3. Elements of singularity theory

Classification of critical points of functions

Denote by $C_o^\infty(x)$ the local algebra of germs at the origin of smooth functions $f(x)$ in $x \in \mathbf{R}^n$.

"Local" means there is a single maximal ideal $\mathcal{M} \subset C_0^\infty$, which is the set of all functions which vanish at the origin $f(0) = 0$.

H'Adamard lemma. A smooth function $f(x,y)$ $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $y \in \mathbf{R}^k$ which vanish on the coordinate subspace $x = 0$ can be written in the form $f = \sum_{i=1}^n x_i h_i(x,y)$ with certain smoooth functions h_i.

Proof. Take $t \in [0,1]$ and differentiate $F(t,x,y) = f(tx,y)$ upon t

$$\frac{\partial F}{\partial t} = \sum_{i=1}^n x_i \frac{\partial \varphi(tx,y)}{\partial x_i}.$$

So

$$f(x,y) = F(1,x,y) = \sum_{i=1}^n x_i (\int_0^1 \frac{\partial f(tx,y)}{\partial x_i} dt).$$

The gradient ideal I_f of a function germ $f(x), 0$ is the ideal generated by partial derivatives of f (of course the ideal does not depend on cooordinates).

$$I_f = C_0^\infty \left\{ \frac{\partial f}{\partial x_i} \right\}$$

Define the local algebra of the germ (f,0) as the factor-algebra $Q_f = C_0^\infty / I_f$.

The dimension μ_f over R of this algebra is called the multiplicity (or Milnor number) of the germ $(f,0)$.

When μ_f is finite - the singularity is isolated (a punctured neighbourhood of the origin contains no critical points).

In the complex analytic case the inverse statement is also true. An isolated critical point of holomorphic function has finite multiplicity.

Any very small deformation of f in a small neighbourhood of the origin cointains at most μ critical points.

The multiplicity μ_f is finite if and only if for certain k $\mathcal{M}^k \subset I_f$. Moreover the following holds

Tougeron's jet sufficiency lemma.

1).$\mathcal{M}^\mu \subset I_f$.

2). Any function germ $\tilde{f}, 0$ with $j^{\mu+1}\tilde{f} = j^{\mu+1} f$ is R-equivalent to $f, 0$. (In particular f is equivalent to a polynomial of degree at most $\mu + 1$.

Proof. Since Q_f is μ-dimensional R-vector space than any $\mu+1$ elements of Q_f are linearly dependent. Take any sequence of germs $\varphi_0, \ldots, \varphi_j, \ldots$ $j = 1, \ldots, \mu$, with $\varphi_j \in \mathcal{M}$ for $j = 1, \ldots \mu$. Consider the sequence of products $\alpha_0 = \varphi_0$, $\alpha_1 = \varphi_0\varphi_1, \alpha_2 = \varphi_0\varphi_1\varphi_2, \ldots$, note that each of the following classes is divisible by the previous one.

These $\mu+1$ classes are linearly dependend so some of them α_j plus a linear combination $c_{j+1}\alpha_{j+1} + \ldots$ of the subsequent ones gives zero class in Q_f. But α_j is a factor of this combination and remaining function is invertible. So α_j is zero in the local algebra.

Thus we see that a product of any μ germs from $\mathcal{M}$ belong to I_f, so any monomial of degree μ belong to I_f. These monomials are generators of $\mathcal{M}^\mu$.

To prove 2). we use Moser homotopic method. Take a deformation $f_t = f + t(\tilde{f} - f), t \in [0, 1]$ and try to find a t-depending diffeomorphism θ_t such that

$$f_t \circ \theta_t = f_0.$$

Differentiation of this equality by t provides so-called "homological" equation

$$\frac{\partial f_t}{\partial t} \circ \theta_t + \left(\frac{\partial f_t}{\partial x} \circ \theta_t\right) \frac{d\theta_t}{dt} = 0.$$

Represent $\frac{d\theta_t}{dt} = v_t(\theta_t(x)) = v_t \circ \theta_t$ where $v_t(x)$ is a (time-dependent) vector field, the flow of which consists of difeomorphisms θ_t. Thus

$$-\frac{\partial f_t}{\partial t} \circ \theta_t + \left(\frac{\partial f_t}{\partial x} \circ \theta_t\right) v_t \circ \theta_t = 0.$$

Since θ_t is a diffeomorphism the homological equation is equivalent to

$$-\frac{\partial f_t}{\partial t} = \frac{\partial f_t}{\partial x} v_t(x) = \sum_{i=1}^{n} \frac{\partial f_t}{\partial x_i} v_t, i(x).$$

We have simplified the problem (instead of solving non-linear equations for θ we have to solve a linear equation for v (however the answer will be not explicit).

Since $\frac{\partial(\tilde{f}-f)}{\partial x} \in \mathcal{M}^{\mu+1}$ then for any t $I_f = I_{f_t}$ and it contains $\mathcal{M}^{\mu}$. Hence the lefthand side has a required decomposition, so the components of v_t exist. They are smooth functions in $t \in [0,1]$ and x from a small neighbourhood of 0. Moreover $v_t(0) = 0$ since the lefthand side $-\frac{\partial f_t}{\partial t}$ belongs to $\mathcal{M}^{\mu+2}$. Thus the origin is a fixed point for the flow, and flow is well defined in a neighbourhoood of it.

Remarks. 1.This lemma holds also in parameter depending case.

In particular, the following parameter depending Morse lemma holds. If the family $f(x,y)$ has non-degenerate critical points with respect to x ($\frac{\partial f}{\partial x} = 0$), at any point $(0,y)$ and $\det(\frac{\partial^2 f}{\partial x^2}) \neq 0$ then there is a diffeomorphism φ of the form $\varphi(x,y) \mapsto (X(x,y),y)$ which reduces the function f near coordinate plane $x=0$ to the sum of $f(0,y)$ with the non-degenerate quadratic form in x with the coefficients depending on y.

In fact, for any y_0 the germ $f(x,y_0) - f(0,y_0), 0$ is Morse function with the multiplicity $\mu = 1$.

Now an appropriate linear change of x coordinates depending on y reduce the function to the required form.

In particular, let $k = n - rank(d^2 f)$ be the corank of the second differential of a function $f(x)$ germ (with the critical point at the origin) Then $f(x)$ is right equivalent to $g(x_1,\ldots,x_k) \pm x_{k+1}^2 \pm \ldots \pm x_n^2$ with a germ $g(x_1,\ldots,x_k) \in \mathcal{M}^3$. (One can prove that the reduced germ g is defined by f up to R-equivalence.

According to Morse lemma it sufice to classify only germs $f(x) \in \mathcal{M}^3$. R-orbits of functions in one variable form an ordered sequence $A_k : f = \pm x^{k+1}$.

R-orbits of functions in two variables (x,y) with non-zero 3-jet form an infinite series $D_k : f = y^2 x \pm x^{k-1}$, three simple orbits $E_6 : x^3 \pm y^4$, $E_7 : x^3 + xy^3$, $E_8 : x^3 + y^5$, or are adjacent to $X_9 : x^4 + y^4 + aX^2y^2, a^2 \neq 4$($X_9$ is adjacent to E_7) or to $J_1 0 : x^3 + ax^2y^2 + y^6, 4a^3 + 27 \neq 0$. (which is adjacent to E_8). All orbits except A, D, E are non-simple (any neighbourhood of them intersects with infinite number of orbits).

A family $F(x,q)$ is called induced from a family $G(x,\lambda)$ if there is a smooth mapping from $\varphi : (q) \mapsto \lambda(q)$ such that $G(x,\lambda(q)$ is R-equivalent to $F(x,q)$.

A germ at the origin of a family $F(x,\lambda)$ is called a versal deformation of the function $f(x) = F(x,o)$ if any deformation $G(x,q)$ of $f(x)$ is induced from F.

Versality theorem. A deformation $F(x,q)$ of a function germ $f(x) = F(x,0)$ is versal if and only if the deformation velocities $\varphi_i(x) = \frac{\partial F}{\partial q_i}|_{q=0}$ in the directions q_i span local algebra Q_f.

The proof is based on homotopy method and the following basic result

Malgrange preparation theorem. Let $g : \mathbf{R}^m \to \mathbf{R}^n, \ g(0) = 0$ $g : x \mapsto y$ be a germ of smooth mapping. Let A be a finitely generated module over the $C_0^\infty(x)$. Denote by $g_*\mathcal{M}_y$ the ideal in $C_0^\infty(x)$ generated by the components of g. Assume that $A = g_*\mathcal{M}_y A + \mathbf{R}\{\varphi_1, \dots \varphi_s\} + \mathcal{M}_x^{s+1}A$, where φ_i are certain elements of A.

Then φ_i generates A as $f_*(C_0^\infty(y))$ module.

In many cases the following easy lemma replaces Malgrange preparation theorem:

Nakayama lemma Let A be a finite generated C_o^∞ module, B - submodule of A. assume that $A = B + \mathcal{M}A$ then $A = B$.

Families of functions which are versal at any point are generic due to an apppropriate version of transversality theorem, which we discuss now.

In the jet space $J^N(\mathbf{R}^k, \mathbf{R}) = \mathbf{R}^n \times J_o^N(n,1)$ with sufficiently high N($N > \mu(f) + 1$) consider the product $\hat{O}_f$ of $\mathbf{R}^n$ with the R-orbit O_f in the space of jets a the origin of a germ $f, 0$ with simple singularity under the action of diffeomorphisms which preserve the origin. The codimension of O_f in $j^N(n,1)$ is $k + \mu(f)$. So the codimension of $\hat{O}_f$ in $J^N(\mathbf{R}^n, \mathbf{R})$ is also $k + \mu(f)$.

The infinitesimal condition of versality of a family $F(x,q)$ is equivalent to transversality of the mapping $J^N F : (x_0, q_0) \mapsto j^N F_{x_0}(x, q_0)$ to $\hat{O}_f$.

To have versality condition at each point we need $J^N F$ be transversal to all orbits. So we need transversality to a stratification.

Stratification of a manifold is a partition of M into open $X_i^{n_i}$ submanifolds (strata) of dimensions n_i such that the boundary condition holds: the closure of a stratum is a union of strata of low dimensions.

A (locally finite) stratification satisfies first regularity (Whitney) condition: if for any sequence of points $x_i, \ i = 1, \dots$ of a stratum X tending to a point y from the boundary stratum Y such that the tangent subspaces $T_{x_i}X$ converge (in some metrics on M) to a subspace T of a tangent space to M at y then $T_y Y \subset T$.

A mapping $f : N \mapsto M$ is called transversal to a stratification S of M if it is transversal to each stratum of S.

Thom transversality theorem. Let S be a Whitney regular stratification of the jet space $J^K(N, M)$ Asuume that manifold N is compact. Then the set of mappings $f : N \to M$ whose jet extension $j^K f : x \mapsto J^K_x f$ is transversal to S form an open and dense subset in the space of all smooth mapping f.

Versality theorem implies that simple R-orbits and appropriate stratification of the union of the non-simple orbits form a regular stratification in any jet space. So families of functions which are versal at any point form open and dense (in the compact case) or countable intersection of open dense subsets in the space of all families.

So generical family of functions is versal at any point. In small dimnensions the germs of generic wavefronts and caustics are diffeomorphic to that of versal deformations of corresponding A, D, E singularities of functuions.

Considering germs (jets) of functions at sets of finitely many points (called respectively multigerms and multijets) the multigerms version of transversality theorem imply that generically the germ of a caustic or wavefront at any point is determined by versal multigerm (it is a transversal intersection of finitely many caustics or wavefronts of versal germs of families).

References

1. V.I.Arnold, A.N.Varchenko, S.M.Gusein-Zade, Singularities of differentiable Maps, vol.1, Bikkhauser,1985

2. V.I.Arnold, V.V.Goryunov, O.V.lyashko, V.A.Vasiliev, Singularity theory.II, Classfication and applications, Dynamical systems VIII, Encyclopaedia Math.Sci 39, Springer,Berlin,1993, 1-235.

3. V.I.Arnold, Singualrities of caustics and wavefronts, Kluwer, 1993.

4. I.A. Bogaevsky, Methamorphoses of singularities of minimum functions, and bifurcations of shock waves of Burgers equation with vanishing viscosity, St.Petersburg MAth. J., 1(4),1990, 807-823.

5. I.A. Bogaevsky, Perestroikas of shocks and singularities of minimum functions, Physica D, 173 (2002), 1-28.