Lagrangian and Legendre Singularities

1 Symplectic and contact geometry

1.1 Symplectic geometry

A symplectic form ω on a manifold M is a closed two-form ω, non-degenerate as a skew-symmetric bilinnear form on the tangent space at each point, so $d\omega = 0$ and ω^n is a volume form.

Manifold M endowed with symplectic form is called symplectic, its dimension is even. If the form is exact $\omega = d\lambda$ the symplectic area upon two-chain S equals $\int_{DS} \lambda$. When λ exists and is fixed M is caled exact symplectic.

Examples. 1. Let $K = M = \mathbf{R}^{2n} = \{q_1, \ldots, q_n, p_1, \ldots, p_n\}$, be a vector space, and $\lambda = pdq, \omega = dp \wedge dq$. In these coordinates the form ω is constant. Corresponding bilinear form is given by the matrix

$$E = \begin{pmatrix} 0 & -I \\ I & O \end{pmatrix}$$

Any constant symplectic structure on K in some basis (called Darboux basis) has this matrix.

2. $M = T^*N, \quad \lambda = pdq$ - Liouville form, which is defined in an invariant (coordinate free) way as follows: $\lambda(\alpha) = \pi(\alpha)\rho_*(\alpha)$, where $\alpha \in T(T^*N),\ \pi : T(T^*N) \to T^*N$ and $\rho : T^*N \to N$. This is exact symplect manifold. If q are local coordinates on the base N the dual coordinates p are the the coefficients of decomposition of a covector into linear combination of the differentials dq_i.

3. On Kahler complex manifold M the imaginary part of Hermitian structure $\omega(\alpha, \beta) = Im(\alpha, \beta)$ is a skew-symmetric 2-form which is closed.

4. Product of two symplectic manifolds. Given two symplectic manifolds $M_i, \omega_i \;\; i = 1, 2$ the pair $M \times M_2,\ (\pi_1)_*\omega_1 - (\pi_2)_*\omega_2$ where π_i are projections of the product to the corresponding factors, is a symplectic manifold.

A diffeomorphism $\varphi : M_1 \to M_2$ mapping symplectic structure ω_2 on M_2 to symplectic structure ω_1 on m_1 $\varphi_*\omega_2 = \omega_1$ is called a symplectomorphism from M_1 ω_1 to M_2 ω_2. When M_i, ω_i coincide symplectomorphism preserves symplectic structure. Clearly, symplectomorphism pregerves a volume form ω^n. In the first example the group $Sp(2n)$ of linear symplectomorphisms $\varphi \in Gl(2n)$ is isomorphic to group of matrices S such that $S^{-1} = -ES^tE$ ("t" -stands for transposition).

It is useful to use auxillary scalar product $(\cdot, \cdot)$ on K. Then always it is possible to choose orthnormal Darboux basis. Then we get a well defined operator $\tilde{E}$ (in this basis it has matrix E) such that $\omega(a, b) = (a, \tilde{E}b)$ and K becomes a complex Hermitian space $z = q + ip$ Multiplication by $i = \sqrt{-1}$ is multiplication by $\tilde{E}$. Hermitian structure is $(a, b) + i\omega(a, b)$. Then we get $Gl(n, \mathbf{C}) \bigcap O(2n) = Gl(n, \mathbf{C}) \bigcap Sp(2n) = O(2n) \bigcap Sp(2n) = U(n)$.

The characteristic polynomial of such φ is reciprocal: if a is an eigenvalue, then a^{-1} also is. Jourdan structure for a and a_1 are the same.

Remark. The image of a unitary sphere $S^{2n-1} : q^2 + p^2 = 1$ under a linear symplectomorphism can belong to a cylinder $q_1^2 + p_2^2 \leq r$ only if $r \geq 1$.

The non-linear analog of this result is rather non trivial: (This is Gromov's theorem on symplectic camel) $S^{2n-1} \in T^*R^n$ (in standard Euclidean structure) can not be symplectically embedded into the cylinder $\{p_1^2 + q_1^2 \leq 1\} \times T^*R^{n-1}$.

Thus symplectomorphisms form a "thin" subset in the space of volume ω^n preserving diffeomorphisms (when $n \geq 2$).

The dimension k of a linear subspace $L^k \subset K$ and the rank r_L of the restriction of bilinear form ω on it are the complete set of invariants of L with respect to $Sp(2n)$.

Define a skew orthogonal $L^\angle$ of L as the set of $v \in L$ such that $\omega(v, u) = 0$ for all $u \in L$. (Note that $L^\angle = \tilde{E}(L)$). So dimension of $L^\angle$ is $2n - k$. The kernel subspace of the ω restriction to L is $L \bigcap L^\angle$. Its dimension is $k - r$.

Subspace is called isotropic if $L \subset L^\angle$ (hence $dimL \leq n$). Any line is isotropic. Subspace is called coisotropic if $L^\angle \subset L$ (hence $dimL \geq n$). Any hypersurface H is coisotropic. $H^\angle$ is called a characteristic direction on H. Subspace is called Lagrangian if $L^\angle = L$ (hence $dimL = n$). ($\tilde{E}(L)$ is orthogonal to L).

Lemma Each Lagrangian subspace $L \subset K$ has regular projection at least to one of the 2^n coordinate Lagrangian planes (p_I, q_J) along the complement lagrangian plane (p_J, q_I). Here $I \bigcup J = \{1, \ldots, n\}$ and $I \bigcap J = \emptyset$.

Proof. Let L_q^k be the intersection of L with q-space and L_q has regular projection on q_I (along q_J) with $|I| = k$. If L has no regular projection to p_J (along (q, p_I)) then L contains a vector $v \in (q, p_I)$. The intersection with q-space of skew-orthogonal complement $v^{\angle}$ has $k-1$ dimensional projection to q_I (along q_J) and so does not coincide with L_q. This contradicts the claim that L is Lagrangian.

A Lagrangian subspace L which has regular projection to q plane is the graph of a self-ajoint operator S from q to p space with symmetric matrix in (orthonormal) Darboux basis.

Splitting of $K = L_1 \bigoplus L_2$ with Lagrangian L_1, L_2 is called a polarization. Any two polarizations are symplectomorphic.

Lagrangian Grassmanian $Gr_L(2n)$ is diffeomorphic to $U(n)/O(n)$. Its foundamental group is $\pi_1(\Lambda) = \mathbf{Z}$.

Grassmanian $Gr_k(2n)$ of k-isotropic spaces is isomorphic to $U(n)/(O(k)+U(n-k))$.

Even in non-linear setting symplectic structure has no local invariants (e.g.,in contrast to Riemannian structure) according to the follwing theorems:

Darboux's theorem. Any two symplectic manifolds of equal dimensions are locally symplectomorphic.

Weinstein's theorem. A tubular neighbourhood of two submanifolds $N_i \subset M$ $i = 1, 2$ are symplectomorphic iff the restrictions of ω to tangent subbundles $T_{N_i}M$ are diffeomorphic.

Givental's theorem. Two germs of submanifolds $N_i \subset M$ (of equal dimensions) are symplectomorhic iff the restrictions of ω to TN_i are diffeomorphic.

Proof of Givental's theorem. It suffices to prove that if the restrictions of two symplectic forms ω_i $i = 1, 2$ on tangent spaces to a germ at the origin of submanifold $G \subset M$ coinside then the forms are locally diffeomorphic. Also we may assume that at the origin forms coincide on total tangent space to M. Using homotopy method find a family of diffeomorphisms $g_t,\ t \in [0, 1]$ such that

$$g_t|_G = id, \quad g_t^*(\omega_t) = \omega_1$$

where $\omega_t = \omega_1 + (\omega_2 - \omega_1)t$. Differentiating by t this is equivalent to

$$\omega_1 - \omega_2 = L_v(\omega_t) = d(i_v \omega_t)$$

where L_v is the Lie derivative along the vector field, whose phase flow is g_t. Since ω_t are closed, the left hand side is locally exact form. Using relative Poincare lemma it is possible to find a primitive α of $\omega_1 - \omega_2$ which vanishes on G. Then the required vector field v exists since ω_t is non-degenerate.

Darboux theorem is a corollary of Givental theorem: (take a point as a submanifold).

If at each point x the tangent subspace to a submanifold L in symplectic space M is Lagrangian in symplectic space T_xM then L is called Lagrangian.

Examples. Fibers, zero section, graph of the differential of a function on the base are Lagrangian submanifolds of the cotangent bundle of the base.

The graph of a symplectomorphism is a Lagrangian submanifold of the product space (which has regularr projections on factors). Arbitrary Lagrangian submanifold of the product space defines so-called Lagrangian relation.

Weinstein theorem imply that a tubular neighbourhood of a Lagrangian submanifold L in any symplectic space is symplectomorphic to the tubulat neighbourhood of the zero section in T^*N.

Fibration with Lagrangian fibers is called Lagrangian. Locally all Lagrangian fibrations are symplectomorphic (the proof is similar to Darboux theorem). Cotangent bundle is a Lagrangian fibration.

Let $\psi : L \to T^*N$ be a Lagrangian embedding. The product $\rho \circ \psi : L \to N$ is called Lagrangian mapping. It critical values

$$\Sigma_L = \{q \in N | \exists p : (p, q) \in L, \ \mathrm{rank} d(\rho \circ \psi) < n.\}$$

form *caustic* of Lagrangian mapping. The equivalence of Lagrange mappings is defined by a fiber preserving symplectomorphism of ambient symplectic space. Caustics of equivalent Lagrange mappings are diffeomorphic.

Given a real function $h : M \to \mathbf{R}$ define a Hamiltonian vector field v_h by the formula $\omega(\cdot, v_h) = dh$. Clearly v_h is tangent to the level hypersurfaces

$H_c = h^{-1}(c)$. The directions of v_h on $h = 0$ are characteristics of tangent spaces of the level hypersurfaces H_c.

Associating v_h to h we obtain a Lie algebra structure on the space of functions $[v_h, v_f] = v_{\{h,f\}}$ where $\{h, f\} = v_h(f)$ is the Poisson bracket of Hamiltonians h, f.

Hamiltonian flow (even if h depends on time) consists of symplectomorphisms. Locally (or in $\mathbf{R}^{2n}$) any time depending family of symplectomorphisms starting from the identity mapping is a phase flow of a time dependend hamiltonian. However, say,on torus $\mathbf{R}^2/\mathbf{Z}+\mathbf{Z}$ (which is the factorization of the plane by integer lattice) the family of constant velocity displacements are symplectomorphisms but they can't be Hamiltonian since Hamiltonian function on torus has to have critical points.

Given a time dependent hamiltonian $\tilde{h} = \tilde{h}(t, p, q)$ consider an extended space $M \times T^*\mathbf{R}$ with auxillary coordinates (s, t) and the form $pdq - sdt$. An auxillary (extended) hamiltonian $\hat{h} = -s + \tilde{h}$ determines a flow in the extended space generated by vector field

$$\dot{p} = -\frac{\partial \hat{h}}{\partial q}, \quad \dot{q} = -\frac{\partial \hat{h}}{\partial p},$$

$$\dot{t} = -\frac{\partial \hat{h}}{\partial s} = 1, \quad \dot{s} = \frac{\partial \hat{h}}{\partial t}$$

The restrictions of this flow to $t = const$ sections are essentially the flow mappings of $\tilde{h}$.

Integral of extended form upon a closed chain in $M \times \{t_o\}$ is preserved by $\hat{h}$ hamiltonian flow. Hypersurfaces $-s + \tilde{h} = const$ are invariant. When h is autonomous form the pdq is also a relative integral invariant.

The (transversal) intersection of a Lagrangian L with Hamilton level set $H_c = h^{-1}(c)$ is an isotropic submanifold L_c. All Hamiltonian trajectories issued from L_c form a Lagrangian submanifold (denoted by $exp_H(L_c)$). The space Ξ_{H_c} of Hamilton trajectories on H_c inherit (at least locally) an induced symplectic structure. The projection of $exp_H(L_c)$ on Ξ_{H_c} is Lagrangian. This is a particular case of a symplectic reduction (which will be discussed later).

Example: Set of straight lines in $\mathbf{R}^{2n}$ is $T * S^{n-1}$ as a space of characteristics of hamiltonian $p^2 = 1$ on $\mathbf{R}^{2n}$.

1.2 Contact geometry

An odd dimensional manifold M^{2n+1} endowed with a maximally non-integrable distribution of a hyperplanes (contact elements) in its tangent spaces is called contact manifold. Maximal non-integrability means that if locally the distribution is determined by zeros of a 1-form α on M then $\alpha \wedge (d\alpha)^n \neq 0$ (recall that complete integrability Frobenious condition is $\alpha \wedge d\alpha = 0$.)

Examples.

1. Projectivized cotangent bundle PT^*N^{n+1} with the projectivization of Liouville form $\alpha = pdq$. This is also called a space of contact elements on N. Spherization of PT^*N^{n+1} is two-fold covering of PT^*N^{n+1} and consists of cooriened contact elements.

2. Space of 1-jets of functions on N^n. (Two functions have the same m-jet at a point x if their Taylor polynomials of degree k at x coincide). The space of all 1-jets at all poins has local coordinaters $q \in N$ $p = df(q)$ are partial derivatives of a function at q), and $z = f(q)$. Contact form is $pdq - dz$.

Contactomorphisms are diffeomorphisms preserving distribution of contact elements.

Contact Darboux theorem. Locally all equidimensional contact manifolds are contactomorphic.

Also an analog of Givental's theorem holds.

Symplectization. Let $\tilde{M}^{2n+2}$ be the space of all linear forms vanishing on contact elements of M. The space $\tilde{M}^{2n+2}$ is a "line" bundle over M (fibers do not contain zero form). Denote the corresponding projection by $\tilde{\pi}$. The symplectic structure (which is homogeneous of degree 1 with respect to fibers) is the differential of canonical $\tilde{\alpha}$ 1-form on $\tilde{M}$ defined as follows: $\tilde{\alpha}(\chi) = p(\tilde{\pi}_*\chi)$, where χ is tangent vector to $\tilde{M}$ at a point p.

To any contactomorphism F of M it corresponds a symplectomorphism defined by $\tilde{F}(p) = (F^*_{F(x)})^{-1}p$. It commute with multiplication by constants in the fibers and preserves $\tilde{\alpha}$. The symplectization of contact vector fields (infinitezimal contactomorphisms) lead to hamiltonian vector fields possessing homogeneous (of degree 1) Hamilton functions $h(rx) = rh(x)$.

Assume the contact structure is defined by zeros of a fixed 1-form β, then M has natural embedding $x \mapsto x_\beta$ into $\tilde{M}$.

Using the local model $J^1\mathbf{R}^n, \;\; pdq - dz$ of a contact space we get the following formulas for components of contact vector field with homogeneous hamiltonian function $K(x) = h(x_\beta)$. (Note that $K = \beta(X)$, where X is the corresponding contact vector field.)

$$\dot{z} = pK_p - K, \;\; \dot{p} = -K_q - pK_z, \;\; \dot{q} = K_p.$$

(here subscripts mean partial derivations).

Various homogeneous conterparts of symplectic properties imply hold in contact geometry (the analogy is similar to that of affine and projective geometries).

In particular, a hypersurface (transversal to contact distribution) in a contact space inherits a field of characteristics.

Contactization. To an exact symplectic space M^{2n} associate $\mathbf{R} \times M$ with extra coordinate z and take the form $\alpha = \lambda - dz$. We get a contact space.

Here vector field $\xi = -\frac{\partial}{\partial z}$ is such that $i_\xi \alpha = 1, i_\xi d\alpha = 0$. Such a field is called Reeb vector field. Its direction is uniquely defined by contact structure.It is transversal to contact distribution. Locally projection along ξ produce a symplectic manifold.

Legendre submanifold $\hat{L}$ of M^{2n+1} is an n-dimensional integral submanifold of contact distribution. This dimension is maximal possible.

Examples.

1. To a Lagrangian $L \subset T^*M$ associate $\hat{L} \subset J^1M$ as follows

$$\hat{L} = \{(z, p, q) \mid z = \int pdq, \;\; (p, q) \in L.$$

Here the integration is taken along a pass joining a distinguished point on L with given point (p, q). $\hat{L}$ is Legendre.

2. The set of all cotangent vectors vanishing on tangent spaces to a submanifold $W_0 \subset N$ (or variety) form a Legendre submanifold (variety) in PT^*N.

3. If an intersection I of Legendre submanifold $\hat{L}$ with a hypersurface G in a contact space is transversal then I is tranversal to the characteristic vector

field on G. The collection of characteristics issuied from I form a Legendre submanifold.

Legendre fibration of a contact space is a fibration with Legendre fibers. For example $PT^*N \to N$ and $J^1N \to J^0N$ are Legendre. Locally any two Legendre fibrations (of course, of equal dimensions) are contactomorphic.

Projection to the base of Legendre fibration of an embedded Legendre submanifold $\hat{L}$ is called a Legendre mapping. Its image is called *wave front* of $\hat{L}$.

Example:
1. Embed Legendre $\hat{L}$ into J^1N. Its projection to J^0N is a graph (wave front $W(\hat{L})$ of a multivalued action-function $\int pdq + c$ (again we integrate along passes on Lagrangian submanifold $L = \pi_1(\hat{L})$, where $\pi_1 : J^1N \to T^*N$ is a projection dropping z coordinate). If q does not belong to Σ_L then over q the wave front $W(\hat{L})$ is a collection of smooth branches. If at two distinct point (p_1, q), (p_2, q) from L and non-caustic value q the values of action function are equal to the same value z then at (z, q) wave front contain transversal intersection of graphs of regular functions on q.
The projection $(z, q) \mapsto q$ of singular and transversal self-intersection locus of $W(\hat{L})$ consists of caustic Σ_L and so-called Maxwell (conflict) set.

2. To a function $f(q)$, $q \in \mathbf{R}^n$ associate ils Legendre lifting $\hat{L} = j^1(f)$ (also called 1-jet extension of f) to $J^1\mathbf{R}^n$. Project $\hat{L}$ along fibers parallel to q space of another Legendre fibration $\pi_1^\wedge(z, p, q) \mapsto (z - pq, p)$ of the same contact space $pdq - dz = -qdp - d(z - pq)$. The image of $\pi_1^\wedge(\hat{L})$ is called *Legendre transform* of the function f. It has singularities if f is not convex.
This is an affine version of the projective duality (which is also related to Legendre mappings). The space PT^*P^n (P^n is the projective space) is isomorphic to the projectivised cotangent bundle $PT^*P^{n\wedge}$ of the dual space $P^{n\wedge}$. Elements of both are the pairs consisting of a point and a hyperplane, containing the point. Natural contact structures coincide. Set of all tangent hyperplanes to a submanifold S in P is a front of dual projection of Legendre lifting of S.

Given a Hamilton function $h : T^*N \to \mathbf{R}$ take exact Lagrangian $L = exp_H(I) \subset H$ of Hamilton level hypersurface $H = h^{-1}(c)$ which is flow invariant and consists of all characteristics issued from the intersection I

with H of homogeneous initial Lagrangian submanifold L_0 formed by all covectors annihilating tangent spaces to initial wave front $W_0 \subset N$.

The intersections of the Legendre lifting $\hat{L}$ of L into $J^1 N$ ($z = \int pdq$) with coordinate hypersurfaces $z = const$ project to Legendre submanifolds (varieties) $\hat{L}_z \subset PT^*N$ (by projectivization with respect to p.) In fact, the form pdq vanishes on each tangent vector to $\hat{L}_z$. Generically the dimension of $\hat{L}_z$ is $n-1$.

The wave front in $J^0 N$ of $\hat{L}$ is called big wave front. It is swept by a family of fronts W_z of $\hat{L}_z$ shifted to correspondin slice of z coordinate. Note that up to a constant the value of z at a point over a point (p, q) is equal to $z = \int p \frac{\partial h}{\partial p} dt$ along a segment of Hamilton trajectory joining certain point from initial I with (p, q).

When h is homogeneous of degree k with respect to p in each fiber then $z_t = kct$. Let $I_t \subset L$ be the image of I under the flow transformation g_t for time t. The projectived I_t are Legendrian in PT^*N. The family of their fronts in N is W_{kct}. So W_t are momentary wavefronts propagating from initial W_0. Their singular locii swept caustic Σ_L.

The case of time depending Hamiltonian $h = h(t, p, q)$ can be reduced to the previous one considering an extended phase space $J^1(N \times \mathbf{R})$, $\alpha = pdq - rdt - dz$. The image of initial Legendrian $\hat{L}_O \subset J^1(N \times \{0\})$ under g_t is a Legendre $L_t \subset J^1(N \times \{t\})$.

When locally z can be written as a regular function in q, t it satisfies the Hamilton-Jacobi equation $-\frac{\partial z}{\partial t} + h(t, \frac{\partial z}{\partial q}, q) = 0$.