2. Generating families

2.1.Lagrangian case

The skew-orthogonal complements $T^{\angle}_{\cdot}C$ of tangent spaces to coisotropic submanifold $C^{n+k} \subset M^{2n}$ determine integrable distribution of C by isotropic fibers I_c. In fact, take two regular functions, whose zero level set contains C. Their Hamilton vector fields belong at each point of C to $T^{\angle}_{\cdot}C$ So the corresponding flows commute. Trajectories of all such fields issued from a point $x \in C$ form a smooth submanifold which coincide with I_c.

Now by Givental theorem locally any coisotropic sumnanifold is symplectomorphic to the coordinate subspace $p_I = 0,\ I = \{1, \dots, n-k\}$. The fibers are $q_J = const$ spaces.

Proposition. Let $L^n \subset M^{2n}$ be Lagrangian and $C^{n+k} \subset M^{2n}$ be coisotropic submanifolds of symplectic space. Asuume L intersects C transversally at a point a. Then $L \bigcap C = X_0$ is transversal to the isotropic fibers I_c near a.

The proof is immediate. If T_aX_0 contais a vector $v \in T_aI_c$ then v is skew-orthogonal to T_aL and also to T_aC that is to any vector from T_aM. Hence $v = 0$.

Isotropic fibers determine the projection $\xi : C \to B$ to certain manifold B of dimension $2k$ (defined at least locally) We can say that B is the manifold of isotropic fibers. It has well defined induced symplectic structure ω_B Given any two vectors u, v tangent to B at some point b take their liftings that is vectors $\tilde{u}, \tilde{v}$. tangent to C at some point of $\xi^{-1}(b)$ such that their projections to B are u and v. The value $\omega(\tilde{u}, \tilde{v})$ depends only on the vectors u, v. For any other choice of liftings te result will be the same. This value is the value of a two-form ω_B on B. So base B gets a symolectic structure, which is called a symplectic reduction of the C.

Example. Consider a Lagrangian section L of (trivial) Lagrangian fibration $T^*(\mathbf{R}^k \times \mathbf{R}^n$. This submanifold $L$is the graph of the differential of a function $f(x, q),\ x \in \mathbf{R}^k, q \in \mathbf{R}^n$. The dual coordinates, y, p on L are equal to $y = \frac{\partial f}{\partial x},\ p = \frac{\partial f}{\partial q}$. The intersection $\tilde{L}$ of L with coisotropic subspace $y = 0$ is defined by the equations $\frac{\partial f}{\partial x} = 0$. The intersection is transversal iff the rank of the matrix of the derivatives of these equations with respect to x and q

equals k. If so, the symplectic reduction of $\tilde{L}$ is a Lagrangian submanifold L_r in $T^*\mathbf{R}^n$ (which can be not a section of $T^*\mathbf{R}^n \to \mathbf{R}^n$). This example leads to the following definition of generating function (the idea is due to Hormander).

Definition. A generating family for the Lagrangian mapping of submanifold $L \subset T^*N$ is a function $F : E \to \mathbf{R}$ defined on a vector bundle E over N such that

$$L = \left\{ (p,q) \,|\exists x : \frac{\partial F}{\partial x} = 0, p = \frac{\partial F(x,q)}{\partial q} \right\}$$

(where $q \in N$, and x is in the fiber over q) We also assume that the following Morse condition is satisfied: 0 is a regular value for mapping $(x,q) \mapsto \frac{\partial F}{\partial x}$. (If so L is smooth manifold).

Remark. The points of intersection of L with the zero section of T^*N are in one-to-one correspondence with the critical points of S. In symplectic topology when interested in such points people avoid the possibility on having no critical points at all (as it may happen on a non-compact manifold E).

So dealing with global generating families defining Lagrangian submanifolds globally people consider Generating family with good behaviour at infinity. A generating family F is said to be Quadratic at Infinity if it coincides with fiberwise quadratic nondegenerate form $Q(x,q)$ outside a compact.

(the topological properties of such families and on their role in symplectic topology see the papers of C.Viterbo, e.g.

C.Viterbo, Generating functions,symplectic geometry and applications, Proc. of Intern.congr. of Math, Zurich,1994))

Existence and (uniqueness up to certain equivalence relation involving Serre fibration) of QI for Lagrangian submanifolds Hamiltonian isotopic to zero section in T^*N of compact N was proved by Viterbo, Laundeback,Sikorav in 80th. Given any two QI for L there is a unique integer m and real l such that $H^k(F_b, F_a) = H^{k-m}(F_{b-l}, F_{a-l})$ for any pair of $a < b$.

However we need local result which is older and easier.

Existence:

Any germ (L, m) of Lagrangian $L \subset T^*\mathbf{R}^n$ has regular projection to some (q_I, p_J) coordinate spaces. In this case there is a function $f(q_I, p_J)$ (defined up to a constant) such that

$$L = \left\{ (p,q) \,|\, q_J = -\frac{\partial f}{\partial p_J}, p_I = \frac{\partial f}{\partial q_I} \right\}.$$

Then the family $F_J = p_I q_I + f(q_I, p_J)$ is generating for L, m. If $|J|$ is minimal possible then $\mathrm{Hess}_{p_J p_J} F_J$ vanishes.

Uniqueness:

Two germs of familes $F_i(x,q), x \in \mathbf{R}^k, q \in \mathbf{R}^n, \ i = 1,2$ at the origin are called R_0 equivalent if there is a diffeomorphism $\mathcal{T} : (x,q) \mapsto (X(x,q), q)$ (preserving fibration $\mathbf{R}^k \times \mathbf{R}^n \to \mathbf{R}^n$) such that $F_2 = F_1 \circ \mathcal{T}$.

Family $\tilde{F}(x,y,q) = F(x,q) \pm y_1^2 \pm \ldots, \pm y_m^2$ is called a stabilization of F.

Two germs of familes $F_i(x,q), x \in \mathbf{R}^k, q \in \mathbf{R}^n, \ i = 1,2$ at the origin are called R_0-stable equivalent if they are R_0 equivalent to appropriate stabilizations of a single family (in lower number of variables).

Lemma. Any two generating families of L, m are (up to addition of a constant) R_0 stable equivalent.

Proof. Morse lemma with parameters implies that any function $F(x,q)$ (with zero value at the origin) is stable R_0 equivalent to $F_n(y,q) \pm z^2$ where $x = (y,z)$ and $\mathrm{Hess}_{p_J p_J} F_n \,|_0 = 0$.

Clearly $F_n(y,q)$ is also generating for (L,m). Let L has regular projection to (p_J, q_I) with $|J| = \dim \ker \rho_* |_{TL}$. Submanifold $X_0 \subset \{y, q\}$ is diffeomorphic to $L \subset \{p, q\}$. Hence there is a local diffeomorphism $\mathcal{T}_1(y,q) \mapsto (p_J, q)$ such that for a new generating family $G = F_n \circ \mathcal{T}_1^{-1}$ the following holds on $\pi_J L$ (here $\pi_J : (p,q) \mapsto (p_J, q)$) :

$$\frac{\partial G}{\partial p_J} = 0, \quad \text{and} \quad p_J = \frac{\partial G}{\partial q_J}$$

Hence $d(G - F_J)$ vanishes on $\pi_J L$. Since this is a regular submanifold for some constant c the family $G - F_J + c$ belongs to the square of the ideal of smooth functions vanishing on $\pi_J L$ with the generators $\frac{\partial F_J}{\partial p_J}$. Now the homotopy method applied to the family $F_j + (G - F_J - c)t, \ t \in [0,1]$. shows that $G - c$ and F_J are R_0 equivalent.

2.2. Legendre case

Definition. A generating family for the Legendre mapping $\pi|_L$ of a Legendre submanifold $L \subset PT^*(N)$ is a function $F : E \to \mathbf{R}$ defined on a vector bundle E over N such that

$$L = \left\{ (p,q) \,|\exists x : F(x,q) = 0, \; \frac{\partial F}{\partial x} = 0, p = \frac{\partial F(x,q)}{\partial q} \right\}$$

where $q \in N$, x is in the fiber over q provided that the following Morse condition is satisfied: 0 is a regular value for mapping $(x,q) \mapsto \{F, \frac{\partial F}{\partial x}\}$.

Definition. Two families of functions $F_i(x,q)$ $i = 1,2$ are called V-equivalent if there is a fiber preserving diffeomorphism $\Theta : (x,q) \mapsto (X(x,q),q)$ and a nonzero function $\Psi(x,q)$ such that $F_2 \circ \Theta = \Psi F_1$.

Two families of functions $F_i(x,q)$ $i = 1,2$ are called V-stable-equivalent if they are stabilization of a pair of V-equivalent functions (may be in lower number of x variables).

Theorem. Any germ $\pi|_L, o$ of Legendre mapping has a generating family. All generating families of a given germ are V-stable equivalent.

Proof. Using $\pi_0 : J^1\hat{N} \to J^0\hat{N}$ model ($\hat{N} = \mathbf{R}^{n-1}$) of Legendre fibration consider the projection $\pi_1 : J^1 \to T^*N$ restricted to L. Its image is a Lagrangian germ $L_S \subset T^*N$. If $F(x,q)$ is a generating family for L_s then $F(x,q) - z$ considered as a family of functions in x with parameters $(q,z) \in J^0\hat{N} = N$ is generating family for L and vice versa. Now the theorem follows from the Lagrange result and evident properies: A multiplication of a Legendre generating family by a a nonzero function provides a generating family. After an appropriate multiplication by $\Psi \neq 0$ an generating family (satisfying the regularity conditions) takes the form $F(x,q) - z$ where (q,z) form local coordinate system in N.

Remarks: Sympectomorphism φ reserving bundle structure of a standard Lagrangian fibration $T^*\mathbf{R}^n \to \mathbf{R}^n$ $(p,q) \mapsto q$ has very simple form

$$\varphi : (q,p) \mapsto \Big(Q(q), DQ^{-1*}(q)(p + df(q)) \Big)$$

Where $DQ^{-1*}(q)$ is the conjugation of the derivative of the inverse mapping of the base of the fibration $Q \circ \pi = \pi \circ \varphi$, and f is a certain function on the base.

To see this it suffice to write in coordinates the formula $\varphi_* \lambda - \lambda = df$.

This formula implies that fibers of any Lagrangian fibration posess a well defined affine structure.

Consequently, contactomorphisms Ψ of standard Legendre fibration $PT^*\mathbf{R}^n \to \mathbf{R}^n$ acts by linear transformations in the fibers $\Psi : (q,p) \mapsto (Q(q), DQ^{-1*}(q)p)$. So there is a well defined projective structure on the fibers.

So Lagrange equivalences act on generating familes as R-equivalences $(x,q) \mapsto (X(x,q), Q(q))$ and additions with functions in parameters Q.

Legendre equivalences act on Legendre generating families just by R-equivalences.

2.3. Examples of generating families

1. Consider a Hamiltonian $hT^*\mathbf{R}^n \to \mathbf{R}$ which is homogeneous of degree k with respect to impulses $p : h(\tau p, q) = \tau^k h(p,q),\ \tau \in \mathbf{R}$.

An initial submanifold $W_0 \subset \mathbf{R}^n$ initial wavefront defines an exact isotropic $I \subset H_c = H^{-1}(c)$. Asuume that I is a manifold transversal to v_h. Put $c = 1$.

The exact Lagrangian flow invariant submanifold $L = exp_H(I)$ is a cylinder over I with local coordinates $\alpha \in I$, and time t belongs to a certain segment (where the flow is defined).

Assume that in certain domain $U \subset T^*R^n \times \mathbf{R}$ the h phase flow g_t restricted to L is given by mapping $(\alpha, t) \mapsto (Q(\alpha,t), P(\alpha,t))$ with $\frac{\partial P}{\partial \alpha, t} \neq 0$. Then the following holds.

Proposition.

1.The family $F = P(\alpha,t)(q - Q(\alpha,t)) + kt$ of functions in α, t with parameters $q \in \mathbf{R}^n$ is the generating family of L in the domain U.

2. The family $\tilde{F_t} = P(\alpha,t)(q - Q(\alpha,t))$ for any fixed t is Legendre generating family for momentary wavefront W_t

The proof is an immediate verification of Hormander definition using the fact that value of the form pdq on each vector tangent to $g_t(I)$ vanishes and on the vector v_h it is equal to $p\frac{\partial h}{\partial p} == kh = k$.

2. Let $\varphi : T^*\mathbf{R}^n \to T^*\mathbf{R}^n\ \varphi : (q,p) \mapsto (Q,P)$ be a symplectomorphism close to the identity. Thus the system of equations $q' = Q(q,p)$ is solvable for q. Write it solution as $q = \tilde{q}(q', p)$.

Let Lagrangian mapping of Lagrangian submanifold L_0 has generating family $F(x, q)$, then the following family G of functions in x, q, p with parameters q' is the generating family for $\varphi(L)$:

$$G(x, p, q) = F(x, \tilde{q}) + p(\tilde{q} - q) + S(p, q')$$

where $S(q', p)$ is the "generating function" in the sense of Hamiltonian mechanics of the canonical transformation φ that is

$$dS = PdQ - pdq.$$

Note that if φ coincides with the identity mapping outside a compact, then G is a quadratic form at infinity with respect to variables (q, p).

The expression $(\tilde{q} - q) + S(p, q')$ from the formula above is the generating family of the symplectomorphism φ.

3. Represent any symplectomorphism $\varphi : T^*\mathbf{R}^n$ homotopic to the identity as a product of symplectomorphisms each of which is close to the identity. Then by iteration of the previous construction we construct a generating family of $\varphi(L)$ as a sum of initial generating family with the generating families of each of these transformations. The number of variables become very large $dim(x) + 2mn$, where m is the number of iterations. Namely, consider a partition of $[0, T]$ time interval into m small pieces $[t_i, t_{i+1}], i = 0, \dots, m-1$ Let $\varphi = \varphi_m \circ \varphi_{m-1} \circ \dots \varphi_1$ where $\varphi_i : (Q_i, P_i) \mapsto (Q_{i+1}, P_{i+1})$ is the flow map on the interval $[t_i, t_{i+1}]$ Then the generating family is equal to

$$G(x, Q, P, q) = F(x, Q_0) + \sum_{i=0}^{m-1} (P_i(U(Q_{i+1}, P_i) - Q_i) + S_i(P_i, Q_{i+1}))$$

where $Q = Q_0, \dots,\ Q_{m-1},\ q = Q_m,\ Q_i \in \mathbf{R}^n,\ q \in \mathbf{R}^n$, S_i is the generating function of φ_i and $U_i(P_i, Q_{i+1})$ are the solutions of the system of equations $Q_{i+1} = Q_{i+1}(Q_i, P_1)$ given by φ_i.

One can show that if φ is a flow map for time $t = 1$ of a hamiltonian function which is convex with respect to impulses then the generating family G is also convex with respect to P_i and by stabilization procedure these variables can be removed. This provides the generating family of $\varphi(L)$ depending only on x, Q, q. which usually belong to a compact domain. So there is a minimal and maximal value of this function on the fiber over point q. this property is impostant in applications.