim \int_0^T \left[h \sum_{j=1}^{N} \left| \frac{u_{j,N}(t)}{h} \right|^2 + h \sum_{k=1}^{N} \left| \frac{u_{N,k}(t)}{h} \right|^2 \right] dt. \tag{6.10}$$

The discrete version of (6.5) is then an inequality of the form

$$E_h(0) \leq C_h(T) \int_0^T \left[h \sum_{j=1}^{N} \left| \frac{u_{j,N}(t)}{h} \right|^2 + h \sum_{k=1}^{N} \left| \frac{u_{N,k}(t)}{h} \right|^2 \right] dt. \tag{6.11}$$

This inequality holds for any $T > 0$ and $h > 0$ as in (5.7), for a suitable constant $C_h(T) > 0$. As in the 1D case this can be proved in two different ways: a) Using the characterization of observability/controllability by means of Kalman's rank condition [51]; b) Using a propagation argument. Indeed, using the information that the right side of (6.11) provides and the semi-discrete system, this information can be propagated to all the nodes $j, k = 1, \ldots, N$ and an inequality of the form (6.11) can be obtained. This argument was developed at the end of subsection 5.2 in 1D. Of course, in 2D one has to be much more careful, since one has to take into account also the direction of propagation on the numerical mesh. However, as proved in [17], this argument can be successfully applied if one uses the information on the nodes in the neighborhood of one of the sides of the square to get a global estimate of the energy.

The problem we discuss here is the 2D version of the 1D one we analyzed in section 5 and can be formulated as follows: *Assuming $T > 2\sqrt{2}\pi$, is the constant $C_h(T)$ in* (6.11) *uniformly bounded as $h \to 0$? In other words, can we recover the observability inequality* (6.5) *as the limit as $h \to 0$ of the inequalities* (6.11) *for the semi-discrete systems* (6.7)?

As in the 1D case the constants $C_h(T)$ in (6.11) necessarily blow up as $h \to 0$, for all $T > 0$.

Theorem 6.1 *For any $T > 0$ we have*

$$\sup_{u \text{ solution of } (6.7)} \left[\frac{E_h(0)}{\int_0^T \left[h \sum_{j=1}^N \left| \frac{u_{j,N}(t)}{h} \right|^2 + h \sum_{k=1}^N \left| \frac{u_{N,k}}{h} \right|^2 \right] dt} \right] \to \infty \text{ as } h \to 0. \tag{6.12}$$

The proof of this negative result can be carried out as in the 1D case, analyzing the solutions in separated variables corresponding to the eigenvector with largest eigenvalue.

In order to prove the positive counterpart of Theorem 6.1 we have to filter the high frequencies. To do this we consider the eigenvalue problem associated with (6.7):

$$\begin{cases} -\frac{1}{h^2}\left[\varphi_{j+1,k}+\varphi_{j-1,k}-4\varphi_{j,k}+\varphi_{j,k+1}+\varphi_{j,k-1}\right]=\lambda\varphi_{j,k}, & j,k=1,\ldots,N \\ \varphi_{j,k}=0, & j=0,N+1; k=0,\ N+1. \end{cases} \tag{6.13}$$

This system admits N^2 eigenvalues that can be computed explicitly (see [44], p. 459):

$$\lambda^{p,q}(h)=4\left[\frac{1}{h^2}\sin^2\left(\frac{ph}{2}\right)+\frac{1}{h^2}\sin^2\left(\frac{qh}{2}\right)\right], \quad p,q=1,\ldots,N \tag{6.14}$$

with corresponding eigenvectors

$$\varphi^{p,q}=\left(\varphi^{p,q}_{j,k}\right)_{1\le j,k\le N}, \quad \varphi^{p,q}_{j,k}=\sin(jph)\sin(kqh). \tag{6.15}$$

The following is a sharp upper bound for the eigenvalues of (6.13):

$$\lambda\le 4\left[\frac{1}{h^2}+\frac{1}{h^2}\right]=8\left[\frac{1}{h^2}\right]. \tag{6.16}$$

Let us also recall what the spectrum of the continuous system is. The eigenvalue problem associated with (6.1) is

$$-\Delta\varphi=\lambda\varphi \text{ in } \Omega;\ \varphi=0 \text{ on } \partial\Omega, \tag{6.17}$$

and its eigenvalues are

$$\lambda^{p,q}=p^2+q^2, \quad p,q\ge 1 \tag{6.18}$$

with corresponding eigenfunctions

$$\varphi^{p,q}(x_1,x_2)=\sin(px_1)\sin(qx_2). \tag{6.19}$$

Once again the discrete eigenvalues and eigenvectors converge as $h\to 0$ to those of the continuous Laplacean. Moreover, in the particular case under consideration, the discrete eigenvectors are actually the restriction to the mesh of the eigenfunctions of the continuous Laplacean. But, again, this is a non-generic fact.

Solutions of (6.7) can be developed in Fourier series of the form

$$u=\sum_{\lambda}\left[a^+_\lambda e^{i\sqrt{\lambda}t}+a^-_\lambda e^{-i\sqrt{\lambda}t}\right]\varphi_\lambda, \tag{6.20}$$

where the sum runs over all eigenvalues of (6.13), $a^\pm_\lambda$ are complex coefficients, and φ_λ are the eigenvectors of (6.13). We then introduce the following classes of solutions of (6.7) in which the high frequencies have been truncated or filtered. For any $0<\gamma\le 8$ we set

$$\mathcal{C}_\gamma(h)=\left\{u \text{ solution of (6.7): } u=\sum_{\lambda\le\gamma h^{-2}}\left[a^+_\lambda e^{i\sqrt{\lambda}t}+a^-_\lambda e^{-i\sqrt{\lambda_k}t}\right]\varphi_\lambda\right\}. \tag{6.21}$$

According to the upper bound (6.16), when $\gamma = 8, \mathcal{C}_\gamma(h) = \mathcal{C}_8(h)$ coincides with the space of all solutions of (6.21). However, when $0 < \gamma < 8$, solutions in the class $\mathcal{C}_\gamma(h)$ do not contain the contribution of the high frequencies $\lambda > \gamma\, h^{-2}$ that have been truncated or filtered.

The observability inequality is not uniform as $h \to 0$ in the classes $\mathcal{C}_\gamma(h)$ when $\gamma \geq 4$. This is due to the fact that, in those classes, there exist solutions corresponding to high frequency oscillations in one direction and very slow oscillations in the other one. These are the separated variable solutions corresponding to the eigenvectors $\varphi^{(p,q)}$ with indices $(p, q) = (N, 1)$ and $(p, q) = (1, N)$, for instance. Thus, further filtering is needed in order to guarantee the uniform observability inequality. Roughly speaking, one needs to filter efficiently in both space directions, and this requires taking $\gamma < 4$.

The following holds (see [91]):

Theorem 6.2 *For any $0 < \gamma < 4$ there exists $T(\gamma) > 0$ such that for all $T > T(\gamma)$ there exists a constant $C = C(\gamma, T)$ independent of h such that*

$$E_h(0) \leq C \int_0^T \left[h \sum_{j=1}^{N} \left| \frac{u_{j,N}(t)}{h} \right|^2 + h \sum_{k=1}^{N} \left| \frac{u_{N,k}(t)}{h} \right|^2 \right] dt, \tag{6.22}$$

for every solution u of (6.7) in the class $\mathcal{C}_\gamma(h)$ and for all $h > 0$. Moreover, $T(\gamma) \to 2\sqrt{2\pi}$ when $\gamma \to 0$ and $T(\gamma) \to \infty$ as $\gamma \to 4$.

Note that, in the class $\mathcal{C}_\gamma(h)$ with $\gamma < 4$ the high frequency oscillations in both space variables have been filtered out. In particular, in what concerns the indices (p, q) of the eigenvectors entering in the Fourier expansion in the class $\mathcal{C}_\gamma(h)$, we have necessarily $p < \beta N$ and $q < \beta N$ for some $\beta = \beta(\gamma) < 1$. In [91] this inequality has been proved by means of discrete multipliers and Fourier series.

In order to better understand the necessity of filtering and getting sharp observability times it is convenient to adopt the approach of [59], [60] based on the use of discrete Wigner measures. The symbol of the semi-discrete system (6.7) for solutions of wavelength h is

$$\tau^2 - 4\left(\sin^2(\xi_1/2) + \sin^2(\xi_2/2)\right) \tag{6.23}$$

and can be easily obtained as in the von Neumann analysis of the stability of numerical schemes by taking the Fourier transform of the semi-discrete equation: the continuous one in time and the discrete one in space. [19]

Note that, in the symbol in (6.23) the parameter h disappears. This is due to the fact that we are analyzing the propagation of waves of wavelength of the order of h.

[19] This argument can be easily adapted to the case where the numerical approximation scheme is discrete in both space and time by taking discrete Fourier transforms in both variables.

The bicharacteristic rays are then defined as follows

$$\begin{cases} x_j'(s) = -2\text{sin}(\xi_j/2)\text{cos}(\xi_j/2) = -\text{sin}(\xi_j), \quad j = 1, 2 \\ t'(s) = \tau \\ \xi_j'(s) = 0, \quad j = 1, 2 \\ \tau'(s) = 0. \end{cases} \tag{6.24}$$

It is interesting to note that the rays are straight lines, as for the constant coefficient wave equation, as a consequence of the fact that the coefficients of the equation and the numerical discretization are both constant. We see however that in (6.24) both the direction and the velocity of propagation change with respect to those of the continuous wave equation.

Let us now consider initial data for this Hamiltonian system with the following particular structure: x_0 is any point in the domain Ω, the initial time $t_0 = 0$ and the initial microlocal direction (τ^*, ξ^*) is such that

$$(\tau^*)^2 = 4\left(\sin^2(\xi_1^*/2) + \sin^2(\xi_2^*/2)\right). \tag{6.25}$$

Note that the last condition is compatible with the choice $\xi_1^* = 0$ and $\xi_2^* = \pi$ together with $\tau^* = 2$. Thus, let us consider the initial microlocal direction $\xi_2^* = \pi$ and $\tau^* = 2$. In this case the ray remains constant in time, $x(t) = x_0$, since, according to the first equation in (6.24), x_j' vanishes both for $j = 1$ and $j = 2$. Thus, the projection of the ray over the space x does not move as time evolves. This ray never reaches the exterior boundary $\partial\Omega$ where the equation evolves and excludes the possibility of having a uniform boundary observability property. More precisely, this construction allows one to show that, as $h \to 0$, there exists a sequence of solutions of the semi-discrete problem whose energy is concentrated in any finite time interval $0 \le t \le T$, as much as one wishes in a neighborhood of the point x_0.

Note that this example corresponds to the case of very slow oscillations in the space variable x_1 and very rapid ones in the x_2-direction and it can be ruled out, precisely, by taking the filtering parameter $\gamma < 4$. In [91] it was proved, using Fourier series and multiplier techniques, that when the filtering parameter is chosen such that $\gamma < 4$, uniform observability holds.

In view of the structure of the Hamiltonian system, it is clear that one can be more precise when choosing the space of filtered solutions. Indeed, it is sufficient to exclude by filtering the rays that do not propagate at all to guarantee the existence of a minimal velocity of propagation (see Figure 14 below). Roughly speaking, this suffices for the observability inequality to hold uniformly in h for a sufficiently large time [59], [60]. This ray approach makes it possible to obtain the optimal uniform observability time depending on the class of filtered solutions under consideration. The optimal time is simply that needed by all characteristic rays entering in the class of filtered solutions to reach the controlled region. It is in fact the discrete version of the Geometric Control Condition (GCC) for the continuous wave equation. Moreover, if the filtering is done so that the wavelength of the solutions under consideration is of an order strictly less than h, then one recovers the classical observability result for the constant coefficient continuous wave equation with the optimal observability time.

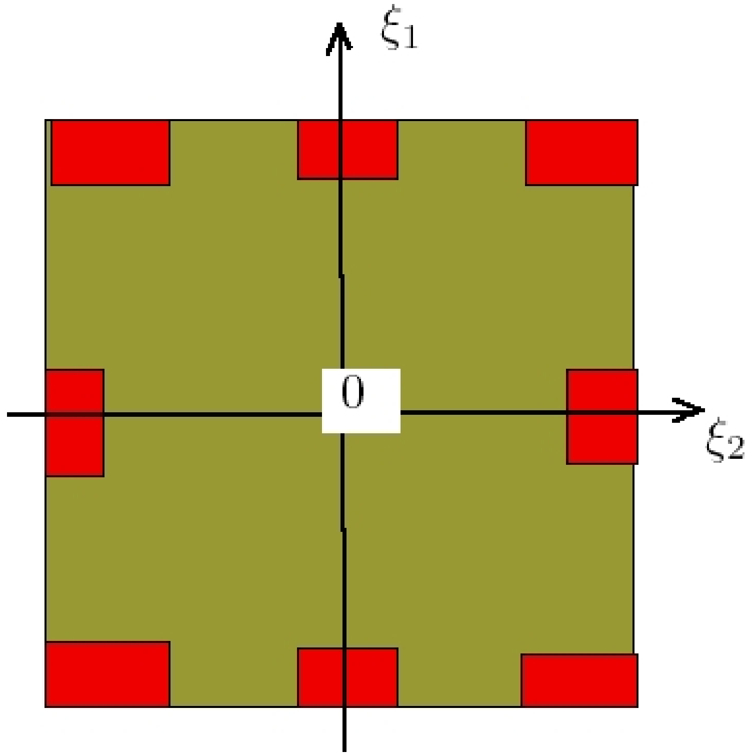

Figure 14: This figure represents the zones in the frequency space that need to be filtered out in order to guarantee a uniform minimal velocity of propagation of rays as $h \to 0$. When the filtering excludes the areas within the eight small neighborhoods of the distinguished points on the boundary of the frequency cell, the velocity of propagation of rays is uniform. Obviously the minimal velocity depends on the size of these patches that have been removed by filtering and, consequently, so does the observation/control time.

As we explained in subsection 5.4 in one space dimension, all the results we have presented in this section have their counterpart in the context of controllability. When uniform observability does not hold, there are initial data for which the controls diverge as h tends to zero. When the filtering of high frequencies reestablishes the uniform observability property we infer the uniform controllability of the corresponding projections, but one does not get the optimal control time for the wave equation (Theorem 6.2 above describes the dependence of the uniform control time as a function of the filtering parameter).

As far as we know, the 2D counterpart of the 1D positive result in [64], showing that initial data involving a finite number of Fourier components are uniformly controllable as

$h \to 0$, has not been proved.

7 Other remedies for high frequency pathologies

In the previous sections we have described the high frequency spurious oscillations that arise in finite difference space semi-discretizations of the wave equation and how they produce divergence of the controls as the mesh size tends to zero. We have also shown that there is a remedy for this, which consists in filtering the high frequencies by truncating the Fourier series. However, this method, which is natural from a theoretical point of view, can be hard to implement in numerical simulations. Indeed, solving the semi-discrete system provides the nodal values of the solution. One then needs to compute its Fourier coefficients and, once this is done, to recalculate the nodal values of the filtered/truncated solution. Therefore, it is convenient to explore other ways of avoiding these high frequency pathologies and to guarantee the convergence of the controls as $h \to 0$ that does not require going back and forth from the physical space to the frequency one. Here we shall briefly discuss other cures that have been proposed in the literature.

7.1 Tychonoff regularization

Glowinski et al. in [36] proposed a Tychonoff regularization technique that allows one to recover the uniform (with respect to the mesh size) coercivity of the functional that one needs to minimize to get the controls in the HUM approach.

Let us recall that the lack of uniform observability makes the functionals (5.40) not uniformly coercive, as we mentioned in section 5.4. As a consequence of this, for some initial data, the controls v_h diverge as $h \to 0$. In order to avoid this lack of uniform coercivity, the functional J_h can be reinforced by means of a Tychonoff regularization procedure.[20] Consider the new funct ional

$$J_h^*((u_j^0, u_j^1)_{j=1,\dots,N}) = \frac{1}{2}\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt + h^3 \sum_{j=0}^{N} \int_0^T (\frac{u'_{j+1} - u'_j}{h})^2 dt + + h\sum_{j=1}^{N} y_j^0 u_j^1 - h\sum_{j=1}^{N} y_j^1 u_j^0. \tag{7.1}$$

This functional is coercive when $T > 2$ and, more importantly, its coercivity is uniform in h. This is a consequence of the following observability inequality, which can be immediately

[20]This functional is a variant of the one proposed in [36] where the added term was $h^2||(\vec{u}^0, \vec{u}^1)||^2_{H^2\times H^1}$ instead of $h^3 \sum_{j=0}^N \int_0^T \left(\frac{u'_{j+1}-u'_j}{h}\right)^2 dt$. Both terms have the same scales, so that both are negligible at low frequencies but are of the order of the energy for the high ones. The one introduced in (7.1) arises naturally when applying discrete multipliers [81].

derived from (2.40) in [42]:

$$E_h(0) \leq C(T)\Big[\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt + h^3 \sum_{j=0}^{N} \int_0^T (\frac{u'_{j+1}-u'_j}{h})^2 dt\Big]. \tag{7.2}$$

This inequality holds for all $T > 2$ for a suitable $C(T) > 0$ which is independent of h and of the solution of the semi-discrete problem (5.2) under consideration.

Note that in (7.2) we have the extra term

$$h^3 \sum_{j=0}^{N} \int_0^T (\frac{u'_{j+1}-u'_j}{h})^2 dt, \tag{7.3}$$

which has also been used in the regularization of the functional J_h^* in (7.1). By inspection of the solutions of (5.2) in separated variables it is easy to understand why this added term is a suitable one to reestablish the uniform observability property. Indeed, consider the solution of the semi-discrete system $u = \exp(\pm i\sqrt{\lambda_j} t) w_j$. The extra term we have added contributes a quantity of the order of $h^2\lambda_j E_h(0)$. Obviously this term is negligible as $h \to 0$ for the low frequency solutions (for j fixed), but becomes relevant for the high frequency ones when $\lambda_j \sim 1/h^2$. Accordingly, when inequality (5.7) fails, i.e. for the high frequency solutions, the extra term in (7.2) reestablishes the uniform character of the estimate with respect to h. It is important to underline that both terms are needed for (7.2) to hold. Indeed, (7.3) by itself does not suffice since its contribution vanishes as $h \to 0$ for the low frequency solutions, for instance, for the separated solution above associated with λ_1.

As we said above, this uniform observability inequality guarantees the uniform boundedness of the minima of J_h^* and the corresponding controls. But there is an important price to pay. The control that J_h^* yields is not only at the boundary but also distributed everywhere in the interior of the domain. The corresponding control system reads as follows:

$$\begin{cases} y''_j - \frac{1}{h^2}\left[y_{j+1} + y_{j-1} - 2y_j\right] = h^2 g'_{h,j}, & 0 < t < T,\ j = 1, \ldots, N \\ y_0(0,t) = 0;\ y_{N+1}(1,t) = v_h(t), & 0 < t < T \\ y_j(0) = y_j^0,\ y'_j(0) = y_j^1, & j = 1, \ldots, N. \end{cases} \tag{7.4}$$

In this case, roughly speaking, when the initial data are fixed independently of h (for instance we consider initial data in $L^2(0,1) \times H^{-1}(0,1)$ and we choose the initial data in (7.4) as the corresponding Fourier truncation) then there exist controls $v_h \in L^2(0,T)$ and g_h such that the solution of (7.4) reaches equilibrium at time T with the following uniform bounds:

$$v_h \text{ is uniformly bounded in } L^2(0,T), \tag{7.5}$$

$$||\overrightarrow{(A_h)^{-1/2} g_h}||_h \text{ is uniformly bounded in } L^2(0,T) \tag{7.6}$$

where A_h is the matrix in (5.4), and $||\cdot||_h$ stands for the standard euclidean norm

$$||\overrightarrow{f_h}||_h = \left[h\sum_{j=1}^{N}|f_{h,j}|^2\right]^{1/2}. \tag{7.7}$$

The role that the two controls play is of different nature: The internal control $h^2 g_h'$ takes care of the high frequency spurious oscillations, and the boundary control deals with the low frequency components. In fact, it can be shown that, as $h \to 0$, the boundary control v_h converges to the control v of (2.4) in $L^2(0,T)$. In this sense, the limit of the control system (7.4) is the boundary control problem for the wave equation. To better understand this fact it is important to observe that, due to the h^2 multiplicative factor on the internal control, its effect vanishes in the limit. Indeed, in view of the uniform bound (7.6), roughly speaking,[21] the internal control is of the order of h^2 in the space $H^{-1}(0,T;H^{-1}(0,1))$ and therefore, tends to zero in the distributional sense. The fact that the natural space for the internal control is $H^{-1}(0,T;H^{-1}(0,1))$ comes from the nature of the regularizing term introduced in the functional J_h^*. Indeed, its continuous counterpart is

$$\int_0^T\int_0^1 |\nabla u_t|^2 dxdt$$

and it can be seen that, by duality, it produces controls of the form $\partial_t\partial_x(f)$ with $f \in L^2((0,1)\times(0,T))$. The discrete internal control reproduces this structure.

It is also easy to see that the control $h^2 g_{h,j}'$ is bounded in L^2 with respect to both space and time. This is due to two facts: a) the total norm of the operator $(A_h)^{1/2}$ is of order $1/h$, and b) taking one time derivative produces multiplicative factors of order $\sqrt{\lambda}$ for the solutions in separated variables. Since the maximum of the square roots of the eigenvalues at the discrete level is of order $1/h$, this yields a contribution of order $1/h$ too. These two contributions are balanced by the multiplicative factor h^2. Now recall that the natural space for the controlled trajectories is $L^\infty(0,T;L^2(0,1))\cap W^{1,\infty}(0,T;H^{-1}(0,1))$ at the continuous level, with the corresponding counterpart for the discrete one. However, the right-hand side terms in L^2 for the wave equation produces finite energy solutions in $L^\infty(0,T;H^1(0,1))\cap W^{1,\infty}(0,T;L^2(0,1))$. Thus, the added internal control only produces a compact correction on the solution at the level of the space $L^\infty(0,T;L^2(0,1))\cap W^{1,\infty}(0,T;H^{-1}(0,1))$. As a consequence of this one can show, for instance, that, using only boundary controls, one can reach states at time T that weakly (resp. strongly) converge to zero as $h\to 0$ in $H^1(0,1)\times L^2(0,1)$ (resp. $L^2(0,1)\times H^{-1}(0,1)$).

Summarizing, we may say that a Tychonoff regularization procedure may allow controlling uniformly the semi-discrete system at the price of adding an extra internal control but in

[21] To make this more precise we should introduce Sobolev spaces of negative order at the discrete level as in (5.43). This can be done using Fourier series representations or extension operators form the discrete grid to the continuous space variable.

such a way that the boundary component of the controls converge to the boundary control for the continuous wave equation. Thus, the method is efficient for computing approximations of the boundary control for the wave equation.

7.2 A two-grid algorithm

Glowinski in [34] introduced a two-grid algorithm that also makes it possible to compute efficiently the control of the continuous model. The relevance and impact of using two grids can be easily understood in view of the analysis above of the 1D semi-discrete model. In section 5 we have seen that that all the eigenvalues of the semi-discrete system satisfy $\lambda \leq 4/h^2$. We have also seen that the observability inequality becomes uniform when one considers solutions involving eigenvectors corresponding to eigenvalues $\lambda \leq 4\gamma/h^2$, with $\gamma < 1$. Glowinski's algorithm is based on the idea of using two grids: one with step size h and a finer one corresponding to $h/2$. In the finer mesh the eigenvalues obey the sharp bound $\lambda \leq 16/h^2$. Thus, the oscillations in the h mesh that correspond to the largest eigenvalues $\lambda \sim 4/h^2$ correspond in the finer mesh to eigenvalues in the class of filtered solutions with parameter $\gamma = 1/4$. Then, according to Theorem 5.2 and Remark 5.3, this corresponds to a situation where the observability inequality is uniform for $T > 2/\cos(\pi/8)$. Note however that, once again, the time needed for this procedure to work is greater than the minimal control time for the wave equation.

This explains the efficiency of the two-grid algorithm for computing the control of the continuous wave equation.

This method was introduced by Glowinski [34] in the context of the full finite difference and finite element discretizations in 2D. It was then further developed in the framework of finite differences by M. Asch and G. Lebeau in [1], where the Geometric Control Condition for the wave equation in different geometries was tested numerically. The convergence of this method has recently been proved rigorously by M. Negreanu [66] for finite difference and finite element semi-discrete approximation in one space dimension.

7.3 Mixed finite elements

Let us now discuss a different approach that is somewhat simpler than the previous ones. It consists in using mixed finite element methods rather than finite differences or standard finite elements, which require some post-processing by means of filtering, Tychonoff regularization or multigrid techniques, as we have shown.

First of all, it is important to underline that the analysis we have developed in section 5 for the finite difference space semi-discretization of the 1D wave equation can be carried out with minor changes for finite element semi-discretizations as well. In particular, due to the high frequency spurious oscillations, uniform observability does not hold [42]. It is thus natural to consider mixed finite element (m.f.e.) methods. This idea was introduced by Banks et al. [3] in the context of boundary stabilization of the wave equation. Here we adapt that approach to the analysis of controllability and observability. A variant of this method was introduced in [35].

The starting point is writing the adjoint wave equation (2.1) in the system form

$$u_t = v, \quad v_t = u_{xx}.$$

We now use two different Galerkin basis for the approximation of u and v. Since u lies in H_0^1, we use classical piecewise linear finite elements, and for v piecewise constant ones.

In these bases, and after some work which is needed to handle the fact that the left- and right-hand side terms of the equations in this system do not have the same regularity, one is led to the following semi-discrete system:

$$\begin{cases} \frac{1}{4}\left[u''_{j+1} + u''_{j-1} + 2u''_j\right] = \frac{1}{h^2}\left[u_{j+1} + u_{j-1} - 2u_j\right], & 0 < t < T,\ j = 1, \dots, N \\ u_j(t) = 0, & j = 0,\ N+1 \\ u_j(0) = u_j^0,\ u'_j(0) = u_j^1, & j = 1, \dots, N. \end{cases} \tag{7.8}$$

This system is a good approximation of the wave equation and converges in classical terms. Moreover, the spectrum of the mass and stiffness matrices involved in this scheme can be computed explicitly and the eigenvectors are those of (5.10), i.e. the restriction of the sinusoidal eigenfunctions of the Laplacean to the mesh points. The eigenvalues are now

$$\lambda_k = \frac{4}{h^2}\tan^2(k\pi h/2), \quad k = 1, ..., N. \tag{7.9}$$

For this spectrum the gap between the square roots of consecutive eigenvalues is uniformly bounded from below, and in fact tends to infinity for the highest frequencies as $h \to 0$ (Figure 15). According to this and applying Ingham's inequality, the uniform observability property can be easily proved. Note however that, due to the fact that the observability inequality is not even uniform for the eigenvectors (see Lemma 5.1) one can not expect the inequality (5.7) to hold. One gets instead that, for all $T > 2$, there exists $C(T) > 0$ such that

$$E_h(0) \le C_h(T) \int_0^T \left[\left| \frac{u_N(t)}{h} \right|^2 + h^2 \left| \frac{u'_N(t)}{h^2} \right|^2 \right] dt \tag{7.10}$$

for every solution of (7.8) and for all $h > 0$. As a consequence, the corresponding systems are also uniformly controllable and the controls converge as $h \to 0$. In [12] similar results have also been proved for a suitable $2D$ mixed finite element scheme.

As pointed out by J. Rasmussen [73], one of the drawbacks of this method is that the CFL stability condition that is required when dealing with fully discrete approximations

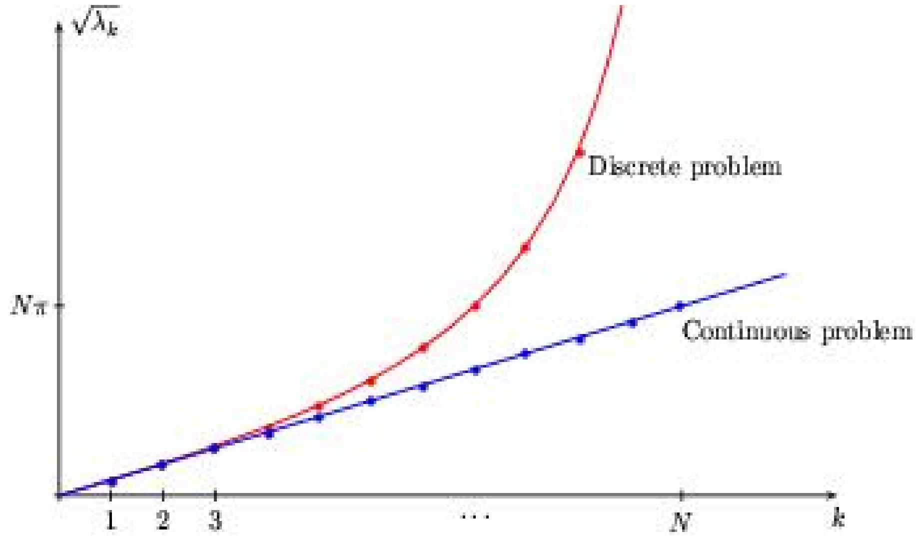

Figure 15: Square roots of the eigenvalues in the continuous and discrete cases with mixed finite elements (compare with Figure 5). The gaps are uniformly bounded from below for the discrete problem, and in fact tend to infinity for the highest frequencies as $h \to 0$.

based on this method is much stronger than for classical finite difference or finite element methods because of the sparsity of the spectrum. In this case, for instance, when considering centered time discretization, one requires $\Delta t \leq c(\Delta x)^2$, in opposition to the classical stability condition $\Delta t \leq c\Delta x^2$ for the other schemes. Thus, applying this method in numerical simulations requires the use of implicit time-discretization schemes.

8 Full 1D space-time discretizations

In this section we briefly present the results by M. Negreanu and the author in [67].

Given $M, N \in \mathbf{N}$ and $T > 0$ we set $\Delta x = 1/(N+1)$ and $\Delta t = T/(M+1)$ and introduce the nets

$$x_0 = 0 < x_1 = \Delta x < ... < x_N = N\Delta x < x_{N+1} = 1,$$

$$t_0 = 0 < t_1 = \Delta t < ... < t_M = M\Delta t < t_{M+1} = T,$$

with $x_j = j\Delta x$ and $t_n = n\Delta t$, $j = 0, 1, ..., N+1$, $n = 0, ..., M+1$. We consider the following finite difference discretization of the controlled wave equation:

$$\begin{cases} \dfrac{y_j^{n+1} - 2y_j^n + y_j^{n-1}}{(\Delta t)^2} = \dfrac{y_{j+1}^n - 2y_j^n + y_{j-1}^n}{(\Delta x)^2}, \quad j = 1, 2, ..., N;\ n = 1, 2, ..., M, \\ y_0^n = 0,\ y_{N+1}^n = v_{\Delta t}^n, \quad n = 1, 2, ..., M, \\ \text{the initial data} y_j^0,\ y_j^1, j = 1, 2, ..., N \quad \text{are also given.} \end{cases} \tag{8.1}$$

We denote by $\overrightarrow{y}^n = (y_1^n, ..., y_N^n)$ the solution at the time step n.

In practice one typically considers $y_j^0 = y^0(x_j)$ and $y_j^1 = y^0(x_j) + \Delta t y^1(x_j)$, $j = 1, 2, ..., N$, (y^0, y^1) being the initial data of the continuous wave equation to be approximated. When (y^0, y^1) are not continuous, one may consider any of the possible variants we have discussed above: Fourier series truncation, averages of the initial data around the mesh points, etc.

As in the context of the continuous wave equation, we consider the uncontrolled system

$$\begin{cases} \dfrac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{(\Delta t)^2} = \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}, \quad j = 1, 2, ..., N;\ n = 1, 2, ..., M, \\ u_0^n = u_{N+1}^n = 0, \quad n = 1, 2, ..., M, \\ \text{the initial data } u_j^0,\ u_j^1, j = 1, 2, ..., N \quad \text{are also given.} \end{cases} \tag{8.2}$$

The numerical schemes (8.1), (8.2) are consistent. In addition, they are stable if and only if $\Delta t \leq \Delta x$ (the CFL condition). Thus, if $\Delta t \leq \Delta x$, schemes (8.1), (8.2) converge as $\Delta t \to 0$, $\Delta x \to 0$ to (2.1) and (2.2), respectively.

The energy of solutions of (8.2) is constant in time:

$$E_n = \frac{\Delta x}{2} \sum_{j=0}^{N} \left[\left(\frac{u_j^{n+1} - u_j^n}{\Delta t} \right)^2 + \left(\frac{u_{j+1}^{n+1} - u_j^{n+1}}{\Delta x} \right) \left(\frac{u_{j+1}^n - u_j^n}{\Delta x} \right) \right] = E_0 \tag{8.3}$$

for all $n \geq 1$. Moreover, E_n is non-negative and vanishes if and only if the solution is identically zero.

We now look for solutions of (8.2) in separated variables of the form $\vec{u}^n = e^{i\omega_{\Delta x,k}\Delta t n} \vec{w}_k^{\Delta x}$ with $\vec{w}_k^{\Delta x}$ the eigenvectors of the space discretization of the Laplacean in (5.8) which are given by (5.10) with Δx instead of h. The corresponding eigenfrequencies are then

$$\omega_{\Delta x,k} = \pm \frac{2}{\Delta t} \arcsin \left(\frac{\Delta t \sin(k\pi\Delta x/2)}{\Delta x} \right).$$

In the particular case $\Delta t = \Delta x = h$ we have

$$\omega_{h,k} = \pm \frac{2}{h} \arcsin \left(\sin \frac{k\pi h}{2} \right) = \pm k\pi.$$

Thus, when $\Delta t = \Delta x = h$ the eigenvalues are the same as those of the continuous wave equation. Using a discrete Fourier decomposition of solutions of (8.2) it can be shown, as we did in the context of the continuous wave equation, that the observability inequality is uniform, i.e., the observability constant remains bounded as $\Delta t = \Delta x = h \to 0$. Once this is done, one can show that the HUM controls for the discrete model (8.1) converge to the control of the continuous wave equation.

As a conclusion of this analysis we see that the high frequency pathologies we observed in the space semi-discretization no longer arise in this case. But this is a very exceptional situation which is due to the fact that the solutions of the continuous wave equation are also exact solutions of the discrete scheme when $\Delta t = \Delta x$. This property is well known and can be easily checked by integrating the wave equation over a characteristic square with vertices (x_{j-1}, t_n), (x_j, t_{n+1}), (x_{j+1}, t_n), (x_j, t_{n-1}).[22] The detailed proof of these results can be found in [67].

Moreover, when $\Delta t/\Delta x = \alpha < 1$, as in the semi-discrete case, the uniform gap condition is lost and this produces the divergence of the observability constant and the controls. Thus, filtering of high frequencies becomes necessary to guarantee the uniform observability as the mesh-size tends to zero. This problem has been addressed by two different techniques: a) The methods developed in [59], [60] allow to address this issue by means of discrete Wigner measures in any space dimension; b) The discrete version of the Ingham inequality in [66] allows obtaining uniform (with respect to the mesh-size) observability estimates in filtered classes of solutions for $1D$ problems.

In several space dimensions the situation is even worse, since there is no possible choice of the space and time steps guaranteeing the convergence of the numerical scheme together with the uniform observability property. This can be seen easily following the analysis in [59], , [60] by means of discrete Wigner measures. If the semi-discrete equation (6.7) is further discretized in time with time step Δt, we get the fully discrete scheme

$$\frac{1}{\Delta t^2}\left[u_{j,k}^{n+1} + u_{j,k}^{n-1} - 2u_{j,k}^{n}\right] - \frac{1}{h^2}\left[u_{j+1,k}^{n} + u_{j-1,k}^{n} - 4u_{j,k}^{n} + u_{j,k+1}^{n} + u_{j,k-1}^{n}\right] = 0.$$

This system is convergent to the continuous wave equation when $h, \Delta t \to 0$, when the CFL condition $\Delta t \leq h/\sqrt{2}$ is satisfied.

The corresponding symbol is

$$4\sin^2(\tau\Delta t/2h) - 4\left(\sin^2(\xi_1/2) + \sin^2(\xi_2/2)\right). \tag{8.4}$$

For $\Delta t \sim \mu h$, for any $\mu \leq \frac{1}{\sqrt{2}}$, there are rays that, as described in section 6 in the context of the semi-discrete scheme, do not propagate at all as time increases. This makes the uniform (as $h, \Delta t \to 0$) boundary observability property impossible for the full discretization. Therefore, the fully discrete schemes need also to be complemented with one of the cures described in the previous sections to guarantee the convergence towards the control of the continuous wave equation: Fourier filtering, Tychonoff regularization, multigrid techniques or mixed finite elements.

9 Other models

In this article we have seen that most numerical schemes for the wave equation produce high frequency pathologies that make the boundary observability inequalities nonuniform

[22] In several space dimensions this exact discrete scheme for the wave equation does not exist.

and produce divergence of the controls of the semi-discrete or discrete systems as the mesh size tends to zero. We have also seen some possible remedies.

However, other equations behave much better due to diffusive or dispersive effects. As we shall see in the present section, these high frequency pathologies do not arise when dealing with the 1D heat and beam equation.

9.1 Finite difference space semi-discretizations of the heat equation

Let us consider the following 1D heat equation with control acting at the boundary point $x = L$:

$$\begin{cases} y_t - y_{xx} = 0, & 0 < x < L,\ 0 < t < T \\ y(0,t) = 0,\ y(L,t) = v(t), & 0 < t < T \\ y(x,0) = y^0(x), & 0 < x < L. \end{cases} \tag{9.1}$$

This is the so called boundary control problem. It is by now well known that (9.1) is null controllable in any time $T > 0$ (see for instance D.L. Russell [76], [77]). To be more precise, the following holds: *For any* $T > 0$*, and* $y^0 \in L^2(0,L)$ *there exists a control* $v \in L^2(0,T)$ *such that the solution* y *of* (9.1) *satisfies*

$$y(x,T) \equiv 0 \ \text{in}\ (0,L). \tag{9.2}$$

This null controllability result is equivalent to a suitable observability inequality for the adjoint system:

$$\begin{cases} u_t + u_{xx} = 0, & 0 < x < L,\ 0 < t < T, \\ u(0,t) = u(L,t) = 0, & 0 < t < T \\ u(x,T) = u^0(x), & 0 < x < L. \end{cases} \tag{9.3}$$

Note that, in this case, due to the time irreversibility of the state equation and its adjoint, in order to guarantee that the latter is well-posed, we take the initial conditions at the final time $t = T$.

The corresponding observability inequality is as follows: *For any* $T > 0$ *there exists* $C(T) > 0$ *such that*

$$\int_0^L u^2(x,0)dx \le C \int_0^T |u_x(L,t)|^2 \, dt \tag{9.4}$$

holds for every solution of (9.3).[23]

[23] This inequality has been greatly generalized (see for instance [31], [29]) to heat equations with potentials in several space dimensions, with explicit observability constants depending on the potentials, etc.

Let us consider now semi-discrete versions of (9.1) and (9.3):

$$\begin{cases} y_j' - \frac{1}{h^2}\left[y_{j+1} + y_{j-1} - 2y_j\right] = 0, & 0 < t < T, \quad j = 1, \dots, N \\ y_0 = 0,\ y_{N+1} = v, & 0 < t < T \\ y_j(0) = y_j^0, & j = 1, \dots, N; \end{cases} \tag{9.5}$$

$$\begin{cases} u_j' + \frac{1}{h^2}\left[u_{j+1} + u_{j-1} - 2u_j\right] = 0, & 0 < t < T,\ j = 1, \dots, N \\ u_0 = u_{N+1} = 0, & 0 < t < T \\ u_j(T) = u_j^0, & j = 1, \dots, N. \end{cases} \tag{9.6}$$

In this case, in contrast with the results we have described for the wave equation, the discretizations are uniformly controllable and observable, respectively, as $h \to 0$. More precisely, the following results hold:

Theorem 9.1 [58] *For any $T > 0$ there exists a positive constant $C(T) > 0$ such that*

$$h \sum_{j=1}^{N} \left|u_j(0)\right|^2 \leq C \int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt \tag{9.7}$$

holds for any solution of (9.6) *and any $h > 0$.*

Theorem 9.2 [58] *For any $T > 0$ and $\{y_1^0, \dots, y_N^0\}$ there exists a control $v \in L^2(0,T)$ such that the solution of* (9.5) *satisfies*

$$y_j(T) = 0,\ j = 1, \dots, N. \tag{9.8}$$

Moreover, there exists a constant $C(T) > 0$, independent of $h > 0$, such that

$$\| v \|_{L^2(0,T)}^2 \leq Ch \sum_{j=1}^{N} \left|y_j^0\right|^2. \tag{9.9}$$

These results were proved in [58] using Fourier series and a classical result on the sums of real exponentials (see for instance Fattorini-Russell [28]) that plays the role of Ingham's inequality in the context of parabolic equations.

Let us recall briefly: Given $\xi > 0$ and a decreasing function $M : (0, \infty) \to \mathbf{N}$ such that $M(\delta) \to \infty$ as $\delta \to 0$, we introduce the class $\mathcal{L}(\xi, M)$ of increasing sequences of positive real numbers $\{\mu_j\}_{j \geq 1}$ such that

$$\mu_{j+1} - \mu_j \geq \xi > 0, \quad \forall j \geq 1, \tag{9.10}$$

$$\sum_{k \geq M(\delta)} \mu_k^{-1} \leq \delta, \qquad \forall \delta > 0. \tag{9.11}$$

The following holds:

Proposition 9.1 *Given a class of sequences $\mathcal{L}(\xi, M)$ and $T > 0$ there exists a constant $C > 0$ (which depends on ξ, M and T) such that*

$$\int_0^T \left| \sum_{k=1}^{\infty} a_k e^{-\mu_k t} \right|^2 dt \geq \frac{C}{\left(\sum_{k\geq 1} \mu_k^{-1} \right)} \sum_{k\geq 1} \frac{|\, a_k \,|^2 \, e^{-2\mu_k T}}{\mu_k} \tag{9.12}$$

for all $\{\mu_j\} \in \mathcal{L}(\xi, N)$ and all sequence $\{a_k\} \in \ell^2$.

The eigenvalues of the semi-discrete Laplacean $\{\lambda_j^h\}_{j=1,...,N}$ in (5.9) belong to one of these uniform classes $\mathcal{L}(\xi, M)$. Indeed, in view of the explicit form of these eigenvalues there exists $c > 0$ such that $\lambda_j^h \geq cj^2$ for all $h > 0$ and $j = 1, ..., N$. On the other hand, the uniform gap condition (9.10) is also satisfied. Recall that, in the context of the wave equation, the lack of gap for the square roots of these eigenvalues was observed for the high frequencies. In particular it was found that $\sqrt{\lambda_N^h} - \sqrt{\lambda_{N-1}^h} \sim h$. But then, $\lambda_N^h - \lambda_{N-1}^h \sim (\sqrt{\lambda_N^h} - \sqrt{\lambda_{N-1}^h})(\sqrt{\lambda_N^h} + \sqrt{\lambda_{N-1}^h}) \sim 1$, since $\sqrt{\lambda_N^h} + \sqrt{\lambda_{N-1}^h} \sim 1/h$. This fact describes clearly why the gap condition is fulfilled in this case.

Taking into account that all the spectra of the semi-discrete systems belong to the same class $\mathcal{L}(\xi, M)$, applying the uniform inequality (9.12) together with the Fourier representation of solutions of (9.6), one gets, for all $T > 0$, a uniform observability inequality of the form (9.7) for the solutions of the semi-discrete systems (9.6). There is one slight difficulty when doing this. As observed in Lemma 5.1, the boundary observability property is not uniform for the high frequency eigenvectors. However, this is compensated in this case by the strong dissipative effect of the heat equation. Indeed, note that solutions of (9.6) can be written in Fourier series as

$$\vec{u}(t) = \sum_{j=1}^{N} a_j e^{-\lambda_j^h (T-t)} \vec{w}_j^h, \tag{9.13}$$

where $\{a_j\}_{j=1,...,N}$ are the Fourier coefficients of the initial data of (9.6) at $t = T$. The solution at the "final"[24] time can be represented as follows:

$$\vec{u}(0) = \sum_{j=1}^{N} a_j e^{-\lambda_j^h T} \vec{w}_j^h. \tag{9.14}$$

We see in this formula that the high frequencies are damped out by an exponentially small factor that compensates for the lack of uniform boundary observability of the high frequency eigenvectors.

Once the uniform observability inequality of Theorem 9.1 is proved, the controls for the semi-discrete heat equation (9.5) can be easily constructed by means of the minimization method described in section 5.4. The fact that the observability inequality is uniform implies

[24] Note that $t = 0$ is the final time for the adjoint equation (9.6), which is solved backwards from $t = T$ to $t = 0$.

the uniform bound (9.9) on the controls. The null controls for the semi-discrete equation (9.6) one obtains in this way are such that, as $h \to 0$, they tend to the null control for the continuous heat equation (9.1) (see [58]).

9.2 The beam equation

In a recent work by L. León and the author [52] the problem of boundary controllability of finite difference space semi-discretizations of the beam equation

$$y_{tt} + y_{xxxx} = 0$$

was addressed. This model has important differences from the wave equation even in the continuous case. First of all, at the continuous level, it turns out that the gap between consecutive eigenfrequencies tends to infinity. For instance, with the boundary conditions

$$y = y_{xx} = 0, \quad x = 0, \pi,$$

the solution admits the Fourier representation formula

$$y(x,t) = \sum_{k \in \mathbf{Z}} a_k e^{i\lambda_k t} \sin(kx),$$

where

$$\lambda_k = \mathrm{sgn}(k) k^2.$$

Obviously, the gap between consecutive eigenvalues is uniformly bounded from below. More precisely,

$$\lambda_{k+1} - \lambda_k = 2k + 1 \to \infty, \quad \text{as } k \to \infty.$$

This allows use to apply a variant of Ingham's inequality for an arbitrarily small control time $T > 0$ (see [65]).[25] As a consequence, boundary exact controllability holds for any $T > 0$ too.

When considering finite difference space semi-discretizations things are better than for the wave equation too. Indeed, as it is proved in [52], roughly speaking, the asymptotic gap[26] also

[25] Although in the classical Ingham inequality the gap between consecutive eigenfrequencies is assumed to be uniformly bounded from below for all indices k, in fact, in order for Ingham inequality to be true, it is sufficient to assume that all eigenfrequencies are distinct and that there is an asymptotic gap as $k \to \infty$. We refer to [65] for a precise statement where explicit estimates of the constants arising in the inequalities are given.

[26] In fact one needs to be more careful since, for $h > 0$ fixed, the gap between consecutive eigenfrequencies is not increasing. Indeed, in order to guarantee that the gap is asymptotically larger than any constant $L > 0$ one has to filter not only a finite number of low frequencies but also the highest ones. However, the methods and results in [65] apply in this context too (see [52]).

tends to infinity as $k \to \infty$, uniformly on the parameter h. This allows proving the uniform observability and controllability (as $h \to 0$) of the finite difference semi-discretizations. However, as we mentioned in section 7, due to the bad approximation that finite differences provide at the level of observing the high frequency eigenfunctions, the control has to be split in two parts. The main part that strongly converges to the control of the continuous equation in the sharp $L^2(0,T)$ space and the oscillatory one that converges to zero in a weaker space $H^{-1}(0,T)$. Thus, in the context of the beam equation, with the most classical finite difference semi-discretization, we get what we got for the wave equation with mixed finite elements. This fact was further explained by means of tools related with discrete Wigner measures in [59] , [60].

The same results apply for the Schrödinger equation.

10 Further comments and open problems

10.1 Further comments

a) We have considered finite difference space semi-discretizations of the wave equation. We have addressed the problem of boundary observability and, more precisely, the problem of whether the observability estimates are uniform when the mesh size tends to zero.

We have proved the following results:

- Uniform observability does not hold for any time T.
- Uniform observability holds if the time T is large enough provided we filter appropriately the high frequencies.

We have also mentioned the main consequences concerning controllability. In this article we have collected the existing work in this subject, to the best of our knowledge.

b) The problem of controllability has been addressed. Nevertheless, similar developments could be carried out, with the same conclusions, in the context of stabilization. For instance, it is well known that the Geometric Control Condition suffices to guarantee the uniform exponential decay of solutions of the linear wave equation when a damping term, supported in the control region, is added to the system. It is then natural to analyze whether the decay rate is uniform with respect to the mesh size for numerical discretizations. The answer is in general negative. Indeed, due to spurious high frequency oscillations, the decay rate fails to be uniform, for instance, for the classical finite difference semi-discrete approximation of the wave equation. This was established rigorously by F. Macià [59], [60] using Wigner measures. This negative result also has important consequences in many other issues related with control theory like infinite horizon control problems, Riccati equations for the optimal stabilizing feedback, etc. But these issues have not been studied so far (see open problem 8 below).

We shall simply mention here that, even if the most natural semi-discretization fails to be uniformly exponentially stable, the uniformity of the exponential decay rate can be reestablished if we add an internal viscous damping term to the equation (see [81]). This is closely related to the enhanced observability inequality (7.2) in which the extra internal viscous term added in the observed quantity guarantees the observability constant to be uniform. We shall return to this issue in open problem # 5 below.

c) According to the analysis above it could seem that most control problems behave badly with respect to numerical approximations. However, this is no longer true for classical optimal control problems or even for approximate controllability problems in which the objective is to drive the solution to any state of size less than a given ε, as shown in section 5.5. This is even true for more sophisticated problems arising in the context of homogenization (see, for instance [94] and [15]).

d) There is a very closely connected problem that we have not addressed in this article that concerns the controllability of PDE with rapidly oscillating coefficients in the context of the theory of homogenization. We refer to [95] for a discussion of these connections. Here we shall simply mention that the ε parameter arising in homogenization theory to denote the small period of the oscillating coefficient and the h parameter describing the mesh size in numerical approximation problems, play a similar role. The problem is more difficult to handle in the context of homogenization because computations are less explicit than in numerical problems. However, surprisingly, the numerical approximation problems only became understood once the main features of the behavior of controllability problems in homogenization had been discovered [13].

10.2 Open problems

1. *Semilinear equations.* The questions we have addressed in this article are completely open in the case of the semilinear heat and wave equations with globally Lipschitz nonlinearities. In the context of continuous models there are a number of fixed point techniques that allow one to extend the results of controllability of linear waves and heat equations to semilinear equations with moderate nonlinearities (globally Lipschitz ones, for instance) [93]. These techniques need to be combined with Carleman or multiplier inequalities allow one to estimate the dependence of the observability constants on the potential of the linearized equation. However, the analysis we have pursued in this article relies very much on the Fourier decomposition of solutions, which does not suffice to obtain explicit estimates on the

observability constants in terms of the potential of the equation. Thus, extending the results of uniform controllability presented in this paper to the numerical approximation schemes of semilinear PDE is a completely open subject of research.

2. *Wavelets and spectral methods.* In the previous sections we have described how filtering of high frequencies can be used to get uniform observability and controllability results. It would be interesting to develop the same analysis in the context of numerical schemes based on wavelet analysis in which the filtering of high frequencies can be easy to implement in practice. In a recent paper [7] it has been shown that the superconvergence properties that the spectral methods provide may help at the level of controlling the wave equation in the sense that less filtering of high frequencies is required. A complete investigation of the use of spectral methods for the observation and control of the wave equation remains to be carried out.

Matache, Negreanu and Schwab in [63] have developed a wavelet based algorithm which is inspired in the multi-grid ideas described in section 7.2. Their method is extremely efficient in numerical experiments giving in practice the optimal control time. The convergence of this algorithm has not been proved so far.

3. *Discrete unique-continuation.* In the context of the continuous wave equation we have seen that the observability inequality and, consequently, exact controllability holds if and only if the domain where the control is being applied satisfies the Geometric Control Condition. However, very often in practice, it is natural to consider controls that are supported in a small subdomain. In those cases, when the control time is large enough, one obtains approximate controllability results as discussed in section 5.5. Approximate controllability is equivalent to a uniqueness or unique-continuation property for the adjoint system: *If the solution u of* (3.4) *vanishes in $\omega \times (0,T)$, then it vanishes everywhere.* We emphasize that this property holds whatever the open subset ω of Ω may be, provided T is large enough, by Holmgren's Uniqueness Theorem.

One could expect the same result to hold also for semi-discrete and discrete equations. But the corresponding theory has not been developed. The following example due to O. Kavian [46] shows that, at the discrete level, one can not expect the unique continuation property to hold if the set ω where the solution is assumed to vanish is not large enough, in contrast with the unique continuation property above for the continuous wave equation. It concerns the eigenvalue problem (6.13) for the 5-point finite difference scheme for the Laplacean in the square. A grid function taking alternating values ± 1 along a diagonal and vanishing everywhere else is an eigenvector of (6.13) with eigenvalue $\lambda = 4/h^2$. According to this example, even at the level of the elliptic equation, the domain ω where the solution vanishes has to be assumed to be large enough to guarantee the unique continuation property. In [17] it was proved that when ω is a "neighborhood of one side of the boundary", then unique continuation holds for the discrete Dirichlet problem in any discrete domain. Here by a "neighborhood of one side of the boundary" we refer to the nodes of the mesh that are located immediately to one side of the boundary nodal points (left, right, top or bottom). Indeed, if one knows that the solution vanishes at the nodes immediately to one side of the boundary, taking into account that they vanish in the boundary too, the 5-point numerical scheme allows propagating the information and showing that the solution vanishes at all nodal points of the whole domain.

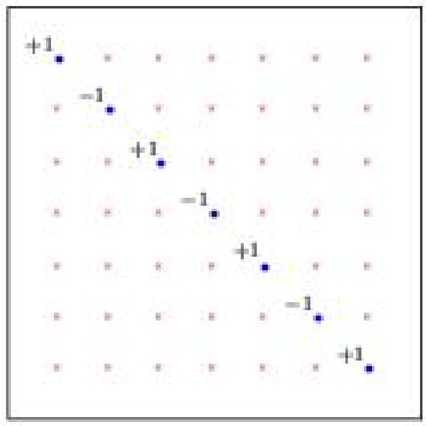

Figure 16: The eigenvector for the 5-point finite difference scheme for the Laplacean in the square, with eigenvalue $\lambda = 4/h^2$, taking alternating values ± 1 along a diagonal and vanishing everywhere else in the domain.

Getting optimal geometric conditions on the set ω depending on the domain Ω where the equation holds, the discrete equation itself, the boundary conditions and, possibly, the frequency of oscillation of the solution for the unique continuation property to hold at the discrete level is an interesting and widely open subject of research.

One of the main tools for dealing with unique continuation properties of PDE are the so called *Carleman inqualities*. It would be interesting to develop the corresponding discrete theory.

4. *Hybrid hyperbolic-parabolic equations.* We have discussed discretizations of the wave equation and have seen that, for most schemes, there are high frequency spurious oscillations that need to be filtered to guarantee uniform observability and controllability. However, we have seen that the situation is much better for the 1D heat equation. Nevertheless, it should also be taken into account that, according to the counterexample above showing that unique continuation may fail for the 2D eigenvalue problem, one can not expect the uniform observability property to hold uniformly for the semi-discretized 2D heat equation for any control sub-domain. Understanding the need of filtering of high frequencies in parabolic equations is also an interesting open problem, very closely related to the unique continuation

problem above.

Of course, the unique continuation problem can be also formulated for the semi-discrete and fully discrete heat equation.

It would also be interesting to analyze mixed models involving wave and heat components. There are two examples of such systems: a) Systems of thermoelasticity and b) Models for fluid-structure interaction (see [50] for the system of thermoelasticity and [88], [89] and [92] for the analysis of a system coupling the wave and the heat equation along an interface.) In particular, it would be interesting to analyze to which extent the presence of the parabolic component makes unnecessary the filtering of the high frequencies for the uniform observability property to hold for space or space-time discretizations.

5. *Viscous numerical damping.* In [81] we analyzed finite difference semi-discretizations of the damped wave equation

$$u_{tt} - u_{xx} + \chi_\omega u_t = 0, \tag{10.1}$$

where χ_ω denotes the characteristic function of the set ω where the damping term is effective. In particular we analyzed the following semi-discrete approximation in which an extra numerical viscous damping term is present:

$$\begin{cases} u_j'' - \frac{1}{h^2}\left[u_{j+1} + u_{j-1} - 2u_j\right] - & \left[u_{j+1}' + u_{j-1}' - 2u_j'\right] - u_j'\chi_\omega = 0, \\ & 0 < t < T,\ j = 1, \ldots, N \\ u_j(t) = 0, & 0 < t < T,\ j = 0,\ N+1 \\ u_j(0) = u_j^0, \quad u_j^1(0) = u_j^1, & j = 1, \ldots, N. \end{cases} \tag{10.2}$$

It was proved that this type of scheme preserves the uniform stabilization properties of the wave equation (10.1). To be more precise we recall that solutions of the 1D wave equation (10.1) in a bounded interval with Dirichlet boundary conditions decay exponentially uniformly as $t \to \infty$ when a damping term as above is added, ω being an open non-empty subinterval. Using the numerical scheme above, this exponential decay property is kept with a uniform rate as h tends to zero. The extra numerical damping that this scheme introduces adding the term $\left[u_{j+1}' + u_{j-1}' - 2u_j'\right]$ damps out the high frequency spurious oscillations that the classical finite difference discretization scheme introduces and that produce a lack of uniform exponential decay in the presence of damping.

The problem of whether this numerical scheme is uniformly observable or controllable as h tends to zero is an interesting open problem.

Note that the system above, in the absence of the damping term lozalized in ω, can be written in the vector form

$$\vec{u}'' + A_h\vec{u} + h^2 A_h \vec{u}' = 0. \tag{10.3}$$

Here $\vec{u}$ stands, as usual, for the vector unknown $(u_1, ..., u_N)^T$ and A_h for the tridiagonal matrix associated with the finite difference approximation of the Laplacean (5.4). In this form it is clear that the scheme above corresponds to a viscous approximation of the wave equation. Indeed, taking into account that A_h provides an approximation of $-\partial_x^2$, the presence of the extra multiplicative factor h^2 in the numerical damping term guarantees that it vanishes asymptotically as h tends to zero. This is true for the classical convergence theory but it remains to be proved for observability and controllability.

6. *Multigrid methods.* In section 7 we presented the two-grid algorithm introduced by R. Glowinski [34] and we explained heuristically why it is a remedy for high frequency spurious oscillations. In [34] the efficiency of the method was exhibited in several numerical examples. As far as we know, a rigorous proof of the convergence of the method proposed in [34] to compute the control of the wave equation remains to be done, except for space semi-discretizations in 1D by finite differences and finite elements [66].

7. *Uniform control of the low frequencies.* As we mentioned in the end of section 6, the 2D counterpart of the 1D positive result in [64] showing that the initial data involving a finite number of Fourier components are uniformly controllable as $h \to 0$ has not been proved in the literature. Such a result is very likely to hold for quite general approximation schemes and domains. But, up to now, it has only been proved in 1D for finite difference semi-discretizations. The methods involving Wigner measures developed in [59], [60] do not seem sufficient to address this issue. On the other hand, moment problems techniques, which require quite technical developments in [64] to deal with the 1D case, also seem to be hard to adapt to a more general setting. This is an interesting (and, very likely, difficult) open problem.

8. *Stabilization and other control theoretical issues.* As we have mentioned above, the topics we have discussed make sense in other contexts of Control Theory. In particular, similar questions arise concerning problems of stabilization, the infinite horizon optimal control problem, the Riccati equations for optimal feedback operators or the reciprocal systems discussed by R. Curtain in [24]. The phenomena we have discussed in this paper, related to high frequency spurious oscillations, certainly affect the results we can expect in these other problems too. But the corresponding analysis has not been done. We recall however that, concerning the problem of stabilization, the necessity and also the sufficiency of adding a numerical viscous damping term was pointed out in [81] in order to get results on the uniform decay of solutions, as discussed in open problem #5 above. The result in [81] has been recently extended in [71] to a class of abstract systems under the assumption of spectral gap. Thus, essentially, the result in [71] applies to a large class of models of vibrations in $1D$.

9. *Extending the Wigner measure theory.* As we mentioned above, F. Macià in [59], [60] has developed a discrete Wigner measure theory to describe the propagation of semi-discrete

and discrete waves at high frequency. However, this was done for regular grids and without taking into account boundary effects. Therefore, a lot has to be done in order to fully develop the Wigner measure tools. The notion of polarization developed in [11] remains also to be analyzed in the discrete setting.

10. *Theory of inverse problems and Optimal Design.* This paper has been devoted mainly to the property of observability and its consequences for controllability. But, as we mentioned from the beginning, most of the results we have developed have consequences in other fields. This is the case for instance for the theory of inverse problems, where one of the most classical problems is the one of reconstructing the coefficients of a given PDE in terms of boundary measurements (see [45]). Assuming that one has a positive answer to this problem in an appropriate functional setting it is natural to consider the problem of numerical approximation. Then, the following question arises: *Is solving the discrete version of the inverse problem for a discretized model an efficient way of getting a numerical approximation of the solution of the continuous inverse problem?*. Thus, as in the context of control we are analyzing whether the procedure of numerical approximation and that of solving the inverse problem commute.

According to the analysis above we can immediately say that, in general, the answer to this problem is negative. Consider for instance the wave equation

$$\begin{cases} \rho u_{tt} - u_{xx} = 0, & 0 < x < 1,\ 0 < t < T \\ u(0,t) = u(1,t) = 0, & 0 < t < T, \end{cases} \tag{10.4}$$

with a constant but unknown density $\rho > 0$. Solutions of this equation are time-periodic of period $2\sqrt{\rho}$ and this can be immediately observed on the trace of normal derivatives of solutions at either of the two boundary points $x = 0$ or $x = 1$, by inspection of the Fourier series representation of solutions of (10.4). Thus, roughly speaking, we can assert that the value of ρ can be determined by means of the boundary measurements.

Let us now consider the semi-discrete version of (10.4). In this case, according to the analysis above, the solutions do not have any well-defined time-periodicity property. On the contrary, for any given values of ρ and h, (10.4) admits a whole range of solutions that travel at different group velocities, ranging from $h/\sqrt{\rho}$ (for the high frequencies) to $1/\sqrt{\rho}$ (for the low frequency ones). In particular, the high frequency numerical solutions do behave more like a solution of the wave equation with an effective density ρ/h^2. This argument shows that the mapping that allows determining the value of the constant density from boundary measurements is unstable under numerical discretization.

Of course, most of the remedies that have been introduced in this paper to avoid the failure of uniform controllability and/or observability can also be used in this context of inverse problems. But developing these ideas remains to be done.

The same can be said about optimal design problems. Indeed, in this context very little is known about the convergence of the optimal designs for the numerical discretized models towards the optimal design of the continuous models and, to a large extent, the difficulties one has to face in this context is very similar that those we addressed all along this paper.

11. *Finite versus infinite-dimensional nonlinear control.* Most of this work has been devoted to analyzing linear problems. There is still a lot to be done to understand the connections between finite-dimensional and infinite-dimensional control theory, and, in particular, concerning numerical approximations and their behavior with respect to the control property. According to the analysis above, the problem is quite complex even in the linear case. Needless to say, one expects a much higher degree of complexity in the nonlinear frame.

There are a number of examples in which the finite-dimensional versions of important nonlinear PDE have been solved from the point of view of controllability. Among them the following are worth mentioning:

a) The Galerkin approximations of the bilinear control problem for the Schrödinger equation arising in Quantum Chemistry ([72] and [83]).

b) The control of the Galerkin approximations of the Navier-Stokes equations [56].

In both cases nothing is known about the possible convergence of the controls of the finite-dimensional system to the control of a PDE as the dimension of the Galerkin subspace tends to infinity. This problem seems to be very complex. However, the degree of difficulty may be different in both cases. Indeed, in the case of the continuous Navier-Stokes and Euler equations for incompressible fluids there are a number of results in the literature indicating that they are indeed controllable ([31], [21], [22]). However, for the bilinear control of the Schrödinger equations, it is known that the reachable set is very small in general, which indicates that one can only expect very weak controllability properties. This weakness of the controllability property at the continuous level makes it even harder to address the problem of passing to the limit on the finite-dimensional Galerkin approximations as the dimension tends to infinity.

This problem is certainly one of the most relevant ones in the frame of controllability of PDE and its numerical approximations.

12. *Wave equations with irregular coefficients.* The methods we have developed do not suffice to deal with wave equations with non-smooth coefficients. However, at the continuous level, in one space dimension, as we described in section 4, observability and exact controllability hold for the wave equation with piecewise constant coefficients with a finite number of jump discontinuities. It would be interesting to see if the main results presented in this paper hold in this setting too. This seems to be a completely open problem. We refer to the book by G. Cohen [19] for the analysis of reflection and transmission indices for numerical schemes for wave equations with interfaces.

References

[1] Asch, M. and Lebeau, G. (1998). "Geometrical aspects of exact boundary controllability of the wave equation. A numerical study". *ESAIM:COCV,* **3**, 163–212.

[2] Banks, H. T.; Smith, R. C., and Wang, Y. (1996) *Smart material structures. Modeling, estimation and control.* Research in Applied Mathematics, Wiley, Masson.

[3] Banks, H.T., Ito, K. and Wang, C. (1991). "Exponentially stable approximations of weakly damped wave equations", *International Series in Numerical Analysis*, **100**, 1–33.

[4] Bardos, C., Bourquin, F., Lebeau, G. (1991). "Calcul de dérivées normales et méthode de Galerkin appliquée au problème de contrôlabilité exacte", *C. R. Acad. Sci. Paris Sér. I Math.,* **313** (11) 757–760.

[5] Bardos, C., Lebeau, G. and Rauch, J. (1992). "Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary", *SIAM J. Cont. Optim.*, **30**, 1024–1065.

[6] Bender, C.M. and Orszag, S.A. (1978). *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill.

[7] Boulmezaoud T. Z. and Urquiza, J. M. (2002). "On the eigenvalues of the spectral second order differentiation operator: Application to the boundary observability of the wave operator", preprint.

[8] Brumer, P. and Shapiro, M. (1995) "Laser control of chemical reactions". *Scientific American*, p. 34–39.

[9] Burq, N. (1997). "Contrôle de l'équation des ondes dans des ouverts peu réguliers". *Asymptotic Analysis*, **14**, 157–191.

[10] Burq, N. and Gérard, P. (1997). "Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes", *C. R. Acad. Sci. Paris*, **325**, 749–752.

[11] Burq, N. and Lebeau, G. (2001). "Mesures de défaut de compacité, application au système de Lamé". *Ann. Sci. École Norm. Sup.* **34** (6), 817–870.

[12] Castro, C. and Micu, S. (2003) "Boundary controllability of a semi-discrete wave equation with mixed finite elements", *preprint.*

[13] Castro, C. and Zuazua, E. (1997). "Contrôle de l'équation des ondes à densité rapidement oscillante à une dimension d'espace", *C. R. Acad. Sci. Paris*, **324**, 1237–1242.

[14] Castro, C. and Zuazua, E. (2002). "Localization of waves in 1D highly heterogeneous media". *Archive Rational Mechanics and Analysis*, **164** (1) 39–72.

[15] Castro, C. and Zuazua, E. (2002). "Control and homogenization of wave equations", *preprint.*

[16] Charpentier, I. and Maday, Y. (1996). "Identification numérique de contrôles distr ibués pour l'equation des ondes", *C. R. Acad. Sci. Paris*, **322** (8), 779–784.

[17] Chenais, D. and Zuazua, E. (2001). "Controllability of an elliptic equation and its finite difference approximation by the shape of the domain", *Numersiche Mathematik*, to appear.

[18] Cioranescu, P. and Donato, P. (1999). *An introduction to homogenization.* The Clarendon Press, Oxford University Press, New York.

[19] Cohen, G. (2001). *Higher-order numerical methods for transient wave equations.* Scientific Computation, Springer.

[20] Conrad, F., Leblond, J. and Marmorat, J.P. (1992). "Boundary control and stabilization of the one-dimensional wave equation". *Boundary control and boundary variation.* Lecture Notes in Control and Inform. Sci., **178**, Springer, Berlin, pp. 142–162.

[21] Coron, J.-M. (1996). "On the controllability of 2-D incompressible perfect fluids". *J. Maths Pures Appl.* **75**(2), 155–188.

[22] Coron, J.-M. (1996). "On the controllability of the 2-D incompressible Navier Stokes equations with the Navier slip boundary conditions."*ESAIM:COCV.* **1**, 35–75.

[23] Crépeau, E. (2002). "Contrôlabilité exact d'équations dispersives issues de la mécanique." *Ph. D. Thesis, Université de Paris-Sud, Orsay.*

[24] Curtain, R. (2002). "Reciprocals of linear regular systems: a survey." *MTNS, 2002.* , South Bend, Indiana, August 12–16.

[25] Dáger, R. and Zuazua, E. (2002). "Spectral boundary controllability of networks of strings". *C. R. Acad. Sci. Paris.* **334**, 545–550.

[26] Eastham, M.S.P. (1973). *The Spectral Theory of Periodic Differential Equations*, Scottish Academic Press, Edinburgh.

[27] Fattorini, H.O. (1999) *Infinite Dimensional Optimization and Control Theory*, Encyclopedia of Mathematics and its Applications **62**, Cambridge University Press.

[28] Fattorini, H. and Russell, D.L. (1974). "Uniform bounds on biorthogonal functions for real exponentials with an application to the control theory of parabolic equations", *Quart. Appl. Math.,* **32**, 45–69.

[29] Fernández-Cara E. and Zuazua E. (2000)."The cost of approximate controllability for heat equations: The linear case", *Advances Diff. Eqs.*, **5** (4–6), 465–514.

[30] Fernández-Cara E. and Zuazua E. (2000). "Null and approximate controllability for weakly blowing-up semilinear heat equations". *Annales Inst. Henri Poincaré, Analyse non-linéaire*, **17** (5), 583–616.

[31] Fursikov A. V. and Imanuvilov O. Yu. (1996). *Controllability of evolution equations*, Lecture Notes Series # 34, Research Institute of Mathematics, Global Analysis Research Center, Seoul National University.

[32] Gérard, P. (1991). Microlocal defect measures, *Comm. P.D.E.*, **16**, 1761–1794.

[33] Gérard, P., Markowich, P.A., Mauser, N.J., Poupaud, F. (1997). "Homogenization Limits and Wigner Transforms", *Comm. Pure and Apple. Math.*, **50**, 323–378.

[34] Glowinski, R. (1992). "Ensuring well-posedness by analogy; Stokes problem and boundary control of the wave equation". *J. Compt. Phys.*, **103**(2), 189–221.

[35] Glowinski, R., Kinton, W. and Wheeler, M.F. (1989) "A mixed finite element formulation for the boundary controllability of the wave equation". Int. J. Numer. Methods Engineering. **27**, 623–635.

[36] Glowinski, R., Li, C. H. and Lions, J.-L. (1990). "A numerical approach to the exact boundary controllability of the wave equation (I). Dirichlet controls: Description of the numerical methods". *Japan J. Appl. Math.*, **7**, 1–76.

[37] Grisvard, P. (1989) "Contrôlabilité exacte des solutions de l'équation des ondes en présence de singularités. *J. Math. Pures Appl.,* **68** (2), 215–259.

[38] Gunzburger, M. D., Hou, L. S. and Ju. L. (2002). "A numerical method for exact boundary controllability problems for the wave equation", *preprint.*

[39] Haraux, A., and Jaffard, S. (1991). "Pointwise and spectral controllability for plate vibrations", *Revista Matemática Iberoamericana,* **7** (1), 1–24.

[40] Ho, L. F. (1986). Observabilité frontière de l'équation des ondes, *C. R. Acad. Sci. Paris*, **302**, 443–446.

[41] Infante, J.A. and Zuazua, E. (1998). "Boundary observability of the space-discretizations of the 1D wave equation", *C. R. Acad. Sci. Paris*, **326**, 713–718.

[42] Infante, J.A. and Zuazua, E. (1999). "Boundary observability for the space-discretizations of the one-dimensional wave equation", *Mathematical Modelling and Numerical Analysis,* **33**, 407–438.

[43] Ingham, A.E. (1936). "Some trigonometric inequalities with applications to the theory of series", *Math. Zeits.,* **41**, 367–379.

[44] Isaacson, E. and Keller, H.B. (1966). *Analysis of Numerical Methods*, John Wiley & Sons.

[45] Isakov, V. (1988). *Inverse Problems for Partial Differential Equations*, Springer-Verlag, Berlin.

[46] Kavian. O. (2001). *Private communication.*

[47] Komornik, V. (1994). *Exact Controllability and Stabilization. The Multiplier Method*, Wiley, Chichester; Masson, Paris.

[48] Lebeau, G. (1992). "Contrôle analytique I: Estimations a priori". *Duke Math. J.*, **68** (1), 1–30.

[49] Lebeau, G. (2000). "The wave equation with oscillating density: observability at low frequency ", *ESAIM: COCV*, **5**, 219–258 .

[50] Lebeau, G. and Zuazua, E. (1998) "Null controllability of a system of linear thermoelasticity", *Archives for Rational Mechanics and Analysis*, **141**(4), 297–329.

[51] Lee, E.B. and Markus, L. (1967). *Foundations of Optimal Control Theory*, The SIAM Series in Applied Mathematics, John Wiley & Sons.

[52] León, L. and Zuazua, E. (2002). "Boundary controllability of the finite difference space semi-discretizations of the beam equation", *ESAIM:COCV, A Tribute to Jacques-Louis Lions,* Tome 2, 827–862.

[53] W.S. Levine. (2000). *Control System and Applications*, CRC Press.

[54] Lions, J.L. (1988). *Contrôlabilité Exacte, Stabilisation et Perturbations de Systèmes Distribués. Tome 1. Contrôlabilité Exacte*, Masson, Paris, RMA 8.

[55] Lions, J.L. (1988). "Exact controllability, stabilizability and perturbations for distributed systems", *SIAM Rev.*, **30** , 1-68.

[56] Lions, J.L. and Zuazua, E. (1998)."Exact boundary controllability of Galerkin's approximations of Navier Stokes equations." *Annali Scuola Norm. Sup. Pisa*, XXVI (98), 605–621.

[57] Lions, P.L. and Paul, Th. (1993). "Sur les mesures de Wigner". *Rev. Mat. Iberoamericana.* **9** (3), 553–618.

[58] López, A. and Zuazua, E. (1997). "Some new results related to the null controllability of the 1D heat equation", *Seminaire EDP, 1997-1998*, Ecole Polytechnique, Paris, VIII-1-VIII-22.

[59] Macià, F. (2002). *Propagación y control de vibraciones en medios discretos y continuos*, PhD Thesis, Universidad Complutense de Madrid.

[60] Macià, F. (2003). *Wigner measures in the discrete setting: high-frequency analysis of sampling & reconstruction operators*, *preprint*

[61] Macià, F. and Zuazua, E. (2002). "On the lack of observability for wave equations: a Gaussian beam approach". *Asymptotic Analysis*, **32** (1), 1–26.

[62] Maday, Y. and Turinici, G. "A parareal in time procedure for the control of partial differential equations", *C. R. Acad. Sci. Paris, Ser. I,* **335** (2002), 1–6.

[63] Matache, A. M., Negreanu, M. and Schwab, C. (2003), 'Wavelet Filtering for Exact Controllability of the Wave Equation", *preprint.*

[64] Micu, S. (2002). "Uniform boundary controllability of a semi-discrete 1-D wave equation", *Numerische Mathematik*, **91**, (4), 723–768.

[65] Micu, S. and Zuazua, E. (1997). "Boundary controllability of a linear hybrid system arising in the control of noise", *SIAM J. Cont. Optim.*, **35**(5), 1614–1638.

[66] Negreanu, M. Ph D Thesis, Universidad Complutense de Madrid, in preparation.

[67] Negreanu, M. and Zuazua, E. (2003). "Uniform boundary controllability of a discrete 1D wave equation. *Systems and Control Letters*, **48** (3-4) , 261-280.

[68] Oleinick, O.A., Shamaev, A.S. and Yosifian, G.A. (1992). *Mathematical Problems in Elasticity and Homogenization*, North-Holland.

[69] Osses, A. (1988). Une nouvelle famille de multiplicateurs et ses applications à la contrôlabilité exacte des ondes. *C. R. Acad. Sci. Paris*, **326**, 1099–1104.

[70] Ralston, J. (1982) "Gaussian beams and the propagation of singularities". *Studies in Partial Differential Equations*, MAA Studies in Mathematics, **23**, W. Littman ed., pp. 206–248.

[71] Ramdani, K., Takahashi, T. and Tucsnak, M. (2003) "Uniformly exponentially stable approximations for a class of second order evolution equations". *preprint.*

[72] Ramakrishna, V., Salapaka, M., Dahleh, M., Rabitz, H., Pierce, A. (1995) "Controllability of molecular systems". Phys. Rev. A, **51** (2), 960–966.

[73] Rasmussen, J. (2002), *Private communication.*

[74] Robbiano, L. (1995) "Fonction de coût et contrôle des solutions des équations hyperboliques". Asymptotic Anal., **10** (2), 95–115.

[75] Rodellar, J. et al. (1999) *Advances in Structural Control.* CIMNE, Barcelona.

[76] Russell, D. L. (1978). "Controllability and stabilizability theory for linear partial differential equations. Recent progress and open questions". *SIAM Rev.,* **20** , 639–739.

[77] Russell, D. L. (1973). "A unified boundary controllability theory for hyperbolic and parabolic partial differential equations". *Studies in Appl. Math.,* **52**, 189–221.

[78] Sanz-Serna, J. (1985). "Stability and convergence in numerical analysis. I: linear problems—a simple, comprehensive account". *Res. Notes in Math.*, **132**, Pitman, Boston, MA, pp. 64–113.

[79] S.I.A.M. (1988). *Future Directions in Control Theory*, Report of the Panel of Future Directions in Control Theory, SIAM Report on Issues in Mathematical Sciences, Philadelphia.

[80] Sontag, E.D. (1998) *Mathematical Control Theory. Deterministic Finite-Dimensional Systems*, 2nd edition, Texts in Applied Mathematics, **6**, Springer-Verlag, New York.

[81] Tcheougoué Tebou, L. R. and Zuazua, E. (2001). "Uniform exponential long time decay for the space semi-discretization of a locally damped wave equation via an artificial numerical viscosity", *Numerische Mathematik,* to appear.

[82] Trefethen, L. N. (1982). "Group velocity in finite difference schemes", *SIAM Rev.,* **24** (2), pp. 113–136.

[83] Turinici, G. and Rabitz H. (2001). "Quantum Wavefunction Controllability." *Chem. Phys.* **267**, 1–9.

[84] Turinici, G. (2000). "Controllable quantities for bilinear quantum systems." *Proceedings of the 39th IEEE Conference on Decision and Control*, Sydney Convention & Exhibition Centre, 1364–1369.

[85] Vichnevetsky, R. and Bowles, J.B. (1982). *Fourier Analysis of Numerical Approximations of Hyperbolic Equations.* SIAM Studies in Applied Mathematics, **5**, SIAM, Philadelphia.

[86] Young, R. M. (1980). *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York.

[87] Zhang, X. (2000). "Explicit observability estimate for the wave equation with potential and its application". *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.,* **456**, 1101–1115.

[88] Zhang, X. and Zuazua, E. (2003). "Polynomial decay and control for a 1-d model of fluid-structure interaction". *C. R. Acad. Sci. Paris,* **336**, 745–750.

[89] Zhang, X. and Zuazua, E. (2003). "Control, observation and polynomial decay for a 1-d heat-wave system". *C. R. Acad. Sci. Paris, I,* **336**, 823–828.

[90] Zuazua, E. (1998). "Some problems and results on the controllability of partial differential equations", *Progress in Mathematics*, **169**, Birkhäuser Verlag, pp. 276–311.

[91] Zuazua, E. (1999). "Boundary observability for the finite-difference space semi-discretizations of the 2D wave equation in the square", *J. Math. Pures et Appliquées*, **78**, 523–563.

[92] Zuazua, E. (2001). "Null control of a 1D model of mixed hyperbolic-parabolic type". *Optimal Control and Partial Differential Equations.* J. L. Menaldi et al., eds., IOS Press, pp. 198–210.

[93] Zuazua, E. (2002). "Controllability of Partial Differential Equations and its Semi-Discrete Approximations". *Discrete and Continuous Dynamical Systems*, **8** (2), 469–513.

[94] Zuazua, E. (1994). "Approximate controllability for linear parabolic equations with rapidly oscillating coefficients". *Control and Cybernetics,* **23** (4), 1–8.

[95] Zuazua, E. (1999). "Observability of 1D waves in heterogeneous and semi-discrete media". *Advances in Structural Control.* J. Rodellar et al., eds., CIMNE, Barcelona, 1999, pp. 1–30.

[96] Zuazua, E. (1993). “Exact controllability for the semilinear wave equation in one space dimension”. *Ann. IHP. Analyse non linéaire*, **10**, 109–129.