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CONTROLLABILITY OF PARTIAL DIFFERENTIAL EQUATIONS AND ITS SEMI-DISCRETE APPROXIMATIONS

ENRIQUE ZUAZUA

Dedicated to the memory of Jacques-Louis Lions.

Abstract. In these notes we analyze some problems related to the controllability and observability of partial differential equations and its space semi-discretizations. First we present the problems under consideration in the classical examples of the wave and heat equations and recall some well known results. Then we analyze the $1-d$ wave equation with rapidly oscillating coefficients, a classical problem in the theory of homogenization. Then we discuss in detail the null and approximate controllability of the constant coefficient heat equation using Carleman inequalities. We also show how a fixed point technique may be employed to obtain approximate controllability results for heat equations with globally Lipschitz nonlinearities. Finally we analyze the controllability of the space semi-discretizations of some classical PDE models: the Navier-Stokes equations and the $1-d$ wave and heat equations. We also present some open problems.

1. **Introduction.**

In these lectures we address some topics related to the controllability of partial differential equations and its space semi-discretizations.

The controllability problem may be formulated as follows. Consider an evolution system (either described in terms of Partial or Ordinary Differential Equations). We are allowed to act on the trajectories of the system by means of a suitable choice of the control (the right hand side of the system, the boundary conditions, etc.). Then, given a time interval $t \in (0, T)$, and initial and final states we have to find a control such that the solution matches both the initial state at time $t = 0$ and the final one at time $t = T$.

This is a classical problem in Control Theory and there is a large literature on the topic. We refer for instance to the book of Lee and Marcus [83] for an introduction to the topic in the context of finite-dimensional systems described in terms of Ordinary Differential Equations (ODE). We also refer to the survey paper by Russell [129] and to the book of Lions [88] for an introduction to the case of systems modelled by means of PDE, also refered to as Distributed Parameter Systems.

There has been a very intensive research in this area in the last two decades and it would be impossible in these Notes to report on the main progresses that have been made. For this reason we have chosen a number of specific topics to

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present some recent results that exhibit the variety of problems arising in this field and some of the mathematical tools that have been used and developed to deal with them. Of course, the list of topics we have chosen is rather limited and it is not intended to represent the whole field. We hope however that, through these notes, the reader will become familiar with some of the main research topics in this area. We have also included a long (but still incomplete) list of references for those readers interested is pursuing the study of these problems.

As we have mentioned above, when dealing with controllability problems, to begin with, one has to distinguish between finite-dimensional systems modelled by ODE and infinite-dimensional distributed systems described by means of PDE. Most of these notes will deal with problems related to PDE.

However, even if we work in the context of PDE, for computational purposes one is obliged to analyze also the corresponding numerical approximations. In this way one is driven naturally to analyze discrete or semi-discrete systems. Therefore, in these notes we shall also address the space semi-discrete approximations of the PDE under consideration. Therefore, when doing this, we are back to the controllability problem for finite-dimensional systems of ODE. As we shall see, understanding the limit behavior of the controllability properties as the size of the finite-dimensional system tends to infinity to recover the controllability of the PDE model is not a simple issue.The purely discrete systems will not be addressed here.

Even in the context of PDE, in order to address controllability problems, one has still to distinguish between linear and non-linear systems, time-reversible and time-irreversible systems, etc. For this reason in these notes we discuss both the wave and the heat equation as well as some semilinear variants.

Recently important progresses have been made also in the context of the systems of thermoelasticity in which the parabolic nature of the heat equation and the time-reversibility of the wave equation are coupled. We do not address this topic here. We refer the interested reader to the works of Hansen [52], Zuazua [145] and Lebeau and Zuazua [82].

The content of this paper is as follows. In section 2 we describe the main issues related to the controllability of the linear heat and wave equations. In section 3 we discuss the controllability of the wave equation with rapidly oscillating coefficients following the works [7, 15, 16] and [18]. In section 4 we discuss in detail the controllability of the heat equation. In section 5 we present a fixed point method allowing to prove the approximate controllability of the semilinear heat equation with globally Lipschitz nonlinearity. In section 6 we analyze the controllability of some space semi-discretizations. In particular we discuss the controllability of the Galerkin approximations of the Navier-Stokes equations following [96]-[98], the finite-difference approximation of the 1-d wave equation as in [62, 63] and the finite-difference space approximation of the heat equation as in [104]. We end up with a section devoted to present some open problems.

2. Problem formulation: The linear wave and heat equations.

2.1. The wave equation.

Let Ω be a bounded domain of $\mathbb{R}^n, n \geq 1$, with boundary Γ of class C^2. Let ω be an open and non-empty subset of Ω and $T > 0$.

Consider the linear controlled wave equation in the cylinder $Q = \Omega \times (0, T)$:

$$\begin{cases} u_{tt} - \Delta u = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x, 0) = u^0(x), u_t(x, 0) = u^1(x) & \text{in} \quad \Omega. \end{cases} \tag{2.1}$$

In (2.1) Σ represents the lateral boundary of the cylinder Q, i.e. $\Sigma = \Gamma \times (0,T), 1_\omega$ is the characteristic function of the set $\omega, u = u(x,t)$ is the state and $f = f(x,t)$ is the control variable. Since f is multiplied by 1_ω the action of the control is localized in ω.

When $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $f \in L^2(Q)$ system (2.1) has an unique solution $u \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right)$.

The problem of *controllability* consists roughly on *describing the set of reachable final states*

$$R\left(T; (u^0, u^1)\right) = \left\{(u(T), u_t(T)) : f \in L^2(Q)\right\}.$$

One may distinguish the following degrees of controllability:

(a) *Approximate controllability*: System (2.1) is said to be approximately controllable in time T if the set of reachable states is dense in $H_0^1(\Omega) \times L^2(\Omega)$ for every $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

(b) *Exact controllability*: System (2.1) is said to be exactly controllable at time T if

$$R\left(T; (u^0, u^1)\right) = H_0^1(\Omega) \times L^2(\Omega)$$

for all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

(c) *Null controllability*: System (2.1) is said to be null controllable at time T if $(0,0) \in R\left(T; (u^0, u^1)\right)$ for all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

Remark 2.1. (a) Since we are dealing with solutions of the wave equation, for any of these properties to hold the control time T has to be sufficiently large due to the finite speed of propagation.

(b) Since system (2.1) is linear and reversible in time null and exact controllability are equivalent notions. As we shall see, the situation is completely different in the case of the heat equation.

(c) Clearly every exactly controllable system is approximately controllable too. However, system (2.1) may be approximately but not exactly controllable. In those cases it is natural to study the *cost of approximate controllability*, or, in other words, the size of the control needed to reach to an ε-neighborhood of a final state which is not exactly reachable. This problem was analyzed by Lebeau in [77]. We shall address this problem below in the case of the heat equation.

(e) The controllability problem above may also be formulated in other function spaces in which the wave equation is well posed.

(f) Null controllability is a physically interesting notion since the state $(0,0)$ is an equilibrium for system (2.1).

(g) Most of the literature on the controllability of the wave equation has been written on the framework of the *boundary control* problem. The control problems formulated above for system (2.1) are usually refered to as *internal controllability* problems since the control acts on the subset ω of Ω.

■

Let us now briefly discuss the *approximate controllability problem.*

It is easy to see that approximate controllability is equivalent to an *unique continuation property* of the adjoint system:

$$\begin{cases} \varphi_{tt} - \Delta\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x), \varphi_t(x,T) = \varphi^1(x) & \text{in} \quad \Omega. \end{cases} \tag{2.2}$$

Indeed, system (2.1) is approximately controllable if and only if the following holds:

$$\varphi \equiv 0 \text{ in } \omega \times (0,T) \Rightarrow \left(\varphi^0, \varphi^1\right) \equiv (0,0). \tag{2.3}$$

By using Holmgren's Uniqueness Theorem it can be easily seen that (2.3) holds if T is large enough. We refer to [88], chapter 1 and [20] for a discussion of this problem.

There are at least two ways of checking that (2.3) implies the approximate controllability:

(a) The application of Hahn-Banach Theorem.
(b) The variational approach developed in [89].

The second approach will be presented below in the context of the heat equation. We refer to [88] for the application of the first approach.

When approximate controllability holds, then the following (apparently stronger) statement also holds:

Theorem 2.2. ([149]) *Let E be a finite-dimensional subspace of $H_0^1(\Omega)\times L^2(\Omega)$ and let us denote by π_E the corresponding orthogonal projection. Then, if approximate controllability holds, for any $\left(u^0, u^1\right)$, $\left(v^0, v^1\right) \in H_0^1(\Omega) \times L^2(\Omega)$ and $\varepsilon > 0$ there exists $f \in L^2(Q)$ such that the solution of* (2.1) *satisfies*

$$\left\| \left(u(T) - v^0, u_t(T) - v^1\right) \right\|_{H_0^1(\Omega)\times L^2(\Omega)} \leq \varepsilon;\ \pi_E\left(u(T), u_t(T)\right) = \pi_E\left(v^0, v^1\right). \tag{2.4}$$

This result, that will be referred to as the *finite-approximate controllability property*, may be proved at least in two different ways:

(a) By a suitable modification of the variational approach introduced in [89].
(b) As a direct consequence of the approximate controllability and the following Theorem of functional Analysis (we refer to [94] for a proof):

Theorem 2.3. ([94]) *Let V and H be two Hilbert spaces and L a bounded linear operator from V to H with dense range. Let E be a finite-dimensional subspace of H and π_E the corresponding orthogonal projection.*

Then, given any $e_0 \in E$, when v runs over the set of elements of v such that $\pi_E Lv = e_0$, Lv describes a dense set in $e_0 + E^\perp$.

The results above hold for wave equations with analytic coefficients too. Indeed, the control problem can be reduced to the unique continuation one and the latter may be solved by means of Holmgren's Uniqueness Theorem.

However, the problem is not completely solved in the frame of the wave equation with lower order potentials $a \in L^\infty(Q)$ of the form

$$u_{tt} - \Delta u + a(x,t)u = f 1_\omega \text{ in } Q.$$

Once again the problem of approximate controllability of this system is equivalent to the unique continuation property of its adjoint. We refer to Alinhac [1], Tataru [134] and Robbiano-Zuilly [127] for deep results in this direction.

Let us now discuss the *exact controllability problem.*

The unique continuation property (2.3) by itself does not allow to address the exact controllability problem. As it was shown by Lions [88], using the so called HUM (Hilbert Uniqueness Method), exact controllability is equivalent to the following inequality:

$$\|(\varphi(0), \varphi_t(0))\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt \tag{2.5}$$

for all solutions of (2.2).

This inequality allows to estimate the total energy of the solution of (2.2) by means of a measurement in the control region $\omega \times (0,T)$. Thus, it establishes the *continuous observability* of system (2.2). The energy $\|(\varphi(t), \varphi_t(t))\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}$ of solutions of (2.2) is conserved along trajectories. Thus, (2.5) is equivalent to

$$\left\|\left(\varphi^0, \varphi^1\right)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt. \tag{2.6}$$

When (2.5) holds one can minimize the functional

$$J\left(\varphi^0, \varphi^1\right) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \left\langle (\varphi(0), \varphi_t(0)), \left(u^1, -u^0\right)\right\rangle \tag{2.7}$$

in the space $L^2(\Omega) \times H^{-1}(\Omega)$. Indeed, the following is easy to prove: *When the observability inequality (2.5) holds, the functional J has an unique minimizer $\left(\widehat{\varphi}^0, \widehat{\varphi}^1\right)$ in $L^2(\Omega)\times H^{-1}(\Omega)$ for all $\left(u^0, u^1\right) \in H^1_0(\Omega) \times L^2(\Omega)$. The control $f = \widehat{\varphi}$ with $\widehat{\varphi}$ solution of (2.2) corresponding to $\left(\widehat{\varphi}^0, \widehat{\varphi}^1\right)$ is such that the solution of (2.1) satisfies*

$$u(T) \equiv u_t(T) \equiv 0. \tag{2.8}$$

Consequently, in this way, the exact controllability problem is reduced to the analysis of inequality (2.6).

Let us now discuss what is known about the observability inequality (2.6):

(a) Using multiplier techniques in the spirit of C. Morawetz [113], Ho in [58] proved that if one considers subsets of Γ of the form

$$\Gamma(x^0) = \left\{x \in \Gamma : (x - x^0)\cdot n(x) > 0\right\}$$

for some $x^0 \in \mathbb{R}^n$ (by $n(x)$ we denote the outward unit normal to Ω in $x \in \Gamma$ and by $\cdot$ the scalar product in $\mathbb{R}^n$) and if $T > 0$ is large enough, the following boundary observability inequality holds:

$$\|(\varphi(0), \varphi_t(0))\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \leq C \int_0^T \int_{\Gamma(x^0)} \left|\frac{\partial\varphi}{\partial n}\right|^2 d\Gamma dt \tag{2.9}$$

for all $\left(\varphi^0, \varphi^1\right) \in H^1_0(\Omega) \times L^2(\Omega)$.

This is the observability inequality that is required to solve the boundary controllability problem mentioned above.

Later on inequality (2.9) was proved in [87, 88], for any $T > T(x^0) = 2 \parallel x - x^0 \parallel_{L^\infty(\Omega)}$. This is the optimal observability time that one may derive by means of multipliers.

Proceeding as in [88], vol. 1, one can easily prove that (2.9) implies (2.5) when ω is a neighborhood of $\Gamma(x^0)$ in Ω, i.e. $\omega = \Omega \cap \Theta$ where Θ is a neighborhood of $\Gamma(x^0)$ in $\mathbb{R}^n$, with $T > 2 \parallel x - x^0 \parallel_{L^\infty(\Omega\setminus\omega)}$. More recently Osses in [Os] has introduced a new multiplier which is basically a rotation of the previous one and he has obtained a larger class of subsets of the boundary for which observability holds.

(b) C. Bardos, G. Lebeau and J. Rauch [9] proved that, in the class of C^∞ domains, the observability inequality (2.5) holds if and only if (ω, T) satisfy the following *geometric control condition* in Ω*: Every ray of geometric optics that propagates in Ω and is reflected on its boundary Γ enters ω in time less than T.*

This result was proved by means of microlocal Analysis techniques. Recently the microlocal approach has been greatly simplified by N. Burq [12] by using the microlocal defect measures introduced by P. Gerard [47] in the context of the homogenization and the kinetic equations. In [12] the geometric control condition was shown to be sufficient for exact controllability for domains Ω of class C^3 and equations with C^2 coefficients.

We have described here the HUM and some tools to prove observability inequalities. In particular, we have explained how HUM can be combined systematically with multiplier methods. One can also combine HUM with the theory of nonharmonic Fourier series when the coefficients of the system are time independent. We refer to Avdonin and Ivanov [6] and Komornik [67] for a complete presentation of this approach.

However, other methods have been developed to address controllability problems: Moment problems, fundamental solutions, controllability via stabilization, etc. We will not present them here. We refer to the survey paper by D. L. Russell [129] for the interested reader.

2.2. **The heat equation.** With the same notations as above we consider the linear controlled heat equation:

$$\begin{cases} u_t - \Delta u = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega. \end{cases} \tag{2.10}$$

We assume that $u^0 \in L^2(\Omega)$ and $f \in L^2(Q)$ so that (2.10) admits an unique solution $u \in C\left([0,T]\,; L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right)$.

We set $R(T;u^0) = \left\{u(T) : f \in L^2(Q)\right\}$. The controllability problems can be formulated as follows:

(a) System (2.10) is said to be approximately controllable if $R(T;u^0)$ is dense in $L^2(\Omega)$ for all $u^0 \in L^2(\Omega)$.
(b) System (2.10) is exactly controllable if $R(T;u^0) = L^2(\Omega)$ for all $u^0 \in L^2(\Omega)$.
(c) System (2.10) is null controllable if $0 \in R(T;u^0)$ for all $u^0 \in L^2(\Omega)$.

Remark 2.4. (a) Approximate controllability holds for every open non-empty subset ω of Ω and for every $T > 0$.
(b) It is easy to see that exact controllability may not hold except possibly in the case in which $\omega = \Omega$. Indeed, due to the regularizing effect of the heat equation, solutions of (2.10) at time $t = T$ are smooth in $\Omega \backslash \overline{\omega}$. Therefore $R(T;u^0)$ is strictly contained in $L^2(\Omega)$ for all $u^0 \in L^2(\Omega)$.
(c) Null controllability implies that all the range of the semigroup generated by the heat equation is reachable too.
(d) Proving that null controllability implies approximate controllability requires the use of the density of $S(T)[L^2(\Omega)]$ in $L^2(\Omega)$. In the case of the linear heat equation this can be seen easily developing solutions in Fourier series. However, if the equation contains time dependent coefficients this is no longer true. In those cases the density of the range of the semigroup, by duality, may be reduced to a backward uniqueness property in the spirit of Lions and Malgrange [92] (see also Ghidaglia [46]).

■

Let us now discuss the *approximate controllability problem.*

System (2.10) is approximately controllable for any open, non-empty subset ω of Ω and $T > 0$. To see this one can apply Hahn-Banach's Theorem or the variational approach developed in [89]. In both cases the approximate controllability is reduced to the unique continuation property of the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega. \end{cases} \tag{2.11}$$

More precisely, approximate controllability holds if and only if the following uniqueness property is true: *If* φ *solves* (2.11) *and* $\varphi = 0$ *in* $\omega \times (0,T)$ *then, necessarily,* $\varphi \equiv 0$, i.e. $\varphi^0 \equiv 0$.

This uniqueness property holds for every open non-empty subset ω of Ω and $T > 0$ by Holmgren's Uniqueness Theorem.

Following the variational approach of [89] the control can be constructed as follows. First of all we observe that it is sufficient to consider the particular case $u^0 \equiv 0$. Then, for any u^1 in $L^2(\Omega), \varepsilon > 0$ and E finite-dimensional subspace of $L^2(\Omega)$ we introduce the functional

$$J_\varepsilon(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \varepsilon \left\|(I - \pi_E)\varphi^0\right\|_{L^2(\Omega)} - \int_\Omega \varphi^0 u^1 dx \tag{2.12}$$

where π_E denotes the orthogonal projection from $L^2(\Omega)$ over E.

The functional J_ε is continuous and convex in $L^2(\Omega)$. On the other hand, in view of the unique continuation property above, one can prove that

$$\lim_{\|\varphi^0\|_{L^2(\Omega)} \to \infty} \frac{J_\varepsilon(\varphi^0)}{\| \varphi^0 \|_{L^2(\Omega)}} \geq \varepsilon \tag{2.13}$$

(we refer to [149] for the details of the proof).

Then, J_ε admits an unique minimizer $\widehat{\varphi}^0$ in $L^2(\Omega)$. The control $f = \widehat{\varphi}$ where $\widehat{\varphi}$ solves (2.11) with $\widehat{\varphi}^0$ as data is such that the solution u of (2.11) with $u^0 = 0$ satisfies

$$\| u(T) - u^1 \|_{L^2(\Omega)} \leq \varepsilon,\ \pi_E\left(u(T)\right) = \pi_E(u^1). \tag{2.14}$$

A slight change on the functional J_ε allows to build *bang-bang* controls. Indeed, we set

$$\widetilde{J}_\varepsilon(\varphi^0) = \frac{1}{2}\left(\int_0^T \int_\omega | \varphi | \, dxdt\right)^2 + \varepsilon \parallel (I - \pi_E)\varphi^0 \parallel_{L^2(\Omega)} - \int_\Omega u^1\varphi^0 dx. \tag{2.15}$$

The functional $\widetilde{J}_\varepsilon$ is continuous and convex in $L^2(\Omega)$ and satisfies the coercivity property (2.13) too.

Let $\widehat{\varphi}^0$ be the minimizer of $\widetilde{J}_\varepsilon$ in $L^2(\Omega)$ and $\widehat{\varphi}$ the corresponding solution of (2.11). We set

$$f = \int_0^T \int_\omega | \widehat{\varphi} | \, dxdt \, \mathrm{sgn}(\widehat{\varphi}) \tag{2.16}$$

where sgn is the multivalued sign function: $\mathrm{sgn}(s) = 1$ if $s > 0$, $\mathrm{sgn}(0) = [-1, 1]$ and $\mathrm{sgn}(s) = -1$ when $s < 0$. The control f given in (2.16) is such that the solution u of (2.10) with null initial data satisfies (2.14).

Due to the regularizing effect of the heat equation, the zero set of non-trivial solutions of (2.11) is of zero $(n+1)-$dimensional Lebesgue measure. Thus, the control f in (2.16) is of *bang-bang* form, i.e. $f = \pm\lambda$ a.e. in Q where

$$\lambda = \int_0^T \int_\omega \mid \widehat{\varphi} \mid dxdt.$$

We have proved the following result:

Theorem 2.5. ([149]) *Let ω be any open non-empty subset of Ω and $T > 0$ be any positive control time. Then, for any $u^0, u^1 \in L^2(\Omega)$, $\varepsilon > 0$ and finite-dimensional subspace E of $L^2(\Omega)$ there exists a bang-bang control $f \in L^\infty(Q)$ such that the solution u of* (2.10) *satisfies* (2.14).

Remark 2.6. The control (2.16) obtained by minimizing $\widetilde{J}_\varepsilon$ is the one of minimal $L^\infty(Q)$-norm among the admissible ones (we refer to [35] for the details of the proof in the particular case where $E = \{0\}$).

■

Let us now analyze the *null controllability* problem.

The null controllability problem for system (2.10) is equivalent to the following observability inequality for the adjoint system (2.11):

$$\| \varphi(0) \|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt, \quad \forall \varphi^0 \in L^2(\Omega). \tag{2.17}$$

Due to the irreversibility of the system, (2.17) is not easy to prove. For instance, multiplier methods do not apply.

In [129] the boundary null controllability of the heat equation was proved in one space dimension using moment problems and classical results on the linear independence in $L^2(0,T)$ of families of real exponentials. Later on in [130] it was shown that *if the wave equation is exactly controllable for some $T > 0$ with controls supported in ω, then the heat equation* (2.10) *is null controllable for all $T > 0$ with controls supported in ω*. As a consequence of this result and in view of the controllability results above, it follows that the heat equation (2.10) is null controllable for all $T > 0$ provided ω satisfies the geometric control condition.

However, the geometric control condition does not seem to be natural at all in the context of the heat equation.

More recently Lebeau and Robbiano [81] have proved that *the heat equation* (2.10) *is null controllable for every open, non-empty subset ω of Ω and $T > 0$*. This result shows, as expected, that the geometric control condition is unnecessary in the context of the heat equation.

A simplified proof of this result from [81] was given in [82] where the linear system of thermoelasticity was addressed. Let us describe briefly this proof. The main ingredient of it is an observability estimate for the eigenfunctions of the Laplace operator:

$$\begin{cases} -\Delta\psi_j = \lambda_j\psi_j & \text{in} \quad \Omega \\ \psi_j = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.18}$$

Recall that the eigenvalues $\{\lambda_j\}$, repeated according to their multiplicity, form an increasing sequence of positive numbers such that $\lambda_j \to \infty$ as $j \to \infty$ and that the eigenfunctions $\{\psi_j\}$ may be chosen such that they form an orthonormal basis of $L^2(\Omega)$.

The following holds:

Theorem 2.7. ([81, 82]) *Let Ω be a bounded domain of class C^∞. For any non-empty open subset ω of Ω there exist positive constants $C_1, C_2 > 0$ such that*

$$\int_\omega \left| \sum_{\lambda_j \le \mu} a_j \psi_j(x) \right|^2 dx \ge C_1 e^{-C_2\sqrt{\mu}} \sum_{\lambda_j \le \mu} \mid a_j \mid^2 \tag{2.19}$$

for all $\{a_j\} \in \ell^2$ and for all $\mu > 0$.

This result was implicitly used in [81] and it was proved in [82] by means of Carleman's inequalities.

As a consequence of (2.19) one can prove that the observability inequality (2.17) holds for solutions of (2.11) with initial data in $E_\mu = \mathrm{span}\,\{\varphi_j\}_{\lambda_j \le \mu}$, the constant being of the order of $\exp\left(C\sqrt{\mu}\right)$. This shows that the projection of solutions over E_μ can be controlled to zero with a control of size $\exp\left(C\sqrt{\mu}\right)$. Thus, when controlling the frequencies $\lambda_j \le \mu$ one increases the $L^2(\Omega)$-norm of the high frequencies $\lambda_j > \mu$ by a multiplicative factor of the order of $\exp\left(C\sqrt{\mu}\right)$. However, solutions of the heat equation (2.10) without control ($f = 0$) and such that the projection of the initial data over E_μ vanishes, decay in $L^2(\Omega)$ at a rate of the order of $\exp(-\mu t)$. This can be easily seen by means of the Fourier series decomposition of the solution. Thus, if we divide the time interval $[0, T]$ in two parts $[0, T/2]$ and $[T/2, T]$, we control to zero the frequencies $\lambda_j \le \mu$ in the interval $[0, T/2]$ and then allow the equation to evolve without control in the interval $[T/2, T]$, it follows that, at time $t = T$, the projection of the solution u over E_μ vanishes and the norm of the high frequencies does not exceed the norm of the initial data u^0.

This argument allows to control to zero the projection over E_μ for any $\mu > 0$ but not the whole solution. To do that an iterative argument is needed in which the interval $[0, T]$ has to be decomposed in a suitably chosen sequence of subintervals $[T_k, T_{k+1})$ and the argument above is applied in each subinterval to control an increasing range of frequencies $\lambda_j \le \mu_k$ with μ_k going to infinity at a suitable rate. We refer to [81] and [82] for the proof.

Remark 2.8. (a) Once (2.17) is known to hold one can obtain the control with minimal $L^2(Q)$-norm among the admissible ones. To do that it is sufficient to minimize the functional

$$J(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \int_\Omega \varphi(0) u^0 dx \tag{2.20}$$

over the Hilbert space

$$H = \{\varphi^0 : \text{ the solution } \varphi \text{ of (2.11) satisfies } \int_0^T \int_\omega \varphi^2 dxdt < \infty\}.$$

To be more precise, H is the completion of $L^2(\Omega)$ with respect to the norm $[\int_0^T \int_\omega \varphi^2 dxdt]^{1/2}$. In fact, H is much larger than $L^2(\Omega)$. We refer to Theorem 4.5 for precise estimates on the nature of this space.

Observe that J is convex and, according to (2.17), it is also continuous in H. On the other hand (2.17) guarantees the coercivity of J and the existence of its minimizer.

(b) As a consequence of the internal null controllability property of the heat equation one can deduce easily the null boundary controllability with controls in an arbitrarily small open subset of the boundary. To see this it is sufficient to extend the domain Ω by a little open subset attached to the subset of the boundary where the control needs to be supported. The arguments above allow to control the system in the large domain by means of a control supported in this small added domain. The restriction of the solution to the original domain satisfies all the requirements and its restriction or trace to the subset of the boundary where the control had to be supported, provides the control we were looking for.

(c) The method of proof of the null controllability we have described is based on the possibility of developing solutions in Fourier series. Thus it can be applied in a more general class of heat equations with variable but time-independent coefficients. The same can be said about the methods of [130].

■

The null controllability of the heat equation with lower order time-dependent terms of the form

$$\begin{cases} u_t - \Delta u + a(x,t)u = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega \end{cases} \tag{2.21}$$

has been studied by Fursikov and Imanuvilov (see for instance [21, 42, 43, 44, 45, 60] and [61]). Their approach, based on the use of Carleman inequalities, is different to the one we have presented here. As a consequence of their results on null controllability it follows that an observability inequality of the form (2.17) holds for the solutions of the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi + a(x,t)\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega \end{cases} \tag{2.22}$$

when ω is any open subset of Ω.

We shall return to this method in section 4 below.

3. **The wave equation with rapidly oscillating coefficients.**

In practice, the equation under consideration often depends on some parameter. In those cases it is natural to analyze whether the controls depend continuously on this parameter or not. When the parameter enters in the system as a singular perturbation, this continuous dependence may be lost. This is for instance the case in the wave equation with rapidly oscillating coefficients, a classical problem in the theory of homogenization that we address here.

Let us consider the wave equation

$$\begin{cases} \rho(x/\varepsilon)u_{tt} - u_{xx} = 0, \; 0 < x < 1, \; 0 < t < T \\ u(0,t) = u(1,t) = 0, \; 0 < t < T \\ u(x,0) = u^0(x), \; u_t(x,0) = u^1(x), \; 0 < x < 1. \end{cases} \tag{3.1}$$

Here $\rho \in L^\infty(\mathbb{R})$ is a periodic function of period $\ell > 0$ such that

$$0 < \rho_m \le \rho(x) \le \rho_M < \infty \text{ a.e. } x \in \mathbb{R}$$

where ρ_m, ρ_M are two positive constants.

The parameter ε ranges in the interval $0 < \varepsilon < 1$ and is devoted to tend to zero.

System (3.1) is a simple model for the vibrations of a string with rapidly oscillating density.

Let us denote $\rho_\varepsilon(x) = \rho(x/\varepsilon)$. It is easy to check that

$$\rho_\varepsilon \rightharpoonup \bar{\rho} \text{ weakly } -* \text{ in } L^\infty(0,1); \text{ as } \varepsilon \to 0, \tag{3.2}$$

i.e.,

$$\int_0^1 \rho_\varepsilon(x)\varphi(x)dx \to \bar{\rho}\int_0^1 \varphi(x)dx \text{ as } \varepsilon \to 0,\ \forall \varphi \in L^1(0,1)$$

where

$$\bar{\rho} = \frac{1}{\ell}\int_0^\ell \rho(x)dx \tag{3.3}$$

is the average density.

The energy of solutions of (3.1) is given by

$$E_\varepsilon(t) = \frac{1}{2}\int_0^1 \left[\rho_\varepsilon(x) \mid u_t(x,t) \mid^2 + \mid u_x(x,t) \mid^2\right] dx \tag{3.4}$$

and it is constant in time.

The observability problem for (3.1) can be formulated as follows: *To find $T > 0$ and $C_\varepsilon(T) > 0$ such that*

$$E_\varepsilon(0) \le C_\varepsilon(T) \int_0^T \mid u_x(1,t) \mid^2 dt \tag{3.5}$$

holds for any solution of (3.1).

If ρ is regular enough, say $\rho \in C^1(\mathbb{R})$, one can show that observability holds for all $T > 2\sqrt{\rho_M}$ and for all $0 < \varepsilon < 1$.

We address here the problem of *uniform observability*: Given $T > 2\sqrt{\rho_M}$, is the observability constant $C_\varepsilon(T)$ in (3.5) bounded as $\varepsilon \to 0$?

This question arises naturally since the limit of system (3.1) as $\varepsilon \to 0$, in view of (3.2), is given by the wave equation with constant density $\bar{\rho}$:

$$\begin{cases} \bar{\rho} u_{tt} - u_{xx} = 0,\ 0 < x < 1,\ 0 < t < T \\ u(0,t) = u(1,t) = 0,\ 0 < t < T \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x),\ 0 < x < 1 \end{cases} \tag{3.6}$$

and the later is observable for any $T \ge \sqrt{\bar{\rho}}$.

Solutions of (3.1) can be developed in Fourier series. Indeed, for any $\varepsilon > 0$ the eigenvalue problem

$$\begin{cases} -\frac{\partial^2 w_k^\varepsilon}{\partial x^2} = \lambda_k^\varepsilon w_k^\varepsilon,\ 0 < x < 1 \\ w_k^\varepsilon(0) = w_k^\varepsilon(1) = 0 \end{cases} \tag{3.7}$$

admits a sequence of eigenvalues

$$0 < \lambda_1^\varepsilon < \lambda_2^\varepsilon < \cdots < \lambda_k^\varepsilon < \cdots \to \infty$$

with corresponding eigenfunctions $\{w_k^\varepsilon\}_{k\ge1}$ that may be chosen to constitute and orthonormal basis of $L^2(0,1)$ with the scalar product

$$(\varphi,\psi)_\varepsilon = \int_0^1 \varphi(x)\psi(x)\rho_\varepsilon(x)dx. \tag{3.8}$$

Then, solutions of (3.1) may be written in the form

$$u_\varepsilon = \sum_{k\ge1}\left(a_k^\varepsilon \cos\left(\sqrt{\lambda_k^\varepsilon}t\right) + \frac{b_k^\varepsilon}{\sqrt{\lambda_k^\varepsilon}}\sin\left(\sqrt{\lambda_k^\varepsilon}t\right)\right) w_k^\varepsilon(x) \tag{3.9}$$

where a_k^ε and b_k^ε are the Fourier coefficients of the initial data

$$u^0 = \sum_{k\geq 1} a_k^\varepsilon w_k^\varepsilon;\ u^1 = \sum_{k\geq 1} b_k^\varepsilon w_k^\varepsilon. \tag{3.10}$$

On the other hand, solutions of (3.6) can be written as

$$u = \sum_{k\geq 1}\left[a_k \cos\left(\frac{k\pi t}{\sqrt{\bar\rho}}\right) + \frac{\sqrt{\bar\rho} b_k}{k\pi}\sin\left(\frac{k\pi t}{\sqrt{\bar\rho}}\right)\right]\sin(k\pi x). \tag{3.11}$$

Using the mini-max characterization of the eigenvalues of (3.7) one can deduce that, for each $k\geq 1$,

$$\lambda_k^\varepsilon \to \frac{k^2\pi^2}{\bar\rho} \text{ as } \varepsilon\to 0, \tag{3.12}$$

$$w_k^\varepsilon \to \sqrt{\frac{2}{\bar\rho}}\sin(k\pi x) \text{ in } H_0^1(0,1), \text{ as } \varepsilon\to 0. \tag{3.13}$$

However, these convergences are far from being sufficient to address the uniform observability problem. Indeed, in order to attack the problem of uniform observability, one has to know how uniform the convergences (3.12)-(3.13) are with respect to the index k.

Classical results in the theory of homogenization provide convergence rates for (3.12)-(3.13) (see for instance Oleinick et al. [117]) and this allows to show that for any $\delta>0$ there exists $c_\delta>0$ such that

$$\left|\sqrt{\lambda_k^\varepsilon} - \frac{k\pi}{\sqrt{\bar\rho}}\right| \leq \delta,\ \forall k\leq c_\delta/\sqrt{\varepsilon}. \tag{3.14}$$

However, this result is far from being sufficient for our purposes. Indeed, the critical scale for the problem under consideration is $k\sim 1/\varepsilon$ which corresponds to the case where the wavelength of the solutions is of the order of the microstructure. Obviously, this critical size is much beyond the range $k\leq C/\sqrt{\varepsilon}$ in which (3.14) applies.

In fact, as we shall see, whatever $T>0$ is, the uniform observability fails because "spurious" oscillations occur when $k\sim C/\varepsilon$ if C is large enough. Using the WKB method (see [10]) we shall exhibit a complete asymptotic description of the spectrum in the range $k\leq c/\varepsilon$ with $c>0$ small enough. We shall show how uniform observability results may be obtained provided the high frequencies are filtered in an appropriate way. The results of this section are proved in detail by Castro and Zuazua [18, 19]. We refer to [15] for a complete analysis of the limit behavior of the controllability properties as $\varepsilon\to 0$ and to [16] for the analysis of the spectrum at the critical frequencies $k\sim C/\varepsilon$.

In order to analyze the behavior of the eigenvalues of the order of $\lambda\sim 1/\varepsilon^2$ it is natural to introduce the change of variables $y=x/\varepsilon$, so that equation (3.7) becomes

$$\begin{cases} -w_{yy} = \mu\rho(y)w, & 0<y<1/\varepsilon \\ w(0)=w(1/\varepsilon)=0 \end{cases} \tag{3.15}$$

with

$$\mu = \lambda\varepsilon^2. \tag{3.16}$$

Consider the problem in the whole line associated to (3.15):

$$-w_{yy} = \mu\rho(y)w,\ y\in\mathbb{R}. \tag{3.17}$$

Taking into account that ρ is periodic, the behavior of solutions of (3.17) depending on the different values of μ may be analyzed by means of Floquet Theorem (see [32]). This analysis suffices to prove the following result:

Theorem 3.1. ([7, 18]) *Assume that $\rho \in L^\infty(\mathbb{R})$ is ℓ-periodic and such that*

$$0 < \rho_m \leq \rho(x) \leq \rho_M < \infty, \text{a.e. } x \in \mathbb{R}. \tag{3.18}$$

Then, there exists a sequence $\varepsilon_j \to 0$ and a sequence of indexes $k_j \to \infty$ of the order of ε_j^{-1} such that the corresponding eigenfunctions $w_{k_j}^{\varepsilon_j}$ of (3.7) satisfy

$$\int_0^1 \left|\partial_x w_{k_j}^{\varepsilon_j}(x)\right|^2 dx \Big/ \left|\partial_x w_{k_j}^{\varepsilon_j}(1)\right|^2 \geq \exp(C/\varepsilon_j) \quad \text{as} \quad \varepsilon_j \to 0 \tag{3.19}$$

for some $C > 0$.

As an immediate corollary the following holds:

Corollary 3.2. *Under the assumptions of Theorem 3.1, there exists a sequence $\varepsilon_j \to 0$ such that*

$$\sup_{u \text{ solution of (3.1)}} \left[\frac{E_\varepsilon(0)}{\displaystyle\int_0^T |u_x(1,t)|^2 \, dt}\right] \to \infty, \quad \text{as} \quad \varepsilon_j \to \infty \tag{3.20}$$

for all $T > 0$.

Proof of Corollary 3.2. Assuming for the moment that Theorem 3.1 holds we consider solutions of (3.1) of the form

$$u_{\varepsilon_j}(x,t) = \cos\left(\sqrt{\lambda_{k_j}^{\varepsilon_j}}t\right) w_{k_j}^{\varepsilon_j}(x)$$

where the sequence $\varepsilon_j \to 0$ and the sequence of eigenvalues $\lambda_{k_j}^{\varepsilon_j}$ and eigenfunctions $w_{k_j}^{\varepsilon_j}$ are as in Theorem 3.1.

We have

$$E_\varepsilon(0) = \frac{1}{2}\int_0^1 \left|\partial_x w_{k_j}^{\varepsilon_j}(x)\right|^2 dx \tag{3.21}$$

and

$$\int_0^T \left|\partial_x w_{k_j}^{\varepsilon_j}(1,t)\right|^2 dt = \left|\partial_x w_{k_j}^{\varepsilon_j}(1)\right|^2 \int_0^T \cos^2\left(\sqrt{\lambda_{k_j}^{\varepsilon_j}}t\right) dt. \tag{3.22}$$

Since

$$\lambda_{k_j}^{\varepsilon_j} \to \infty \text{ as } \varepsilon_j \to 0, \tag{3.23}$$

we have

$$\int_0^T \cos^2\left(\sqrt{\lambda_{k_j}^{\varepsilon_j}}t\right) dt \to T/2. \tag{3.24}$$

Note that (3.23) holds since, by using the min-max characterization of eigenvalues, one has

$$\frac{k^2\pi^2}{\rho_M} \leq \lambda_k^\varepsilon \leq \frac{k^2\pi^2}{\rho_m}$$

for all $0 < \varepsilon < 1$ and $k \geq 1$.

Combining (3.19), (3.21), (3.22) and (3.24) we deduce that (3.20) holds.

■

Let us now give a *sketch of the proof of Theorem 3.1.*

Since ρ is assumed not to be constant, by Floquet's Theorem we deduce the existence of some $\mu > 0$ such that (3.17) has a solution of the form $\omega_1(y) = e^{-\alpha y} p_1(y)$ with $\alpha > 0$, p_1 being ℓ-periodic. Let us assume for simplicity that $p_1(0) = 0$. We refer to [19] for the case where $p_1(0) \neq 0$.

The function p_1 has an unbounded sequence of zeroes. Let us denote by $0 < z_1 < z_2 < \cdots < z_n < \cdots \to \infty$ the positive zeroes of p_1. Given a zero z_k of p_1 we set $w_k(x) = e^{-\alpha z_k x} p_1(z_k x)$. Note that w_k may also be written as

$$w_k(x) = e^{-\alpha x/\varepsilon_k} p_1\left(\frac{x}{\varepsilon_k}\right)$$

with $\varepsilon_k = 1/z_k$. Taking into account that $p_1(0) = p_1(z_k) = 0$ we deduce that $w_k(0) = w_k(1) = 0$. On the other hand, in view of (3.17) we also have

$$-\partial_x^2 w_k = \lambda_k \rho\left(\frac{x}{\varepsilon_k}\right) w_k,\ 0 < x < 1$$

with $\lambda_k = \mu/\varepsilon_k^2$.

Moreover, in view of the explicit form of w_k it is easy to see that (3.19) holds along the sequence $\varepsilon_k \to 0$.

This completes the proof of Theorem 3.1.

■

Remark 3.3. Note that the proof of Theorem 3.1 provides necessarily sequences of eigenvalues $\lambda_{k_j}^{\varepsilon_j}$ such that $\lambda_{k_j}^{\varepsilon_j} \geq c/\varepsilon_j^2$ with $c > 0$. As we shall see below, this is a sharp estimate since uniform observability estimates hold for eigenfunctions corresponding to eigenvalues λ such that $\lambda \leq c/\varepsilon^2$ with $c > 0$ small enough.

■

Let us now analyze the asymptotic behavior of the spectrum following [18, 19].

Theorem 3.4. ([18, 19]) *For any $\delta > 0$ there exists $C(\delta) > 0$ such that*

$$\sqrt{\lambda_{k+1}^{\varepsilon}} - \sqrt{\lambda_k^{\varepsilon}} \geq \frac{\pi}{\sqrt{\bar\rho}} - \delta \tag{3.25}$$

for all $k \leq C(\delta)/\varepsilon$ and $0 < \varepsilon < 1$.

Moreover, there exist $C, c > 0$ such that

$$\frac{1}{C}\left|\partial_x w_k^{\varepsilon}(1)\right|^2 \leq \int_0^1 \left|\partial_x w_k^{\varepsilon}\right|^2 dx \leq C\left|\partial_x w_k^{\varepsilon}(1)\right|^2 \tag{3.26}$$

for all $k \leq c/\varepsilon$ and $0 < \varepsilon < 1$.

Remark 3.5. The first statement of this theorem guarantees that the *gap* between consecutive eigenvalues remains uniformly bounded below in the range $k \leq C/\varepsilon$ for $C > 0$ small enough (which is equivalent to $\lambda \leq C'/\varepsilon^2$ for a suitable C'). In fact, the gap corresponding to the limit spectrum $\lambda_k = k^2\pi^2/\bar\rho$ is precisely $\pi/\sqrt{\bar\rho}$, i.e.

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = \frac{(k+1)\pi}{\sqrt{\bar\rho}} - \frac{k\pi}{\sqrt{\bar\rho}} = \frac{\pi}{\sqrt{\bar\rho}}.$$

According to (3.25) the gap may be made to be arbitrarily close to $\pi/\sqrt{\bar\rho}$ for all $0 < \varepsilon < 1$ provided $k \leq C(\delta)/\varepsilon$ with $C(\delta)$ small enough.

The second statement of Theorem 3.4 guarantees the uniform observability of the eigenfunctions from the extreme $x = 1$ provided $k \leq c/\varepsilon$ with $c > 0$ small enough. This result is sharp since, according to Theorem 3.1, there exists $\varepsilon_j \to 0$ and a sequence of eigenvalues of the order of $\lambda_{k_j}^{\varepsilon_j} \sim C/\varepsilon_j^2$ for a sufficiently large C such that

$$\int_0^1 \left|\partial_x w_{k_j}^{\varepsilon_j}(x)\right|^2 dx \Big/ \left|\partial_x w_{k_j}^{\varepsilon_j}(1)\right|^2 \to 0, \text{ as } \varepsilon_j \to 0.$$

■

Let us now briefly comment the *proof of Theorem 3.4.*

We perform the change of variables $y = x/\varepsilon$ so that the differential equation corresponding to the eigenfunctions becomes (3.17). We employ the *shooting method.* Thus we solve (3.17) under the "initial conditions"

$$w(1/\varepsilon) = 0; \partial_y w(1/\varepsilon) = 1. \tag{3.27}$$

Finding the eigenvalues λ_k^ε is then equivalent to find the values of μ_k^ε such that the solution of (3.17)-(3.27) satisfies

$$w(0) = 0. \tag{3.28}$$

We employ the WKB asymptotic expansion method (see [10]) to analyze the structure of the solutions of (3.17)-(3.27). This asymptotic expansion turns out to converge in the interval $(0, 1/\varepsilon)$ when $\mu > 0$ is small enough, i.e. when $\lambda \leq C/\varepsilon^2$ for $C > 0$ small enough.

This allows us to rewrite equation (3.28) as finding the zeroes of a infinite series. We conclude that in the range $\lambda \leq c/\varepsilon^2$, with $c > 0$ sufficiently small, λ_k^ε is an eigenvalue if and only if λ_k^ε is the root of

$$\sqrt{\lambda_k^\varepsilon \bar{\rho}} + \sum_{n \geq 1} \left(\varepsilon^{2n} d_{2n-1} + \varepsilon^{2n+1} c_{2n}\left(\varepsilon^{-1}\right)\right) \left(\lambda_k^\varepsilon\right)^{(2n+1)/2} = k\pi \tag{3.29}$$

where $\{d_{2j-1}\}_{j \geq 1}$ are constants and $\{c_{2j}\}_{j \geq 1}$ are ℓ-periodic functions that may be computed explicitly.

The same method provides and asymptotic expansion of the eigenfunctions w_k^ε as well.

In order to illustrate how the gap condition (3.25) arises let us consider the second order approximation of λ_k^ε. According to (3.29) it follows that

$$\sqrt{\lambda_k^\varepsilon} \sim \frac{k\pi}{\sqrt{\bar{\rho}}} - \varepsilon^2 (k\pi)^3 \frac{d_1}{\bar{\rho}^2}. \tag{3.30}$$

Therefore

$$\begin{aligned} \sqrt{\lambda_{k+1}^\varepsilon} - \sqrt{\lambda_k^\varepsilon} &\sim \frac{\pi}{\sqrt{\bar{\rho}}} - \frac{\varepsilon^2 \pi^3 d_1}{\bar{\rho}^2}\left((k+1)^3 - k^3\right) \\ &= \frac{\pi}{\sqrt{\bar{\rho}}} - \frac{\varepsilon^2 \pi^3 d_1}{\bar{\rho}^2}\left(\varepsilon k^2 + 3k + 1\right) \\ &\sim \frac{\pi}{\sqrt{\bar{\rho}}} - \frac{3\pi^3 d_1}{\bar{\rho}^2}(\varepsilon k)^2. \end{aligned} \tag{3.31}$$

Thus, according to the second order approximation, (3.25) holds if

$$\varepsilon k \leq \frac{\bar{\rho}\sqrt{\delta}}{\sqrt{3\pi^3 d_1}}. \tag{3.32}$$

To summarize, one can say that the WKB expansion method allows to prove that the uniform gap condition (3.25) is guaranteed up to the critical level $k \leq C\varepsilon^{-1}$ with $C > 0$ small enough.

As we shall see, this uniform gap condition together with the uniform observability of the eigenfunctions (3.26) is sufficient to prove the uniform observability of the solutions whose spectrum lies in the range $\lambda \leq C\varepsilon^{-2}$ with $C > 0$ small enough.

To do this we need a classical result due to Ingham in the theory of non-harmonic Fourier series (see [142]).

Ingham's Theorem. Let $\{\mu_k\}_{k\in\mathbb{Z}}$ be a sequence of real numbers such that

$$\mu_{k+1} - \mu_k \geq \gamma > 0, \, \forall k \in \mathbb{Z}. \tag{3.33}$$

Then, for any $T > 2\pi/\gamma$ there exists a positive constant $C(T,\gamma) > 0$ such that

$$\frac{1}{C(T,\gamma)} \sum_{k\in\mathbb{Z}} \mid a_k \mid^2 \leq \int_0^T \left| \sum_{k\in\mathbb{Z}} a_k e^{i\mu_k t} \right|^2 dt \leq C(T,\gamma) \sum_{k\in\mathbb{Z}} \mid a_k \mid^2 \tag{3.34}$$

for all sequence of complex numbers $a_k \in \ell^2$.

Remark 3.6. Ingham's inequality may be viewed as a generalization of the orthogonality property of trigonometric functions. Indeed, assume that

$$\mu_k = k\gamma, \, k \in \mathbb{Z}$$

for some $\gamma > 0$. Then (3.34) holds with equality for all k. We set $T = 2\pi/\gamma$. Then

$$\int_0^{2\pi/\gamma} \left| \sum_{k\in\mathbb{Z}} a_k e^{i\gamma k t} \right|^2 dt = \frac{2\pi}{\gamma} \sum_{k\in\mathbb{Z}} \mid a_k \mid^2 . \tag{3.35}$$

Note that under the weaker gap condition (3.33) we obtain upper and lower bounds instead of identity (3.35). It is also important to note that the Ingham inequality is in general false in the criticial case $T = 2\pi/\gamma$.

■

The uniform observability result we have proved is as follows:

Theorem 3.7. [18, 19]) *Assume that* $\rho \in L^\infty(\mathbb{R})$ *is* $\ell-$*periodic and such that*

$$0 < \rho_m \leq \rho(x) \leq \rho_M < \infty, \, a.e. \, x \in \mathbb{R}.$$

Then, for any $T > 2\sqrt{\bar\rho}$ *there exist positive constant* $c(T)$, $C(T) > 0$ *such that*

$$\frac{1}{C(T)} \int_0^T |\partial_x u(1,t)|^2 \, dt \leq E_\varepsilon(0) \leq C(T) \int_0^T |\partial_x u(1,t)|^2 \, dt \tag{3.36}$$

for all $0 < \varepsilon < 1$ *and all solution* u *of* (3.1) *in the class*

$$u \in \operatorname{span}\left\{ w_k^\varepsilon : k \leq c(T)\varepsilon^{-1} \right\}. \tag{3.37}$$

Remark 3.8. Observe that the minimal time needed to apply Theorem 3.7 is $2\sqrt{\bar\rho}$ which is the observability time for the limit wave equation

$$\begin{cases} \bar\rho u_{tt} - u_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\ u(0,t) = u(1,t) = 0, & 0 < t < T. \end{cases} \tag{3.38}$$

■

Proof of Theorem 3.7. We set

$$\mu_k^\varepsilon = \sqrt{\lambda_k^\varepsilon};\ \mu_{-k}^\varepsilon = -\mu_k^\varepsilon;\ w_{-k}^\varepsilon = w_k^\varepsilon,\ \text{for } k \geq 1.$$

Then, according to (3.9) u can also be written as follows

$$u = \sum_{k\in\mathbb{Z}\setminus\{0\}} c_k^\varepsilon e^{i\mu_k^\varepsilon t} w_k^\varepsilon \tag{3.39}$$

where

$$c_k^\varepsilon = \frac{a_k^\varepsilon - ib_k^\varepsilon/\mu_k^\varepsilon}{2}$$

and $a_k^\varepsilon = a_{-k}^\varepsilon$, $b_k^\varepsilon = b_{-k}^\varepsilon$.

According to (3.39) we have

$$\partial_x u(1,t) = \sum_{k\in\mathbb{Z}\setminus\{0\}} c_k^\varepsilon e^{i\mu_k^\varepsilon t} \partial_x w_k^\varepsilon(1). \tag{3.40}$$

We now consider solutions with frequencies in the range $k \leq c\varepsilon^{-1}$. Then

$$u = \sum_{\substack{|k|\leq c/\varepsilon \\ k\neq 0}} c_k^\varepsilon e^{i\mu_k^\varepsilon t} w_k^\varepsilon \tag{3.41}$$

and

$$\partial_x u(1,t) = \sum_{\substack{|k|\leq c/\varepsilon \\ l\neq 0}} c_k^\varepsilon e^{i\mu_k^\varepsilon t} \partial_x w_k^\varepsilon(1). \tag{3.42}$$

Given $T > 2\sqrt{\rho}$ we have

$$T > 2\pi/\gamma \tag{3.43}$$

for some $\gamma < \pi/\sqrt{\bar\rho}$. Let $\delta > 0$ be such that

$$\frac{\pi}{\sqrt{\bar\rho}} - \delta \geq \gamma > 0. \tag{3.44}$$

According to Theorem 3.4, there exists $c(\delta) > 0$ such that

$$\sqrt{\lambda_{k+1}^\varepsilon} - \sqrt{\lambda_k^\varepsilon} \geq \frac{\pi}{\sqrt{\bar\rho}} - \delta \geq \gamma \tag{3.45}$$

for all $0 < \varepsilon < 1$ and $k \leq C(\delta)\varepsilon^{-1}$.

In view of (3.45) we can apply Ingham's Theorem to the series in (3.42) provided $c \leq c(\delta)$. It then follows that there exists $C > 0$ such that

$$\begin{aligned} \frac{1}{C}\sum_{\substack{|k|\leq c/\varepsilon \\ k\neq 0}} |\ c_k^\varepsilon\ |^2|\ \partial_x w_k^\varepsilon(1)\ |^2 &\leq \int_0^T \left| \sum_{\substack{|k|\leq c/\varepsilon \\ k\neq 0}} c_k^\varepsilon e^{i\mu_k^\varepsilon t} w_k^\varepsilon \right|^2 dt \\ &\leq C \sum_{\substack{|k|\leq c/\varepsilon \\ k\neq 0}} |\ c_k^\varepsilon\ |^2|\ \partial_x w_k^\varepsilon(1)\ |^2 . \end{aligned} \tag{3.46}$$

On the other hand, taking into account that

$$\int_0^1 |\partial_x w_k^\varepsilon(x)|^2\, dx = \lambda_k^\varepsilon \int_0^1 |\ w_k^\varepsilon\ |\ \rho(x/\varepsilon) dx = \lambda_k^\varepsilon,$$

according to (3.26) we deduce that, for $k \leq c/\varepsilon$,

$$\frac{\lambda_k^\varepsilon}{C} \leq |\partial_x w_k^\varepsilon(1)|^2 \leq C\lambda_k^\varepsilon. \tag{3.47}$$

Combining (3.45) and (3.47) and choosing $c>0$ possibly smaller such that both (3.46) and (3.47) apply in the range $k \leq c\varepsilon^{-1}$ we deduce that

$$\begin{aligned} \frac{1}{C} \sum_{\substack{|k| \leq c/\varepsilon \\ k \neq 0}} \lambda_k^\varepsilon \mid c_k^\varepsilon \mid^2 \quad &\leq \quad \int_0^T \left| \sum_{\substack{|k| \leq c/\varepsilon \\ k \neq 0}} c_k^\varepsilon e^{i\mu_k^\varepsilon t} \partial_x w_k^\varepsilon(1) \right|^2 dt \\ &\leq \quad C \sum_{\substack{|k| \leq c/\varepsilon \\ k \neq 0}} \lambda_k^\varepsilon \mid c_k^\varepsilon \mid^2 . \end{aligned} \tag{3.48}$$

On the other hand,

$$\sum_{\substack{|k| \leq c/\varepsilon \\ k \neq 0}} \lambda_k^\varepsilon \mid c_k^\varepsilon \mid^2 = \frac{1}{2} \sum_{1 \leq k \leq c/\varepsilon} \left[\lambda_k^\varepsilon \mid a_k^\varepsilon \mid^2 + \mid b_k^\varepsilon \mid^2 \right] = E_\varepsilon(0). \tag{3.49}$$

Combining (3.48)-(3.49) inequality (3.36) follows.

■

Let us summarize the content of this section. We have shown that the uniform observability does not hold when the wavelength of solutions is of the order ε of the microstructure. We have also shown that uniform observability holds in the class of solutions whose spectrum is in the range $\lambda \leq c\varepsilon^{-2}$ for $c>0$ small enough.

Analyzing carefully the proof above it can be seen that if

$$k \leq c\varepsilon^{-1} \tag{3.50}$$

with

$$0 < c < c^* \tag{3.51}$$

and

$$c^* = \left(\frac{\bar{\rho}^{3/2}}{3\pi^2 d_1} \right)^{1/2} \tag{3.52}$$

uniform observability holds for $T > T(c)$ where $T(c)$ is such that

- $T(c) \searrow 2\sqrt{\bar{\rho}}$ as $c \searrow 0$;
- $T(c) \nearrow \infty$ as $c \nearrow c^*$.

This result shows that the observability inequality for the limit wave equation (3.38) may be viewed as the limit when ε and c tend to zero of observability inequalities for systems (3.1) in the range (3.50).

All this section has been devoted to the analysis of the low frequencies $k \leq c/\varepsilon$. Note however that, under suitable regularity assumptions on the density ρ, the WKB method allows to obtain an asymptotic expansion for the high frequencies $k >> \varepsilon^{-1}$ as well. We refer to [19] for a careful analysis of this problem. At this respect, it is worth mentioning that, in a first approximation, the effective wave equation for the high frequencies is

$$\begin{cases} \rho^* u_{tt} - u_{xx} = 0, & 0<x<1, \quad 0<t<T \\ u(0,t) = u(1,t) = 0, & 0<t<T \end{cases}$$

where

$$\rho^* = \left(\frac{1}{\ell} \int_0^\ell \sqrt{\rho} dx \right)^2 .$$

Note that $\rho^* < \bar{\rho}$. This indicates that the time of observability for the high frequencies $k >> \varepsilon^{-1}$ is smaller than for the low ones.

We refer to [16] for a detailed discussion of the behavior of eigenvalues and eigenfunctions corresponding to the critical case $\lambda \sim c\varepsilon^{-2}$.

4. **Approximate and null controllability of the linear heat equation.**

4.1. **The cost of approximate controllability.**

In this section we present some of the results of [41] on the cost of approximate controllability of the system (2.10). In [41] we address the more general case of the heat equation perturbed by a potential depending both in space and time. But here, to simplify the presentation, we shall focus on the constant coefficient heat equation.

We shall use in an essential manner the fact that solutions of (2.10) and of its adjoint (2.11) may be developed in Fourier series. Therefore the case in which potentials depending both on x and t arise needs further developments (we refer to [41]).

As we said above, without loss of generality, we may assume that $u^0 \equiv 0$. Given $u^1 \in L^2(\Omega)$ and $\varepsilon > 0$ we set

$$\mathcal{C}(u^1, \varepsilon) = \min_{f \in \mathcal{U}_{ad}} \| f \|_{L^2(\omega \times (0,T))} \tag{4.1}$$

where $\mathcal{U}_{ad}$ is the set of admissible controls $v \in L^2(\omega \times (0,T))$ such that u the solution of (2.10) with $u^0 \equiv 0$ satisfies

$$\| u(T) - u^1 \|_{L^2(\Omega)} \leq \varepsilon. \tag{4.2}$$

Obviously $\mathcal{C}(u^1, \varepsilon)$ represents the *cost* (the size of the control) needed to drive the solution of (2.10) from the initial state $u^0 \equiv 0$, to a ball of radius ε around u^1.

We have the following result

Theorem 4.1. ([41]) *Let $T > 0$ and ω be a non-empty open subset of Ω. Then there exists a constant $C > 0$ such that*

$$\mathcal{C}(u^1, \varepsilon) \leq \exp\left(C \| u^1 \|_{H_0^1(\Omega)} \Big/ \varepsilon \right) \| u^1 \|_{L^2(\Omega)} \tag{4.3}$$

for all $u^1 \in H_0^1(\Omega)$ and $0 < \varepsilon < \| u^1 \|_{L^2(\Omega)}$.

In order to prove this result we first need suitable observability estimates for the adjoint system (2.11). Using the methods developed in [45] and [42] based on Carleman inequalities, in [41] the following is proved:

Proposition 4.2. ([41]) *There exists a constant $C > 0$ depending only on Ω and ω such that*

$$\| \varphi(x, 0) \|^2_{L^2(\Omega)} \leq \exp\left(C \left(1 + \frac{1}{T} \right) \right) \int_0^T \int_\omega \varphi^2 dxdt \tag{4.4}$$

holds for all solution of (2.11) and for all $T > 0$.

Note that (4.4) provides the observability inequality (2.17), but that it also provides the dependence of the observability constant on T.

However to prove Theorem 4.1 we need a more precise result providing global information about φ in all of Q. The following holds:

Proposition 4.3. ([41]) *Let $T > 0$ and ω be an open non-empty subset of Ω. There exist positive constants c, $C > 0$ which are independent of T such that*

$$\int_0^T \int_\Omega e^{-\frac{c(1+T)}{T-t}} \varphi^2 dxdt \leq e^{C(1+\frac{1}{T})} \int_0^T \int_\omega \varphi^2 dxdt \tag{4.5}$$

holds for all solution φ of (2.11).

Estimate (4.5) is a direct consequence of the Carleman estimate proved in [41] following the method introduced in [45]. Let us recall it briefly.

We introduce a function $\eta^0 = \eta^0(x)$ such that

$$\left\{\begin{array}{lll} \eta^0 \in C^2(\bar{\Omega}) & & \\ \eta^0 > 0 & \text{in} & \Omega,\ \eta^0 = 0 \quad \text{in} \quad \partial\Omega \\ \nabla\eta^0 \neq 0 & \text{in} & \overline{\Omega\backslash\omega}. \end{array}\right. \tag{4.6}$$

The existence of this function was proved in [45]. In particular cases, for instance when Ω is star-shaped with respect to a point in ω, it can be built explicitly without difficulty.

Let $k > 0$ such that

$$k \geq 5 \max_{\bar{\Omega}} \eta^0 - 6 \min_{\bar{\Omega}} \eta^0$$

and let

$$\beta^0 = \eta^0 + k, \bar{\beta} = \frac{5}{4} \max \beta^0,\ \rho^1(x) = e^{\lambda\bar{\beta}} - e^{\lambda\beta^0}$$

with $\lambda, \bar{\beta}$ sufficiently large. Let be finally

$$\gamma = \rho^1(x)/(t(T-t)); \rho(x,t) = \exp(\gamma(x,t))$$

and the space of functions

$$Z = \left\{q : Q \to \mathbb{R} : q \in C^2(\bar{Q}),\ q = 0 \text{ en } \Sigma\right\}.$$

The following Carleman inequality holds:

Proposition 4.4. ([41]) *There exist positive constants $C_*, s_1 > 0$ such that*

$$\begin{aligned} &\frac{1}{s}\int_Q \rho^{-2s} t(T-t)\left[|q_t|^2 + |\Delta q|^2\right] dxdt \\ &+s\int_Q \rho^{-2s} t^{-1}(T-t)^{-1} |\nabla q|^2 dxdt + s^3 \int_Q \rho^{-2s} t^{-3}(T-t)^{-3} q^2 dxdt \\ &\leq C_*\left[\int_Q \rho^{-2s} |\partial_t q + \Delta q|^2 dxdt + s^3 \int_0^T \int_\omega \rho^{-2s} t^{-3}(T-t)^{-3} q^2 dxdt\right] \end{aligned} \tag{4.7}$$

for all $q \in Z$ and $s \geq s_*$.

Moreover, C_* depends only on Ω and ω and s_1 is of the form

$$s_1 = s_0(\Omega, \omega)(T + T^2),$$

where $s_0(\Omega, \omega)$ only depends on Ω and ω.

From (4.7) we deduce (4.5) immediately taking into account that the first term on the right hand side of (4.7) vanishes when φ is the solution of (2.11) and making use only of the third term on the left hand side of (4.7).

From (4.5) we easily obtain an observability result for φ at time $t = 0$ which improves substantially (4.4). To do this we consider the eigenvalues and eigenfunctions of the Laplacian:

$$\left\{\begin{array}{rll} -\Delta w_k = \lambda_k w_k & \text{in} & \Omega \\ w_k = 0 & \text{on} & \partial\Omega. \end{array}\right. \tag{4.8}$$

We normalize the eigenfunctions $\{w_k\}$ such that they constitute an orthonormal basis of $L^2(\Omega)$. Then the solution φ of (2.11) may be written as

$$\varphi = \sum_{k=1}^{\infty} a_k e^{-\lambda_k(T-t)} w_k(x)$$

where $\{a_k\}$ are the coefficients of the initial datum

$$\varphi^0 = \sum_{k=1}^{\infty} a_k w_k.$$

Let us now analyze the left hand side of (4.5). We have

$$\int_0^T \int_\Omega e^{-\frac{c(1+T)}{T-t}} \varphi^2 dxdt = \sum_{k=1}^{\infty} a_k^2 \int_0^T e^{-\frac{c}{T-t}} e^{-2\lambda_k (T-t)} dt.$$

It is easy to check that

$$\int_0^T e^{-\frac{c(1+T)}{T-t}} e^{-2\lambda_k (T-t)} dt \geq e^{-c^* \sqrt{\lambda_k}} \quad \text{when} \quad \lambda_k \to \infty$$

for a suitable c^* depending on T.

In this way we obtain the following result:

Theorem 4.5. ([41]) *Let $T > 0$ and ω an open non-empty subset of Ω. There exist $C, c > 0$ such that*

$$\sum_{k=1}^{\infty} |a_k|^2 e^{-c\sqrt{\lambda_k}} \leq C \int_0^T \int_\omega \varphi^2 dxdt \tag{4.9}$$

for all solution of (2.11).

Note that the left hand side of (4.9) defines a norm of φ^0 that corresponds to the one in the domain of the operator $\exp\left(-c\sqrt{-\Delta}\right)$.

It is also worth to note that (4.9) is much stronger than (2.17) since the later provides an upper bound on the quadratic the quantity

$$\sum_{k=1}^{\infty} a_k^2 e^{-T\lambda_k}.$$

Let us now proceed to the proof of Theorem 4.1 from Theorem 4.5.

Given $u^1 \in H_0^1(\Omega)$ we introduce its Fourier series expansion:

$$u^1 = \sum_{k=1}^{\infty} b_k w_k. \tag{4.10}$$

We have

$$\| u^1 \|^2_{H_0^1(\Omega)} = \sum_{k=1}^{\infty} b_k^2 \lambda_k; \ \| u^1 \|^2_{L^2(\Omega)} = \sum_{k=1}^{\infty} b_k^2. \tag{4.11}$$

Given N we fix the following projection of u^1:

$$u_N^1 = \sum_{k=1}^{N} b_k w_k. \tag{4.12}$$

We look for a control f_N such that the solution of (2.10) satisfies exactly

$$u(T) = u_N^1. \tag{4.13}$$

This is possible since $u_N^1 \in S(T) L^2(\Omega)$.

In view of (4.12)-(4.13) we have:

$$\| u(T) - u^1 \|^2_{L^2(\Omega)} = \sum_{k=N+1}^{\infty} b_k^2 \leq \frac{1}{\lambda_{N+1}} \sum_{k=1}^{\infty} b_k^2 \lambda_k = \| u^1 \|^2_{H_0^1(\Omega)} \Big/ \lambda_{N+1}. \tag{4.14}$$

Then we choose $N = N(\varepsilon)$ such that

$$\| u^1 \|^2_{H^1_0(\Omega)} \Big/ \lambda_{N+1} \leq \varepsilon^2 . \tag{4.15}$$

In this way, we will have

$$\| u(T) - u^1 \|_{L^2(\Omega)} \leq \varepsilon \tag{4.16}$$

and

$$\mathcal{C}\left(u^1, \varepsilon\right) \leq \| f_{N(\varepsilon)} \|_{L^2(\omega\times(0,T))} . \tag{4.17}$$

Let us finally see how we may get upper bounds on $f_{N(\varepsilon)}$. It is easy to prove that the control $f_{N(\varepsilon)}$ satisfying (4.13) can be chosen in the form

$$f_{N(\varepsilon)} = \varphi \text{ in } \omega \times (0,T)$$

where φ is the solution of (2.11) with initial datum φ^0, the minimizer of the functional

$$J(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 - \int_\Omega u^1_N \varphi^0 dx$$

in the Hilbert space

$$H = \left\{ \varphi^0 : \text{ the solution } \varphi \text{ of (2.11) is such that } \int_0^T \int_\omega \varphi^2 dxdt < \infty \right\}.$$

In view of inequality (4.9) it is easy to see that the minimizer φ^0 of J exists. Moreover, the minimum satisfies

$$\begin{aligned} \| \varphi \|^2_{L^2(\omega\times(0,T))} &\leq C \sum_{k=1}^N b_k^2 e^{2c\sqrt{\lambda_k}} \\ &\leq Ce^{2c\sqrt{\lambda_N}} \| u^1 \|^2_{L^2(\Omega)} \leq Ce^{2c\sqrt{\lambda_{N+1}}} \| u^1 \|^2_{L^2(\Omega)} . \end{aligned} \tag{4.18}$$

In view of (4.15), (4.17) and (4.18) we immediately deduce the result in Theorem 4.1.

■

Let us now see that the estimates above on the cost of approximate controllability in Theorem 4.1 are optimal.

The following holds:

Theorem 4.6. ([41]) *Let $T > 0$ and ω be an open non-empty subset of Ω such that $\omega \neq \Omega$. Then, there exists a sequence $\left\{u^1_\varepsilon\right\}$ of data in $H^1_0(\Omega)$ such that*

$$\| u^1_\varepsilon \|_{H^1_0(\Omega)} = 1, \forall \varepsilon > 0 \tag{4.19}$$

and

$$\mathcal{C}\left(u^1_\varepsilon, \varepsilon\right) \geq \exp(c/\varepsilon) \text{ when } \varepsilon \to 0 \tag{4.20}$$

for a suitable positive constant $c > 0$.

Proof. It is sufficient to build a sequence of solutions $\{\varphi_\varepsilon\}$ of the adjoint problem (2.11) such that

$$\| \varphi_\varepsilon^0 \|_{L^2(\Omega)} \left[\frac{\| \varphi_\varepsilon^0 \|_{L^2(\Omega)}}{\| \varphi_\varepsilon^0 \|_{H_0^1(\Omega)}} - \varepsilon \right] \geq \exp(c/\varepsilon) \tag{4.21}$$

and

$$\int_0^T \int_\omega \varphi_\varepsilon^2 dxdt = 1, \forall \varepsilon > 0. \tag{4.22}$$

Indeed, once this sequence is built, it is sufficient to take

$$u_\varepsilon^1 = \varphi_\varepsilon^0 / \| \varphi_\varepsilon^0 \|_{H_0^1(\Omega)} . \tag{4.23}$$

Then, if f_ε is the control of (2.10) such that

$$\| u_\varepsilon(T) - u_\varepsilon^1 \|_{L^2(\Omega)} \leq \varepsilon,$$

we have

$$\int_0^T \int_\omega f_\varepsilon \varphi_\varepsilon dxdt = \int_\Omega u_\varepsilon(T) \varphi_\varepsilon^0 dx = \int_\Omega \left(u_\varepsilon(T) - u_\varepsilon^1 \right) \varphi_\varepsilon^0 dx + \int_\Omega u_\varepsilon^1 \varphi_\varepsilon^0 dx$$

and therefore, in view of (4.21) and (4.23):

$$\begin{aligned} \| f_\varepsilon \|_{L^2(\omega\times(0,T))} &\geq \int_\Omega u_\varepsilon^1 \varphi_\varepsilon^0 dx - \varepsilon \| \varphi_\varepsilon^0 \|_{L^2(\Omega)} \\ &= \| \varphi_\varepsilon^0 \|_{L^2(\Omega)} \left[\frac{\| \varphi_\varepsilon^0 \|_{L^2(\Omega)}}{\| \varphi_\varepsilon^0 \|_{H_0^1(\Omega)}} - \varepsilon \right] + \exp(c/\varepsilon). \end{aligned}$$

In order to build the sequence of solutions of (2.11) satisfying (4.21) and (4.22) we assume, without loss of generality, that $0 \notin \Omega \backslash \bar{\omega}$. Then, for a suitable $A > 0$ we have $| x | > A$ in $\bar{\omega} \cup (\mathbb{R}^n \backslash \Omega)$. We then introduce the function

$$\psi(x,t) = \cos\left(\frac{Ax_1}{2t}\right) e^{A^2/4t} G(x,t)$$

where G is the fundamental solution of the heat equation

$$G(x,t) = (4\pi t)^{-N/2} \exp\left(- | x |^2 / 4t\right).$$

It is then easy to see that ψ is a solution of the heat equation in the whole space.

We then define $\psi_\varepsilon(x,t) = \psi(x, t + \delta(\varepsilon))$ with $0 < \delta(\varepsilon) < 1$ that will be chosen later on. It is easy to check that

$$0 < C_1 \leq \int_0^T \int_\omega | \psi_\varepsilon |^2 \, dxdt < C_2, \forall \varepsilon > 0.$$

On the other hand

$$| \psi_\varepsilon | \leq C, \text{ on } \Sigma, \forall \varepsilon > 0.$$

Let then h_ε be the solution of

$$\begin{cases} \partial_t h_\varepsilon - \Delta h_\varepsilon = 0 & \text{in} & Q \\ h_\varepsilon = -\psi_\varepsilon & \text{on} & \Sigma \\ h_\varepsilon(0) = 0 & \text{in} & \Omega. \end{cases}$$

From the maximum principle we have

$$| h_\varepsilon | \leq C \text{ in } Q, \forall \varepsilon > 0.$$

Finally we set $\chi_\varepsilon = \psi_\varepsilon - h_\varepsilon$ that satisfies

$$\begin{cases} \partial_t \chi_\varepsilon - \Delta \chi_\varepsilon = 0 & \text{in} \quad Q \\ \chi_\varepsilon = 0 & \text{on} \quad \Sigma \\ \chi_\varepsilon(0) = \psi(\delta(\varepsilon)) & \text{in} \quad \Omega. \end{cases}$$

We normalize the solution χ_ε such that

$$\int_0^T \int_\omega \chi_\varepsilon^2 dxdt = 1$$

and we make the change of variables $t \to T - t$. We then obtain a sequence of solutions φ_ε of (2.11) satisfying (4.22). On the other hand, an explicit computation shows that

$$\| \varphi_\varepsilon^0 \|_{L^2(\Omega)} / \| \varphi_\varepsilon^0 \|_{H_0^1(\Omega)} \sim \| \psi(\delta(\varepsilon)) \|_{L^2(\Omega)} / \| \psi(\delta(\varepsilon)) \|_{H_0^1(\Omega)} \sim c\delta(\varepsilon)$$

with $c > 0$, while

$$\| \varphi_\varepsilon^0 \|_{L^2(\Omega)} \sim \| \psi(\delta(\varepsilon)) \|_{L^2(\Omega)} \sim C\delta(\varepsilon)^{n/2} e^{A^2/2\delta(\varepsilon)}.$$

In this way we obtain (4.21).

4.2. **Convergence rates in the penalization procedure.** It is rather natural to build approximate controls by penalizing a suitable optimal control problem. This has been done systematically for instance for numerical simulations in the works by Glowinski [49] and Glowinski et al. [50]. This method has also been used to prove the approximate controllability for some linear and semilinear heat equations in [91] and [39] respectively.

Let us briefly describe this procedure in the example under consideration. First of all, without loss of generality, we set $u^0 \equiv 0$. Given $u^1 \in L^2(\Omega)$ we introduce the functional

$$J_k(f) = \frac{1}{2} \int_0^T \int_\omega f^2 dxdt + \frac{k}{2} \| u(T) - u^1 \|_{L^2(\Omega)}^2 \tag{4.24}$$

which is well defined in $L^2(\omega \times (0,T))$ for all $k > 0$, where u is the solution of (2.10) with $u^0 \equiv 0$.

It was proved in [91] that J_k has a unique minimizer $f_k \in L^2(\omega \times (0,T))$ for all $k > 0$ and that this sequence of controls is such that

$$u_k(T) \to u^1 \text{ in } L^2(\Omega),\ k \to \infty. \tag{4.25}$$

In view of (4.25), to compute the control f satisfying (4.2) it is sufficient to take $f = f_k$ where $k = k(\varepsilon)$ is sufficiently large.

Using the results above it is easy to get explicit estimates of the rate of convergence in (4.25) (we refer to [41] for the details of the proof):

Theorem 4.7. ([41]) *Given $T > 0$, ω an open non-empty subset of Ω and $u^1 \in H_0^1(\Omega)$, there exists $C > 0$ such that*

$$\| u_k(T) - u^1 \| \leq C/\log k \tag{4.26}$$

and

$$\| f_k \|_{L^2(\omega\times(0,T))} \leq C\sqrt{k}/\log k \tag{4.27}$$

when $k \to \infty$.

Note that (4.26)-(4.27) provide logarithmic (and therefore very slow) convergence rates. This fact agrees with the extremely high cost (exponentially depending on $1/\varepsilon$) that approximate controllability requires.

4.3. **Unbounded domains.** It is well known that one of the most relevant features of the heat equation is the infinite speed of propagation. This property has also important consequences on the controllability of heat like equations. Indeed, due to infinite speed of propagation, the heat equation is approximately controllable also in unbounded domains, in an arbitrarily small time and with controls supported in any non empty open subset of the domain or of the boundary. This result can be also extended to the semilinear setting (see [137]).

The situation is completely different in the context of null controllability. Indeed, as it was shown in [110], the heat equation in the half line is not controllable to zero by means of boundary controls. Even more, it was proved that none C^∞ and compactly supported initial data may be driven to zero in any time $T > 0$. This result was later extended to several space dimensions in [111]. According to these results, roughly speaking, in order to obtain the null controllability property for the heat equation in an unbounded domain one may only leave a bounded domain without control. In this geometrical setting the null controllability property was proved in [14].

5. **The semilinear heat equation.** In section 4 we have analyzed in some detail the approximate and null controllability of the constant coefficient linear heat equation and the corresponding observability estimates.

In this section, following [157], we discuss the approximate controllability of the semilinear heat equation with a globally Lipschitz nonlinearity. We shall apply the fixed point method introduced in [147] and later adapted to the heat equation in [34] and [149].

Consider the semilinear heat equation

$$\begin{cases} u_t - \Delta u + f(u, \nabla u) = v 1_\omega & \text{in} \quad \Omega \times (0,T) \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u^0(x) & \text{in} \quad \Omega. \end{cases} \tag{5.1}$$

The function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is assumed to be *globally Lipschitz* all along the paper, i.e.

$$\begin{aligned} \exists L > 0 : \mid f(y,\xi) - f(z,\eta) \mid \leq L \left[\mid y - z \mid + \mid \xi - \eta \mid \right], \\ \forall y,\, z \in \mathbb{R}; \xi, \eta \in \mathbb{R}^n. \end{aligned} \tag{5.2}$$

We observe that, for any $y \in L^2(0,T;\, H_0^1(\Omega))$ the following identity holds:

$$\begin{aligned} f(y, \nabla y) - f(0,0) &= \int_0^1 \frac{d}{d\sigma}(f(\sigma y, \sigma \nabla y)) d\sigma \qquad (5.3) \\ &= \int_0^1 \frac{\partial f}{\partial y}(\sigma y, \sigma \nabla y) d\sigma y + \int_0^1 \frac{\partial f}{\partial \eta}(\sigma y, \sigma \nabla y) d\sigma \cdot \nabla y. \end{aligned}$$

In (5.3) $\partial f/\partial y$ and $\partial f/\partial \eta$ denote respectively the partial derivatives of f with respect to the variables y and ∇y.

We set

$$F(y) = \int_0^1 \frac{\partial f}{\partial y}(\sigma y, \sigma \nabla y) d\sigma; \quad G(y) = \int_0^1 \frac{\partial f}{\partial \eta}(\sigma y, \sigma \nabla y) d\sigma. \tag{5.4}$$

In view of the globally Lipschitz assumption (5.2) on f, F and G map $L^2\big(0,T; H_0^1(\Omega)\big)$ into a bounded set of $L^\infty(\Omega \times (0,T))$. Moreover,

$$\| F(y) \|_{L^\infty(\Omega\times(0,T))} \leq L,\ \forall y \in L^2\left(0,T; H_0^1(\Omega)\right), \tag{5.5}$$

$$\| G(y) \|_{(L^\infty(\Omega\times(0,T)))^n} \leq L,\ \forall y \in L^2\left(0,T; H_0^1(\Omega)\right), \tag{5.6}$$

L being the Lipschitz constant of f.

Using these notations system (5.1) can be rewritten as follows

$$\begin{cases} u_t - \Delta u + F(u)u + G(u)\cdot\nabla u + f(0,0) = v1_\omega & \text{in } \Omega\times(0,T) \\ u = 0 & \text{on } \partial\Omega\times(0,T) \\ u(x,0) = u^0(x) & \text{in } \Omega. \end{cases} \tag{5.7}$$

Given $y\in L^2\left(0,T;H_0^1(\Omega)\right)$ we now consider the "linearized" system

$$\begin{cases} u_t - \Delta u + F(y)u + G(y)\cdot\nabla u + f(0,0) = v1_\omega & \text{in } \Omega\times(0,T) \\ u = 0 & \text{on } \partial\Omega\times(0,T) \\ u(x,0) = u^0(x) & \text{in } \Omega. \end{cases} \tag{5.8}$$

Observe that (5.8) is a linear system on the state u with potentials

$$a = F(y)\in L^\infty(\Omega\times(0,T)) \quad \text{and } b = G(y)\in\left(L^\infty(\Omega\times(0,T))\right)^n$$

satisfying the following uniform bound

$$\| a \|_{L^\infty(\Omega\times(0,T))}\le L,\ \| b \|_{(L^\infty(\Omega\times(0,T)))^n}\le L. \tag{5.9}$$

With this notation system (5.8) may be rewritten in the form

$$\begin{cases} u_t - \Delta u + au + b\cdot\nabla u + f(0,0) = v1_\omega & \text{in} \quad \Omega\times(0,T) \\ u = 0 & \text{on} \quad \partial\Omega\times(0,T) \\ u(x,0) = u^0(x) & \text{in} \quad \Omega. \end{cases} \tag{5.10}$$

We now fix the initial datum $u^0\in L^2(\Omega)$, the target $u^1\in L^2(\Omega)$, $\varepsilon>0$ and the finite-dimensional subspace E of $L^2(\Omega)$.

Using the variational approach to approximate controllability introduced by Lions in [89], further developed in [34] and adapted to the problem of finite-approximate controllability in [149], we build a control v for the linear system (5.8) such that

$$\begin{cases} \| u(T) - u^1 \|_{L^2(\Omega)}\le\varepsilon, \\ \pi_E(u(T)) = \pi_E(u^1). \end{cases} \tag{5.11}$$

Thus, for any $y\in L^2\left(0,T;\,H_0^1(\Omega)\right)$ we define a control $v=v(x,t;\,y)\in L^2(\omega\times(0,T))$ such that the solution $u=u(x,t;\,y)\in C\left([0,T];\,L^2(\Omega)\right)\cap L^2\left(0,T;\,H_0^1(\Omega)\right)$ of (5.10) satisfies (5.11). This allows to build a non-linear mapping

$$\mathcal{N}: L^2\left(0,T;\,H_0^1(\Omega)\right)\to L^2\left(0,T;\,H_0^1(\Omega)\right),\ \mathcal{N}(y)=u. \tag{5.12}$$

We claim that the problem is then reduced to finding a fixed point of $\mathcal{N}$. Indeed, if $y\in L^2\left(0,T;\,H_0^1(\Omega)\right)$ is such that $\mathcal{N}(y)=u=y$, the solution u of (5.10) is actually solution of (5.7). Then, the control $v=v(y)$ is the one we were looking for since, by construction, $u=u(y)$ satisfies (5.11).

As we shall see, the nonlinear map $\mathcal{N}:L^2\left(0,T;\,H_0^1(\Omega)\right)\to L^2\left(0,T;\,H_0^1(\Omega)\right)$ satisfies the following two properties:

$$\mathcal{N} \text{ is continuous and compact;} \tag{5.13}$$

$$\begin{cases} \text{the range of } \mathcal{N} \text{ is bounded, i.e. } \exists R>0: \\ \| \mathcal{N}(y) \|_{L^2\left(0,T;\,H_0^1(\Omega)\right)}\le R,\ \forall y\in L^2\left(0,T;\,H_0^1(\Omega)\right). \end{cases} \tag{5.14}$$

In view of these two properties and as a consequence of Schauder's fixed point Theorem, the existence of a fixed point of $\mathcal{N}$ follows immediately.

The uniform bound (5.14) on the range of $\mathcal{N}$ is a consequence of the uniform bound (5.9) on the potentials a and b which, in turn, is a consequence of the globally Lipschitz assumption (5.2).

Roughly speaking, the control problem for the semilinear equation (5.1) through this fixed point method, is reduced to the obtention of a uniform controllability result for the family of linear control problems (5.10) under the constraint (5.9). At this level the unique continuation result of C. Fabre [33] for equations of the form

$$\varphi_t - \Delta\varphi + a\varphi + \mathrm{div}(b\varphi) = 0$$

with L^∞-coefficients a and b plays a crucial role.

■

As a consequence of these developments the following result is proved:

Theorem 5.1. ([39, 157]) *Assume that f satisfies* (5.2). *Then, for all $T > 0$, system* (5.1) *is finite-approximately controllable.*

More precisely, for any finite-dimensional subspace E of $L^2(\Omega)$, u^0, $u^1 \in L^2(\Omega)$ and $\varepsilon > 0$ there exists a control $v \in L^2(\omega \times (0,T))$ such that the solution u of (5.1) *satisfies* (5.11).

Remark 5.2. This result was proved in [39] by means of a suitable penalization of an optimal control problem. The proof based on the fixed point technique we present here was given in [157].

■

The rest of this section is devoted to give a brief sketch of the proof of this result. We proceed in several steps.

Step 1. Let us first analyze in more detail the controllability of the linearized systems.

Given L^∞-potentials $a \in L^\infty(\Omega \times (0,T))$, $b \in (L^\infty(\Omega \times (0,T)))^n$ and a real constant $\lambda \in \mathbb{R}$ we consider the control problem

$$\begin{cases} u_t - \Delta u + au + b \cdot \nabla u + \lambda = v 1_\omega & \text{in} \quad \Omega \times (0,T) \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u(0) = u^0 & \text{in} \quad \Omega. \end{cases} \tag{5.15}$$

Let E be a finite-dimensional subspace of $L^2(\Omega)$. Given $u^0 \in L^2(\Omega)$, $u^1 \in L^2(\Omega)$ and $\varepsilon > 0$ we look for a control $v \in L^2(\omega \times (0,T))$ such that the solution u of (5.15) satisfies (5.11).

The following holds:

Proposition 5.3. *Let $T > 0$. Then, there exists a control $v \in L^2(\omega \times (0,T))$ such that the solution $u \in C\left([0,T];\, L^2(\Omega)\right) \cap L^2\left(0,T;\, H_0^1(\Omega)\right)$ of* (5.15) *satisfies* (5.11).

Moreover, for any $R > 0$ there exists a constant $C(R) > 0$ such that

$$\| v \|_{L^2(\omega\times(0,T))} \leq C(R) \tag{5.16}$$

for any $a \in L^\infty(\Omega \times (0,T))$, $b \in (L^\infty(\Omega \times (0,T)))^n$ satisfying

$$\| a \|_{L^\infty(\Omega\times(0,T))} \leq R, \; \| b \|_{(L^\infty(\Omega\times(0,T)))^n} \leq R. \tag{5.17}$$

Remark 5.4. Proposition 5.3 does not provide any estimate on how the norm of the control v depends on E, u^0, u^1 and $\varepsilon > 0$. However (5.16) guarantees that v remains uniformly bounded when the potentials a, b remain bounded in L^∞.

The control v is not unique. The construction we develop below provides the control of minimal L^2-norm. It is this control of minimal norm which satisfies the uniform boundedness condition (5.16).

■

Proof of Proposition 5.3. Without loss of generality we may assume that $\lambda = 0$ and $u^0 \equiv 0$.

Consider the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi + a\varphi - \operatorname{div}(b\varphi) = 0 & \text{in} \quad \Omega \times (0,T) \\ \varphi = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ \varphi(T) = \varphi^0 & \text{in} \quad \Omega. \end{cases} \tag{5.18}$$

Taking into account that the potentials a and b are bounded it is easy to see that for any $\varphi^0 \in L^2(\Omega)$ system (5.18) has a unique solution in the class $\varphi \in C\left([0,T];\, L^2(\Omega)\right) \cap L^2\left(0,T;\, H_0^1(\Omega)\right)$.

We now consider the functional $J : L^2(\Omega) \to \mathbb{R}$ defined as follows:

$$J(\varphi^0) = \frac{1}{2}\int_0^T\int_\omega \varphi^2 dxdt + \varepsilon \parallel (I - \pi_E)\varphi^0 \parallel_{L^2(\Omega)} - \int_\Omega u^1\varphi^0 dx. \tag{5.19}$$

It is easy to see that

$$J : L^2(\Omega) \to \mathbb{R} \text{ is continuous;} \tag{5.20}$$

$$J : L^2(\Omega) \to \mathbb{R} \text{ is convex.} \tag{5.21}$$

Moreover

$$J : L^2(\Omega) \to \mathbb{R} \text{ is strictly convex .} \tag{5.22}$$

This property is a consequence of the following unique continuation result due to Fabre [33]:

Proposition 5.5. ([33]). *Assume that $a \in L^\infty(\Omega\times(0,T))$ and $b \in (L^\infty(\Omega\times(0,T)))^n$. Let $\varphi^0 \in L^2(\Omega)$ be such that the solution φ of (5.18) satisfies*

$$\varphi = 0 \text{ in } \omega \times (0,T). \tag{5.23}$$

Then, necessarily, $\varphi^0 \equiv 0$.

Using this unique continuation result it can be shown that the functional $J : L^2(\Omega) \to \mathbb{R}$ is also coercive. More precisely, the following holds:

Proposition 5.6. *Under the assumptions above*

$$\liminf_{\|\varphi^0\|_{L^2(\Omega)}\to\infty} \frac{J(\varphi^0)}{\parallel \varphi^0 \parallel_{L^2(\Omega)}} \geq \varepsilon. \tag{5.24}$$

Proof of Proposition 5.6. The proof of this Proposition follows the argument in [34] and [145] combined with the unique-continuation result of Proposition 5.5. Let us recall it for the sake of completeness.

Let $\{\varphi_j^0\}$ be a sequence in $L^2(\Omega)$ such that

$$\left\|\varphi_j^0\right\|_{L^2(\Omega)} \to \infty \text{ as } j \to \infty. \tag{5.25}$$

We denote by $\{\varphi_j\}$ the corresponding sequence of solutions of (5.18).

We also set

$$\widehat{\varphi}_j^0 = \varphi_j^0 / \parallel \varphi_j^0 \parallel_{L^2(\Omega)},\ \widehat{\varphi}_j = \varphi_j / \parallel \varphi_j^0 \parallel_{L^2(\Omega)}\ . \tag{5.26}$$

Obviously $\widehat{\varphi}_j$ is the solution of (5.18) with the normalized initial data $\widehat{\varphi}_j^0$.

We have

$$\begin{aligned} \frac{J\left(\varphi_j^0\right)}{\parallel \varphi_j^0 \parallel_{L^2(\Omega)}} &= \frac{\parallel \varphi_j^0 \parallel_{L^2(\Omega)}}{2}\int_0^T\int_\omega \left|\widehat{\varphi}_j\right|^2 dxdt \\ &+ \varepsilon\left\|(I-\pi_E)\widehat{\varphi}_j^0\right\|_{L^2(\Omega)} - \int_\Omega u^1\widehat{\varphi}_j^0 dx. \end{aligned} \tag{5.27}$$

We distinguish the following two cases:

Case 1. $\liminf_{j\to\infty} \int_0^T \int_\omega |\widehat{\varphi}_j|^2 \, dxdt > 0$;

Case 2. $\liminf_{j\to\infty} \int_0^T \int_\omega |\widehat{\varphi}_j|^2 \, dxdt = 0$.

In the first case, due to (5.25), the first term in (5.27) tends to $+\infty$ while the other two remain bounded. We deduce that

$$\liminf_{j\to\infty} \frac{J\left(\varphi_j^0\right)}{\left\|\varphi_j^0\right\|_{L^2(\Omega)}} = +\infty$$

in this case.

Let us now analyze the second case. Let us consider a subsequence (still denoted by the index j to simplify the notation) such that

$$\int_0^T \int_\omega |\widehat{\varphi}_j|^2 \, dxdt \to 0, \text{ as } j \to \infty. \tag{5.28}$$

By extracting subsequences we may deduce that

$$\widehat{\varphi}_j^0 \rightharpoonup \widehat{\varphi}^0 \text{ weakly in } L^2(\Omega). \tag{5.29}$$

Consequently

$$\widehat{\varphi}_j \rightharpoonup \widehat{\varphi} \text{ weakly in } L^2\left(0, T; H_0^1(\Omega)\right) \tag{5.30}$$

where $\widehat{\varphi}$ is the solution of (5.18) with datum $\widehat{\varphi}^0$. According to (5.28) we deduce that

$$\widehat{\varphi} \equiv 0 \text{ in } \omega \times (0, T)$$

and, as a consequence of Proposition 5.5, that $\widehat{\varphi}^0 \equiv 0$. Therefore, if (5.28) holds, necessarily

$$\widehat{\varphi}_j^0 \rightharpoonup 0 \text{ weakly in } L^2(\Omega). \tag{5.31}$$

But then, E being finite-dimensional, $\pi_E \widehat{\varphi}_j^0 \to 0$ in $L^2(\Omega)$ and therefore

$$\left\|(I - \pi_E)\widehat{\varphi}_j^0\right\|_{L^2(\Omega)} \to 1 \tag{5.32}$$

since $\left\|\widehat{\varphi}_j^0\right\|_{L^2(\Omega)} = 1$ for all j.

As a consequence of (5.31) and (5.32) we deduce that

$$\liminf_{j\to\infty} \frac{J\left(\varphi_j^0\right)}{\left\|\varphi_j^0\right\|_{L^2(\Omega)}} \geq \liminf_{j\to\infty} \left[\varepsilon \left\|(I - \pi_E)\widehat{\varphi}_j^0\right\|_{L^2(\Omega)} - \int_\Omega u^1 \widehat{\varphi}_j^0 dx\right] = \varepsilon.$$

This concludes the proof of Proposition 5.6.

■

In view of the properties (5.20), (5.21) and (5.24) of the functional J we deduce that J achieves its minimum at a unique $\widehat{\varphi}^0 \in L^2(\Omega)$, i.e.

$$\begin{cases} J\left(\widehat{\varphi}^0\right) = \min\limits_{\varphi^0 \in L^2(\Omega)} J\left(\varphi^0\right) \\ J\left(\widehat{\varphi}^0\right) < J\left(\varphi^0\right), \, \forall \varphi^0 \in L^2(\Omega), \, \varphi^0 \neq \widehat{\varphi}^0. \end{cases} \tag{5.33}$$

It is easy to see that the control

$$v = \widehat{\varphi} \text{ in } \omega \times (0, T), \tag{5.34}$$

$\widehat{\varphi}$ being the solution of (5.18) with the minimizer $\widehat{\varphi}^0$ as datum is such that the solution u of

$$\begin{cases} u_t - \Delta u + au + b \cdot \nabla u = v 1_\omega & \text{in} \quad \Omega \times (0,T) \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u(0) = 0 & \text{in} \quad \Omega \end{cases} \tag{5.35}$$

satisfies (5.11) (see [157] for the details of the proof).

This concludes the proof of the finite-approximate controllability of the linear equation (5.15).

In order to prove the uniform bound (5.16), we first observe that the problem may be reduced to the case $u^0 \equiv 0$ and $\lambda = 0$, provided u^1 is allowed to vary in a relatively compact set of $L^2(\Omega)$.

The following holds:

Proposition 5.7. *Let $R > 0$ and K be a relatively compact set of $L^2(\Omega)$. Then, the coercivity property* (5.24) *holds uniformly on $u^1 \in K$ and potentials a and b satisfying* (5.17).

Remark 5.8. Note that the functional J depends on the potentials a and b and the target u^1. Proposition 5.7 guarantees the uniform coercivity of these functionals when $u^1 \in K$, K being a compact set of $L^2(\Omega)$ and the potentials a and b are uniformly bounded.

■

As a consequence of Proposition 5.7 we deduce that the minimizers $\widehat{\varphi}^0$ of the functionals J are uniformly bounded when $u^1 \in K$ and the potentials a and b are uniformly bounded. Consequently, the controls $v = \widehat{\varphi}$ are uniformly bounded as well.

The proof of Proposition 5.7 is similar to the one we have given for Proposition 5.5.

This completes the proof of Proposition 5.3.

■

Step 2. As indicated above, in order to conclude the existence of a fixed point of $\mathcal{N}$ by means of Schauder's fixed point method it is sufficient to check the following three facts:

$$\mathcal{N} : L^2\left(0,T;\, H_0^1(\Omega)\right) \to L^2\left(0,T;\, H_0^1(\Omega)\right) \text{ is continuous;} \tag{5.36}$$

$$\mathcal{N} : L^2\left(0,T;\, H_0^1(\Omega)\right) \to L^2\left(0,T;\, H_0^1(\Omega)\right) \text{ is compact;} \tag{5.37}$$

$$\exists R > 0 : \| \mathcal{N}(y) \|_{L^2\left(0,T;\, H_0^1(\Omega)\right)} \leq R,\ \forall y \in L^2\left(0,T;\, H_0^1(\Omega)\right). \tag{5.38}$$

Let us prove these three properties.

Continuity of $\mathcal{N}$. Assume that $y_j \to y$ in $L^2\left(0,T;\, H_0^1(\Omega)\right)$. Then the potentials $F(y_j)$, $G(y_j)$ are such that

$$F(y_j) \to F(y) \text{ in } L^p(\Omega \times (0,T)) \tag{5.39}$$

$$G(y_j) \to G(y) \text{ in } \left(L^p(\Omega \times (0,T))\right)^n \tag{5.40}$$

for all $1 \leq p < \infty$ and

$$\| F(y_j) \|_{L^\infty(\Omega\times(0,T))} \leq L;\ \| G(y_j) \|_{(L^\infty(\Omega\times(0,T)))^n} \leq L. \tag{5.41}$$

According to Proposition 5.3 the corresponding controls are uniformly bounded:

$$\| v_j \|_{L^2(\omega\times(0,T))} \leq C,\ \forall j \geq 1 \tag{5.42}$$

and, more precisely,

$$v_j = \widehat{\varphi}_j \text{ in } \omega \times (0,T) \tag{5.43}$$

where $\widehat{\varphi}_j$ solves

$$\begin{cases} -\varphi_t - \Delta\varphi + F(y_j)\varphi - \operatorname{div}(G(y_j)\varphi) = 0 & \text{in} \quad \Omega \times (0,T) \\ \varphi = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ \varphi(T) = \widehat{\varphi}_j^0 & \text{in} \quad \Omega \end{cases} \tag{5.44}$$

with the datum $\widehat{\varphi}_j^0$ minimizing the corresponding functional J_j. We also have

$$\left\|\widehat{\varphi}_j^0\right\|_{L^2(\Omega)} \leq C. \tag{5.45}$$

By extracting subsequences we have

$$\widehat{\varphi}_j^0 \rightharpoonup \widehat{\varphi}^0 \text{ weakly in } L^2(\Omega) \tag{5.46}$$

and, in view of (5.39)-(5.40), arguing as in the proof of Proposition 5.7, we deduce that

$$\widehat{\varphi}_j \rightharpoonup \widehat{\varphi} \text{ weakly in } L^2\left(0,T;\, H_0^1(\Omega)\right) \tag{5.47}$$

where $\widehat{\varphi}$ solves

$$\begin{cases} -\varphi_t - \Delta\varphi + F(y)\varphi - \operatorname{div}(G(y)\varphi) = 0 & \text{in} \quad \Omega \times (0,T) \\ \varphi = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ \varphi(T) = \widehat{\varphi}^0 & \text{in} \quad \Omega. \end{cases} \tag{5.48}$$

We also have that

$$\partial_t \widehat{\varphi}_j \text{ is bounded in } L^2\left(0,T;\, H^{-1}(\Omega)\right), \tag{5.49}$$

and, once again, by Aubin-Lions compactness Lemma, it follows that

$$\widehat{\varphi}_j \to \widehat{\varphi} \text{ strongly in } L^2(\Omega \times (0,T)). \tag{5.50}$$

Consequently

$$v_j \to v \text{ in } L^2(\omega \times (0,T)) \tag{5.51}$$

where

$$v = \widehat{\varphi} \text{ in } \omega \times (0,T). \tag{5.52}$$

It is then easy to see that

$$u_j \to u \text{ in } L^2\left(0,T;\, H_0^1(\Omega)\right) \tag{5.53}$$

where

$$\begin{cases} u_t - \Delta u + F(y)u + G(y)\cdot\nabla u + f(0,0) = v1_\omega & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(0) = u^0 & \text{in } \Omega \end{cases} \tag{5.54}$$

and (5.11) holds.

To conclude the continuity of $\mathcal{N}$ it is sufficient to check that the limit $\widehat{\varphi}^0$ in (5.46) is the minimizer of the functional J associated to the limit control problem (5.54), (5.11).

To do this, given $\psi^0 \in L^2(\Omega)$ we have to show that $J\left(\widehat{\varphi}^0\right) \leq J\left(\psi^0\right)$. But this is immediate since, by lower semicontinuity, we have

$$J\left(\widehat{\varphi}^0\right) \leq \liminf_{j\to\infty} J_j\left(\widehat{\varphi}_j^0\right),$$

on one hand,

$$J\left(\psi^0\right) = \liminf_{j\to\infty} J_j\left(\psi^0\right)$$

on the other one, and finally

$$J_j\left(\widehat{\varphi}_j^0\right) \le J_j\left(\psi^0\right)$$

since $\widehat{\varphi}_j^0$ is the minimizer of J_j.

Compactness of $\mathcal{N}$. The arguments above show that when y lies in a bounded set B of $L^2\left(0,T;H_0^1(\Omega)\right)$, $u=\mathcal{N}(y)$ also lies in a bounded set of $L^2\left(0,T;\, H_0^1(\Omega)\right)$. We have to show that $\mathcal{N}(B)$ is relatively compact in $L^2\left(0,T;\, H_0^1(\Omega)\right)$. But this can be obtained easily by means of the regularizing effect of the heat equation.

Indeed, we have

$$\begin{cases} u_t-\Delta u=h & \text{in} \quad \Omega\times(0,T)\\ u=0 & \text{on} \quad \partial\Omega\times(0,T)\\ u(0)=u^0 & \text{in} \quad \Omega,\end{cases}$$

with

$$h=v1_\omega-F(y)u-G(y)\cdot\nabla u-f(0,0)$$

which is uniformly bounded in $L^2(\Omega\times(0,T))$.

Then, u can be decomposed as

$$u=p+q$$

where

$$\begin{cases} p_t-\Delta p=0 & \text{in} \quad \Omega\times(0,T)\\ p=0 & \text{on} \quad \partial\Omega\times(0,T)\\ p(0)=u^0 & \text{in} \quad \Omega\end{cases}$$

and

$$\begin{cases} q_t-\Delta q=h & \text{in} \quad \Omega\times(0,T)\\ q=0 & \text{on} \quad \partial\Omega\times(0,T)\\ q(0)=0 & \text{in} \quad \Omega.\end{cases}$$

Obviously, p is a fixed element of $L^2\left(0,T;\, H_0^1(\Omega)\right)$. On the other hand, by classical regularity results on the heat equation we deduce that q lies in a bounded set of $L^2\left(0,T;\, H^2(\Omega)\right)\cap H^1\left(0,T;\, L^2(\Omega)\right)$, which, as a consequence of Aubin-Lions compactness Lemma, is a relatively compact set of $L^2\left(0,T;\, H_0^1(\Omega)\right)$.

This completes the proof of the compactness of $\mathcal{N}$.

Boundedness of the range of $\mathcal{N}$. Proposition 5.3 shows that there exists $C>0$ such that the control $v=v(y)$ satisfies

$$\| v(y) \|_{L^2(\omega\times(0,T))}\le C.$$

Classical energy estimates for system (5.54) show that

$$\| u(y) \|_{L^2\left(0,T;\, H_0^1(\Omega)\right)}\le C$$

as well, since the potentials involved in it are uniformly bounded.

This concludes the proof of Theorem 5.1.

■

6. **Space semi-discretizations.**

6.1. **Galerkin approximations of the Navier-Stokes equations.** Let Ω be a bounded open set of $\mathbb{R}^2$ or $\mathbb{R}^3$. Let ω be an open non-empty subset of Ω. Given $T>0$ we consider the Navier-Stokes equations

$$\begin{cases} y_t - \mu\Delta y + y\cdot\nabla y = -\nabla p + v\chi_\omega & \text{in} \quad Q_T, \\ y = 0 & \text{on} \quad \Sigma_T, \\ \operatorname{div} y = 0 & \text{in} \quad Q_T, \\ y(0) = y^0 & \text{in} \quad \Omega. \end{cases} \tag{6.1}$$

In (6.1) $y=y(x,t)$ is the velocity field (the state), $p=p(x,t)$ is the pressure, $v=v(x,t)$ the control and χ_ω denotes the characteristic function of the set ω. Thus the control acts on the system through the subset ω.

We denote by V the Hilbert space

$$V = \left\{\varphi \in \left(H_0^1(\Omega)\right)^3 : \operatorname{div}\varphi = 0 \text{ in } \Omega\right\}$$

endowed with the norm induced by $\left(H_0^1(\Omega)\right)^3$.

Let E be a finite-dimensional subspace of V: $E=\operatorname{span}[e_1,\cdots,e_N]$.

The Galerkin approximation of system (6.1) is as follows:

$$\begin{cases} y \in C([0,T];E) \\ y(0) = \pi_E(y^0) \\ (y_t,e) + \mu(\nabla y,\nabla e) + (y\cdot\nabla y, e) = (v\chi_\omega, e)\,,\ \forall e \in E. \end{cases} \tag{6.2}$$

In (6.2), π_E denotes the orthogonal projection from V over E and $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$.

System (6.2) is a set of N ordinary differential equations which are non-linear. Global existence of solutions is insured by the fact that $(e\cdot\nabla e, e)=0$ for all $e\in E$.

The following holds:

Theorem 6.1. ([96]-[98]) *Assume that*

$$\textit{the dimension of span } \{e_j\,|_\omega\}_{j=1,\cdots,N} \textit{ equals } N. \tag{6.3}$$

Then, for any $T>0$ system (6.2) *is exactly controllable. More precisely, for any $y^0\in E$ and $y^1\in E$ there exists a control $v\in L^2(\omega\times(0,T))$ such that the solution y of* (6.2) *satisfies*

$$y(T) = y^1. \tag{6.4}$$

Assumption (6.3) guarantees that N linearly independent controls act on the $N-$dimensional system (6.2). This is a natural sufficient condition for controllability, but very possibly it is not a necessary condition.

The existence of Galerkin basis satisfying (6.3) is proved in [95]. In fact, in [95] it is proved that condition (6.3) is fulfilled generically among the set of Riesz basis of V.

The proof of Theorem 6.1 uses the fixed point argument of section 5. Given $z\in C([0,T];E)$ we analyze the exact controllability of the linearized system

$$\begin{cases} y \in C([0,T];E) \\ y(0) = \pi_E(y^0) \\ (y_t,e) + \mu(\nabla y,\nabla e) + (z\cdot\nabla y, e) = (v\chi_\omega, e)\,,\ \forall e \in E. \end{cases} \tag{6.5}$$

Note that in system (6.5) the non-linear term $(y\cdot\nabla y,e)$ of (6.2) has been replaced by the linear one $(z\cdot\nabla y,e)$.

We prove the exact controllability of system (6.5) using the Hilbert Uniqueness Method (HUM) (see J.-L. Lions [88]). At this level the assumption (6.3) plays a crucial role. We then obtain bounds on the control v and the state y which are independent of z and finally we apply Schauder's fixed point Theorem as in section 5 to the map $z \mapsto y$ from $C([0,T];E)$ into itself. To do this we use the cancellation property of the non-linearity of the Navier-Stokes equations, namely, the fact that $(z \cdot \nabla e, e) = 0$. The fixed point y of this map solves the non-linear system (6.2) and, by construction, satisfies the control constraint at time $t = T$.

We refer to [96] for the details of the proof. We also refer to [97, 98] for the case where the control acts on the boundary of Ω.

There are by now a number of significant results on the null-controllability of the Navier-Stokes equations (see for instance [28] and [29]). The problem of passing to the limit in the controls obtained in Theorem 6.1 as the dimension N of the finite-dimensional system tends to infinity is open.

6.2. **Finite-difference space discretizations of the $1-d$ wave equation.** Let us consider the $1-d$ wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0,\ 0 < x < L,\ t > 0 \\ u(0,t) = u(L,t) = 0,\ t > 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ 0 < x < L. \end{cases} \tag{6.6}$$

The energy

$$E(t) = \frac{1}{2}\int_0^L \left[|u_t(x,t)|^2 + |u_x(x,t)|^2 \right] dx \tag{6.7}$$

remains constant in time.

Let us consider now the wave equation with a control acting on the extreme $x = L$ of the boundary

$$\begin{cases} y_{tt} - y_{xx} = 0,\ 0 < x < L,\ t > 0 \\ y(0,t) = 0,\ y(L,t) = v(t),\ t > 0 \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x),\ 0 < x < L. \end{cases} \tag{6.8}$$

It is by now well known that the wave equation (6.8) is exactly controllable. More precisely, the following holds: *If $T \geq 2L$, for any $(y_0, y_1) \in L^2(0,L) \times H^{-1}(0,L)$ there exists a control $v \in L^2(0,T)$ such that the solution of* (6.8) *satisfies*

$$y(x,T) \equiv y_t(x,T) \equiv 0. \tag{6.9}$$

This exact controllability result is equivalent to the following boundary observability property of the adjoint system: *For any $T \geq 2L$ there exists a positive constant $C(T) > 0$ such that*

$$E(0) \leq C(T) \int_0^T |u_x(L,t)|^2 \, dt \tag{6.10}$$

holds for every solution of (6.6).

In this section we report on the work of [62, 63] in which we analyze the semi-discrete version of (6.10).

Let us take $N \in \mathbb{N}$ and set $h = L/(N+1)$. We consider the following finite-difference space semi-discretization of (6.6):

$$\begin{cases} u_j'' = \left[u_{j+1} + u_{j-1} - 2u_j\right] / h^2,\ t > 0,\ j = 1, \cdots, N \\ u_0 = u_{N+1} = 0,\ t > 0 \\ u_j(0) = u_{0,j},\ u_j'(0) = u_{1,j},\ j = 1, \cdots, N. \end{cases} \tag{6.11}$$

The energy of system (6.11) is given by

$$E_h(t) = \frac{h}{2}\sum_{j=1}^{N}\left|u_j'(t)\right|^2 + \frac{h}{2}\sum_{j=0}^{N}\frac{\left|u_{j+1}(t)-u_j(t)\right|^2}{h^2} \tag{6.12}$$

and it is also conserved in time.

We analyze the following semi-discrete version of (6.10):

$$E_h(t) \le C\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt. \tag{6.13}$$

More precisely, we are interested on the existence of a positive constant $C > 0$ such that (6.13) holds. Moreover, we want to analyze whether (6.13) holds with a constant C which is independent of h so that the observability inequality (6.10) for the continuous wave equation (6.6) might be viewed as the limit as $h \to 0$ of observability inequalities of the form (6.13) for the semi-discrete systems (6.11).

Let us first analyze the spectrum of system (6.11). The corresponding eigenvalue problem is of the form:

$$\begin{cases} -\left[\omega_{j+1}+\omega_{j-1}-2\omega_j\right]/\,h^2 = \lambda\omega_j,\ j=1,\cdots,N \\ \omega_0 = \omega_{N+1} = 0. \end{cases} \tag{6.14}$$

The eigenvalues and eigenvectors of (6.14) may be computed explicitly (see [64]):

$$\begin{cases} \lambda_j(h) = \frac{4}{h^2}\sin^2\left(\frac{\pi j h}{2L}\right),\ j=1,\cdots,N \\ \omega_j \equiv \left(\omega_{j,1},\cdots,\omega_{j,N}\right);\ \omega_{j,k} = \sin\left(\frac{j\pi h k}{L}\right),\ j,k=1,\cdots,N \end{cases} \tag{6.15}$$

The following identity holds:

$$h\sum_{k=0}^{N}\left|\frac{\omega_{k+1}-\omega_k}{h}\right|^2 = \frac{2L}{4-\lambda h^2}\left|\frac{\omega_N}{h}\right|^2 \tag{6.16}$$

for any eigenvector of (6.15). Observe that this identity provides the ratio between the total energy of the eigenvectors (represented by the quantity on the left hand side of (6.16)) and the energy concentrated on the boundary (represented by $|\omega_N/h|^2$).

It is also easy to check that

$$\lambda_N(h)h^2 \to 4 \text{ as } h \to 0. \tag{6.17}$$

Combining (6.16)-(6.17) it is immediate to see that the following negative result holds:

Theorem 6.2. ([62, 63]) *For any* $T > 0$

$$\sup_{u \text{ solution of } (6.11)}\left[\frac{E_h(0)}{\displaystyle\int_0^T \left|u_N(t)/h\right|^2 dt}\right] \to \infty,\ \textit{as } h \to 0. \tag{6.18}$$

In order to state the positive counterpart of Theorem 6.18 we develop solutions of (6.11) in Fourier series:

$$u = \sum_{j=1}^{N}\left(a_j\sin\left(\sqrt{\lambda_j}t\right) + b_j\cos\left(\sqrt{\lambda_j}t\right)\right)\omega_j. \tag{6.19}$$

The fact that the constant in (6.13) may not remain uniformly bounded as $h \to 0$ (as stated in (6.18)) is due to the pathological behavior of the high frequencies. Indeed, as a consequence (6.16)-(6.17), it follows that the ratio

$$h \sum_{k=0}^{N} \left| \frac{\omega_{k+1} - \omega_k}{h} \right|^2 \Bigg/ \left| \omega_N / h \right|^2 \tag{6.20}$$

tends to infinity as $h \to 0$ for the N-th eigenvectors. This allows to prove that the uniform observability inequality fails simply by considering solutions of (6.11) of the form

$$u = \sin\left(\sqrt{\lambda_N(h)} t\right) \omega_N \tag{6.21}$$

in separated variables corresponding to the N-th eigenvectors, which is associated to the largest eigenvalue $\lambda_N(h)$.

This indicates that, in order to obtain uniform observability inequalities, the high frequencies have to be filtered or truncated. To do that, given any $0 < \gamma < 1$ we introduce the following class of solutions $\mathcal{C}_\gamma(h)$ of (6.11) of the form:

$$u = \sum_{j=1}^{\gamma N} \left(a_j \sin\left(\sqrt{\lambda_j} t\right) + b_j \cos\left(\sqrt{\lambda_j} t\right)\right) \omega_j. \tag{6.22}$$

Note that in (6.22) the eigenvectors corresponding to the indexes $j > \gamma N$ do not enter.

The following holds:

Theorem 6.3. ([62, 63]) *For any $0 < \gamma < 1$ there exists $T(\gamma) > 2L$ such that for all $T > T(\gamma)$ there exists $C = C(T, \gamma)$ such that* (6.13) *holds for any solution of* (6.11) *in the class $\mathcal{C}_\gamma(h)$ and any $h > 0$.*

Moreover, $T(\gamma) \nearrow \infty$ as $\gamma \nearrow 1$ and $T(\gamma) \searrow 2L$ as $\gamma \searrow 0$.

This result was proved in [62, 63] using two different methods: Discrete multiplier techniques and Ingham's inequalities for series of complex exponentials.

Note that, as indicated in Theorem 6.3, the time needed for the uniform observability to hold tends to infinity as $\gamma \nearrow 1$. This is due to the fact that the gap between the roots of the consecutive highest eigenvalues entering in the Fourier development of solutions in $\mathcal{C}_\gamma(h)$ tends to zero as $\gamma \nearrow 1$. On the other hand as $\gamma \searrow 0$ the time needed for the uniform observability converges to the observability time of the continuous wave equation. Therefore, as a consequence of Theorem 6.3, the observability of the wave equation (6.6) may be obtained as limit of uniform observability inequalities as $h \to 0$ provided $\gamma \to 0$ as well.

There are clear analogies between the results of section 3 on the wave equation with rapidly oscillating coefficients and those of this section. We refer to [158] for a detailed discussion of this issue.

Very recently a fundamental contribution to this subject has been made by S. Micu [107]. He has proved that, in particular, if the initial data to be controlled for the wave equation has only a finite number of non trivial Fourier components, then the controls of the semi-discrete systems remain bounded as the mesh size tends to zero. This result has been proved by a technical analysis of the behavior of the biorthogonal families to the sequences of complex exponentials involved in the Fourier expansion of solutions of the semi-discrete systems.

Note that this positive result by S. Micu is compatible with the negative ones we presented above. Indeed, as shown above, the boundary observability inequalities for the semi-discrete systems are not uniform when the mesh size tends to zero. This, according to the Uniform Boundedness Principle, indicates that there exist initial data for the wave equation in $L^2(0,L) \times H^{-1}(0,L)$, for which the controls of the semi-discrete problem diverge. According to the result by S. Micu this pathological initial data have necessarily an infinite number of non trivial Fourier components.

6.3. **Finite-difference space semi-discretizations of the heat equation.** Let us consider now the following $1-d$ heat equation with control acting on the extreme $x = L$:

$$\begin{cases} u_t - u_{xx} = 0, \; 0 < x < L, \; 0 < t < T \\ u(0,t) = 0, \; u(L,t) = v(t), \; 0 < t < T \\ u(x,0) = u_0(x), \; 0 < x < L. \end{cases} \tag{6.23}$$

This is the so called boundary control problem.

It is by now well known that system (6.23) is null controllable (see for instance D.L. Russell [129, 130]). To be more precise, the following holds: *For any $T > 0$, and $u_0 \in L^2(0,L)$ there exists a control $v \in L^2(0,T)$ such that the solution y of* (6.1) *satisfies*

$$u(x,T) \equiv 0 \;\; in \;\; (0,L). \tag{6.24}$$

As indicated in section 4, this null controllability result is equivalent to a suitable observability inequality for the adjoint system:

$$\begin{cases} \varphi_t + \varphi_{xx} = 0, \; 0 < x < L, \; 0 < t < T, \\ \varphi(0,t) = \varphi(L,t) = 0, \; 0 < t < T \\ \varphi(x,T) = \varphi_0(x), \; 0 < x < L. \end{cases} \tag{6.25}$$

The corresponding observability inequality is as follows: *For any $T > 0$ there exists $C(T) > 0$ such that*

$$\int_0^L \varphi^2(x,0)dx \leq C \int_0^T |\varphi_x(L,t)|^2 \, dt \tag{6.26}$$

holds for every solution of (6.25).

Let us consider now the semi-discrete versions of systems (6.23) and (6.25):

$$\begin{cases} u_j' - [u_{j+1} + u_{j-1} - 2u_j] \big/ h^2 = 0, \; 0 < t < T, \; j = 1, \cdots, N \\ u_0 = 0, \; u_{N+1} = v, \; 0 < t < T \\ u_j(0) = u_{0,j}, \; j = 1, \cdots, N; \end{cases} \tag{6.27}$$

$$\begin{cases} \varphi_j' + [\varphi_{j+1} + \varphi_{j-1} - 2\varphi_j] \big/ h^2 = 0, \; 0 < t < T, \; j = 1, \cdots, N \\ \varphi_0 = \varphi_{N+1} = 0, \; 0 < t < T \\ \varphi_j(T) = \varphi_{0,j}, \; j = 1, \cdots, N. \end{cases} \tag{6.28}$$

In this case, in contrast with the results we have described on the wave equation, systems (6.27) and (6.28) are uniformly controllable and observable respectively as $h \to 0$.

More precisely, the following results hold:

Theorem 6.4. ([103, 104]) *For any $T > 0$ there exists a positive constant $C(T) > 0$ such that*

$$h \sum_{j=1}^N |\varphi_j(0)|^2 \leq C \int_0^T \left| \frac{\varphi_N(t)}{h} \right|^2 dt \tag{6.29}$$

holds for any solution of (6.28) *and any $h > 0$.*

Theorem 6.5. ([103, 104]) *For any $T > 0$ and $\{u_{0,1}, \cdots, u_{0,N}\}$ there exists a control $v \in L^2(0,T)$ such that the solution of* (6.27) *satisfies*

$$u_j(T) = 0,\ j = 1, \cdots, N. \tag{6.30}$$

Moreover, there exists a constant $C(T) > 0$ independent of $h > 0$ such that

$$\| v \|^2_{L^2(0,T)} \leq Ch \sum_{j=1}^{N} |u_{0,j}|^2 . \tag{6.31}$$

These results were proved in [104] using Fourier series and a classical result on the sums of real exponentials (see for instance Krabs [68] and Fattorini-Russell [37]) that plays the role of Ingham's inequality in the context of parabolic equations.

Let us recall it briefly: Given $\xi > 0$ and a decreasing function $N : (0, \infty) \to \mathbb{N}$ such that $N(\delta) \to \infty$ as $\delta \to 0$, we introduce the class $\mathcal{L}(\xi, N)$ of increasing sequences of positive real numbers $\{\mu_j\}_{j \geq 1}$ such that

$$\mu_{j+1} - \mu_j \geq \xi > 0,\ \forall j \geq 1, \tag{6.32}$$

$$\sum_{k \geq N(\delta)} \mu_k^{-1} \leq \delta,\ \forall \delta > 0. \tag{6.33}$$

The following holds:

Proposition 6.6. *Given a class of sequences $\mathcal{L}(\xi, N)$ and $T > 0$ there exists a constant $C > 0$ (which depends on ξ, N and T) such that*

$$\int_0^T \left| \sum_{k=1}^{\infty} a_k e^{-\mu_k t} \right|^2 dt \geq \frac{C}{\left(\sum_{k \geq 1} \mu_k^{-1} \right)} \sum_{k \geq 1} \frac{|\ a_k\ |^2 \ e^{-2\mu_k T}}{\mu_k} \tag{6.34}$$

for all $\{\mu_j\} \in \mathcal{L}(\xi, N)$ and all bounded sequence of real numbers.

One can even prove that the null controls for the semi-discrete equation (6.27) can be built so that, as $h \to 0$, they tend to the null control for the continuous heat equation (6.23) (see [104]).

7. **Some open problems.** In this section we present some open problems related to the topics we addressed in this paper.

1.- The extension of the results of section 3 on the wave equation with rapidly oscillating coefficients to the multi-dimensional wave equation is a widely open problem.

Nevertheless, the multi-dimensional counterpart of Theorem 3.3 for smooth densities ρ has been proved by G. Lebeau in [80] using Bloch waves decompositions and microlocal analysis techniques.

2.- In section 4 we have proved sharp estimates on the cost of approximate controllability for the constant coefficient heat equation.

In the more general case where the equation under consideration is (2.21) with $a = a(x,t)$ a L^∞ potential depending both on x and t the estimate we get in [41] is worse than in (4.3). Indeed, we get an estimate of the order of $\exp(c/\varepsilon^2)$ as $\varepsilon \to 0$. Whether this estimate is sharp or not is an open problem. Note that the estimate we got in section 4 of the order of $\exp(c/\varepsilon)$ is also valid when $a = a(x)$ since we may use Fourier series developments. The same can be said about the case where

a depends on t only since a simple change of unknown reduces the problem to the constant coefficient heat equation. The problem is open when a depends both on x and t.

3.- In the context of the constant coefficient heat equation we have obtained the sharp estimate

$$\sum_{k=1}^{\infty} |a_k|^2 e^{-c\sqrt{\lambda_k}} \leq C \int_0^T \int_\omega \varphi^2 dxdt \tag{7.1}$$

for the solutions φ of the adjoint heat equation (2.11).

The problem of characterizing the best constant c in (7.1) in terms of the geometric properties of Ω and ω and T is also open. At this respect one should take into account that the construction in Theorem 4.6 provides an explicit lower bound on the constant c in terms of the radius A of the largest ball contained in $\Omega \setminus \omega$.

4.- The results of section 6.2 on the exact controllability of the finite-difference approximations of the wave equation has been extended to two space dimensions in [156]. The results of section 6.3 on the null controllability of the finite-difference approximations of the heat equation to several space dimensions can also be extended to the case where the domain is a square and the control acts on one side of the boundary (see [103]). The problem is open in the case of a general domain.

5.- The questions we have discussed in problem # 4 above are open in the case of the semilinear heat and wave equations with globally Lipschitz nonlinearities.

6.- In section 6.1 we have proved the controllability of the Galerkin approximations of the Navier-Stokes equations under the condition (6.3) on the Galerkin basis. The problem is open when (6.3) does not hold.

7.- The results of section 6.1 do not provide any estimate on the size of the control in terms of the dimension N of the Galerkin approximation. Passing to the limit as $N \to \infty$ to recover the controllability properties of the Navier-Stokes equations that are by now well known (see [28, 29] and [45]) is a completely open problem.

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Departamento de Matemática Aplicada, Universidad Complutense, 28040 Madrid, Spain

E-mail address: `zuazua@eucmax.sim.ucm.es`