SINGULARITIES OF FOLIATIONS AND VECTOR FIELDS

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Contents

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EXTENDED ABSTRACTS TO SECTIONS

1. Differential 1-forms and geometric control theory. Equations $\alpha = 1$.

• Studying certain types of control systems using differential forms is much more effective than using vector fields.

• Simultaneously, I explain the advantage of the canonical (invariant, coordinate-free) point of view on control system based on the definition that a control system on a manifold M is a subset of the tangent bundle TM^n.

• This allows, in particular, to present certain control systems in the form $\alpha = 1$, where α is a differential 1-form, and to identify a control system, defined up to feed-back equivalence, with α. Consequently, the feedback classification of such control systems is exactly the same problem as classification of 1-forms with respect to the group of diffeomorphisms and all results on classification of 1-forms (starting with Darboux theorem obtained more than 100 years ago) can be reformulated in terms of control theory.

• The definition of an integral (= admissible) curve of a general control system $\Sigma \subset TM$. Studying the structure of the set of all integral curves of control systems $\Sigma \subset TM$ of certain types is one of the main tasks of geometric control theory.

• The Appendix to section 1 is devoted to the question whether the feedback equivalence of families of vector fields is always the same property as the diffeomorphness of corresponding subsets of the tangent bundle or this is true only beyond degenerations of infinite codimension.

Section 1 contains 2 exercises.

2. Pfaff equations $\alpha = 0$. Integrating factor. Frobenius theorem.

• **Convention.** In what follows all objects are germs at the origin of a fixed category: C^∞ or real analytic

• A subset of $\Sigma \subset T\mathbb{R}^n$ given by the equation $\alpha = 0$, where α is a differential 1-form, is defined in exactly the same way as the subset of $T\mathbb{R}^n$ given by the equation $\alpha = 1$, see section 1. It is called *the field of kernels of* α. The equation $\alpha = 0$ is called Pfaff equation.

• By definition, a differential 1-form α admits an integrating factor if there exists a function Q, $Q(0) \neq 0$, such that $Q\alpha = dH$ for some function H. Other possible definitions are the same beyond infinitely degenerate cases that can be described in terms of 1-division property.

• In the 2-dimensional case a Pfaff equation has the form $\alpha = A_1(x,y)dx + A_2(x,y)dy = 0$, and it can be found in almost any textbook on ODE's along with some ways of calculating an integral factor. In many though not all texbooks it is explained that the existence of a local integrating factor in the case $\alpha(0) \neq 0$ is, de-facto, a basic theorem of ODE's (straightening of a non-singular vector field). Some text books on ODE's also explain that in the case $\alpha(0) = 0$ there is a necessary condition for an integrating factor: $d\alpha(0) = 0$ (it remains necessary in the n-dimensional case with any n), but just few text books explain that this necessary condition is very far from sufficient and finding and analyzing sufficient conditions is the subject of a number of works of the 20-th and the 21-st centuries, including works on and around center-focus problem, see section 7.

• Return to the n-dimensional case. Frobenius theorem: if $\alpha(0) \neq 0$ then the following conditions are equivalent: (a) α admits an integrating factor; (b) Pfaff equation $\alpha = 0$ defines a non-singular foliation; (c) $\alpha \wedge d\alpha \equiv 0$.

• Proof of Frobenius theorem (reduction to the 2-dimensional case) using the Darboux theorem on normal form for closed 2-forms of a constant rank (not necessarily maximum possible).

Section 2 contains 3 exercises.

3. Integrable 1-forms. Kupka phenomenon.

• Definition of integrable 1-forms α: $\alpha \wedge d\alpha \equiv 0$.

• Singular foliations. Definition of a structurally stable case (within integrable 1-forms α, $\alpha(0) = 0$.)

• The set of singular points $Sing(\alpha)$ and the ideal $I(\alpha)$ generated by the coefficients of α.

• Kupka phenomenon: the case

$d\alpha(0) \neq 0, I(\alpha)$ is generated by two differentially independent functions
(and consequently $Sing(\alpha)$ is a non-singular codimension 2 submanifold)

is C^1-structurally stable

• Theorem (I. Kupka): if α is an integrable 1-form on $\mathbb{R}^n$, $\alpha(0) = 0, d\alpha(0) \neq 0$ then
$\alpha = a_1(x_1, x_2)dx_1 + a_2(x_1, x_2)$ in suitable coordinates

• Proof of Kupka phenomenon and Kupka theorem.

• Corollary: if α is an integrable 1-form on $\mathbb{R}^n$, $\alpha(0) = 0, d\alpha(0) \neq 0$ then $codimSing(\alpha) \leq 2$

• Open book foliation

• Corollary: the local classification of Pfaff equations $\alpha = 0$ (with respect to diffeomorphisms), where α is an integrable 1-form, $\alpha(0) = 0, d\alpha(0) \neq 0$, is exactly the same problem as the local classification with respect to the orbital equivalence of vector fields V on the plane $\mathbb{R}^2$ such that $V(0) = 0$ and $traceV(0) \neq 0$ (relatively easy part of the problem of local orbital classification of vector fields on $\mathbb{R}^2$).

• Corollary: integrable 1-forms $\alpha, \alpha(0) = 0, d\alpha(0) \neq 0$ on $\mathbb{R}^n$ can be described by $(n-1)$ commuting vector fields, $(n-2)$ of them - non-singular.

• Integrability $\not\Rightarrow$ integrating factor. The absence of an integrating factor for integrable 1-forms $\alpha, \alpha(0) = 0, d\alpha(0) \neq 0$.

Section 3 contains 7 exercises.

4. Frobenius through singularities (Reeb, Moussu, Malgrange).

• The Kupka case

K: $d\alpha(0) \neq 0$, the ideal $I(\alpha)$ is generated by two differentially independent functions (and consequently $Sing(\alpha)$ is a codimension 2 non-singular submanifold of $\mathbb{R}^n$)

is structurally stable. Is there a structurally stable case (within integrable $\alpha, \alpha(0) = 0$) such that the Frobenius condition $\alpha \wedge d\alpha \equiv 0$ implies the existence of an integrating factor? This natural question goes back to the 19-th century.

• Consider the case

R: the ideal $I(\alpha)$ is generated by n differentially independent functions (and consequently $Sing(\alpha) = \{0\}$).

It is clear that this case is also structurally stable. Theorem (G. Reeb): in the analytic category with $n \geq 3$ any integrable form α within the case **R** admits an integrating factor.

• The starting point for the analysis of the case **R** is the proposition that in this case $d\alpha(0) = 0$. This is almost direct corollary of Kupka theorem. Kupka theorem implies that the same conclusion $d\alpha(0) = 0$ holds, in the analytic category, in a much wider case

MAL (only in analytic category) $codimSing(\alpha^{\mathbb{C}}) \geq 3$, where $\alpha^{\mathbb{C}}$ is the complexification of α.

The case **MAL** is also structurally stable.

• Theorem (B. Malgrange): in the analytic category any integrable form α within the case **MAL** admits an integrating factor.

The assumption $codimSing(\alpha^{\mathbb{C}}) \geq 3$ cannot be replaced by $codimSing(\alpha^{\mathbb{C}}) \geq 2$, otherwise one has the Kupka case. This assumption also plays crucial role in the proof since it guarantees the 2-division property of α. The proof on the level of formal series is based on obtaining an infinite series of equalities which follow from the Frobenius condition $\alpha \wedge d\alpha \equiv 0$ and the 2-division property of α. The proof on the level of formal series was obtained a bit earlier by R. Moussu (within algebraically isolated singularities, but R. Moussu used the algebraic isolateness for the 2-division property only, therefore in the case **MAL** his proof remains the same). The transition from formal series to analytic category was made by R. Moussu under an additional assumption.

• The first result on the existence of an integrating factor in the C^∞ category was obtained by G.Reeb, the final result - by R.Moussu. Multiplicity of an ideal. Algebraically isolated singularities. The case

M α has algebraically isolated singularity
(and consequently $Sing(\alpha) = \{0\}$)

• **R**$\subset$**M**$\subset$**MAL**. These three cases are structurally stable. In these cases the linear part of an integrable 1-form α is dH_2, H_2 is a quadratic form. This allows to define the invariants $r = r(\alpha)$ (the rank of H_2) and $i(\alpha)$ (the maximum of the index of H_2 and ($r -$ the index).

• Theorem (G. Reeb): in the C^∞ category an integrable 1-form $\alpha \in$**R** admits an integrating factor provided that $i(\alpha) = n$ (in the case **R** $r(\alpha) = n$).

• Moussu criterium of the existence of an integrating factor for integrable $\alpha \in$ **M** in the C^∞ category (either elliptic case or certain geometric condition). Moussu example of an integrable 1-form $\alpha \in$**M** such that an integrating factor does not exist: $\alpha = d(x_1^2 + x_2^2 - x_3^2) + \tau$, where τ is a flat 1-form constructed explicitly.

• Theorem (R. Moussu): two other cases in the C^∞ category (in terms of $r(\alpha)$ and $i(\alpha)$) that an integrable 1-form $\alpha \in$**M** admits an integrating factor: $n \geq 4,\ r(\alpha) = n,\ i(\alpha) = n-1$ and $n \geq 4,\ r(\alpha) = i(\alpha) = n-1$.

• An integrable 1-forms $\alpha \in$**MAL** is not associated to a tuple of vector fields.

Section 4 contains 7 exercises.

5. Integrable 1-forms between Kupka and Malgrange (Camacho, Neto, Cerveau, Chaperon).

• Between Kupka and Malgrange one has the case

$d\alpha(0) = 0, \quad codimSing(\alpha^{\mathbb{C}}) = 2.$

• Example of a structurally stable case between Kupka and Malgrange:

the case C in some local coordinates the 2-jet of α has the form

$\lambda_1 x_2 x_3 dx_1 + \lambda_2 x_1 x_3 dx_2 + \lambda_3 x_1 x_2 dx_3$

(logarithmic integrable 1-form) with $\lambda_1, \lambda_2, \lambda_3$ satisfying certain arithmetic condition.

• complete proof that the case **C** is structurally stable
(based on a work by C. Camacho)

• Logarithmic integrable 1-forms on $\mathbb{R}^n$: the $(n-1)$-jet is

$\lambda_1 x_2 x_3 \cdot \dots \cdot x_n dx_1 + \lambda_2 x_1 x_3 \cdot \dots \cdot x_n dx_2 + \dots + \lambda_n x_1 x_2 \cdot \dots \cdot x_{n-1} dx_n.$

Definition of $(n-1)$-determinacy.

• Describing logarithmic integrable 1-forms via $(n-1)$-tuple of vector fields. The hyperplane

$R_\lambda = \{\mathrm{r} \ = (r_1, ..., r_n) \in \mathbb{R}^n : \quad r_1\lambda_1 + \dots + r_n\lambda_n = 0.\}$

• Explanation of the following results by D.Cerveau - A. Lins Neto:

If $\lambda_1, ..., \lambda_n$ are distinct non-zero numbers and the hyperplane R_λ contains a non-resonant vector (respectively a vector satisfying the Siegel condition) then the logarithmic integrable 1-form is $(n-1)$-determined on the level of formal series (respectively in the analytic category).

• The case $\lambda_1 = \cdots = \lambda_n$. D. Cerveau and A. Lins Neto have results for this case generalizing the classical Lyapunov-Poincare theorem which concerns **exactly** this case with $n = 2$, see section 7.

• Theorem by M. Chaperon on 1-determinacy of modules of vector fields which are closed with respect to the Lie bracket (formal series and C^∞ category).

• Corollary of this theorem:

If $\lambda_1, ..., \lambda_n$ are distinct non-zero numbers, at least two of them have different signs, and the hyperplane R_λ contains a non-resonant vector then the integrable logarithmic 1-form is $(n-1)$-determined in the C^∞ category.

Section 5 contains 4 exercises.

6. Poisson structures and integrable 1-forms on $\mathbb{R}^3$

• Definition of a Poisson structure

• In the 3-dimensional case Poisson structures can be multiplied by a function

• the problem of classification of local Poisson structures on $\mathbb{R}^3$ with respect to diffeomorphisms and multiplication by a non-vanishing function is exactly the same problem as local classification of integrable 1-forms on $\mathbb{R}^3$ with respect to diffeomorphisms and multiplication by a non-vanishing function.

• Therefore the 3-dimensional part of all known results on local classification of Poisson structures (A. Weinstein, J.F. Conn, J.P. Dufour and his students) can be reformulated as local classification results for integrable Pfaff equations $\alpha = 0$ on $\mathbb{R}^3$. This gives, in particular, a good part of results presented in sections 4-5, restricted to the 3-dimensional case.

Section 6 contains 3 exercises.

7. Center-focus problem. Poincare-Lyapunov theorem.

• Return now to the 2-dimensional case. Remind that any 1-form on $\mathbb{R}^2$ is integrable. Consider, in the analytic category, the classical case

$\alpha = d(x_1^2 + x_2^2) + o(||x||)$

• this case can also be treated as the case between Kupka and Malgrange:
$codimSing(\alpha) = 2$ and $d\alpha(0) = 0$.

• A simple proposition: α admits an integrating factor if and only if the Pfaff equation $\alpha = 0$ is diffeomorphic to its linear part $d(x_1^2 + x_2^2) = 0$.

• Resonant normal form for a vector field V associated to α via a volume form
$(\dot{x} = y + \cdots, \dot{y} = -x + \cdots)$

• Orbital resonant normal form for V

• Poincare-Lyapunov theorem and explanation of its formal series part

• Corollaries for Pfaff equations $\alpha = 0$. The following conditions are equivalent:

1. The 1-form α admits an analytic integrating factor;

2. The 1-form α admits a formal integrating factor;

3. The Pfaff equation $\alpha = 0$ is can be reduced to its linear approximation by an analytic change of coordinates;

4. The Pfaff equation $\alpha = 0$ is can be reduced to its linear approximation by a formal change of coordinates;

5. All focus numbers (coefficients in the formal **orbital** normal form) are equal to zero;

6. The foliation defined by the Pfaff equation $\alpha = 0$ is a center.

• Let $A_m = \{d(x_1^2 + x_2^2) + P_m(x_1, x_2)dx_1 + Q_m(x_1, x_2)dx_2\}$, where P_m and Q_m are degree m polynomials without free and linear terms. Proposition: there exists $d = d(m)$ such that a 1-form $\alpha \in A_m$ defines a center if and only if the first $d(m)$ focus numbers are equal to zero. Corollary: for any $m \geq 2$ there exists a finite time algorithm distinguishing center-focus within A_m. Nevertheless, for $m \geq 3$ nobody knows such an algorithm. The thing is that the number $d(m)$ is known for $m = 2$ only. In the 20-th century a number of mathematicians spend much time trying to find $d(m)$ for $m \geq 3$, at least $d(3)$. It seems that in the 21-st century the number of people attacking this problem substantially decreased.

8. Division properties.

• A number of problems of local analysis and in particular singularities of integrable 1-forms require theorems on the following property of a differential 1-form α:

if ω is a k-form such that $\alpha \wedge \omega \equiv 0$ then there exists a $(k-1)$-form γ such that $\omega = \alpha \wedge \gamma$.

This property of α is called the k-division property. If $\alpha(0) \neq 0$ then α has the k-division property for any k. If α is a 1-form on $\mathbb{R}^n$ and $\alpha(0) = 0$ then α does not have the k-division property with $k \geq n$. Therefore in what follows $\alpha(0) = 0$ and $k \leq n - 1$.

• The first results on division properties are due to G. de Rham and D.G. Northcott. The most applicable results were obtained by R. Moussu and B. Malgrange. In now-days this topic is completely understood, and the most general results can be found in some books on commutative rings and algebra (D. Eisenbud, S. Balcerzyk-T.Jozefiak).

• The k-division property of α is the property of the ideal generated by the coefficients of α (this ideal does not depend on the choice of a coordinate system).

• In the analytic category the main result is as follows. Let $I = (f_1, ..., f_p)$ be a p-generated ideal of function germs on $\mathbb{R}^n$. Let $Sing(I^{\mathbb{C}}) = \{z \in \mathbb{C}^n : f_1^{\mathbb{C}}(z) = \cdots = f_p^{\mathbb{C}}(z) = 0\}$. Here $f_i^{\mathbb{C}}$ is the complexification of f_i. The ideal I has the k-division property if and only if $codimSing(I^{\mathbb{C}}) \geq k+1$.

• In the C^∞ category one has the following result. Let $p \leq n$. Assume that a p-generated ideal I in the ring of C^∞ function germs is diffeomorphic to an ideal (in the same ring) generated by p analytic function germs $f_1, ..., f_p$. Denote by $I^{\mathbb{A}}$ the ideal in the ring of analytic function germs generated by $f_1, ..., f_p$. Let $k \geq 1$. The ideal I has the k-division property if and only if the ideal $I^{\mathbb{A}}$ has k-division property. Consequently, the ideal I has k-division property if and only if $codimSing(I^{\mathbb{A},\mathbb{C}}) \geq k + 1$.

• Corollary: a 1-form α on $\mathbb{R}^n$ with algebraically isolated singularity at 0 has the k-division property for any $k \leq n - 1$.

• Regular sequences in the ring of function germs. Theorem by G. de Rham and D.G. Northcott.

• The depth of an ideal. The most general algebraic theorem on division properties (of proper finitely generated ideals in any commutative ring with a unit).

Section 8 contains 3 exercises.

9. References and comments.

This section contains references and few comments. The references are exhausted by the works I used preparing the Notes and the works mentioned (by name, year) in Sections 1-8.

10. Appendix. Vector fields and differential 1-forms.

This appendix is for students who do not know well one of the following:

A. Canonical (coordinate free) definitions of

• vector fields V and differential 1-forms α

• $\Phi_* V$ and $\Phi^* \alpha$, where Φ is a diffeomorphism

• the differential 1-form dH, where H is a function

• the Lie derivative of a function along a vector field

B. Calculation of everything in **A** in a local system of coordinates

1. Differential 1-forms and geometric control theory. Equations $\alpha = 1$.

The most used object in control theory is vector fields. Vector fields is a classical object. Differential forms is also a classical object. In control theory they are used much less than vector fields. Except historical traditions, this is related to the fact that people are mainly interested in the case that the number of controls is much less than the dimension of the phase space. If the number of controls is close to the dimension of the phase space then using differential forms might be much more effective than using vector fields. I will illustrate this considering control systems with the number of controls one less than the dimension of the phase space. Simultaneously, I will explain the advantage of the canonical (invariant, coordinate-free) point of view on control system.

Consider the following control system in usual for control theory notations

$$\dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x), \qquad x \in \mathbb{R}^3 \tag{1.1}$$

where $f(x), g_1(x), g_2(x)$ are smooth (or analytic) vector fields on $\mathbb{R}^3$ and u_1, u_2 denote controls.

When I was involved to some problems of control theory I could not understand the words like u or u_1, u_2 denote controls. What are controls, who controls and how? Are u_1, u_2 arbitrary numbers or functions? Functions of time t or of x or both? And what is control system, is there some definition?

Later I realized that defining control system can be compared with defining fairness or democracy. People understand what it is, but any precise definition would be attacked by many counterexamples - the cases where it does not work. Nevertheless, one of the advantages of math is absolutely precise definitions and language, unlike in everyday life everything should be well-defined.[1] Working with control systems without precise definition of control systems is impossible.

Of course there are many possible definitions, and if a mathematician is involved in control theory then he has to chose for himself a definition that he appreciates, which is good for the circle of problems he wants to attack.

My choice corresponds to the definition given by A. Agrachev in his talk in the International Math Congress in 1994. His talk and later his work in the Proceedings of the Congress started with the definition: *a control system on a manifold M is a subset of the tangent bundle TM.*

For some people this definition sounds too wide, for some people - too restrictive. I appreciate this definition, and, moreover, consider advertising of its advantages as an important task.

[1] if in everyday life at least basic things were also well-defined then, I believe, the life would be better.

Let us see what does the given definition mean for system (1.1). Let us

$$\text{assume that the vector fields } f, g_1, g_2 \text{ are pointwise independent.} \tag{1.2}$$

Fix a point $x \in \mathbb{R}^3$. The tangent space $T_x\mathbb{R}^3$ is the space of all possible velocities at x. System (1.1) says that the velocity at x is not arbitrary: the velocity vector at x belongs to the set $f(x) + span\{g_1(x), g_2(x)\}$. Under assumption (1.2) this set is an affine plane in the 3-space $T_x\mathbb{R}^3$ which does not contain zero velocity vector[2]. So, (1.1) defines the following subset of $T\mathbb{R}^3$:

$$(1.1) \hookrightarrow \Sigma \subset T\mathbb{R}^3 = \{\Sigma(x),\ x \in \mathbb{R}^3\}, \ \ \Sigma(x) = \text{affine plane in } T_x\mathbb{R}^3.$$

How an affine plane $\Sigma(x)$ in the 3-space $T_x\mathbb{R}^3$ can be defined? One of the ways is as follows: we fix a point $a \in \Sigma(x)$ and take the vector $f(x)$ joining 0 to a. Then the set $\{v - f(x),\ \ v \in \Sigma(x)\}$ is a 2-dimensional subspace of $T_x\mathbb{R}^3$. Take any basis $g_1(x), g_2(x)$ of this 2-dimensional subspace. We obtain

$$\Sigma(x) = \{f(x) + span(g_1(x), g_2(x))\}$$

which corresponds to (1.1).

Of course in this construction the vectors $f(x), g_1(x), g_2(x)$ are not unique. One can replace the point a used in the construction by any other point of $\Sigma(x)$ - then the vector $f(x)$ will be replaced by $\tilde{f}(x) = f(x) + \alpha(x)g_1(x) + \beta(x)g_2(x)$ for some $\alpha(x), \beta(x) \in \mathbb{R}$. One also can replace the basis $g_1(x), g_2(x)$ by another basis $\tilde{g}_1(x) = k_{11}(x)g_1(x) + k_{12}(x)g_2(x),\ \tilde{g}_2(x) = k_{21}(x)g_1(x) + k_{22}(x)g_2(x)$, where $(k_{ij}(x))$ is a non-degenerate matrix. Then

$$\text{the same affine plane } \Sigma(x) = \{\tilde{f}(x) + span(\tilde{g}_1(x), \tilde{g}_2(x))\}.$$

In the standard language of control theory the transformation from the tuple $f(x), g_1(x), g_2(x)$ to the tuple $\tilde{f}(x), \tilde{g}_1(x), \tilde{g}_2(x)$ given by the above relations is called feedback transformation. So, if one accepts the definition of a control system as a subset of the tangent bundle then

feedback transformations do not change at all the control system; they play the same role as transformation of a basis of a linear space to another basis.

There is another way to describe an affine plane in a 3-space. In $\mathbb{R}^3(x, y, z)$ it can be given by equation $Ax + By + Cz = D$. If the affine plane is not a subspace (does not contain 0) then D can be reduced to 1 and the affine plane can be given by the equation $L(x, y, z) = 1$, where L is a linear functional on $\mathbb{R}^3(x, y, z)$ sending (x, y, z) to $Ax + By + Cz$.

[2] affine plane = plane, not necessarily 2-dimensional subspace

In the same way an affine plane $\Sigma(x)$ in the 3-space $T_x\mathbb{R}^3$ which does not contain the zero velocity vector can be given by the equation

$$\Sigma(x) = \{v \in T_x\mathbb{R}^3 : \ \alpha(v) = 1\}, \ \alpha \text{ is a linear functional on } \ \mathbb{T}_x R^3.$$

A linear functional on $T_x\mathbb{R}^3$ is called (algebraic) 1-form on $T_x\mathbb{R}^3$. A family of algebraic 1-forms $\alpha(x)$ parameterized in smooth (analytic) way by $x \in \mathbb{R}^3$ is called smooth (analytic) differential 1-form on $\mathbb{R}^3$. Therefore

under assumption (1.2) a control system (1.1) is a subset of $T\mathbb{R}^3$ which can be described by equation of the form $\alpha = 1$, α is a differential 1-form.

It is clear how to pass from (1.1) to a differential 1-form α: α is a differential 1-form such that $\alpha(g_1) = \alpha(g_2) \equiv 0$ and $\alpha(f) \equiv 1$. Under assumption (1.2) such a differential 1-form exists and unique (simple exercise: prove it).

Example. Consider a control system

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = 1 + a(x)u_1 + b(x)u_2.$$

It is a subset of $T\mathbb{R}^3$ which can be given in the form

$$\Sigma = \{\Sigma(x) = f(x) + span\{g_1(x), g_2(x)\}, \quad x \in \mathbb{R}^3\},$$

$$g_1 = \frac{\partial}{\partial x_1} + a(x)\frac{\partial}{\partial x_3}, \qquad g_2 = \frac{\partial}{\partial x_2} + b(x)\frac{\partial}{\partial x_3}, \qquad f = \frac{\partial}{\partial x_3}.$$

The vector fields f, g_1, g_2 are pointwise independent. The given control system can be written down in the form

$$\alpha = dx_3 - b(x)dx_2 - a(x)dx_1 = 1.$$

In fact, it is easy to calculate that α (and only α) annihilates the vector fields g_1, g_2 and takes the value 1 on the vector field f.

In general case, to pass from f, g_1, g_2 to a 1-form α one has to solve a linear system with three equations with respect to three unknowns; a point $x \in \mathbb{R}^3$ is a parameter of the linear system. One can use determinants to solve the system.

The work with determinants can be replaced by operations with differential forms. The latter allows to write down the answer in the form

$$\alpha = \frac{\Omega(g_1, g_2, \cdot)}{\Omega(g_1, g_2, f)},$$

where Ω is a volume form (a non-degenerate 3-form on $\mathbb{R}^3$). The form $\Omega(g_1, g_2, \cdot)$ is a 1-form taking on a tangent vector v the same value as Ω takes on the triple g_1, g_2, v. The denominator $\Omega(g_1, g_2, f)$ is a function, it does not vanish due to assumption (1.2).

Definition 1.1. *Two control systems $\Sigma, \tilde{\Sigma} \subset T\mathbb{R}^3$ are called diffeomorphic if there exists a diffeomorphism $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ sending Σ to $\tilde{\Sigma}$, i.e. $\Phi_{*,x}\Sigma(x) = \tilde{\Sigma}(\Phi(x))$, for any $x \in \mathbb{R}^3$.*

The diffeomorphness of local control systems Σ and $\tilde{\Sigma}$ (germs at points $p \in \mathbb{R}^3$ and $\tilde{p} \in \mathbb{R}^3$ respectively) means the existence of a diffeomorphism $\Phi : (U, p) \to (\tilde{U}, \tilde{p})$ from some neighborhood U of p to some neighborhood $\tilde{U}$ of $\tilde{p}$ such that $\Phi_{*,x}\Sigma(x) = \tilde{\Sigma}(\Phi(x))$, for any $x \in U$. Terminology: two global control systems are locally diffeomorphic at a point p if their germs at p are diffeomorphic.

It is clear that *for any non-vanishing differential* 1*-forms α and $\tilde{\alpha}$ on $\mathbb{R}^3$ the control systems $\{\alpha = 1\} \subset T\mathbb{R}^3$ and $\{\tilde{\alpha} = 1\} \subset T\mathbb{R}^3$ are (locally) diffeomorphic if and only if the* 1*-forms α and $\tilde{\alpha}$ are (locally) diffeomorphic, i.e. there exists a (local) diffeomorphism Φ sending α to $\tilde{\alpha}$: $\Phi^*\alpha = \tilde{\alpha}$.*

Conclusion. *The following problems are* **exactly the same***:*

1. Classification of control systems (1.1) (given in coordinates) under assumption (1.2) with respect to the feedback equivalence

2. Classification of subsets of the tangent bundle $T\mathbb{R}^3$ corresponding to (1.1) with respect to the natural equivalence (equivalent = diffeomorphic)

3. Classification of non-vanishing differential 1*-forms on $\mathbb{R}^3$ with respect to the natural equivalence (equivalent = diffeomorphic).*

Important remark. The local classification Problem 3. (for germs) is a classical problem, first results were obtained more than 100 years ago by Darboux. Therefore Darboux results and other, more recent results on classification of 1-forms can be formulated in terms of control theory.

From $\mathbb{R}^3$ to $\mathbb{R}^n$ or M^n. Of course all constructions, definitions and conclusions given above hold in $\mathbb{R}^n$, or on an n-dimensional manifold M^n (not only in $\mathbb{R}^3$) provided that we start with a control system of the form
$\dot{x} = f(x) + u_1 g_1(x) + \cdots + u_{n-1} g_{n-1}(x)$, $x \in \mathbb{R}^n$ with pointwise linearly independent vector fields $f, g_1, \ldots, g_{n-1}$. The remark about the time when the first results were obtained remains true.

Definition 1.2. Let Σ be a subset of the tangent bundle $T\mathbb{M}^n$. A curve $\gamma : (a, b) \to M^n$ is called *integral* (another name: *admissible*) if the velocity vector $\dot{\gamma}(t) \in T_{\gamma(t)} M^n$ belongs to the set $\Sigma(\gamma(t))$ for any $t \in (a, b)$.

One of the mains task of geometric control theory is studying the structure of the set of all integral curves for subsets of the tangent bundle of a certain type.

Appendix (contains open question):

On the feedback equivalence of arbitrary control systems of the form
$\dot{x} = F(x, u),\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$
and the diffeomorphness of the corresponding subsets of $T\mathbb{R}^n$.

A control system also can be treated as a family of vector fields. Consider a family of vector fields $\dot{x} = F(x; u),\ x \in \mathbb{R}^n$, parameterized by $u \in \mathbb{R}^m,\ \ m \leq n$. For such families one of the tasks of control theory is the classification with respect to the following equivalence, called feedback equivalence. Two families are called feedback equivalent if they can be brought one to the other by a diffeomorphism of $\mathbb{R}^{n+m}$ of the form $(x, u) \to (\phi(x), \psi(x, u))$. Associate to a family $\dot{x} = F(x; u)$ a subset $\Sigma \subset T\mathbb{R}^n$ such that for any $x \in \mathbb{R}^n$ the subset $\Sigma(x) \subset T_x\mathbb{R}^n$ has the form $\Sigma(x) = \{F(x; u)\}_{u \in \mathbb{R}^m}$. In other words, $\Sigma(x)$ is the set of all possible velocities at the point x.

Question. Is it true that the local feedback equivalence of two families $\dot{x} = F(x; u),\ \dot{x} = \tilde{F}(x; u)$ is the same property as the local diffeomorphness of the corresponding subsets $\Sigma, \tilde{\Sigma} \subset T\mathbb{R}^n$?

It follows from the definitions that *the feedback equivalence of the families implies the diffeomorphness of* Σ *and* $\tilde{\Sigma}$ (think about this statement in order to understand well the definitions). Whether or not the converse holds is the same as to ask

Equivalent question. Assume that two families of the form $\dot{x} = F(x, u)$ correspond to the same subset of the tangent bundle. Is it true that the families are feedback equivalent?

Exercise 1.1. Prove that the answer is yes within local families $\dot{x} = F(x, u)$ (local = defined in a sufficiently small neighborhood of $(0, 0)$) which are non-degenerate with respect to controls, i.e. the $n \times m$ matrix $(\frac{\partial F_i}{\partial u_j})(0, 0)$ has maximally possible rank m.

Exercise 1.2. Prove that the answer is no for families

a. $\dot{x} = u$ and $\dot{x} = u^3 \quad (n = m = 1)$

b. $\dot{x} = ux$ and $\dot{x} = ux^2 \quad (n = m = 1)$

c. $\dot{x} = uv(x)$ and $\dot{x} = u(x_1^2 + x_2^2)v(x)$, where $v(x)$ is a vector field on $\mathbb{R}^2$, $v(0) = 0 \ \ (n = 2, m = 1)$

and give few more similar examples.

The examples in the last exercise are "very degenerate". *Conjecturally the answer is yes for all families beyond certain set E_∞ of infinite codimension, but as far as I know neither complete proof nor description of E_∞ are available.*

2. Pfaff equations $\alpha = 0$. Integrating factor. Frobenius theorem.

Important convention. In what follows all objects are germs at $0 \in \mathbb{R}^n$ and belong to a fixed category: C^∞ or real-analytic.

Let α be a differential 1-form on $\mathbb{R}^n$. A subset of $\Sigma \subset T\mathbb{R}^n$ given by the equation $\alpha = 0$ is defined in exactly the same way as the subset of $T\mathbb{R}^n$ given by the equation $\alpha = 1$, see section 1. It is called *the field of kernels of* α. The equation $\alpha = 0$ is called Pfaff equation. (More precisely, a local Pfaff equation is a 1-generated module of differential 1-forms over the ring of function germs.) If $\alpha(0) \neq 0$ then the field of kernels of α is as follows:

$$\{\alpha = 0\} \leftrightarrows \Sigma = \{\Sigma(x)\}_{x \in \mathbb{R}^n}, \quad \Sigma(x) = \text{ subspace of codimension 1 in } T_x\mathbb{R}^n.$$

Such subsets of the tangent bundle are called corank one (vector) distributions.

The sign $\leftrightarrows$ means that locally a corank one distribution and a Pfaff equation $\alpha = 0, \alpha(0) \neq 0$ are two objects that can be identified, but one should understand well what does this mean. This means that any germ of a Pfaff equation defines, in the given above way, a *unique* germ of a corank one distribution and any germ of a corank one distribution is defined by a *unique* germ of a Pfaff equation $\alpha = 0, \alpha(0) \neq 0$. Of course this requires the natural convention that $\alpha = 0$ *and* $Q\alpha = 0$*, where* Q *is a non-vanishing function, is the same Pfaff equation* (since α and $Q\alpha$ generate the same module of differential 1-forms over the ring of function germs).

Definition 2.1. A 1-form α admits an integrating factor if there exists a function Q, $Q(0) \neq 0$, such that $Q\alpha = dH$ for some function H. The function H is called the first integral.

According to the definitions, a differential 1-form α admits an integrating factor if and only if the Pfaff equation $\alpha = 0$ coincides with the Pfaff equation $dH = 0$ for some function H.

Important remark. Here and in further sections Definition 2.1 does **not** exclude the case $\alpha(0) = 0$.

In some works another definition is used:

Definition 2.1.a. α admits an integrating factor if $\alpha \wedge dH \equiv 0$ for some function H (called the first integral) which does not vanish at a generic point of a neighborhood of the origin.

If $\alpha(0) \neq 0$ then the two definitions are equivalent. If $\alpha(0) = 0$ then Definition 2.1.a is a weaker property of α. For example, take two function f, g such that $f(0) = 0, (df \wedge dg)(0) \neq 0$. It is easy to see that the 1-form fdg does not admit an integrating factor according to Definition 2.1, but of course it admits an integrating factor according to Definition 2.1.a. Nevertheless the difference between Definitions 2.1 and 2.1.a holds only in a very degenerate case. In the analytic category the relation $\alpha \wedge dH \equiv 0$ implies that $Q\alpha = dH$ for some function $Q, Q(0) \neq 0$ provided that the

1-forms α and dH are not divisible by a function vanishing at the origin. Equivalent condition: the Taylor power series of the coefficients of the 1-forms α and dH have no non-trivial common factor (are not divisible by a formal power series without free term). Under these conditions the 1-forms α and dH have so-called 1-division property, see section 8.

In the 2-dimensional case a Pfaff equation has the form

$$\alpha = A_1(x,y)dx + A_2(x,y)dy = 0,$$

and it can be found in almost any textbook on ODE's along with some ways of calculating an integral factor. Some textbooks contain the following

Proposition 2.1. (follows from basic theorems on ODE's.) *Any Pfaff equation $\alpha = 0$ on $\mathbb{R}^2$ such that $\alpha(0) \neq 0$ (i.e. either $A_1(0,0) \neq 0$ or $A_2(0,0) \neq 0$) admits a local integrating factor.*

Proof. Take any volume form Ω on $\mathbb{R}^2$ (for example $dx \wedge dy$) and define a vector field V by the relation $\Omega(V, \cdot) = \alpha$, where the dot denotes any vector field. (If α has the given above form and $\Omega = dx \wedge dy$ then $V = A_2(x,y)\frac{\partial}{\partial x} - A_1(x,y)\frac{\partial}{\partial y}$.) Since $\alpha(0) \neq 0$ then $V(0) \neq 0$ and V can be "straightened": in suitable coordinates $V = \frac{\partial}{\partial x}$. Then, in the same coordinates, $\alpha = Q(x,y)dy$, where Q is a non-vanishing function.

Not many textbooks on ODEs explain that the existence of an integrating factor for a Pfaff equation $\alpha = 0$ on $\mathbb{R}^2$, where $\alpha(0) = 0$, requires the following necessary condition.

Proposition 2.2. *Let α be a differential 1-form on $\mathbb{R}^2$ (or on $\mathbb{R}^n$ with any n) such that $\alpha(0) = 0$. If the Pfaff equation $\alpha = 0$ admits an integrating factor then $d\alpha(0) = 0$.*

Proof. Let $Q\alpha = dH$, $Q(0) \neq 0$. Take the differential of this equality. We obtain $dQ \wedge \alpha + Qd\alpha \equiv 0$. Since $\alpha(0) = 0$ and $Q(0) \neq 0$ then $d\alpha(0) = 0$.

In the 2-dimensional case, in terms of a vector field V associated to α via a volume form, the conditions $\alpha(0) = 0, d\alpha(0) = 0$ mean that the origin is a singular point of V and the trace of V at this point (the sum of the eigenvalues of the matrix of linearization) is equal to zero. This means, for example that

(a) there is no integrating factor in the case that the phase portrait of V is a focus or a node

(b) there is no integrating factor if the phase portrait of V is a saddle and the ratio of the eigenvalues of V is different from -1.

Exercise 2.1. Prove (a) using just the picture of a focus or a node, without Proposition 2.2. Hint: prove that if $Q\alpha = dH$ then $V(H) \equiv 0$ where $V(H)$ is the Lie derivative of H along V.

Just few textbooks on ODE's explain that already in the 2-dimensional case with $\alpha(0) = 0$ the necessary condition $d\alpha(0) = 0$ is *very far* from being sufficient, and studying sufficient conditions and related problems is a significant part of qualitative theory of ODE's including the center-focus problem and a number of works of the 20-th and the 21-st centuries. See section 7.

In the 2-dimensional case a Pfaff equation $\alpha = 0$ defines a foliation of a neighborhood of the origin by curves. It is the phase portrait of a vector field V associated to α via a volume form. If $\alpha(0) \neq 0$ then this foliation is non-singular and has the form $\{H = const\}$, where H is a function such that $dH(0) \neq 0$.

Return to the n-dimensional case.

Definition 2.2. A Pfaff equation $\alpha = 0$ on $\mathbb{R}^n$ defines a non-singular foliation (= foliation of a neighbourhood of the origin by non-singular hypersurfaces) if the following holds for some function H such that $dH(0) \neq 0$: a curve $\gamma : (a, b) \to \mathbb{R}^n$ is an integral curve of the field of kernels of α if and only if $H(\gamma(t)) \equiv const, t \in (a, b)$.

If this is the case then of course the non-singular foliation defined by the Pfaff equation $\alpha = 0$ is the foliation $\{H = const\}$.

Theorem 2.1 (Frobenius). Let α be a differential 1-form on $\mathbb{R}^n$. If $\alpha(0) \neq 0$ then the following conditions are equivalent:

(a) α admits an integrating factor

(b) the Pfaff equation $\alpha = 0$ defines a non-singular foliation

(c) $\alpha \wedge d\alpha \equiv 0$.

Note that in the 2-dimensional case (c) holds for *any* differential 1-form α since $\alpha \wedge d\alpha$ is a 3-form. Therefore Proposition 2.1 (de facto - basic theorem of ODE's) is a particular case of the Frobenius theorem.

Exercise 2.2. Prove the equivalence of (a) and (b) (in order to understand well the definitions).

It is also clear that (a) implies (c): the relation $Q\alpha = dH$ with a non-vanishing function Q can be written in the form $\alpha = RdH, R = 1/Q$, and then $d\alpha = dR \wedge dH$ and consequently $\alpha \wedge d\alpha \equiv 0$.

Therefore the only non-trivial part of Theorem 2.1 is (c) $\Longrightarrow$ (a). I will prove this using the Darboux theorem on normal form for closed 2-forms of a constant rank (not necessarily maximum possible).

A differential 2-form ω is closed if $d\omega \equiv 0$. Locally (for germs) this is so if and only if $\omega = d\alpha$ for some 1-form α. If a differential 2-form ω is given in coordinates then one can associate to it an anti-symmetric matrix $A(x)$; the rank of ω at a point $x \in \mathbb{R}^n$ is $rankA(x)$. It is always an even number. The canonical (coordinate-free) definition is as follows: the rank of ω at a point $x \in \mathbb{R}^n$ is equal to $2m$, where m is defined as follows: $\omega^m(x) \neq 0, \omega^{m+1}(x) = 0$ ($\omega^m = \omega \wedge \omega \wedge \cdots \wedge \omega$ m times).

Theorem 2.2. (Darboux). Let ω be a closed 2-form on $\mathbb{R}^n$ of constant rank $2m$. Then in suitable coordinate system $\omega = dx_1 \wedge dy_1 + \cdots + dx_m \wedge dy_m$.

This is a classical theorem, the normal form for symplectic structures is its particular (though the most known) case $n = 2m$.

Proof of the non-trivial part of the Frobenius Theorem 2.1 ((c) $\Longrightarrow$ (a)). We will need the following simple statement.

Exercise 2.3. Let α be a differential 1-form on $\mathbb{R}^n$. Prove that if $\alpha(0) \neq 0$ then there exists a non-vanishing function Q such that $(d(Q\alpha))(0) \neq 0$.

This statement allows to assume that $d\alpha(0) \neq 0$, without loss of generality. Taking the differential of the relation $\alpha \wedge d\alpha \equiv 0$ we obtain $d\alpha \wedge d\alpha \equiv 0$. Since $d\alpha(0) \neq 0$ then the closed 2-form $d\alpha$ has constant rank 2. By Theorem 2.2 in suitable coordinates $(x_1, ..., x_n)$ one has $d\alpha = dx_1 \wedge dx_2$. Then $\alpha = x_1 dx_2 + dS(x_1, ..., x_n)$ for some function S. Return to the relation $\alpha \wedge d\alpha \equiv 0$. We obtain that S depends on x_1, x_2 only. This reduces the Frobenius theorem to the 2-dimensional case: an integrating factor exists by Proposition 2.1.

3. Integrable 1-forms. Kupka phenomenon.

Frobenius theorem leads to the following

Definition 3.1. A differential 1-form α on $\mathbb{R}^n$ is called integrable if $\alpha \wedge d\alpha \equiv 0$. In this definition α is an *arbitrary* differential 1-form, the case that it vanishes at the origin or on any set of points including the origin is *not* excluded.

By Frobenius theorem, if $\alpha(0) \neq 0$ then α admits a local integrating factor if and only if it is integrable. If α is an integrable 1-form and $\alpha(0) = 0$ then a local integrating factor exists in a sufficiently small neighborhood of any point $p \notin Sing(\alpha)$, where

$$Sing(\alpha) = \{x \in \mathbb{R}^n : \ \alpha(x) = 0\}.$$

Consequently, a Pfaff equation $\alpha = 0, \alpha(0) = 0$ defines a singular foliation - a foliation by non-singular hypersurfaces of the set $\mathbb{R}^n - Sing(\alpha)$. Of course in what follows we will assume that the set $\mathbb{R}^n - Sing(\alpha)$ is open and dense. These non-singular hypersurfaces (leaves of the foliation) can approach the set $Sing(\alpha)$ in various ways. Moreover, the set $Sing(\alpha)$ itself can have various structures. The main goal of the local analysis of singular foliations is to distinguish the cases which are locally structurally stable in certain sense. This is not an easy task because of the integrability (Frobenius) condition $\alpha \wedge d\alpha \equiv 0$ - the set of germs of 1-forms satisfying this condition has a very complicated structure (C. Camacho, A. Lins Neto). Some results are from the first look very surprising, and sometimes are called phenomena.

In what follows we will study integrable 1-forms $\alpha, \alpha(0) = 0$ and singular foliations defined by Pfaff equation $\alpha = 0$. I will use the following definition of a structurally stable case.

Definition 3.2 (of a structurally stable case within integrable 1-forms vanishing at $0 \in \mathbb{R}^n$.)[3]
By "a case" I mean a set S of germs at $0 \in \mathbb{R}^n$ of integrable 1-forms $\alpha, \alpha(0) = 0$. I will say that the case S is structurally stable if for any $\alpha^* \in S$ there exists $k < \infty$ such that S contains all germs at 0 of integrable 1-form $\alpha, \alpha(0) = 0$ whose k-jet at 0 is sufficiently close to the k-jet of α^*. If there exists such a k that does not depend on α^* then I will say that the case S is C^k-structurally stable, otherwise it is C^∞ structurally stable.

The first phenomenon was discovered by I. Kupka and probably G. Reeb, because in some works it is called Kupka phenomenon, in some works Kupka-Reeb phenomenon. Let

$$\alpha = \alpha_1(x_1, ..., x_n)dx_1 + \cdots + \alpha_n(x_1, ..., x_n).$$

Then

$$Sing(\alpha) = \{x \in \mathbb{R}^n : \alpha_1(x_1, ..., x_n) = \cdots = \alpha_n(x_1, ..., x_n) = 0\}.$$

Consider also, in the ring of all function germs,

the ideal $I(\alpha) = (\alpha_1(x), ..., \alpha_n(x))$ generated by function germs $\alpha_1, ..., \alpha_n$.

The set $Sing(\alpha)$ is the zero set of the ideal $I(\alpha)$.

Exercise 3.1. Prove that the ideal $I(\alpha)$ does not depend on the choice of the coordinate system.

Definition. Let $f_1, ..., f_r$ be functions vanishing at the origin. They are called differentially independent if $(df_1 \wedge \cdots \wedge df_r)(0) \neq 0$.

It is clear that *the case that the ideal $I(\alpha)$ is generated by n differentially independent functions (and consequently $Sing(\alpha) = \{0\}$) is C^1-structurally stable in the sense of Definition 3.2.*

It seems that the case $codimSing(\alpha) < n$ would be very special.

[3] this definition will be used in sections 3 and 4; in section 5 another definition (structural stability within all integrable 1-forms) will be used.

Theorem 3.1. (Kupka phenomenon, 1964). *The case*

K: *$d\alpha(0) \neq 0$, the ideal $I(\alpha)$ is generated by two differentially independent functions (and consequently $Sing(\alpha)$ is a non-singular codimension 2 submanifold of $\mathbb{R}^n$)*

is C^1 structurally stable in the sense of Definition 3.2.

Theorem 3.1 is a corollary of the following result on normal forms

Theorem 3.2. (I. Kupka, 1964). *Let α be an integrable 1-form on $\mathbb{R}^n$ such that $\alpha(0) = 0,\ d\alpha(0) \neq 0$. Then in suitable local coordinates $\alpha = a_1(x_1, x_2)dx_1 + a_2(x_1, x_2)dx_2$ (the coordinates $x_3, ..., x_n$ are not present).*

Proof of Theorem 3.1 using Theorem 3.2. Let α^* be an integrable 1-form vanishing at 0 and satisfying the assumptions **K**. Let $\alpha, \alpha(0) = 0$ be any integrable 1-form with the linear approximation at 0 sufficiently close to that of α^*. Then $d\alpha(0) \neq 0$. Consider the $n \times n$ matrices $A^*(x) = (\frac{\partial \alpha_i^*}{\partial x_j})$ and $A(x) = (\frac{\partial \alpha_i}{\partial x_j})$. One has $rankA^*(0) = 2$. The closeness of the linear part of α to that of α^* implies $rankA(0) \geq 2$. By Theorem 3.2 α has the form $a(x_1, x_2)dx_1 + b(x_1, x_2)dx_2$ in some coordinate system. The condition $rankA(0) \geq 2$ does not depend on the choice of coordinate system. Therefore the matrix $(\frac{\partial a_i}{\partial x_j})(0), i, j = 1, 2$ has rank ≥ 2 and consequently it has maximal possible rank 2. This means that the functions $a_1(x_1, x_2)$ and $a_2(x_1, x_2)$ are differentially independent. Consequently the 1-form α also satisfies the assumptions **K**.

Exercise 3.2. The proof used the fact that the rank of the matrix $A(0)$ does not depend on the choice of coordinate system. Prove this.

Proof of Theorem 3.1. The integrability condition $\alpha \wedge d\alpha \equiv 0$ implies $d\alpha \wedge d\alpha \equiv 0$, and the assumption $d\alpha(0) \neq 0$ plays the crucial role - it allows to conclude that the closed 2-form $d\alpha$ has **constant** rank 2. This allows to apply the Darboux Theorem 2.2 and to **repeat exactly the same arguments** as in the proof of Frobenius Theorem 2.1.

Theorem 3.2 implies the following obvious corollary.

Corollary 3.3. *If α is an integrable 1-form such that $\alpha(0) = 0, d\alpha(0) \neq 0$ then $codimSing(\alpha) \leq 2$.*

Exercise 3.3. Draw the foliation defined by the Pfaff equation on $\mathbb{R}^3$ of the form $x_1dx_2 - x_2dx_1$.

This exercise and Theorem 3.2 explain the name *open book foliation* in the case that α is an integrable 1-form, $\alpha(0) = 0, d\alpha(0) \neq 0$.

Another corollary of Theorem 3.2 is as follows.

Proposition 3.1. *The local classification of Pfaff equations $\alpha = 0$ (with respect to diffeomorphisms), where α is an integrable* 1*-form, $\alpha(0) = 0, d\alpha(0) \neq 0$, is exactly the same problem as the local classification with respect to the orbital equivalence of vector fields V on the plane $\mathbb{R}^2$ such that $V(0) = 0$ and $traceV(0) \neq 0$.*

I explain this statement. By definition, two Pfaff equations $\alpha = 0$ and $\tilde{\alpha} = 0$ are diffeomorphic if there exists a local diffeomorphism Φ such that $\Phi^*\alpha = Q\tilde{\alpha}$, where Q is a non-vanishing function (i.e. the Pfaff equations $\Phi^*\alpha = 0$ and $\tilde{\alpha} = 0$ are the same). By definition, two vector fields V and $\tilde{V}$ are orbitally equivalent if there exists a local diffeomorphism Φ such that $\Phi_*V = Q\tilde{V}$, where Q is a non-vanishing function. In this definition the sign of $Q(0)$ might be either $+$ or $-$. The diffeomorphism Φ sends the phase portrait of V to the phase portrait of $\tilde{V}$ preserving orientation of phase curves in the case $+$ and reversing it in the case $-$.

The problem of orbital classification of vector fields V on $\mathbb{R}^2$ is a classical problem with a huge number of directions and results obtained in the 19-th, 20-th and the 21-st centuries. The case that V has non-zero trace at the singular point 0 is relatively easy and well-studied.

Proposition 3.1 follows from Theorem 3.2 and the following two simple statements.

Exercise 3.4. Let α and $\tilde{\alpha}$ be differential 1-forms on $\mathbb{R}^2$ and let V and $\tilde{V}$ be vector fields on $\mathbb{R}^2$ corresponding to these 1-forms via a non-degenerate volume form Ω on $\mathbb{R}^2$. Prove that the Pfaff equations $\alpha = 0$ and $\tilde{\alpha} = 0$ are diffeomorphic if and only if the vector fields V and $\tilde{V}$ are orbitally equivalent.

Hint. Apply a local diffeomorphism Φ to the equalities $V \rfloor \Omega = \alpha$ and $\tilde{V} \rfloor \Omega = \tilde{\alpha}$ and take into account that $\Phi^*\Omega$ is always a non-degenerate volume form.

Exercise 3.5 (requires some work). Consider two Pfaff equations on $\mathbb{R}^n$ of the form $\alpha = \alpha_1(x_1, x_2)dx_1 + \alpha_2(x_1, x_2)dx_2 = 0$ and $\tilde{\alpha} = \tilde{\alpha}_1(x_1, x_2)dx_1 + \tilde{\alpha}_2(x_1, x_2)dx_2 = 0$. Prove that if $d\alpha(0) \neq 0$ and $d\tilde{\alpha}(0) \neq 0$ then they are diffeomorphic if and only if they are diffeomorphic as Pfaff equations on $\mathbb{R}^2$. Prove that in the analytic category the assumption $d\alpha(0) \neq 0, d\tilde{\alpha}(0) \neq 0$ in this statement is not needed.

Probably in the C^∞ category the assumption $d\alpha(0) \neq 0, d\tilde{\alpha}(0) \neq 0$ in the statement of Exercise 3.5 is neither needed, but I am not sure (this requires rather technical job with zero divisors). Anyhow, Proposition 3.1 follows from the statements of exercises 3.4 and 3.5.

One more corollary of Theorem 3.1 is as follows.

Proposition 3.2. *An integrable* 1*-form $\alpha, \alpha(0) = 0, d\alpha(0) \neq 0$ on $\mathbb{R}^n$ can be described by $(n - 1)$ commuting vector field such that $(n - 2)$ of them are non-singular.*

I explain. A tuple of vector fields is called commuting if $[V_i, V_j] \equiv 0$ for any vector fields V_i, V_j of the tuple. Here $[V_i, V_j]$ is the Lie bracket. Any $(n-1)$ vector fields on $\mathbb{R}^n$ define a differential 1-form α on $\mathbb{R}^n$ via a non-degenerate volume form Ω on $\mathbb{R}^n$ by the relation

$$\alpha(Y) = \Omega(X_1, ..., X_{n-1}, Y), \text{ where } Y \text{ is an arbitrary vector field.}$$

We will say that α is associated to the tuple $(V_1, ..., V_{n-1})$ via Ω. The following proposition is well-known.

Proposition 3.3. *Let $V_1, ..., V_{n-1}$ be commuting tuple of vector fields on $\mathbb{R}^n$. The 1-form α associated to the tuple $V_1, ..., V_{n-1}$ via a volume form is always integrable. Moreover, α is integrable under a weaker assumption: it is enough to require that $[V_i, V_j](x) \in span(V_1(x), ..., V_n(x))$ for any point $x \in \mathbb{R}^n$.*

Exercise 3.6. Prove Proposition 3.3 in the following way:

1. Prove that $\alpha(V_i) \equiv 0$.

2. Use the particular case of Cartan formula:
if $\alpha(V_i) = \alpha(V_j) \equiv 0$ then $d\alpha(V_i, V_j) = \alpha([V_i, V_j])$.

3. Prove that $(\alpha \wedge d\alpha)(x) = 0$ for any point $x \in \mathbb{R}^n$.

To prove Proposition 3.2 it suffices to note that a 1-form of the form $a_1(x_1, x_2)dx_1 + a_2(x_1, x_2)dx_2$ is associated, via $\Omega = dx_1 \wedge \cdots \wedge dx_n$, to the tuple

$$V_1 = \frac{\partial}{\partial x_3}, ..., V_{n-2} = \frac{\partial}{\partial x_n}, \ V_{n-1} = a_2(x_1, x_2)\frac{\partial}{\partial x_1} - a_1(x_1, x_2)\frac{\partial}{\partial x_2}.$$

Exercise 3.7. Let $V_1, ..., V_{n-1}$ be any tuple of commuting vector fields such that $V_{n-1}(0) = 0$ and $V_1(0), ..., V_{n-2}(0)$ are linearly independent vectors. Prove that in suitable coordinates the 1-form α associated to the tuple via a volume form has the form $a_1(x_1, x_2)dx_1 + a_2(x_1, x_2)dx_2$.

I conclude this section with

Proposition 3.4. *If α is an integrable 1-form, $\alpha(0) = 0, d\alpha(0) \neq 0$ then α does not admit an integrating factor.*

This proposition was already proved in section 2 (Proposition 2.2), and the word "integrable" in Proposition 3.4 can be omitted (I just want to underline that the integrability does not imply integrating factor). The proof is so simple that it is worth to repeat it: if an integrating factor exists then (by Definition 2.1) there exists functions $Q, Q(0) \neq 0$ and H such $Q\alpha = dH$. It follows that $dQ \wedge \alpha + Qd\alpha \equiv 0$ and since $\alpha(0) = 0, Q(0) \neq 0$ then $d\alpha(0) = 0$.

4. Frobenius through singularities (G. Reeb, R. Moussu, B. Malgrange).

Remind the notations introduced in the previous section: $Sing(\alpha) \subset \mathbb{R}^n$ is the set of points at which a 1-form α vanishes, and $I(\alpha)$ is the ideal in the ring of functions generated by the n coefficients of α. This ideal does not depend on the choice of a coordinate system and $Sing(\alpha)$ is its zero set.

In section 3 we discussed, within integrable 1-forms on $\mathbb{R}^n$ such that $\alpha(0) = 0$, the Kupka case

case K: $d\alpha(0) \neq 0$, the ideal $I(\alpha)$ is generated by two differentially independent functions (and consequently $Sing(\alpha)$ is a codimension 2 non-singular submanifold of $\mathbb{R}^n$.)

We proved that this case is C^1-structurally stable (according to Definition 3.2) and within this case integrable 1-forms do not admit an integrating factor. The following question is a natural classical question going back to the 19-th century:

is there a structurally stable case (within integrable $\alpha, \alpha(0) = 0$) such that the Frobenius condition $\alpha \wedge d\alpha \equiv 0$ implies the existence of an integrating factor?

Exercise 4.1. Prove that if a 1-form $\alpha, \alpha(0) = 0$ admits an integrating factor (i.e. $Q\alpha = dH$, where $Q(0) \neq 0$) then the singular foliation defined by the Pfaff equation $\alpha = 0$ is the foliation $\{H = const\}$, i.e. any curve $\gamma : (a, b) \to \mathbb{R}^n$ (including curve passing through 0) is an integral curve of the field of kernels of α if and only if $H(\gamma(t)) \equiv const,\ t \in (a, b)$.

The area related to this question can be called "Frobenius through singularities". The answer to the question is positive. The first result was obtained by G. Reeb. Consider the case

case R: the ideal $I(\alpha)$ is generated by n differentially independent functions (and consequently $Sing(\alpha) = \{0\}$).

It is clear that the case **R** is C^1 structurally stable in the sense of Definition 3.2.

Theorem 4.1. (G. Reeb, 1956) *In the analytic category with $n \geq 3$ any integrable form α within the case* **R** *admits an integrating factor.*

Remark. The condition $n \geq 3$ cannot be taken away (see section 2).

The starting point for the analysis of case **R** is the following

Proposition 4.1. *Let α be an integrable 1-form on $\mathbb{R}^n, n \geq 3$ within the case* **R**. *Then $d\alpha(0) = 0$.*

Proof. Assume that $d\alpha(0) \neq 0$. Then by Kupka Theorem 3.2 one has $codimSing(\alpha) \leq 2$. This contradicts to **R** because $n \geq 3$.

Note that the same conclusion $d\alpha(0) = 0$ can be made under much weaker assumption than **R**. Namely, in the analytic category consider the complexification $\alpha^{\mathbb{C}}$ of α (take any coordinate system and replace coordinates $x_1, ..., x_n$ by complex-valued coordinates $z_1, ..., z_n$). Then the set $Sing(\alpha^{\mathbb{C}})$ is a subset of $\mathbb{C}^n$ which is the zero set of an ideal generated by holomorphic functions. The codimension of such subset of $\mathbb{C}^n$ is defined as follows. Take a point p near which $Sing(\alpha^{\mathbb{C}})$ is a non-singular submanifold of $\mathbb{C}^n$. Let c_p be the (complex) codimension of $Sing(\alpha^{\mathbb{C}})$ near p. Let U be a neighborhood of the origin. Let $c_p(U)$ be the minimum of c_p over all points $p \in U$ near which $Sing(\alpha^{\mathbb{C}})$ is a non-singular submanifold of $\mathbb{C}^n$. Then $codimSing(\alpha^{\mathbb{C}}) = lim c_p(U)$ as $U \to 0$. Consider the case

case MAL (only in analytic category) $codimSing(\alpha^{\mathbb{C}}) \geq 3$

Of course in the analytic category **R**$\subset$**MAL**. It is clear that if the ideal $I(\alpha)$ is 2-generated then $codimSing(\alpha^{\mathbb{C}}) \leq 2$. Therefore Kupka Theorem 3.2 implies the following

Proposition 4.2. *Let α be an integrable analytic 1-form within the case* **MAL**. *Then $d\alpha(0) = 0$.*

The following proposition is intuitively clear, but its precise proof requires rather involved techniques of singularity theory.

Proposition 4.3. *The case* **MAL** *is C^∞ structurally stable in the sense of Definition 3.2.*

Exercise 4.2. Prove that for any $k < \infty$ the case **MAL** is not C^k structurally stable. It suffices to show that for any fixed k there exists an ideal generated by n holomorphic functions with zero k-jets such that the zero set of the ideal has codimension ≤ 3.

The following theorem by B. Malgrange is the strongest and to some extend the final result on the existence of an integrating factor in the analytic category.

Theorem 4.2 (B. Malgrange, 1976) *In the analytic category any integrable form α within the case* **MAL** *admits an integrating factor.*

Proposition 4.2 explains the role of the assumption $codimSing(\alpha^{\mathbb{C}}) \geq 3$: namely due to this assumption one has $d\alpha(0) = 0$. Another role of the assumption $codimSing(\alpha^{\mathbb{C}}) \geq 3$ is as follows.

Proposition 4.4. (see section 8 for more general statements) *Let α be an analytic 1-form (not necessarily integrable) such that codimSing$(\alpha^{\mathbb{C}}) \geq 3$. Then α has the following property (called 2-division property): if ω is an analytic 2-form such that $\alpha \wedge \omega \equiv 0$ then there exists a 1-form β such that $\omega = \alpha \wedge \beta$.*

Exercise 4.3. Prove that a 1-form of the form $\alpha = A_1(x)dx_1 + A_2(x)dx_2$, where $A_1(0) = A_2(0) = 0$, does not have the 2-division property and consequently in Proposition 4.4 the assumption *codimSing*$(\alpha^{\mathbb{C}}) \geq 3$ cannot be replaced by the assumption *codimSing*$(\alpha^{\mathbb{C}}) \geq 2$.

Proposition 4.4 plays the crucial role in the proof of Theorem 4.2. The outline of the proof is as follows. The Frobenius condition $\alpha \wedge d\alpha \equiv 0$ and the 2-division property of α imply that $d\alpha = \alpha \wedge \beta_0$ for some 1-form β_0. Taking the differential of this equality one gets $0 \equiv \alpha \wedge d\beta_0 + d\alpha \wedge \beta_0$. The previous equality implies $d\alpha \wedge \beta_0 \equiv 0$. Consequently $\alpha \wedge d\beta_0 \equiv 0$. Now one can again use the 2-division property of α: $d\beta_0 = \alpha \wedge \beta_1$. Taking the differential and using the previous equalities one obtains $0 \equiv \alpha \wedge (\beta_0 \wedge \beta_1 + d\beta_1)$, and again the 2-division property of α can be used. The infinite series of equalities that can be obtained in this way allows to construct an integrating factor on the level of formal power series, i.e. to prove the existence of a formal power series $\hat{Q}$ with a non-zero free term and a formal power series $\hat{H}$ such that $\hat{Q}\hat{\alpha} = d\hat{H}$, where $\hat{\alpha}$ is the Taylor series of α. After that it can be proved that the formal power series $\hat{Q}$ and $\hat{H}$ can be chosen to be convergent.

Remark. Theorem 4.2 under a stronger assumption than **MAL** was published in the work of R. Moussu, 1976 (earlier than B. Malgrange, 1976) along with results on existence of an integrating factor in the C^∞ category (these results are discussed below). The way of constructing an integrating factor on the level of formal power series using the Frobenius condition along with the 2-division property of α is realized in R. Moussu, 1976. In this work Moussu worked under the case **M**$\subset$**MAL** (the case of algebraically isolated singularity, I present it below). The case **M** is defined in the same way in either the real analytic or in C^∞ category. R. Moussu showed that in any of these categories within the case **M** a 1-form has q-division property, for any $q \leq n-1$, see section 8. The construction of an integrating factor on the level of formal power series requires the 2-division property only. Therefore though the existence of a formal integrating factor was proved by R. Moussu, 1976 in the case **M**$\subset$**MAL**, his proof remains exactly the same for analytic integrable 1-forms within the case **MAL**. The possibility to chose a convergent integrating factor (a task requiring involved techniques) was proved in R. Moussu, 1976 under an additional assumption; a bit later B. Malgrange proved that this assumption can be taken away.

Now I will explain results on the existence of an integrating factor in the C^∞ category; the first result was obtained by G. Reeb and the final result - by R. Moussu.

Definition 4.1. A 1-form $\alpha, \alpha(0) = 0$ has an algebraically isolated singularity (at the origin) if the factor-space

$$\frac{\text{all formal power series}}{\text{Taylor power series of the coefficients } \alpha_1(x), ...\alpha_n(x) \text{ of } \alpha}$$

is a finite number $\mu < \infty$.

Remark. The number μ is called the multiplicity of the ideal $I(\alpha)$. There is a theorem in singularity theory stating that if $\mu < \infty$ then μ does not change if one replaces:

all formal power series $\quad \Rightarrow \quad$ the space of all germs
Taylor power series of $\alpha_1, ..., \alpha_n \quad \Rightarrow \quad \alpha_1, ..., \alpha_n$
(this is true in C^∞ and analytic categories).

Another well-known fact of singularity theory is as follows:

Proposition 4.5. *If a finitely generated ideal has a finite multiplicity then its zero set (the set of points at which all functions of the ideal vanish) is* $\{0\}$.

Consequently, if α has an algebraically isolated singularity then $Sing(\alpha) = \{0\}$ (in the analytic category this remains true after complexification of α.)

Consider the case, within integrable 1-forms $\alpha, \alpha(0) = 0$

case M $\quad \alpha$ has algebraically isolated singularity
(and consequently $Sing(\alpha) = \{0\}$)

Exercise 4.4. Show that if $n \geq 3$ then $\mathbf{R} \subset \mathbf{M} \subset \mathbf{MAL}$. Prove that in the case $\mathbf{R}$ the multiplicity of the ideal $I(\alpha)$ is equal to 1.

Of course, the case $\mathbf{M}$ is much larger than $\mathbf{R}$. The basic results of singularity theory imply the following

Proposition 4.6. *The case* $\mathbf{M}$ *is* C^∞ *structurally stable in the sense of Definition 3.2.*

Exercise 4.5. Explain why the case $\mathbf{M}$ is not C^k structurally stable for any finite k (cf. Exercise 4.2).

If α is an integrable 1-form on $\mathbb{R}^n$, $n \geq 3$ within the case $\mathbf{M}$ then $d\alpha(0) = 0$, see Propositions 4.1 and 4.2. It follows that the linear part of α has the form dH_2, where H_2 is a quadratic form (a homogeneous degree 2 function). It can be brought to the normal form $x_1^2 + \cdots + x_l^2 - x_{l+1}^2 - \cdots - x_r^2$, where $0 \leq l \leq r$ ($l = r$ means that all signs are $+$ and $l = 0$ means that all signs are $-$). The number r is the rank of the quadratic form H_2, and the number l (i.e. the number of $+$) is its index.

Notation. Let α be an integrable 1-form within the case $\mathbf{M}$. Then (as it was proved) the linear part of α has the form dH_2. Denote by $r = r(\alpha)$ the rank of the quadratic form H_2 and by $i(\alpha)$ the number $max(indexH_2,\ r - indexH_2)$.

Exercise 4.6. Prove that $r(\alpha)$ and $i(\alpha)$ are the invariants of the Pfaff equation $\alpha = 0$.

Theorem 4.3 (R. Moussu, 1976). *In the C^∞ category an integrable 1-form $\alpha \in$***M** *admits a C^∞ integrating factor provided that one of the following holds:*
(a) (proved by G. Reeb, 1956) $r(\alpha) = i(\alpha) = n$
(b) $n \geq 4,\ r(\alpha) = n,\ i(\alpha) = n-1$
(c) $n \geq 4,\ r(\alpha) = i(\alpha) = n-1$.

Remark. The case $r(\alpha) = n$ is exactly the case **R**.

The question on existence of an integrating factor in the case **M** in the C^∞ category was completely solved by R. Moussu in 1983. R. Moussu distinguished the elliptic and hyperbolic cases (the elliptic case is a weaker assumption than (a)). He proved the existence of a C^∞ integrating factor in the elliptic case and obtained a criterium (necessary and sufficient condition in geometric terms) for the existence of a C^∞ integrating factor in the hyperbolic case. This geometric condition always holds in the cases (b) and (c); in other cases it might be violated for certain integrable 1-forms that are not C^∞-diffeomorphic to analytic 1-forms. One of examples is as follows.

Remind that a function germ f at the origin is called flat if its Taylor series is zero, i.e. all derivatives of f at the origin are equal to zero. A differential 1-form is called flat if all its coefficients are flat (in some and then any coordinate system). In the analytic category the only flat 1-form is the zero 1-form, but in the C^∞ category this is not so. For example, if β is any 1-form and $f(x) = e^{\frac{-1}{||x||^2}}$ as $x \neq 0$ and $f(0) = 0$ then $\alpha = f\beta$ is a C^∞ flat 1-form.

Theorem 4.4 (R. Moussu, 1976). There exists a flat C^∞ 1-form τ on $\mathbb{R}^3$ such that the 1-form $\alpha = d(x_1^2 + x_2^2 - x_3^2) + \tau$ is integrable and does not admit a C^∞ integrating factor. (Moussu constructed such a flat 1-form τ explicitly).

I conclude this section with the following proposition. Remind that in the Kupka case an integrable 1-form on $\mathbb{R}^n$ can be described by $(n-1)$ vector fields, see Proposition 3.2.

Proposition 4.7. *In the case* **MAL** *an integrable* 1*-form cannot be described by vector fields.*

Proof. Assume that α is associated to a tuple of vector fields $V_1, ..., V_{n-1}$ via a volume form. Then

$$Sing(\alpha) = \{x \in \mathbb{R}^n : \ \ dimspan\{(V_1(x), ..., V_{n-1}(x)\} \leq n-2\}.$$

The contradiction follows from the following fact.

Exercise 4.7. Let $A(x)$ be an $(n-1) \times n$ matrix whose entries are holomorphic functions. Then

$$codim\{x \in \mathbb{C}^n : \ rankA(x) \leq n-2\} \geq 2.$$

5. Integrable 1-forms between Kupka and Malgrange (C. Camacho, A. Lins Neto, D. Cerveau, M. Chaperon).

Let α be an integrable analytic 1-form on $\mathbb{R}^n$, $\alpha(0) = 0$. The Malgrange case, studied in section 4, is the case $codimSing(\alpha^{\mathbb{C}}) \geq 3$. The case $codimSing(\alpha^{\mathbb{C}}) = 1$ is not interesting: in this case α has the form $f\beta$, where $f(0) = 0$ and β is also an integrable 1-form (see section 8). The significant case beyond the Malgrange assumption is the case $codimSing(\alpha^{\mathbb{C}}) = 2$. If $d\alpha(0) \neq 0$ then one has the Kupka case, see section 3. Between Kupka and Malgrange one has the case

$$\text{between Kupka and Malgrange :} \qquad d\alpha(0) = 0, \quad codimSing(\alpha^{\mathbb{C}}) = 2.$$

Are there structurally stable cases between Kupka and Malgrange? The answer is positive provided that in Definition 3.2 of structural stability one does not fix the source point of a germ. A series of results in this direction were obtained by C. Camacho and A. Lins Neto, 1978 - 1982. The most studied structurally stable case between Kupka and Malgrange is the case of **logarithmic** integrable 1-forms. This case was studied in a series of works of D. Cerveau and A. Lins Neto. It also was studied by M. Chaperon in terms of vector fields (see the end of this section). In the first part of this section I will present a result on structural stability in the 3-dimensional case, with a complete proof. In the second part I will present and explain some results on normal forms in the n-dimensional case.

PART 1. Example of a structurally stable case between Kupka and Malgrange.

Let $\lambda_1, \lambda_2, \lambda_3$ be any real non-zero numbers. Consider the 1-form

$$\alpha = x_1x_2x_3\left(\lambda_1 d(lnx_1) + \lambda_2 d(lnx_2) + \lambda_3 d(lnx_3)\right) =$$

$$= \lambda_1 x_2x_3dx_1 + \lambda_2 x_1x_3dx_2 + \lambda_3 x_1x_2dx_3 \tag{5.1}$$

It is clear from this relation that this 1-form is integrable. The set $Sing(\alpha)$ is the union of three lines - the x_1-axes, the x_2-axes, and the x_3-axes. The same holds after complexification of α. Therefore $codimSing(\alpha) = codimSing(\alpha^{\mathbb{C}}) = 2$. On the other hand, the 1-form (5.1) has zero 2-jet, therefore $d\alpha(0) = 0$. So, (5.1) is an example of an integrable 1-form in the case "between Kupka and Malgrange". Consider the following case:

the case C in some local coordinates the 2-jet of the germ α has form (5.1) with $\lambda_1, \lambda_2, \lambda_3$ satisfying the following arithmetic conditions:

Condition A the numbers $\lambda_3 - \lambda_2,\ \lambda_1 - \lambda_3,\ \lambda_2 - \lambda_1$ are distinct;

Condition A$_3$ if k_1, k_2, k_3 are non-negative integer numbers such that $k_1 + k_2 + k_3 \leq 3$ and $k_1(\lambda_3 - \lambda_2) + k_2(\lambda_1 - \lambda_3) + k_3(\lambda_2 - \lambda_1) = 0$ then $(k_1, k_2, k_3) = (1, 1, 1)$.

This case is well-defined in either C^∞ or real-analytic category. The explanation of the conditions **A** and $\mathbf{A}_3$ will be clear from the proof of the following theorem.

Theorem 5.1. (based on work's by C. Camacho, 1978) *The case* **C** *is* C^2 *structurally stable in the following sense. Let* α^* *be the germ at* 0 *of an integrable* 1*-form on* $\mathbb{R}^3$, $\alpha^* \in$**C**. *Let* α *be an integrable* 1*-form defined in a neighborhood of* 0 *such that* $j_0^2\alpha$ *is close to* $j_0^2\alpha^*$. *Denote by* α_p *the germ of* α *at a point* p. *There exists a point* p *close to the origin* ($p \to 0$ *as* $j_0^2\alpha \to j_0^2\alpha^*$) *such that* $\alpha(p) = 0$ *and* $\alpha_p \in$**C**.

Exercise 5.1. Prove that the case **C** is not structurally stable in the sense of Definition 3.2. Hint: the case **C** is "adjoint" to the Kupka case **K**. Given (5.1) consider the germ of this 1-form at a point $p \neq 0$ of one of the axes.

Proof of Theorem 5.1. Take a system of coordinates x_1, x_2, x_3 centered at 0 and such that $j_0^2\alpha^*$ has form (5.1). Let α be an integrable 1-form defined in a neighborhood of 0 and such that $j_0^2\alpha$ is close to $j_0^2\alpha^*$. Let

$\alpha = \alpha_1(x)dx_1 + \alpha_2(x)dx_2 + \alpha_3(x)dx_3,$

$d\alpha = a_1(x)dx_2 \wedge dx_3 + a_2(x)dx_3 \wedge dx_1 + a_3(x)dx_1 \wedge dx_2.$

Then the tuple $(j_0^1a_1(x), j_0^1a_2(x), j_0^1a_3(x))$ is close to the tuple $((\lambda_3 - \lambda_2)x_1, (\lambda_1 - \lambda_3)x_2, (\lambda_2 - \lambda_1)x_3)$. By Condition $\mathbf{A}_3$ the λ's are distinct, therefore $(da_1 \wedge da_2 \wedge da_3)(0) \neq 0$. It follows that there exists a point p near 0 such that

$$a_1(p) = a_2(p) = a_3(p) = 0, \quad (da_1 \wedge da_2 \wedge da_3)(p) \neq 0. \tag{5.2}$$

We will show that this is a required point, i.e. the germ of α at p satisfies **C**.

At first we will prove that $\alpha(p) = 0$. Consider the vector field V associated to the 2-form $d\alpha$ via a volume form Ω:

$$V \rfloor \Omega = d\alpha. \tag{5.3}$$

If $\Omega = dx_1 \wedge dx_2 \wedge dx_3$ then

$$V = a_1(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + a_3(x)\frac{\partial}{\partial x_3}. \tag{5.4}$$

Exercise 5.2. Prove that V is a divergent-free vector field (with respect to Ω) and consequently the sum of the eigenvalues of V at any singular point is zero. Hint: take the differential of (5.3).

Relation (5.3) implies that the 2-form $d\alpha$ annihilates V: $V \rfloor d\alpha \equiv 0$. The integrability condition $\alpha \wedge d\alpha \equiv 0$ implies that α also annihilates V, i.e.

$\alpha_1(x)a_1(x) + \alpha_2(x)a_2(x) + \alpha_3(x)a_3(x) \equiv 0$. Differentiating this relation at the point p and taking into account (5.2) we obtain that $\alpha_1(p) = \alpha_2(p) = \alpha_3(p) = 0$, i.e. $\alpha(0) = 0$.

The sum of the eigenvalues of V at the singular point p is zero (Exercise 5.2) and the eigenvalues are close to $\lambda_3 - \lambda_2, \lambda_1 - \lambda_3, \lambda_2 - \lambda_1$. The latter three numbers are distinct by the condition **A**. Therefore there exists a coordinate system y_1, y_2, y_3 centered at the point p and numbers $\lambda_1', \lambda_2', \lambda_3'$ close to $\lambda_1, \lambda_2, \lambda_3$ such that

$$j_p^1 V = (\lambda_3' - \lambda_2')y_1 \frac{\partial}{\partial y_1} + (\lambda_1' - \lambda_3')y_2 \frac{\partial}{\partial y_2} + (\lambda_2' - \lambda_1')y_3 \frac{\partial}{\partial y_3}. \tag{5.5}$$

Since the scaling the coordinates preserves (5.5) then there is no loss of generality to assume that (5.3) holds with a volume form Ω such that $\Omega(p) = dy_1 \wedge dy_2 \wedge dy_3$. It follows that

$$j_p^1(d\alpha) = (\lambda_3' - \lambda_2')y_1 dy_2 \wedge dy_3 + (\lambda_1' - \lambda_3')y_2 dy_3 \wedge dy_1 + (\lambda_2' - \lambda_1')y_3 dy_1 \wedge dy_2.$$

We have already proved that $\alpha(p) = d\alpha(p) = 0$. It follows that

$$j_p^2 \alpha = dH_2(y) + \alpha_2 + dH_3(y), \quad \alpha_2 = \lambda_1' y_2 y_3 dy_1 + \lambda_2' y_1 y_3 dy_2 + \lambda_3' y_1 y_2 dy_3, \tag{5.6}$$

where $H_2(y)$ and $H_3(y)$ are homogeneous degree 2 and 3 functions.

The Frobenius condition $\alpha \wedge d\alpha \equiv 0$ implies

$$dH_2 \wedge \alpha_2 \equiv 0. \tag{5.7}$$

Lemma 5.1. *(5.7) implies $H_2 \equiv 0$.*

Since $\alpha_2 \wedge d\alpha_2 \equiv 0$ then now the Frobenius condition $\alpha \wedge d\alpha \equiv 0$ implies

$$dH_3 \wedge \alpha_2 \equiv 0. \tag{5.8}$$

Lemma 5.2. *(5.8) implies $H_3 = ay_1y_2y_3,\ a \in \mathbb{R}$*

Exercise 5.3. Show that relations (5.7) and (5.8) imply that the Lie derivative of H_2 and H_3 along the linear vector field (5.5) is equal to zero, i.e.

$$(\lambda_3' - \lambda_2')y_1 \frac{\partial H_i}{\partial y_1} + (\lambda_1' - \lambda_3')y_2 \frac{\partial H_i}{\partial y_2} + (\lambda_2' - \lambda_1')y_3 \frac{\partial H_i}{\partial y_3} \equiv 0, \quad i = 2, 3$$

and prove Lemmas 5.1 and 5.2 using assumption $\mathbf{A}_3$ and the closeness of $\lambda_1', \lambda_2', \lambda_3'$ to $\lambda_1, \lambda_2, \lambda_3$.

By Lemmas 5.1 and 5.2

$$j_p^2 \alpha = \tilde{\lambda}_1 y_2 y_3 dy_1 + \tilde{\lambda}_2 y_1 y_3 dy_2 + \tilde{\lambda}_3 y_1 y_2 dy_3, \quad \tilde{\lambda}_i = \lambda_i' + a.$$

It remains to note that the numbers $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3$ satisfy the assumptions **A** and $\mathbf{A}_3$ since these assumptions respect the operation $\lambda_i \to \lambda_i + a, a \in \mathbb{R}$ and the numbers $\lambda_1', \lambda_2', \lambda_3'$ satisfy the assumptions **A** and $\mathbf{A}_3$ since they are close to $\lambda_1, \lambda_2, \lambda_3$.

PART 2. $(n-1)$**-determinacy of integrable logarithmic** 1**-forms on** $\mathbb{R}^n$.

Of course the logarithmic integrable 1-form (5.1) can also be defined (in the same way) in the n-dimensional case:

$$\alpha = \lambda_1 x_2 x_3 \cdot \cdots \cdot x_n dx_1 + \lambda_2 x_1 x_3 \cdot \cdots \cdot x_n dx_2 + \cdots + \lambda_n x_1 x_2 \cdot \cdots \cdot x_{n-1} dx_n. \quad (5.9)$$

Exercise 5.4. Prove that the integrable 1-form (5.9) admits an integrating factor if and only if $\lambda_1 = \cdots = \lambda_n$, i.e. $\alpha = \lambda d(x_1 \cdot \cdots \cdot x_n)$.

I will formulate a result on the reduction to (5.9) of any integrable 1-form with the $(n-1)$-jet (5.9) in terms of the tuple $\lambda = (\lambda_1, ..., \lambda_n)$. This property is called $(n-1)$-determinacy.

Definition 5.1. An integrable 1-form is k-determined (within integrable 1-forms) if it is diffeomorphic to any integrable 1-form with the same k-jet.

The $(n-1)$-determinacy of the integrable 1-form (5.9) requires certain assumptions on the tuple $\lambda = (\lambda_1, ..., \lambda_n)$. To explain this assumption we should describe (5.9) in terms of vector fields.

Note that the integrable 1-form (5.9) can be defined via a tuple of $(n-1)$ commuting linear vector fields (cf. Proposition 3.2 in the Kupka case and Proposition 4.7 in the Malgrange case).

Notation. Given a vector $\mathbf{r} = (r_1, ..., r_n) \in \mathbb{R}^n$ denote by V_{r} the vector field $V_{\mathrm{r}} = r_1 x_1 \frac{\partial}{\partial x_1} + \cdots + r_n x_n \frac{\partial}{\partial x_n}$.

Consider the hyperplane

$$R_\lambda = \{\mathrm{r} \ = (r_1, ..., r_n) \in \mathbb{R}^n : \quad r_1\lambda_1 + \cdots + r_n\lambda_n = 0.\}$$

Take any basis $\mathrm{r}^{(1)}, ..., \mathrm{r}^{(n-1)}$ of the vector space R_λ. Consider the linear vector fields $V_{\mathrm{r}^{(1)}}, ..., V_{\mathrm{r}^{(n-1)}}$. The form (5.9) is associated to these vector fields via a volume form:

$$\alpha(\cdot) = \Omega\left(V_{\mathrm{r}^{(1)}}, ..., V_{\mathrm{r}^{(n-1)}}, \cdot\right) \quad (5.10)$$

where Ω is some non-degenerate volume form and the dot denotes any vector field.

Definition 5.2. A vector field $V, V(0) = 0$, is 1-determined if it is diffeomorphic to any vector field with the same linear part at the singular point 0.

It turns out that for the $(n-1)$-determinacy of (5.9) with non-zero distinct λ's it suffices to have the 1-determinacy of just one of vector fields in (5.10).

Theorem 5.2 (based on the work D. Cerveau - A. Lins Neto, 1985). *Fix one of the categories: formal (formal power series) or analytic. Assume that $\lambda_1, ..., \lambda_n$ are non-zero distinct numbers and the hyperplane R_λ contains a vector* $\mathbf{r}$ *such that the vector field V_{r} is 1-determined in the chosen category. Then the integrable 1-form (5.9) is $(n-1)$-determined in the same category.*

The results on 1-determinacy of vector fields are classical. They require the following definitions.

Definition 5.3. A tuple of real or complex numbers $r_1, \ldots, r_n$ is called non-resonant if there are no relations of the form

$$r_i = k_1 r_1 + \cdots + k_n r_n, \quad i = 1, ..., n, \quad k_i \text{ are integers } \geq 0, \quad k_1 + \cdots + k_n \geq 2.$$

Definition 5.4. A tuple of real or complex numbers $r_1, ..., r_n$ satisfies Siegel condition if it is non-resonant and on top of that there exist positive C and ν such that $r_i - (k_1 r_1 + \cdots + k_n r_n) > C(k_1 + \cdots + k_n)^{-\nu}$ for all non-negative integers $k_1, ..., k_n$ such that $k_1 + \cdots + k_n \geq 2$.

Classical theorems (formal category: Poincare, analytic category: Siegel). *Let V be a C^∞ or analytic vector field on $\mathbb{R}^n$ with eigenvalues $r_1, ..., r_n$ at the singular point 0. If the tuple $(r_1, ..., r_n)$ is non-resonant then V can be reduced to its linear part by a formal change of coordinates (on the level of formal series). If V is analytic and the tuple $(r_1, ..., r_n)$ satisfies the Siegel condition then V can be reduced to its linear part by an analytic change of coordinates.*

Therefore Theorem 5.2 implies the following result.

Theorem 5.3. (corollary of Theorem 5.2; based on the work D. Cerveau -A. Lins Neto, 1985). *If $\lambda_1, ..., \lambda_n$ are nonzero distinct numbers and the hyperplane R_λ contains a non-resonant vector $(r_1, ..., r_n)$ then the integrable 1-form (5.9) is $(n-1)$-determined in the formal category (on the level of formal series). If the hyperplane R_λ contains a vector satisfying the Siegel condition then the integrable 1-form (5.9) is $(n-1)$-determined in the analytic category.*

The assumption on the hyperplane R_λ in this theorem are very weak. The simplest cases that they hold are as follows. In the formal category it suffices to require the condition

(*) there exist λ_i, λ_j such that $\frac{\lambda_i}{\lambda_j}$ is an irrational number.

In the analytic category it is suffices to require (*) plus the condition that at least two of the $\lambda's$ have different signs.

Of course these conditions are sufficient, but not necessary for the assumptions of Theorem 5.3. For example, if all λ's have the same sign then the assumption in Theorem 5.3 in the analytic category holds provided that the 2-tuple (λ_1, λ_2) satisfies the Siegel condition.

The simplest example that the assumptions of Theorem 5.3 do not hold is the case $\lambda_1 = \cdots = \lambda_n$. In this case $r_1 + \cdots + r_n = 0$ for any vector $(r_1, ..., r_n)$ in the hyperplane R_λ and one has infinitely many resonant relations given in Definition 5.3. D. Cerveau and A. Lins Neto have results for the case $\lambda_1 = \cdots = \lambda_n$ generalizing the classical Lyapunov-Poincare theorem which concerns **exactly** this case with $n = 2$, see section 7. If $\lambda_1 = \cdots = \lambda_n$ then the integrable 1-form (5.9) admits an integrating factor (see Exercise 5.4), but this does not mean at all that any integrable 1-form with the $(n-1)$-jet (5.9) admits an integrating factor. The existence of an integrating factor requires infinitely many independent conditions on terms of order $\geq n$. In the 2-dimensional case (5.9) takes the form $\lambda_1 x_2 dx_1 + \lambda_2 x_1 dx_2$ and if $\lambda_1 = \lambda_2 = \lambda$ then one has the classical case $\alpha = \lambda d(x_1 x_2) + \cdots$, where the dots denote non-linear terms. In fact, the classical case is $d(x_1^2 + x_2^2) + \cdots$, but in the analytic category the main results for these two cases are the same because in the holomorphic category the functions $x_1 x_2$ and $x_1^2 + x_2^2$ are the same up to a change of coordinates.

We see that Theorems 5.2 and 5.3 are tied with the classification of $(n-1)$-generated modules of vector fields which are closed with respect to the Lie bracket. One of the ways to obtain the formal-series part of these theorems is to use the formal-series part of the following result by Chaperon.

Theorem 5.4 (M. Chaperon, 1979.) *Let $I = (V_1, ..., V_p)$ be a module of vector fields over the ring of function germs generated by p vector fields vanishing at 0. Assume that the module I is closed with respect to the Lie bracket: $[V_i, V_j] \in I$ for any i, j. Assume also that the module I contains a vector field with distinct eigenvalues $(r_1, .., r_n)$ such that $(r_1, ..., r_n)$ is a non-resonant tuple. Then there is another tuple of generators $\tilde{V}_1, ..., \tilde{V}_p$ and a formal change of coordinates Φ such that $\Phi_* V_i$ is a linear diagonal vector field, $i = 1, ..., p$. The same holds in the C^∞ category under the additional assumption that the real parts of the eigenvalues $r_1, ..., r_n$ have the same sign.*

Remark. M. Chaperon has results that the latter assumption (in the C^∞ category) can be replaced by a weaker assumption. The formulated theorem in the C^∞ category is the simplest case. To explain this, consider the case $n = 3, p = 2$, $I = (V_1, V_2)$, $[V_1, V_2] = 0$. Assume also that in some coordinates V_1 is a linear diagonal vector field with non-resonant tuple of distinct eigenvalues. It is easy to prove that $V_2 = L + \tau$, where L is a linear diagonal vector field (in the same coordinate system) and τ is a flat vector field. Then $[V_1, \tau] = 0$. It is easy to prove that this relation implies $\tau \equiv 0$ provided that the real parts of the eigenvalues of V_1 have the same sign; for if not one can construct counterexamples. Nevertheless, if the real parts of the eigenvalues are not of the same sign, but satisfy a certain assumption, then the condition $[V_1, \tau] = 0$ implies the possibility to bring the vector field $L + \tau$ to L by a diffeomorphism preserving V_1.

Theorem 5.4 leads (after some work on transition from integrable 1-forms to modules of vector fields, see sect. 9) to the following corollary.

Theorem 5.5. (can be obtained from Theorem 5.4 with $p = n - 1$). *If $\lambda_1, ..., \lambda_n$ are nonzero distinct numbers, at least two of them have different signs, and the hyperplane R_λ contains a non-resonant vector then the integrable 1-form (5.9) is $(n-1)$-determined in the C^∞ category.*

The role of the assumption that at least two of the λ's have different signs is as follows: under this assumption the hyperplane R_λ contains a vector with positive components.

6. Poisson structures and integrable 1-forms on $\mathbb{R}^3$

Let R be the ring of functions germs on $\mathbb{R}^n$. A local Poisson structure P on $\mathbb{R}^n$ is a composition law $(f, g) \mapsto \{f, g\}$ on R which endows R with a Lie algebra structure and satisfies the condition $\{fg, h\} = f\{g, h\} + g\{f, h\}$ for every f, g and h in R. Such a composition law is called a Poisson bracket. It is locally determined by the brackets $\{x_i, x_j\}$ where $(x_1, \ldots, x_n)$ are local coordinates.

Exercise 6.1. Let

$$\{x, y\} = B(x, y, z), \ \{y, z\} = C(x, y, z), \ \{z, x\} = D(x, y, z). \tag{6.1}$$

Under which conditions on the functions B, C, D the relations (6.1) define a Poisson structure on $\mathbb{R}^3$?

Exercise 6.2. Prove that in the 3-dimensional case any Poisson structure can be multiplied by a function: if $(f, g) \mapsto \{f, g\}$ is a Poisson structure then for any function Q the composition law $(f, g) \mapsto Q\{f, g\}$ is also a Poisson structure. Show that this is not so if $n \geq 4$.

A Poisson structure can also be viewed as the 2-vector $P = \sum_{i<j} \{x_i, x_j\} \partial/\partial x_i \wedge \partial/\partial x_j$, i.e. a family of 2-forms on $T_x^*\mathbb{R}^n$ parameterized by $x \in \mathbb{R}^n$. Then the Poisson bracket of two functions f and g is given by the relation $\{f, g\} = P(df, dg)$.

In **the 3-dimensional case** one can associate to a Poisson structure P an integrable 1-form α as follows. Consider P as a field of 2-vectors. Take a non-degenerate volume form Ω. Define α by the relation

$$\alpha(Y) = \Omega(Y \wedge P) \ \text{ for any vector field } Y.$$

The relation $[P, P] = 0$, valid for any Poisson structure, implies the integrability of α: $\alpha \wedge d\alpha = 0$.

The constructed integrable 1-form α depends on the choice of Ω, but it is clear that α is invariantly related to P up to multiplication by a non-vanishing function. Therefore the Pfaff equation $\alpha = 0$ is invariantly related to the Poisson structure P.

Exercise 6.3. Show that the Pfaff equation associated to the Poisson structure (6.1) has the form $B(x,y,z)dz + C(x,y,z)dx + D(x,y,z)dy = 0$.

It is easy to show that any Pfaff equation $\alpha = 0$ with an integrable 1-form α is associated, in the given way, to some Poisson structure P, and this P is defined up to multiplication by a non-vanishing function (see Exercise 6.2!). Therefore we obtain the following

Proposition 6.1. *The problem of classification of local Poisson structures on $\mathbb{R}^3$ with respect to diffeomorphisms and multiplication by a non-vanishing function is exactly the same problem as local classification of Pfaff equations $\alpha = 0$ on $\mathbb{R}^3$ with integrable 1-form α.*

It follows that all known results on local classification of Poisson structures on $\mathbb{R}^3$ can be reformulated as local classification results for Pfaff equations $\alpha = 0$ on $\mathbb{R}^3$, where α is an integrable 1-form.

In particular, in the 3-dimensional case a good part of results of section 4 can be obtained as corollaries of results of A. Weinstein and J.F. Conn on Poisson structures on $\mathbb{R}^n$ (restricted to the 3-dimensional case). Also, in the 3-dimensional case the classification results presented in section 5 can be obtained as corollaries of results of J.P. Dufour (and his students A. Wade and A. Haraki) on Poisson structures on $\mathbb{R}^n$ (restricted to the 3-dimensional case).

7. Center-focus problem. Poincare-Lyapunov theorem.

Return now to the 2-dimensional case. Remind that any 1-form on $\mathbb{R}^2$ is integrable. Consider the classical case

$$\alpha = d(x_1^2 + x_2^2) + o(||x||) \tag{7.1}$$

in the analytic category. The foliation defined by the linear part $d(x_1^2 + x_2^2)$ of α is center. One can expect that the foliation defined by the whole 1-form α is a focus (foliation by spirals) unless the non-linear part of α is very specific. This is true - one has center only if the non-linear part of α satisfies infinitely many independent conditions.

Note that case (7.1) can also be treated as the case between Kupka and Malgrange, see section 5. In this case $Sing(\alpha^{\mathbb{C}}) = \{0\}$, therefore, like in section 5, $codimSing(\alpha) = 2$ and $d\alpha(0) = 0$.

Recall also that the condition $d\alpha(0) = 0$ is necessary for the existence of an integrating factor (see Proposition 2.2 or 3.4).

Proposition 7.1. *A 1-form (7.1) admits an integrating factor if and only if the Pfaff equation $\alpha = 0$ is diffeomorphic to its linear part $d(x_1^2 + x_2^2) = 0$.*

Proof. Assume that α admits an integrating factor. Then, by Definition 2.1, there exist functions Q and H such that $Q(0) \neq 0$ and $Q\alpha = dH$. There is no loss of generality to asssume $Q(0) = 1$. Then the first integral H has the form $x_1^2 + x_2^2 + o(||x||^2)$. Any function H of this form can be reduced to $x_1^2 + x_2^2$ by a local diffeomorphism. This diffeomorphism reduces the 1-form dH to the 1-form $d(x_1^2 + x_2^2)$.

Assume now that the Pfaff equation $\alpha = 0$ is diffeomorphic to its linear part $d(x_1^2 + x_2^2) = 0$, i.e.. there exists a local diffeomorphism Φ and a non-vanishing function Q such that $Q(x)\alpha = \Phi^*(d(x_1^2 + x_2^2))$. Then α has an integrating factor $Q(x)$ with the corresponding first integral $\Phi^*(x_1^2 + x_2^2)$.

Remark. Proposition 7.1 holds in the analytic and the C^∞ categories.

Now we will study the obstacles for the linearization of the Pfaff equation $\alpha = 0$. Let V be a vector field associated to α via a volume form Ω, for example $dx_1 \wedge dx_2$:

$$\alpha = V \rfloor \Omega, \quad V: \ \dot{x}_1 = x_2 + \cdots, \qquad \dot{x}_2 = -x_1 + \cdots, \tag{7.2}$$

where the dots denote non-linear terms.

Exercise 7.1. Prove that the Pfaff equation $\alpha = 0$ is diffeomorphic to its linear part if and only if the vector fields V is orbitally equivalent to its linear part, i.e. can be reduced to its linear part by a change of coordinates and multiplication by a non-vanishing function.

Studying vector fields (7.2) it is convenient to introduce complexly-conjugate coordinates

$$z_1 = x_1 - ix_2, \quad z_2 = \bar{z}_1 = x_1 + ix_2. \tag{7.3}$$

In these coordinates the vector field has diagonal matrix of linear approximation:

$$V: \quad \dot{z}_1 = iz_1 + \cdots, \quad \dot{x}_2 = -iz_2 + \cdots$$

Theorem 7.1. (a particular case of the classical Poincare-Dulac theorem on resonant normal form). *The Taylor series of any vector field of form (7.2) can be reduced by a formal change of coordinates (on the level of formal power series) to a form such that in the complexly-conjugate coordinates (7.3) one has*

$$\begin{aligned} V: \quad \dot{z}_1 &= z_1 \left(i + f_1 \cdot (z_1 z_2) + f_2 \cdot (z_1 z_2)^2 + f_3 \cdot (z_1 z_2)^3 + \cdots \right), \\ \dot{z}_2 &= z_2 \left(i + \bar{f}_1 \cdot (z_1 z_2) + \bar{f}_2 \cdot (z_1 z_2)^2 + \bar{f}_3 \cdot (z_1 z_2)^3 + \cdots \right), \end{aligned} \tag{7.4}$$

where $f_1, f_2, f_3, ... \in \mathbb{C}$.

Theorem 7.2. (Poincare, Lyapunov). *Let V be an analytic vector field of form (7.2). The following conditions are equivalent:*

1. The phase portrait of V is a center

2. The complex numbers f_i in the normal form (7.4) of V satisfy the conditions

$$Re(f_1) = Re(f_2) = Re(f_3) = \cdots = 0. \tag{7.5}$$

3. The vector field V is orbitally equivalent to its linear part on the level of formal series

4. The vector field V is orbitally equivalent to its linear part in the analytic category.

If at least one of the numbers $Re(f_i)$ is not zero then the phase portrait of V is a focus (foliation by spirals). This statement remains true in the C^∞ category. The part 1. ⇒ 2. is a relatively simple part of Theorem 7.2.

I will explain (7.5) and 2. ⇒ 3, this is also simple. The part 4. ⇒ 1 is obvious. The remaining part 3. ⇒ 4. is a difficult and surprising statement.

To explain (7.5) we write (7.4) in the form

$$\dot z_1 = iz_1\left[1 + b_1\cdot(z_1z_2) + b_2\cdot(z_1z_2)^2 + \cdots\right] + z_1\left[a_1\cdot(z_1z_2) + a_2\cdot(z_1z_2)^2 + \cdots\right],$$

$$\dot z_2 = -iz_2\left[1 + b_1\cdot(z_1z_2) + b_2\cdot(z_1z_2)^2 + \cdots\right] + z_2\left[a_1\cdot(z_1z_2) + a_2\cdot(z_1z_2)^2 + \cdots\right],$$

where $f_i = a_i + ib_i$. Divide this vector field by the formal power series $1 + b_1\cdot(z_1z_2) + b_2\cdot(z_1z_2)^2 + \cdots$. One has

$$\frac{a_1\cdot(z_1z_2) + a_2\cdot(z_1z_2)^2 + \cdots}{1 + b_1\cdot(z_1z_2) + b_2\cdot(z_1z_2)^2 + \cdots} = c_1\cdot(z_1z_2) + c_2\cdot(z_1z_2)^2 + \cdots.$$

We obtain the following corollary Theorem 7.1.

Theorem 7.3. (corollary of Theorem 7.1.) *On the level of formal power series any vector field of form (7.2) is orbitally equivalent to a vector field such that in complexly conjugate coordinates it has the form*

$$\dot z_1 = z_1\left[i + c_1\cdot(z_1z_2) + c_2\cdot(z_1z_2)^2 + \cdots\right],$$

$$\dot z_2 = z_2\left[-i + c_1\cdot(z_1z_2) + c_2\cdot(z_1z_2)^2 + \cdots\right] \tag{7.6}$$

where $c_1, c_2, \ldots$ are real numbers.

This explains (7.5) and 2. ⇒ 3. in Theorem 7.2 - it is clear that (7.5) is equivalent to the condition $c_1 = c_2 = c_3 = \cdots = 0$. The latter means that (7.6) is a linear vector field.

Return now to real-valued coordinates and to Pfaff equations. Making in (7.6) a change of coordinates $x_1 = (z_1 + z_2)/2,\ x_2 = (z_2 - z_1)/2i$ (inverse to (7.3)) and transferring the obtained vector field to a Pfaff equation via a volume form, we obtain (using also a simple Proposition 7.1) the following corollary of Theorem 7.2.

Theorem 7.4. (corollary of Theorem 7.2). *Let α be an analytic 1-form of the form $\alpha = d(x_1^2 + x_2^2) + \cdots$. The Pfaff equation $\alpha = 0$ is diffeomorphic on the level of formal series to a Pfaff equation of the form*

$$d(x_1^2 + x_2^2) + \left[c_1 \cdot (x_1^2 + x_2^2) + c_2 \cdot (x_1^2 + x_2^2)^2 + \cdots\right](x_1 dx_2 - x_2 dx_1). \tag{7.7}$$

The following conditions are equivalent:

1. The 1-form α admits an analytic integrating factor;

2. The 1-form α admits a formal integrating factor;

3. The Pfaff equation $\alpha = 0$ can be reduced to its linear approximation by an analytic change of coordinates;

4. The Pfaff equation $\alpha = 0$ can be reduced to its linear approximation by a formal change of coordinates;

5. In the normal form (7.7) of α one has $c_1 = c_2 = c_3 = \cdots = 0$.

6. The foliation defined by the Pfaff equation $\alpha = 0$ is a center.

Does this theorem mean that in the analytic category there exists an algorithm with a finite number of operations to distinguish center and focus? The real parts of the numbers f_i in normal form (7.4) or the numbers c_i in normal form (7.6) or (7.7) are called focus numbers. In fact, there are various definitions of focus numbers, but the condition "all focus numbers are equal to zero" does not depend on the choice of a definition. The i-th focus number can be calculated by a finite number of operations (growing fast when $i \to \infty$), it depends polynomially on the coefficients of the $(2i + 1)$-jet of the 1-form α. If one takes a fixed α starting with $d(x_1^2 + x_2^2)$, calculates the first focus number and sees that it is not zero then no other operations are needed - the phase portrait is a focus. The same is true if the first focus number is zero and the second is not, and so on. Nevertheless, for any fixed $q < \infty$ there exists α_1, α_2 starting with $d(x_1^2 + x_2^2)$, such that for each of them the first q focus numbers are equal to zero and such that α_1 defines a focus and α_2 - a center. Therefore within the set of all analytic 1-forms starting with $d(x_1^2 + x_2^2)$ then there is no a finite-time algorithm distinguishing center-focus.

Restrict now ourselves to polynomial 1-forms of a fixed degree m, i.e. let is work in the set

$$A_m = \{d(x_1^2 + x_2^2) + P_m(x_1, x_2)dx_1 + Q_m(x_1, x_2)dx_2\},$$

$$P_m, Q_m \text{ are degree } m \text{ polynomials without free and linear terms.}$$

The set A_m is parameterized by $s = (m + 1)(m + 2) - 6$ parameters, denote them $\kappa_1, ..., \kappa_s$. The i-th focus number of any $\alpha \in A_m$ has the form $R_i(\kappa_1, ..., \kappa_s)$, where

R_i is a fixed polynomial (the degree of R_i tends to ∞ as $i \to \infty$) and $\kappa_1, ..., \kappa_s$ is the tuple corresponding to α. Therefore α defines a center if and only if

$$R_1(\kappa_1, ..., \kappa_s) = R_2(\kappa_1, ..., \kappa_s) = ... = 0. \tag{7.8}$$

Here $R_1, R_2, ...$ are **fixed** polynomials of a fixed number of variables $s = (m+1)(m+2) - 6$. It is well known that an infinite system (7.8) is equivalent to its finite subsystem, i.e. there exists a number d such that $\kappa_1, ..., \kappa_s$ satisfy (7.8) if and only if they satisfy the first d equalities in (7.8). In this construction everything except m is fixed, therefore d depends on m only: $d = d(m)$.

It follows that there exists a finite time algorithm distinguishing center and focus within the set A_m, for any $m \geq 2$. Namely, compute the first $d(m)$ focus numbers. A 1-form $\alpha \in A_m$ defines a center if and only if all these $d(m)$ numbers are equal to zero.

The existence of a finite time algorithm distinguishing center and focus within the set A_m does not mean that such an algorithm is known. The thing is that the number $d(m)$ is known for $m = 2$ only. In the 20-th century a number of mathematicians spend much time trying to find $d(m)$ for $m \geq 3$, at least $d(3)$. It seems that in the 21-st century the number of people attacking this problem substantially decreased. The numbers $d(3), d(4), ...$ are not known.

8. Division properties.

The first results on division properties are due to G. de Rham and D.G. Northcott. The most applicable results were obtained by R. Moussu and B. Malgrange. In now-days this topic is completely understood, and the most general results can be found in some books on commutative rings and algebra (D. Eisenbud, S. Balcerzyk-T.Jozefiak).

A number of problems of local analysis and in particular classification of integrable 1-forms require the following definition and related theorems.

Definition 8.1. A differential 1-form α on $\mathbb{R}^n$ has k-division property if for any differential k-form ω such that $\alpha \wedge \omega \equiv 0$ there exists a $(k-1)$-form γ such that $\omega = \alpha \wedge \gamma$.

Exercise 8.1. Prove that if $\alpha(0) \neq 0$ then α has the k-division property for any k.

Exercise 8.2. Prove that if α is differential 1-form on $\mathbb{R}^n$ such that $\alpha(0) = 0$ then α does not have the k-division property for $k \geq n$.

Therefore in what follows we will assume that $\alpha(0) = 0$ and will study the k-division property of α with $1 \leq k \leq n-1$.

Example. The 1-division property of a 1-form $\alpha = \alpha_1(x)dx_1 + \cdots + \alpha_n(x)dx_n$ means that if $\beta_1(x), ..., \beta_n(x)$ is a tuple of functions such that

$$rank \begin{pmatrix} \alpha_1(x) & \alpha_2(x) & \cdots & \alpha_n(x) \\ \beta_1(x) & \beta_2(x) & \cdots & \beta_n(x) \end{pmatrix} \leq 1, \quad x \in \mathbb{R}^n$$

then there exists a function $f(x)$ such that $\beta_i(x) = f(x)\alpha_i(x),\ i = 1, ..., n$. It is clear that α does not have the 1-division property if the coefficients $\alpha_1(x), ..., \alpha_n(x)$ are divisible by the same function vanishing at 0.

From this example it is clear that the 1-division property of α is the property of the ideal in the ring of function germs generated by the coefficients of α. One can prove that this is also so for the k-division property with any k.

Proposition 8.1. (J. Martinet, R. Moussu).
The k-division property of a differential 1-form α, $\alpha(0) = 0$, is the property of ideal generated by the coefficients of α (this ideal does not depend on the choice of a coordinate system).

This means that if α and $\tilde{\alpha}$ are 1-forms whose coefficients generate the same ideal then $\tilde{\alpha}$ has the k-division property if and only if α has the k-division property.

Therefore it is worth to give the following definition, which is correct by Proposition 8.1.

Definition 8.2. Let $I = (f_1(x), ..., f_p(x))$ be an ideal in the ring of function germs on $\mathbb{R}^n$ generated by $p \leq n$ function germs. The ideal I has k-division property if the 1-form $f_1(x)dx_1 + \cdots + f_p(x)dx_p$ has the k-division property.

The validity of the k-division property of an ideal I is related to the codimension of the zero set of I.

Exercise 8.3. Let $f_1(0) = \cdots = f_p(0) = 0$. Prove that the ideal $(f_1, ..., f_p)$ does not have the k-division property with $k \geq p$.

In the analytic category, given an ideal $I = (f_1, ..., f_p)$ consider the zero set of its complexification

$$Sing(I^{\mathbb{C}}) = \{z \in \mathbb{C}^n : f_1^{\mathbb{C}}(z) = \cdots = f_p^{\mathbb{C}}(z) = 0\},$$

where $f_i^{\mathbb{C}}$ is the complexification of f_i.

Convention. The codimension of the empty set is ∞.

Theorem 8.1. (analytic category only) (firstly published with a complete proof by B. Malgrange).
Let $p \leq n$. A p-generated ideal I in the ring of function germs on $\mathbb{R}^n$ has k-division property if and only if $codimSing(I^{\mathbb{C}}) \geq k+1$.

Using certain general theorems on the ring of C^∞ function germs and the ring of analytic function germs one can prove the following

Theorem 8.2. (C^∞ category) (R. Moussu).
Let $p \leq n$. Assume that a p-generated ideal I in the ring of C^∞ function germs is diffeomorphic to an ideal (in the same ring) generated by p analytic function germs $f_1, ..., f_p$. Denote by $I^{\mathbb{A}}$ the ideal in the ring of analytic function germs generated by $f_1, ..., f_p$. Let $k \geq 1$. The ideal I has the k-division property if and only if the ideal $I^{\mathbb{A}}$ has k-division property.

Theorems 8.1 and 8.2 give complete answer to the question on the division properties of ideals in the ring of analytic functions and ideals in the ring of C^∞ functions which are C^∞ diffeomorphic to ideals generated by analytic functions.

If, in the C^∞ or analytic category, an ideal I is generated by n functions vanishing at the origin and has finite multiplicity (see Definition 4.1 and the remark after it) then it is always diffeomorphic to an ideal generated by n polynomials and $Sing(I^{\mathbb{C}}) = \{0\}$. Therefore Theorems 8.1 and 8.2 imply the following corollary.

Theorem 8.3. (C^∞ or analytic category) (R. Moussu).
Let I be an ideal in the ring of function germs on $\mathbb{R}^n$ generated by n function germs vanishing at the origin. If I has a finite multiplicity then it has the k-division property for any $k = 1, 2, ..., n-1$.

Remark. Theorem 8.3 is a direct corollary of Theorems 8.1 and 8.2, but R. Moussu proved it without using Theorem 8.1. His proof was based on Theorem 8.2 and results by G. de Rham and D.G. Northcott.

Theorems 8.1 - 8.3 are based on the following

Theorem 8.4. (C^∞ and analytic categories) (G. de Rham, D.G. Northcott).
Let I be an ideal in the ring of function germs on $\mathbb{R}^n$ generated by function germs $f_1, ..., f_p$ vanishing at 0, $p \leq n$. Let $k \leq p-1$. The ideal I has the k-division property provided that f_1 is not a zero divisor in the ring of all function germs and for any $2 \leq i \leq k+1$ the function germ f_i is not a zero divisor in the ring of all function germs factorized by the ideal generated by $f_1, ..., f_{i-1}$.

Theorem 8.4 is a particular case of the following general pure algebraic result.

Definition 8.3. Let R be a commutative ring with a unit. A tuple $r_1, ..., r_s \in R$ is called regular if the element r_1 is not a zero divisor in R and for any $2 \leq i \leq s$ the element r_i is not a zero divisor in the ring R factorized by the ideal generated by $r_1, ..., r_{i-1}$. The number s is the length of the regular tuple.

Definition 8.4. Let I be an ideal in a commutative ring R with a unit. The depth of I in R is the supremum of lengths of regular tuples in the ideal I.

The definition of the k-division property of an ideal in the ring of function germs can be generalized in a natural way to finitely generated ideals I in any commutative ring R with a unit (if I is m-generated then one has to consider the free module M over R of rank m, the spaces $\Lambda^i M$ and the corresponding Koszul complex, see D. Eisenbud and S. Balcerzyk-T.Jozefiak.)

Theorem 8.5. (see D. Eisenbud and S. Balcerzyk-T.Jozefiak).
1. A proper finitely generated ideal I in a commutative ring R with a unit has k-division property if $k < depth(I)$.

2. If R is a local Noetherian ring then the k-division property of I holds if and only if $k < depth(I)$.

Note that Theorem 8.1 is a corollary of Theorem 8.4 and the following

Theorem 8.6. (see, for example, D. Eisenbud). *The depth of a finitely generated ideal I in the ring of analytic function germs is equal to $codimSing(I^{\mathbb{C}})$.*

9. References and comments.

Section 1. This section is based on Appendix C of the book

M.Zhitomirskii, Typical singularities of differential 1-forms and Pfaffian equations. *Transl. of Math. Monographs, Vol. 113, AMS 1992.*

By certain reasons I had to repeat the main message in

M. Zhitomirskii, Darboux and Martinet theorems in control theory, *Singularites et Geometrie Sous-Riemannienne, Travaux en Cours 62, Hermann, Paris (P.Orro, F. Pelletier eds.), 2000, 161-163* .

Section 2. A proof of Frobenius theorem without using normal forms and any coordinates (pure canonical construction) can be found in

V.I. Arnol'd, Mathematical methods of classical mechanics, *Springer-Verlag, Berlin, Heidelberg, New York, 1990*

Section 3.

I. Kupka, Singularities of integrable Pfaffian forms, *Proc. Nat. Acad. Sci. 52, 1964.*

Section 4.

G. Reeb, Sur certaines proprietes topologiques des varietes feuilletees, *Paris, Hermann, 1952*

R. Moussu, Sur l'existence d'integrales premieres pour un germe de forme de Pfaff,
Ann. Inst. Fourier, Grenoble, **26**, *2, 171-220, 1976*

B. Malgrange, Frobenius avec singularites 1 - codimension un,
Publ. Math. IHES, **46**, *163-173, 1976*

R. Moussu, Classification C^∞ des equations de Pfaff completement integrables a singularite isolee,
Invent. math., 73, 419-436, 1983

Section 5.

There is a series of results by C.Camacho and C.Camacho - A. Lins Neto on **structural stability** including:

C. Camacho, Structural stability of foliations with singularities.
Lecture Notes in Math., Vol. 652, Springer, Berlin, 1978.

C. Camacho, Structural stability theorems for integrable differential forms on 3-manifolds,
Topology **17**, *2, 143–155, 1978*

C. Camacho, A. Lins Neto, Stabilite des feuilletages definis par une forme differentielle integrable au voisinage d'une singularite. *C. R. Acad. Sci. Paris Ser. A-B 290, no. 9, 1980*

C. Camacho, A. Lins Neto, The topology of integrable differential forms near a singularity.
Inst. Hautes Etudes Sci. Publ. Math. No. 55, 5-35, 1982

Logarithmic integrable 1-forms. The main work I used working on Section 5, part 2 is

D. Cerveau, A. Lins Neto, Formes tangentes a des actions commutatives,
Annales Faculte des Sciences Toulouse, Vol. VI, 51-85, 1984

The results by M. Chaperon presented in section 5 can be found in

M. Chaperon, Linearisation des germes hyperboliques d'actions differentiables de $R^k \times Z^m$: le domaine de Poincare, *C. R. Acad. Sci. Paris Ser. A-B 289, no. 5, 1979*

Obtaining Theorem 5.5 as a corollary of Theorem 5.4 requires certain techniques. The assumption $\lambda_i \neq 0$ in Theorem 5.5 is very important - if at least one of the coefficients λ_i is equal to 0 then α does not have 1-division property (see section 8) and fails the duality between integrable 1-forms and $(n-1)$-generated modules of vector fields, closed with respect to the Lie bracket. The work

P.Mormul, M.Zhitomirskii, Modules of vector fields, differential forms
and degenerations of differential systems. *Israel Journal of Mathematics 95, 411-428, 1996*

contains a general theorem establishing the duality between 1-forms on $\mathbb{R}^n$ and $(n-1)$-generated modules of vector fields under the 1-division property.

Section 6.

A. Weinstein, The local structure of Poisson manifolds,
J. Differential Geometry 18, 523-557, 1983.

J.F. Conn, Normal forms for analytic Poisson structures,
Ann. of Math., 119 (2), 576 - 601, 1984

J.F. Conn, Normal forms for smooth Poisson structures,
Ann. of Math., 121 (2), 565 - 593, 1985

J.P. Dufour, Linéarisation de certaines structures de Poisson,
J. Differential Geometry 32, 415 - 428, 1990

J.P. Dufour, A. Haraki, Rotationnels et structures de Poisson quadratiques,
C.R. Acad. Sci. Paris 312 I, 137-140, 1991

A. Wade, Normalisation de structures de Poisson,
these (Directeur de these: J.P.Dufour), Universite Montpellier II, 1996

See also section 6 of the work

J.P. Dufour, M. Zhitomirskii, Singularities and bifurcations of 3-dimensional Poisson structures,
Israel Journal of Mathematics, vol. 121, 199 -220, 2001

Section 7. Detailed exposition of the center-focus problem and many related results can be found in the book

K.S. Sibirsky, Introduction to the algebraic theory of invariants of differential equations.
Nonlinear Science: Theory and Applications. Manchester University Press, Manchester, 1988

Theorem 7.4 holds of course in the holomorphic category. Therefore in the real analytic category it can be reformulated for 1-forms of the form
$\alpha = d(x_1x_2) + \cdots$. On the level of formal power series the Pfaff equation $\alpha = 0$ can be reduced to the normal form
$d(x_1x_2) + \left[c_1(x_1x_2) + c_2(x_1x_2)^2 + \cdots\right](x_1dx_2 - x_2dx_1)$.
The conditions 1.-5. given in Theorem 7.4 are equivalent.

The 1-forms $\alpha = d(x_1x_2) + \cdots$ on $\mathbb{R}^2$ are the 2-dimensional case of logarithmic 1-forms (5.9) on $\mathbb{R}^n$ with equal $\lambda_1 = \cdots = \lambda_n$. The work

D. Cerveau, A. Lins Neto, Formes tangentes a des actions commutatives,
Annales Faculte des Sciences Toulouse, Vol. VI, 51-85, 1984

contains, among a series of other results, a result for this case generalizing the Poincare-Lyapunov theorem.

Section 8.

G. de Rham, Sur la division des formes et des courants par une forme lineaire,
Commentari. Math. Hel., 28, 1954

D.G. Northcott, Lessons on rings Modules and Multiplicities,
Cambridge University Press, 1968

R. Moussu, Sur l'existence d'integrales premieres pour un germe de forme de Pfaff,
Ann. Inst. Fourier, Grenoble, **26**, *2, 171-220, 1976*

B. Malgrange, Frobenius avec singularites 1 - codimension un,
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S. Balcerzyk, T. Jozefiak, Commutative rings; dimension, multiplicity, and homological methods,
Polish Scientific Publishers, Warsaw, 1989

D. Eisenbud, Commutative Algebra, *Springer-Verlag, 1994*

See also Appendix 1 of the work

B. Jakubczyk, M. Zhitomirskii, Local reduction theorems and invariants for singular contact structures, *Ann. Inst. Fourier, Grenoble,* **51**, *1, 237-295, 2001*

10. Appendix: Vector fields and differential 1-forms.

Let M^n be a smooth n-dimensional manifold, in particular $\mathbb{R}^n$.

1. Let $p \in M^n$ and let γ be a smooth parameterized curve passing through p:
$\gamma : (-\epsilon, \epsilon) \to M^n,\ \gamma(0) = p$. Take any local coordinate system $x \in \mathbb{R}^n$ centered at the point p. Let $|x|$ be the standard norm. By definition, two curves γ_1, γ_2 of the given above form are tangent at p if $|x(\gamma_1(t)) - x(\gamma_2(t))| = o(t)$ as $t \to 0$. Here $x(q)$ is the tuple of coordinates of a point $q \in M^n$.

Lemma. The definition is correct, i.e. the choice of a coordinate system is irrelevant.

2. A tangent vector to M^n at a point p is the space of all smooth curves passing through p factorized by the following equivalence: two curves are equivalent if they are tangent at p. A tangent vector at p can be understood as a velocity of a point moving on M^n at the moment it goes through p.

Lemma and definitions. The operation $(\gamma_1, \gamma_2) \to (k_1\gamma_1 + k_2\gamma_2)$ "respects" this factorization, therefore the set of all tangent vectors at p is a vector space. It has dimension n. It called the tangent space to M^n at p and denoted by T_pM^n. The collection $T_pM^n,\ p \in M^n$ is denoted by TM^n and is called the tangent bundle to M^n.

3. Take a local coordinate system $(x_1, \dots, x_n)$ near a point $p \in M^n$. Consider the curve
$t \to$ (point with coordinates $(1, 0, ..., 0)$). The tangent vector at p defined by this curve is denoted $\frac{\partial}{\partial x_1}$. In the same way one defines $\frac{\partial}{\partial x_2}$, ..., $\frac{\partial}{\partial x_n}$. For example $\frac{\partial}{\partial x_n}$ is the tangent vector at p corresponding to the curve sending t to the point with coordinates $(0, 0, ..., 0, 1)$. The tuple of tangent vectors $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n} \in T_pM^n$ is a basis of the space T_pM^n (associated to the coordinate system $(x_1, ..., x_n)$).

4. Take a diffeomorphism $\Phi : (U_1, p_1) \to (U_2, p_2)$, where p_1, p_2 are point of M^n and U_1, U_2 are their neighborhoods. Take a vector $v \in T_{p_1}M^n$. Take a curve $\gamma : (-\epsilon, \epsilon) \to M^n$ representing this vector, see 2. Consider now the curve $\tilde{\gamma} : (-\epsilon, \epsilon) \to M^n,\ \tilde{\gamma}(t) = \Phi(\gamma(t))$. This curve passes through the point p_2 and defines a tangent vector $\tilde{v} \in T_{p_2}M^n$. This vector is denoted $\Phi_{*,p_1}v$.

Lemma. The operator $\Phi_{*,p_1} : T_{p_1}M^n \to T_{p_2}M^n$ is well defined (respects the factorization in 2.). This operator is linear.

5. Let Φ be a diffeomorphism as in 4. Let $x = (x_1, ..., x_n)$ be a local coordinate system near the point p_1, centered at p_1 (i.e. the point p_1 has zero coordinates), and let $y = (y_1, ..., y_n)$ be a local coordinate system near the point p_2, centered at p_2. Express Φ in these coordinate systems: $\Phi = (\Phi_1(x), ..., \Phi_n(x))$, which means that Φ brings a point with coordinates x to a point with coordinates $y_1 = \Phi_1(x), ..., y_n = \Phi_n(x)$.

Let $v_1 \in T_{p_1}M^n$ and $v_1 = a_1\frac{\partial}{\partial x_1} + \cdots + a_n\frac{\partial}{\partial x_n}$, where $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n} \in T_{p_1}M^n$ is a basis of the space $T_{p_1}M^n$ associated to the coordinate system $(x_1, ..., x_n)$, see 3. Then $\Phi_{*,p_1}v_1 = b_1\frac{\partial}{\partial y_1} + \cdots + b_n\frac{\partial}{\partial y_n}$, where $\frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_n} \in T_{p_2}M^n$ is a basis of the space $T_{p_2}M^n$ associated to the coordinate system $(y_1, ..., y_n)$. Let

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix},\ b = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix},\ \Phi'(x) = \begin{pmatrix} \frac{\partial \Phi_1(x)}{\partial x_1} & \dots & \frac{\partial \Phi_1(x)}{\partial x_n} \\ \frac{\partial \Phi_2(x)}{\partial x_1} & \dots & \frac{\partial \Phi_2(x)}{\partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial \Phi_n(x)}{\partial x_1} & \dots & \frac{\partial \Phi_n(x)}{\partial x_n} \end{pmatrix} \text{(Jacobi matrix).}$$

Exercise. Use 1.-4. to prove that $b = \Phi'(0)a$.

6. Let Σ be any subset of the tangent bundle TM^n. Then Σ is a collection of subsets $\Sigma(p) \subset T_pM^n$, where p varies in M^n. Let Φ be a diffeomorphism of M^n. Fix $p \in M^n$. Apply the linear operator $\Phi_{*,p}$ to all tangent vectors in $\Sigma(p)$. We obtain a subset of $T_{\Phi(p)}M^n$. Do this for all points $p \in M^n$. We obtain a subset of TM^n. It is denoted $\Phi_*\Sigma$.

7. A smooth vector field V on M^n is a smooth section of the tangent bundle TM^n: $V = \{v_p \in T_pM^n,\ p \in M^n\}$. This means that at any point $p \in M^n$ a tangent vector $v_p \in T_pM^n$ is fixed and v_p depends smoothly on p. Therefore a vector field is a (very important) particular case of a subset of the tangent bundle, and given a diffeomorphism Φ we can apply to V the operation Φ_*, see 6. The obtained subset of the tangent bundle is also a vector field. It is denoted Φ_*V.

8. Given a local coordinate system $x = (x_1, ..., x_n)$ in a neighborhood of a point $p \in M^n$ one can express a vector field V in this neighborhood using, at any point q of the neighborhood, the basis $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}$ of T_qM^n, see 3. Then $V = V_1(x)\frac{\partial}{\partial x_1} + \cdots + V_n(x)\frac{\partial}{\partial x_n}$, which means that at a point q with coordinates x the vector field V defines the tangent vector with coordinates $V_1(x), ..., V_n(x)$ in the basis $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}$ of the space T_qM^n.

Let Φ be a diffeomorphism as in 4, and let $\Phi_*V = \tilde{V}_1(y)\frac{\partial}{\partial y_1} + \cdots + \tilde{V}_n(y)\frac{\partial}{\partial y_n}$ in the local coordinate system y near p_2. Let

$$\Phi = \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \\ \cdots \\ \Phi_n(x) \end{pmatrix}, \ V = \begin{pmatrix} V_1(x) \\ V_2(x) \\ \cdots \\ V_n(x) \end{pmatrix}, \ \tilde{V} = \begin{pmatrix} \tilde{V}_1(y) \\ \tilde{V}_2(y) \\ \cdots \\ \tilde{V}_n(y) \end{pmatrix}.$$

Then by 5.

$$\tilde{V}(\Phi(x)) = \Phi'(x)V(x).$$

To calculate the expression of Φ_*V in coordinates y one has to find the vector function

$$\Psi(y) = \begin{pmatrix} \Psi_1(y) \\ \Psi_2(y) \\ \cdots \\ \Psi_n(y) \end{pmatrix} \text{ corresponding to the inverse diffeomorphism } \Phi^{-1},$$

then $\tilde{V}(y) = \Phi'(\Psi(y))V(\Psi(y)$.

9. The previous section 8. is a local change of coordinates which is denoted $y = \Phi(x)$ or equivalently $x = \Psi(y)$. If the vector function $\Psi(y)$ is known then instead of the formula given at the end of 8. it might be more convenient to use the *equivalent* formula

$$\tilde{V}(y) = (\Psi'(y))^{-1}V(\Psi(y)).$$

10. One should not confuse vector fields or diffeomorphisms with vector functions. In 8. and 9. Φ, Ψ are diffeomorphisms of M^n, $V, \tilde{V}$ are vector fields on M^n, and $\Phi_i(x), \Psi_i(y), V_i(x), \tilde{V}_i(y)$ are functions of n variables. $\Phi(x), \Psi(y), V(x), \tilde{V}(y)$ are vector functions.

11. Saying that a vector field V on M^n is given by a system of ordinary differential equations $\dot{x} = V(x)$, where $V(x)$ is a vector function one should mean the following: there is a fixed local coordinate system x such that the tangent vector defined by the vector field at a point $p \in M^n$ with coordinates $(x_1, ..., x_n)$ is the vector $V_1(x)\frac{\partial}{\partial x_1} + \cdots + V_n(x)\frac{\partial}{\partial x_n}$, where $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}$ is a basis of the space T_pM^n associated to the coordinate system $(x_1, ..., x_n)$. A solution of the ODE

$\dot{x} = V(x)$ is a curve $t \to$ point with coordinates $x(t)$. At any point $t \in \mathbb{R}$ the velocity vector $\dot{x}(t)$ of this curve coincides with tuple of coordinates of the vector of V at the point with coordinates $x(t)$ in the basis of $T_{x(t)}M^n$ associated with the coordinate system x. The change of coordinates in 9. can be expressed, in terms of ODE's, as follows: changing $x = \Psi(y)$ in the system $\dot{x} = V(x)$ we obtain $\Psi'(y)\dot{y} = V(\Psi(y)$ and consequently $\dot{y} = (\Psi'(y))^{-1}V(\Psi(y))$.

12. A linear functional on the space T_pM^n is called (algebraic) 1-form on T_pM^n. The set of all algebraic 1-forms on T_pM^n is denoted $T_p^*M^n$. It is a vector space of dimension n. If one has a local coordinate system $x_1, ..., x_n$ near a point $p \in M^n$ and consequently the associated basis $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}$ of the tangent space T_pM^n then one can define the associated basis $dx_1, ..., dx_n$ of the space $T_p^*M^n$ as follows: the linear functional dx_i takes value 1 on the tangent vector $\frac{\partial}{\partial x_i}$ and annihilates the tangent vectors $\frac{\partial}{\partial x_j}, j \neq i$. Consequently one has

$$(A_1dx_1 + \cdots + A_ndx_n) \text{ applied to } a_1\frac{\partial}{\partial x_1} + \cdots + a_n\frac{\partial}{\partial x_n} = A_1a_1 + \cdots + A_na_n.$$

13. Let $\Phi : (U_1, p_1) \to (U_2, p_2)$ be a diffeomorphism, where p_1, p_2 are point of M^n and U_1, U_2 are their neighborhoods. Let $\mu \in T_{p_2}^*M^n$ and let $v \in T_{p_1}M^n$. Then $\Phi_{*,p_1}(v) \in T_{p_2}M^n$. The map $v \to \mu(\Phi_{*,p_1}v)$ is a linear functional on $T_{p_1}M^n$. It is denoted $\Phi^*\mu$. We obtain a map $\Phi^* : T_{p_2}^*M^n \to T_{p_1}^*M^n$ associated to the diffeomorphism Φ. This map is linear.

Fix a local coordinate system x centered at p_1 and a coordinate system y centered at p_2. Then we have associated basses $dx_1, ..., dx_n$ of the space $T_{p_1}^*M^n$ and $dy_1, ..., dy_n$ of the space $T_{p_2}^*M^n$. Let $\mu = B_1dy_1 + \cdots + B_ndy_n$ and let $\Phi^*\mu = A_1dx_1 + \cdots + A_ndx_n$. Let $\Phi = (\Phi_1(x), ..., \Phi_n(x))$, i.e. the diffeomorphism Φ brings a point with coordinates x to the point with coordinates $y_1 = \Phi_1(x), ..., y_n = \Phi_n(x)$. Let

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_n \end{pmatrix}, \ B = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{pmatrix}, \ (\Phi'(x))^t = \begin{pmatrix} \frac{\partial \Phi_1(x)}{\partial x_1} & \cdots & \frac{\partial \Phi_n(x)}{\partial x_1} \\ \frac{\partial \Phi_1(x)}{\partial x_2} & \cdots & \frac{\partial \Phi_n(x)}{\partial x_2} \\ \cdots & \cdots & \cdots \\ \frac{\partial \Phi_1(x)}{\partial x_n} & \cdots & \frac{\partial \Phi_n(x)}{\partial x_n} \end{pmatrix} (\text{ transpose Jacobi matrix}).$$

Exercise. Show that $A = (\Phi'(0))^tB$.

14. The set $\{T_p^*M^n,\ p \in M^n\}$ is called the cotangent bundle to M^n and it is denoted by T^*M. A smooth section of the cotangent bundle is called a smooth differential 1-form. A smooth differential 1-form α can be understood as a collection of algebraic 1-forms $\alpha_p \in T_p^*M^n$ parameterized by $p \in M^n$ and depending smoothly on p. In local coordinates $\alpha = A_1(x)dx_1 + \cdots A_n(x)dx_n$.
Applying pointwise a differential 1-form α to a vector field V we obtain a function. The map

$$\text{vector field } V \to \alpha(V) = \text{ (another notation) } V \rfloor \alpha = \text{ function on } M^n$$

is linear. In a local coordinate system x one has

$$(A_1(x)dx_1 + \cdots + A_ndx_n) \text{ applied to } (V_1\frac{\partial}{\partial x_1} + \cdots + V_n\frac{\partial}{\partial x_n}) = A_1(x)V_1(x) + \cdots + A_n(x)V_n(x).$$

15. Let Φ be a diffeomorphism as in 13. Applying it pointwise to a differential 1-form α_2 defined in the neighborhood U_2 of the point p_2 we obtain a differential 1-form the neighborhood U_1 of the point p_1. This 1-form is denoted $\Phi^*\alpha_2$. Take local coordinates x and y centered at the points

p_1 and p_2 respectively. Let $\alpha_2 = B_1(y)dy_1 + \cdots + B_n(y)dy_n$, $\alpha_1 = A_1(x)dx_1 + \cdots + A_n(x)dx_n$. Let $\Phi = (\Phi_1(x), ..., \Phi_n(x))$ (see 13.) Let $B(y)$ and $A(x)$ be vector functions with components $B_1(y), ..., B_n(y)$ and $A_1(x), ..., A_n(x)$. Then by 13.

$$A(x) = (\Phi'(x))^t B(\Phi(x)).$$

If $p_1 = p_2$ then this is a formula for change of coordinates. This formula corresponds to notations - instead of using it one can simply substitute $y_1 = \Phi_1(x), ..., y_n = \Phi_n(x)$ in $B_1(y)dy_1 + \cdots + B_n(y)dy_n$ and use the formula for the differential of a function: $df(x) = \frac{\partial f(x)}{\partial x_1}dx_1 + \cdots + \frac{\partial f(x)}{\partial x_n}dx_n$.

16. Of course, the differential of a function can be defined in canonical terms, without using any coordinates. The differential of a function f on M^n is a differential 1-form α on M^n defined as follows. To define α it suffices to define, given an arbitrary vector field V on M^n, the function $g = V \rfloor \alpha = \alpha(V)$. To define the value of the function g at a point $p \in M^n$ take any curve $\gamma : (-\epsilon, \epsilon) \to M^n$, $\gamma(0) = p$, representing the tangent vector $V(p) \in T_pM^n$. Consider the function $\triangle_\gamma : t \to f(\gamma(t)) - f(p)$. It is a function from $(-\epsilon, \epsilon)$ to $\mathbb{R}$. Obviously $\triangle_\gamma(0) = 0$. Set $g(p) = \triangle'_\gamma(0)$.

Lemma and definition. The number $\triangle'_\gamma(0)$ does not depend on the choice of the curve γ representing the tangent vector $V(p)$. Consequently g is a well-defined function on M^n and the construction above leads to a differential 1-form on M^n which depends on the function f only. This differential 1-form is called the differential of the function f and is denoted df.

17. The function g defined in 16. is called the Lie derivative of the function f along the vector field V. One of notations: $g = L_V f$.

According to the construction in 16. one has $L_V f = V \rfloor df$. The map $f \to L_V f$ is linear. It follows that in local coordinates, if $f = f(x), V = a_1(x)\frac{\partial}{\partial x_1} + \cdots + a_n(x)\frac{\partial}{\partial x_n}$, one has

$$L_V f = a_1(x)\frac{\partial f(x)}{\partial x_1} + \cdots + a_n(x)\frac{\partial f(x)}{\partial x_n}.$$

We see that a vector field can be treated as a linear differential first order operator in the space of functions. Therefore the notation $L_V f = V(f)$ is also possible (and used in many works).