Lecture Notes on Hyperbolic Systems of Conservation Laws

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1 - Introduction

A system of conservation laws in one space dimension takes the form

$$u_t + f(u)_x = 0\,. \tag{1.1}$$

Here $u = (u_1, \ldots, u_n)$ is the vector of **conserved quantities** while the components of $f = (f_1, \ldots, f_n)$ are called the **fluxes**. Integrating (1.1) over a fixed interval $[a, b]$ we find

$$\begin{aligned}\frac{d}{dt}\int_a^b u(t,x)\,dx \;&=\; \int_a^b u_t(t,x)\,dx \;=\; -\int_a^b f\big(u(t,x)\big)_x\,dx \\ &= f\big(u(t,a)\big) - f\big(u(t,b)\big) \;=\; [\text{inflow at } a] - \;[\text{outflow at } b]\,.\end{aligned}$$

Each component of the vector u thus represents a quantity which is neither created nor destroyed: its total amount inside any given interval $[a, b]$ can change only because of the flow across boundary points.

Systems of the form (1.1) are commonly used to express the fundamental balance laws of continuum physics, when small viscosity or dissipation effects are neglected. For a comprehensive discussion of conservation laws and their derivation from basic principles of physics we refer to the book of Dafermos [D1].

Smooth solutions of (1.1) satisfy the equivalent quasilinear system

$$u_t + A(u)u_x = 0\,, \tag{1.2}$$

where $A(u) \doteq Df(u)$ is the Jacobian matrix of first order partial derivatives of f. We notice, however, that if u has a jump at some point x_0, then the left hand side of (1.2) contains a product of the discontinuous function $x \mapsto A\big(u(x)\big)$ with the distributional derivative u_x, which in this case contains a Dirac mass at the point x_0. In general, such a product is not well defined. The quasilinear system (1.2) is thus meaningful only within a class of continuous functions. On the other hand, working with the equation in divergence form (1.1) allows us to consider discontinuous solutions as well, interpreted in distributional sense. We say that a locally integrable function $u = u(t,x)$ is a **weak solution** of (1.1) if $t \mapsto u(t,\cdot)$ is continuous as a map with values in $\mathbf{L}^1_{\rm loc}$ and moreover

$$\iint \big\{u\phi_t + f(u)\phi_x\big\}\, dxdt \;=\; 0 \tag{1.3}$$

for every differentiable function with compact support $\phi \in \mathcal{C}^1_c$.

The above system is called **strictly hyperbolic** if each matrix $A(u) \doteq Df(u)$ has n real, distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$. One can then find dual bases of left and right eigenvectors of $A(u)$, denoted by $l_1(u), \ldots, l_n(u)$ and $r_1(u), \ldots, r_n(u)$, normalized according to

$$|r_i| = 1\,, \qquad\qquad l_i \cdot r_j = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{1.4}$$

To appreciate the effect of the non-linearity, consider first the case of a linear system with constant coefficients

$$u_t + Au_x = 0. \tag{1.5}$$

Call $\lambda_1 < \cdots < \lambda_n$ the eigenvalues of the matrix A, and let l_i, r_i be the corresponding left and right eigenvectors as in (1.4). The general solution of (1.5) can be written as a superposition of independent linear waves:

$$u(t,x) = \sum_i \phi_i(x - \lambda_i t) r_i\,, \qquad\qquad \phi_i(y) \doteq l_i \cdot u(0,y)\,.$$

Notice that here the solution is completely decoupled along the eigenspaces of A, and each component travels with constant speed, given by the corresponding eigenvalue of A.

In the nonlinear case (1.2) where the matrix A depends on the state u, new features will appear in the solutions.

(i) Since the eigenvalues λ_i now depend on u, the shape of the various components in the solution will vary in time (fig. 1). Rarefaction waves will decay, and compression waves will become steeper, possibly leading to shock formation in finite time.

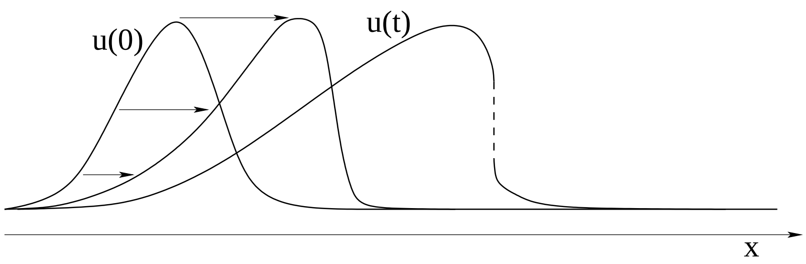


figure 1

(ii) Since the eigenvectors r_i also depend on u, nontrivial interactions between different waves will occur (fig. 2). The strength of the interacting waves may change, and new waves of different families can be created, as a result of the interaction.

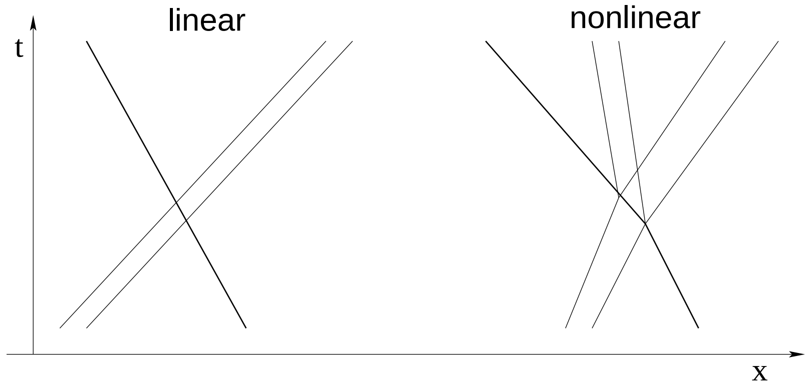


figure 2

The strong nonlinearity of the equations and the lack of regularity of solutions, also due to the absence of second order terms that could provide a smoothing effect, account for most of the difficulties encountered in a rigorous mathematical analysis of the system (1.1). It is well known that the main techniques of abstract functional analysis do not apply in this context. Solutions cannot be represented as fixed points of continuous transformations, or in variational form, as critical points of suitable functionals. Dealing with vector valued functions, comparison principles based on upper or lower solutions cannot be used. Moreover, the theory of accretive operators and contractive nonlinear semigroups works well in the scalar case [C], but does not apply to systems. For the above reasons, the theory of hyperbolic conservation laws has largely developed by *ad hoc* methods, along two main lines.

1. The BV setting, considered by Glimm [G]. Solutions are here constructed within a space of functions with bounded variation, controlling the BV norm by a wave interaction functional.

2. The $\mathbf{L}^\infty$ setting, considered by Tartar and DiPerna [DP2], based on weak convergence and a compensated compactness argument.

Both approaches yield results on the global existence of weak solutions. However, the method of compensated compactness appears to be suitable only for 2×2 systems. Moreover, it is only in the BV setting that the well-posedness of the Cauchy problem could recently be proved, as well as the stability and convergence of vanishing viscosity approximations. Throughout the following we thus restrict ourselves to the study of BV solutions, referring to [DP2] or [Se] for the alternative approach based on compensated compactness.

Since the pioneering work of Glimm, the basic building block toward the construction and the analysis of more general solutions has been provided by the **Riemann problem**, i.e. the initial value problem with piecewise constant data

$$u(0,x) = \bar u(x) = \begin{cases} u^- & \text{if} \quad x < 0, \\ u^+ & \text{if} \quad x > 0. \end{cases} \tag{1.6}$$

This was first introduced by B.Riemann (1860) in the context of isentropic gas dynamics. A century later, P.Lax [Lx] and T.P.Liu [L1] solved the Riemann problem for more general $n \times n$ systems. The new approach of Bianchini [Bi] now applies to all strictly hyperbolic systems, not necessarily in conservation form. Solutions are always found in the self-similar form $u(t,x) = U(x/t)$. The central position taken by the Riemann problem is related to a symmetry of the equations (1.1). If $u = u(t,x)$ is a solution of (1.1), then for any $\theta > 0$ the function

$$u^\theta(t,x) \doteq u(\theta t, \theta x)$$

provides another solution. The solutions which are invariant under these rescalings of the independent variables are precisely those which correspond to some Riemann data (1.6).

For a general Cauchy problem, both the Glimm scheme [G] and the method of front tracking [D1], [DP1], [B1], [BaJ], [HR] yield approximate solutions of a general Cauchy problem by piecing together a large number of Riemann solutions. For initial data with small total variation, this approach is successful because one can provide a uniform a priori bound on the amount of new waves produced by nonlinear interactions, and hence on the total variation of the solution. It is safe to say that, in the context of weak solutions with small total variation, nearly all results on the existence, uniqueness, continuous dependence and qualitative behavior have relied on a careful analysis of the Riemann problem.

In [BiB] a substantially different perspective has emerged from the study of vanishing viscosity approximations. Solutions of (1.1) are here obtained as limits for $\varepsilon \to 0$ of solutions to the parabolic problems

$$u^\varepsilon_t + A(u^\varepsilon)u^\varepsilon_x = \varepsilon u^\varepsilon_{xx} \tag{1.7}$$

with $A(u) \doteq Df(u)$. This approach is very natural and has been considered since the 1950's. However, complete results had been obtained only in the scalar case [O], [K]. For general $n \times n$ systems, the main difficulty lies in establishing the compactness of the approximating sequence. We observe that $u^\varepsilon(t,x)$ solves (1.7) if and only if $u^\varepsilon(t,x) = u(t/\varepsilon,\, x/\varepsilon)$ for some function u which satisfies

$$u_t + A(u)u_x = u_{xx}\,. \tag{1.8}$$

In the analysis of vanishing viscosity approximations, the key step is to derive a priori estimates on the total variation and on the stability of solutions of (1.8). For this parabolic system, the rescaling $(t,x) \mapsto (\theta t,\, \theta x)$ no longer determines a symmetry. Hence the Riemann data no longer hold a privileged position. The role of basic building block is now taken by the **viscous travelling profiles**, i.e. solutions of the form

$$u(t,x) = U(x - \lambda t).$$

Of course, the function U must then satisfy the second order O.D.E.

$$U'' = \big(A(U) - \lambda\big)U'.$$

In this new approach, the profile $u(\cdot)$ of a viscous solution is viewed locally as a superposition of viscous travelling waves. More precisely, let a smooth function $u : I\!R \mapsto I\!R^n$ be given. At each point x, looking at the third order jet (u, u_x, u_{xx}) we seek travelling profiles $U_1, \ldots, U_n$ such that

$$U_i'' = \big(A(U_i) - \sigma_i\big)U_i' \tag{1.9}$$

for some speed σ_i close to the characteristic speed λ_i, and moreover

$$U_i(x) = u(x) \qquad\qquad i = 1, \ldots, n\,, \tag{1.10}$$

$$\sum_i U_i'(x) = u_x(x)\,, \qquad\qquad \sum_i U_i''(x) = u_{xx}(x)\,. \tag{1.11}$$

It turns out that this decomposition is unique provided that the travelling profiles are chosen within suitable center manifolds. We let $\tilde r_i$ be the unit vector parallel to U_i', so that $U_i' = v_i \tilde r_i$ for some scalar v_i. One can show that $\tilde r_i$ remains close to the eigenvector $r_i(u)$ of the Jacobian matrix $A(u) \doteq Df(u)$, but $\tilde r_i \neq r_i(u)$ in general. The first equation in (1.11) now yields the decomposition

$$u_x = \sum_i v_i \tilde r_i\,. \tag{1.12}$$

If $u = u(t,x)$ is a solution of (1.8), we can think of v_i as the density of i-waves in u. The remarkable fact is that these components satisfy a system of evolution equations

$$v_{i,t} + (\tilde{\lambda}_i v_i)_x - v_{i,xx} = \phi_i \qquad\qquad i = 1, \ldots, n\,, \tag{1.13}$$

where the source terms ϕ_i on the right hand side are INTEGRABLE over the whole domain $\{x \in I\!\!R\,,\ t > 0\}$. Indeed, we can think of the sources ϕ_i as new waves produced by interactions between viscous waves. Their total strength is controlled by means of viscous interaction functionals, somewhat similar to the one introduced by Glimm in [G] to study the hyperbolic case. Since the left hand side of (1.13) is in conservation form and the vectors $\tilde{r}_i$ have unit length, for an arbitrarily large time t we obtain the bound

$$\big\|u_x(t)\big\|_{\mathbf{L}^1} \leq \sum_i \big\|v_i(t)\big\|_{\mathbf{L}^1} \leq \sum_i \left(\big\|v_i(t_0)\big\| + \int_{t_0}^t \int \big|\phi_i(s,x)\big|\,dxds \right). \tag{1.14}$$

This argument yields global BV bounds and stability estimates for viscous solutions. In turn, letting $\varepsilon \to 0$ in (1.7), a standard compactness argument yields the convergence of u^ε to a weak solution u of (1.1).

2 - The Cauchy Problem

In most of this section, we shall consider a strictly hyperbolic system of conservation laws satisfying the additional hypothesis

(H) For each $i = 1, \ldots, n$, the i-th field is either **genuinely nonlinear**, so that $D\lambda_i(u) \cdot r_i(u) > 0$ for all u, or **linearly degenerate**, with $D\lambda_i(u) \cdot r_i(u) = 0$ for all u.

We observe that the i-th characteristic field is genuinely nonlinear iff the eigenvalue λ_i is strictly increasing along each integral curve of the corresponding field of eigenvectors r_i. It is linearly degenerate when λ_i is constant along each such curve.

2.1. Centered Rarefaction Waves.

We begin by studying a special type of solutions, in the form of centered rarefaction waves. Fix a state $u_0 \in I\!\!R^n$ and an index $i \in \{1, \ldots, n\}$. Let $r_i(u)$ be a field of i-eigenvectors of the Jacobian matrix $A(u) = Df(u)$. The integral curve of the vector field r_i through the point u_0 is called the ***i*-rarefaction curve** through u_0. It is obtained by solving the initial value problem for the O.D.E. in state space:

$$\frac{du}{ds} = r_i(u), \qquad\qquad u(0) = u_0. \tag{2.1}$$

We shall denote this curve as

$$s \mapsto R_i(s)(u_0).$$

Clearly, the parametrization depends on the choice of the eigenvectors r_i. In particular, if we impose the normalization $\big|r_i(u)\big| \equiv 1$, then the rarefaction curve R_i will be parametrized by arc-length.

Let the i-th field be genuinely nonlinear. Given two states u^-, u^+, assume that u^+ lies on the positive i-rarefaction curve through u^-, i.e. $u^+ = R_i(\sigma)(u^-)$ for some $\sigma > 0$. For each $s \in [0, \sigma]$, define

$$\lambda_i(s) = \lambda_i\big(R_i(s)(u^-)\big).$$

By genuine nonlinearity, the map $s \mapsto \lambda_i(s)$ is strictly increasing. Hence, for every $\lambda \in \big[\lambda_i(u^-),\ \lambda_i(u^+)\big]$, there is a unique value $s \in [0, \sigma]$ such that $\lambda = \lambda_i(s)$. We claim that, for $t > 0$, the function

$$u(t,x) = \begin{cases} u^- & \text{if} \quad x/t < \lambda_i(u^-), \\ R_i(s)(u^-) & \text{if} \quad x/t = \lambda_i(s) \in \big[\lambda_i(u^-),\ \lambda_i(u^+)\big], \\ u^+ & \text{if} \quad x/t > \lambda_i(u^+), \end{cases} \tag{2.2}$$

is a continuous solution of the Riemann problem (1.1), (1.6). Indeed, by construction it follows

$$\lim_{t \to 0+} \big\|u(t, \cdot) - \bar{u}\big\|_{\mathbf{L}^1} = 0.$$

Moreover, the equation (1.2) is trivially satisfied in the sectors where $x/t < \lambda_i(u^-)$ or $x/t > \lambda_i(u^+)$, since here $u_t = u_x = 0$. Next, consider what happens in the intermediate sector $\big\{\lambda_i(u^-) < x/t < \lambda_i(u^+)\big\}$. Since u is constant along each ray through the origin $\{x/t = c\}$, we have

$$u_t(t,x) + \frac{x}{t}\, u_x(t,x) = 0.$$

Our construction at (2.2) now yields $x/t = \lambda_i\big(u(t,x)\big)$. Moreover, u_x is parallel to $r_i(u)$, hence it is an eigenvector of the Jacobian matrix $A(u)$ with eigenvalue $\lambda_i(u)$. This implies $u_t + A(u)u_x = 0$, proving our claim. Notice that the assumption $\sigma > 0$ is essential for the validity of this construction. In the opposite case $\sigma < 0$, the definition (2.2) would yield a triple-valued function in the region where $x/t \in \big]\lambda_i(u^+),\ \lambda_i(u^-)\big[$.

2.2. Shocks and Admissibility Conditions.

The simplest type of discontinuous solution of (1.1) is given by a **shock** :

$$U(t,x) = \begin{cases} u^+ & \text{if} \quad x > \lambda t, \\ u^- & \text{if} \quad x < \lambda t, \end{cases} \tag{2.3}$$

for some left and right states $u^-, u^+ \in I\!\!R^n$ and a speed $\lambda \in I\!\!R$. Using the divergence theorem, one checks that the identity (1.2) is satisfied for all test functions ϕ if and only if the following Rankine-Hugoniot conditions hold:

$$\lambda\,(u^+ - u^-) = f(u^+) - f(u^-)\,. \tag{2.4}$$

As usual, we denote by $A(u) = Df(u)$ the $n \times n$ Jacobian matrix of f at u. For any u, v, consider the averaged matrix

$$A(u,v) \doteq \int_0^1 A\big(\theta u + (1-\theta)v\big)\; d\theta$$

and call $\lambda_i(u,v)$, $i = 1, \ldots, n$ its eigenvalues. We can then write (2.4) in the equivalent form

$$\lambda\,(u^+ - u^-) = \int_0^1 Df\big(\theta u^+ + (1-\theta)u^-\big)\cdot(u^+ - u^-)\; d\theta = A(u^+, u^-)\cdot(u^+ - u^-). \tag{2.5}$$

In other words, the Rankine-Hugoniot conditions hold if and only if the jump $u^+ - u^-$ is an eigenvector of the averaged matrix $A(u^+, u^-)$ and the speed λ coincides with the corresponding eigenvalue.

The representation (2.5) is very useful for studying the set of pairs (u^-, u^+) that can be connected by a shock. Given a left state $u^- = u_0$, for each $i = 1, \ldots, n$ one can prove that there exists a one-parameter curve of right states $u^+ = S_i(s)(u_0)$ which satisfy the Rankine-Hugoniot conditions (2.4) for a suitable speed $\lambda = \lambda_i(s)$. This curve can also be parametrized by arc-length:

$$s \mapsto S_i(s)(u_0).$$

It has a second order tangency with the corresponding rarefaction curve R_i at the point u_0 (see fig. 3).

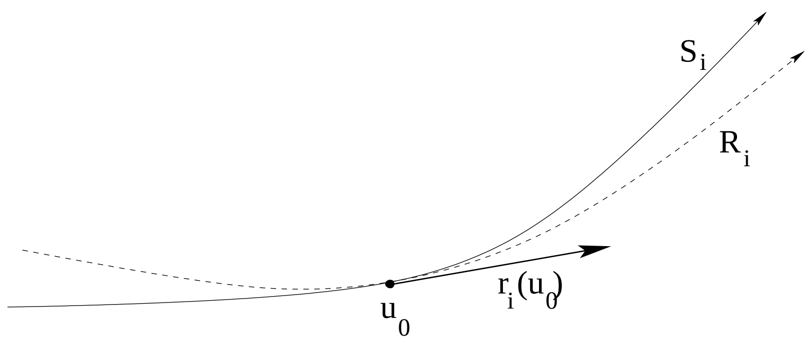


figure 3

To study the behavior of a general weak solution at a point of discontinuity, we first recall some definitions.

A function $u = u(t,x)$ has an **approximate jump discontinuity** at the point (τ,ξ) if there exists vectors $u^+ \neq u^-$ and a speed λ such that

$$\lim_{r\to 0+} \frac{1}{r^2}\int_{-r}^{r}\int_{-r}^{r} \big|u(\tau+t,\,\xi+x) - U(t,x)\big|\,dxdt = 0\,, \tag{2.6}$$

with U as in (2.3). We say that u is **approximately continuous** at the point (τ,ξ) if the above limit hold with $u^+ = u^-$ (and λ arbitrary).

If u is now a solution of the system of conservation laws (1.1) having an approximate jump at a point (τ,ξ), one can prove that the states u^-, u^+ and the speed λ again satisfy the Rankine-Hugoniot conditions (see Theorem 4.1 in [B3]).

In order to achieve uniqueness of solutions to initial value problems, one needs to supplement the conservation equations (1.1) with additional admissibility conditions, to be satisfied at points of jump. We recall here two basic approaches.

A continuously differentiable function $\eta : I\!R^n \mapsto I\!R$ is called an **entropy** for the system of conservation laws (1.1), with **entropy flux** $q : I\!R^n \mapsto I\!R$, if it satisfies the identity

$$D\eta(u)\cdot Df(u) = Dq(u). \tag{2.7}$$

An immediate consequence of (2.7) is that, if $u = u(t,x)$ is a $\mathcal{C}^1$ solution of (2.1), then

$$\eta(u)_t + q(u)_x = 0. \tag{2.8}$$

For a smooth solution u, not only the quantities $u_1,\ldots,u_n$ are conserved, but the additional conservation law (2.8) holds as well. However one should be aware that, for a discontinuous solution u, the quantity $\eta(u)$ may not be conserved. A standard admissibility condition for weak solutions can now be formulated as follows.

Let η be a convex entropy for the system (1.1), with entropy flux q. A weak solution u is **entropy-admissible** if

$$\eta(u)_t + q(u)_x \leq 0 \tag{2.9}$$

in distribution sense, i.e.

$$\iint \big\{\eta(u)\varphi_t + q(u)\varphi_x\big\}\,dxdt \geq 0 \tag{2.10}$$

for every function $\varphi \geq 0$, continuously differentiable with compact support.

In analogy with (2.4), if u is an entropy admissible solution, at every point of approximate jump one can show that (2.9) implies

$$\lambda\big[\eta(u^+) - \eta(u^-)\big] \geq q\big(u^+) - q(u^-). \tag{2.11}$$

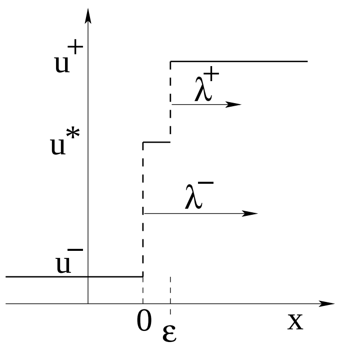
The above admissibility condition can be useful only if some nontrivial convex entropy exists. This is not always the case. Indeed, according to (2.7), to construct an entropy we must solve a system of n equations for the two scalar functions η, q. This system is overdetermined whenever $n \geq 3$. In continuum physics, however, one finds various systems of conservation laws endowed with significant entropies.

An alternative admissibility condition, due to Liu [L2], is particularly useful because it can be applied to any system, even in the absence of nontrivial entropies., To understand its meaning, consider first a scalar conservation law. In this case, any two states u^-, u^+ satisfy the equations (2.4), provided that we choose the speed

$$\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-}\,. \tag{2.12}$$

To test the stability of the shock (2.3) w.r.t. a small perturbation, fix any intermediate state $u^* \in [u^-, u^+]$. We can slightly perturb the initial data by splitting the initial jump in two smaller jumps (fig. 4), say

$$u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0, \\ u^* & \text{if} \quad 0 < x < \varepsilon, \\ u^+ & \text{if} \quad x > \varepsilon. \end{cases}$$



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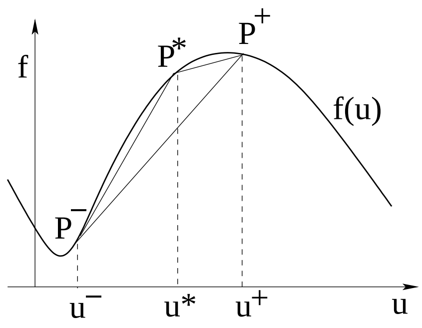

figure 5

Clearly, the original shock can be stable only if the speed λ^- of the jump behind is faster than the speed λ^+ of the jump ahead, i.e.

$$\lambda^- \doteq \frac{f(u^*) - f(u^-)}{u^* - u^-} \;\geq\; \frac{f(u^+) - f(u^*)}{u^+ - u^*} = \lambda^+\,. \tag{2.13}$$

In fig. 5, the first speed corresponds to the slope of the segment P^-P^* while the second speed is the slope of P^*P^+. On the other hand, by (2.12) the speed λ of the original shock is the slope of the segment P^-P^+. Therefore the inequality (2.13) holds if and only if

$$\frac{f(u^*)-f(u^-)}{u^*-u^-} \geq \frac{f(u^+)-f(u^-)}{u^+-u^-}. \tag{2.14}$$

The condition (2.13) cannot be extended to the vector valued case, because in general one cannot find intermediate states u^* such that both couples (u^-,u^*) and (u^*,u^+) are connected by a shock. However, the condition (2.14) has a straightforward generalization to $n\times n$ systems. Denote by $s\mapsto S_i(s)(u^-)$ the i-shock curve through the point u^-, and let $\lambda_i(s)$ be the speed of the shock connecting u^- with $S_i(s)(u^-)$. Now consider any right state

$$u^+ = S_i(\sigma)(u^-)$$

on this shock curve. Following [L2], we say that the shock (2.3) satisfies the **Liu admissibility condition** if

$$\lambda_i(s) \geq \lambda_i(\sigma) \qquad \text{for all} \quad s\in[0,\sigma] \tag{2.15}$$

(or for all $s\in[\sigma,0]$, in case where $\sigma<0$). In other words, if $u^*=S_i(s)(u^-)$ is any intermediate state along the shock curve through u^-, the speed of the shock (u^-,u^*) should be larger or equal to the speed of the original shock (u^-,u^+).

2.3. Solution of the Riemann Problem.

Let the assumptions (H) hold, so that each characteristic field is either genuinely nonlinear or linearly degenerate. Relying on the previous analysis, the solution of the general Riemann problem

$$u_t+f(u)_x=0\,, \qquad u(0,x)=\begin{cases} u^- & \text{if} \quad x<0,\\ u^+ & \text{if} \quad x>0.\end{cases} \tag{2.16}$$

can be obtained by finding intermediate states $\omega_0=u^-,\ \omega_1,\ldots,\ \omega_n=u^+$ such that each pair of adiacent states ω_{i-1},ω_i can be connected by an i-shock, or by a centered rarefaction i-wave. By the implicit function theorem, this can always be done provided that the two states u^-,u^+ are sufficiently close. Following [Lx], the complete solution is now obtained by piecing together the solutions of the n Riemann problems

$$u_t+f(u)_x=0, \qquad u(0,x)=\begin{cases} \omega_{i-1} & \text{if} \quad x<0,\\ \omega_i & \text{if} \quad x>0,\end{cases} \tag{2.17}$$

on different sectors of the t-x plane. A typical situation is illustrated in fig. 6.

If the assumption (H) is removed, one can still solve the Riemann problem in terms of $n+1$ constant states $\omega_0,\ldots,\omega_n$. However, in this case each couple of states ω_{i-1},ω_i can be connected

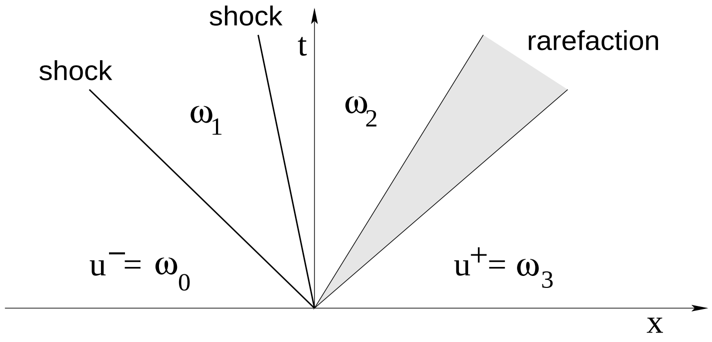

figure 6

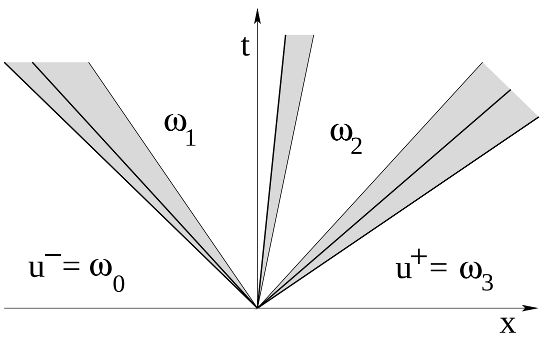

figure 7

by a large number of adiacent shocks and rarefaction waves (fig. 7). The analysis of the Riemann problem thus becomes far more complicated [L1].

2.4. Glimm and front tracking approximations.

Approximate solutions for the general Cauchy problem can be constructed by patching together solutions of several Riemann problems. In the Glimm scheme (fig. 8), one works with a fixed grid in the t-x plane, with mesh sizes Δt, Δx. At time $t=0$ the initial data is approximated by a piecewise constant function, with jumps at grid points. Solving the corresponding Riemann problems, a solution is constructed up to a time Δt sufficiently small so that waves emerging from different nodes do not interact. At time $t_1 \doteq \Delta t$, we replace the solution $u(\Delta t,\cdot)$ by a piecewise constant function having jumps exactly at grid points. Solving the new Riemann problems at every one of these points, one can prolong the solution to the next time interval $[\Delta t,\ 2\Delta t]$. At time $t_2 \doteq 2\Delta t$, the solution is again approximated by a piecewise constant functions with jumps exactly at grid points, etc... A key ingredient of the Glimm scheme is the restarting procedure. At each time $t_j \doteq j\,\Delta t$, a natural way to approximate a BV function with a piecewise constant one is by taking its average value on each subinterval $J_i \doteq [x_{i-1},\ x_i]$. However, this procedure may generate an arbitrarily large amount of oscillations. Instead, the Glimm scheme is based on random sampling: a point y_i is selected at random inside each interval J_i and the old value $u(t_j-,\ y_i)$ at this particular point is taken as the new value of $u(t_j, x)$ for all $x \in J_i$. An excellent introduction to the Glimm scheme can be found in the book by J. Smoller [Sm].

An alternative technique for contructing approximate solutions is by wave-front tracking (fig. 9). This method was introduced by Dafermos [D1] in the scalar case and later developed by various authors [DP1], [B1], [BaJ]. It now provides an efficient tool in the study of general $n\times n$ systems of conservation laws, both for theoretical and numerical purposes [B3], [HR].

The initial data is here approximated with a piecewise constant function, and each Riemann

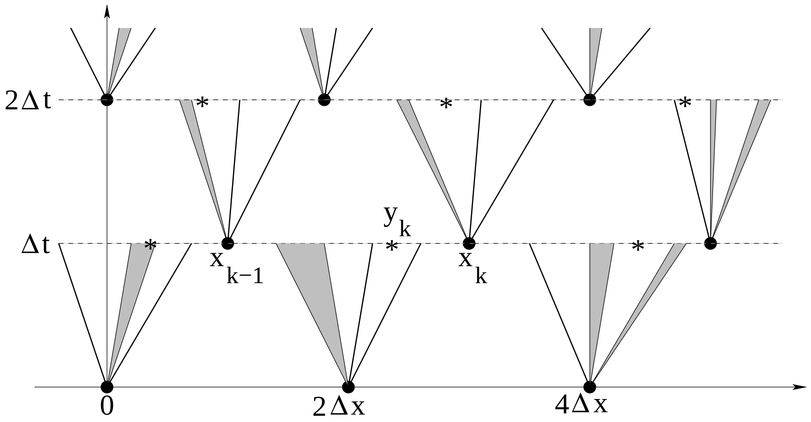

figure 8

problem is solved approximately, within the class of piecewise constant functions. In particular, if the exact solution contains a centered rarefaction, this must be approximated by a *rarefaction fan*, containing several small jumps. At the first time t_1 where two fronts interact, the new Riemann problem is again approximately solved by a piecewise constant function. The solution is then prolonged up to the second interaction time t_2, where the new Riemann problem is solved, etc...

Comparing this method with the previous one of Glimm, we see that in the Glimm scheme one specifies a priori the nodal points where the the Riemann problems are to be solved. On the other hand, in a solution constructed by wave-front tracking the locations of the jumps and of the interaction points depend on the solution itself. No restarting procedure is needed and the map $t \mapsto u(t,\cdot)$ is thus continuous with values in $\mathbf{L}^1_{\rm loc}$.

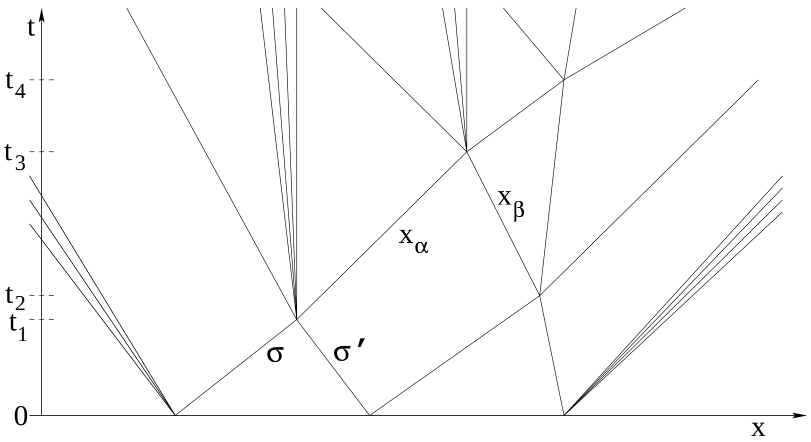

figure 9

In the end, both algorithms produce a sequence of approximate solutions, whose convergence relies on a compactness argument based on uniform bounds on the total variation. We sketch the main idea involved in these a priori BV bounds. Consider a piecewise constant function $u : I\!\!R \mapsto I\!\!R^n$, say with jumps at points $x_1 < x_2 < \cdots < x_N$. Call σ_α the amplitude of the jump at x_α. The *total strength of waves* is then defined as

$$V(u) \doteq \sum_\alpha |\sigma_\alpha| \,. \tag{2.18}$$

Clearly, this is an equivalent way to measure the total variation. Along a solution $u = u(t,x)$ constructed by front tracking, the quantity $V(t) = V\big(u(t,\cdot)\big)$ may well increase at interaction times. To provide global a priori bounds, following [G] one introduces a *wave interaction potential*, defined as

$$Q(u) = \sum_{(\alpha,\beta)\in\mathcal{A}} |\sigma_\alpha\, \sigma_\beta|, \tag{2.19}$$

where the summation runs over the set $\mathcal{A}$ of all couples of approaching waves. Roughly speaking, we say that two wave-fronts located at $x_\alpha < x_\beta$ are *approaching* if the one at x_α has a faster speed than the one at x_β (hence the two fronts are expected to collide at a future time). Now consider a time τ where two incoming wave-fronts interact, say with strengths σ, σ' (for example, take $\tau = t_1$ in fig. 9). The difference between the outgoing waves emerging from the interaction and the two incoming waves σ, σ' is of magnitude $\mathcal{O}(1)\cdot|\sigma\sigma'|$. On the other hand, after time τ the two incoming waves are no longer approaching. This accounts for the decrease of the functional Q in (2.19) by the amount $|\sigma\sigma'|$. Observing that the new waves generated by the interaction could approach all other fronts, the change in the functionals V, Q across the interaction time τ is estimated as

$$\Delta V(\tau) = \mathcal{O}(1)\cdot|\sigma\sigma'|\,, \qquad\qquad \Delta Q(\tau) = -|\sigma\sigma'| + \mathcal{O}(1)\cdot|\sigma\sigma'|\, V(\tau-).$$

If the initial data has small total variation, for a suitable constant C_0 the quantity

$$\Upsilon(t) \doteq V\big(u(t,\cdot)\big) + C_0\, Q\big(u(t,\cdot)\big)$$

is monotone decreasing in time. This argument provides a uniform BV bound on all approximate solutions. Using Helly's compactness theorem, one obtains the strong convergence in $\mathbf{L}^1_{\rm loc}$ of a subsequence of approximate solutions, and hence the existence of a weak solution.

Theorem 1. *Consider the Cauchy problem*

$$u_t + f(u)_x = 0\,, \qquad\qquad u(0,x) = \bar u(x)\,, \tag{2.20}$$

for a system of conservation laws with a smooth flux f defined in a neighborhood of the origin. Assume that the system is strictly hyperbolic and satisfies the conditions (H). Then, for a sufficiently small $\delta > 0$ the following holds. For every initial condition $u(0,x) = \bar u(x)$ with

$$\|\bar u\|_{\mathbf{L}^\infty} < \delta\,, \qquad\qquad Tot.Var.\{\bar u\} < \delta\,, \tag{2.21}$$

the Cauchy problem has a weak solution, defined for all times $t \geq 0$, obtained as limit of front tracking approximations.

2.5. A semigroup of solutions.

For solutions obtained as limits of Glimm or front tracking approximations, the uniqueness and the continuous dependence on the initial data were first established in [BC] for systems of two equations and then in [BCP] for $n \times n$ systems. Relying on some major new ideas introduced by Liu and Yang in [LY1], [LY2] a much simpler proof could then be worked out in [BLY]. Here the key step is the construction of a Lyapunov functional $\Phi(u,v)$, equivalent to the $\mathbf{L}^1$ distance

$$\frac{1}{C}\,\|u-v\|_{\mathbf{L}^1} \leq \Phi(u,v) \leq C\|u-v\|_{\mathbf{L}^1}\,,$$

and almost decreasing in time along each pair of ε-approximate solutions constructed by front tracking

$$\frac{d}{dt}\Phi\big(u(t),\, v(t)\big) \leq \mathcal{O}(1)\cdot\varepsilon\,.$$

A comprehensive treatment of the above existence-uniqueness theory for BV solutions can be found in [B3], [HR]. The main result is as follows.

Theorem 2. *With the same assumptions as Theorem 1, the weak solution of the Cauchy problem (2.20) obtained as limit of Glimm or front tracking approximations is unique and satisfies the Liu entropy conditions. Adopting a semigroup notation, this solution can be written as*

$$u(t,\cdot) = S_t\bar u\,.$$

The semigroup $S : \mathcal{D}\times[0,\infty[\,\mapsto \mathcal{D}$ can be defined on a closed domain $\mathcal{D}\subset\mathbf{L}^1$ containing all functions with sufficiently small total variation, It is uniformly Lipschitz continuous w.r.t. both time and the initial data:

$$\big\|S_t\bar u - S_s\bar v\big\|_{\mathbf{L}^1} \leq L\,\|\bar u-\bar v\|_{\mathbf{L}^1} + L'\,|t-s|\,. \tag{2.22}$$

We refer to the flow generated by a system of conservation laws as a *Riemann semigroup*, because it is entirely determined by specifying how Riemann problems are solved. As proved in [B2], if two semigroups S, S' yield the same solutions to all Riemann problems, then they coincide, up to the choice of their domains.

From (2.22) one can deduce the error bound (fig. 10)

$$\big\|w(T) - S_T w(0)\big\|_{\mathbf{L}^1} \leq L\cdot\int_0^T \left\{\liminf_{h\to0+}\frac{\big\|w(t+h)-S_h w(t)\big\|_{\mathbf{L}^1}}{h}\right\}\,dt\,, \tag{2.23}$$

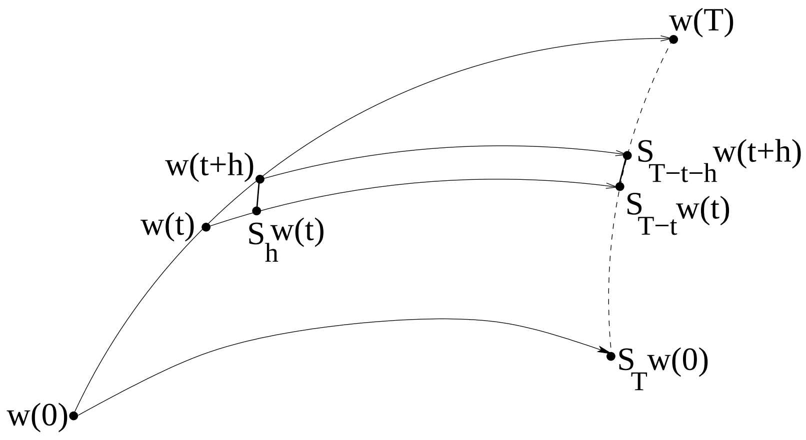

figure 10

valid for every Lipschitz continuous map $w : [0,T] \mapsto \mathcal{D}$ taking values inside the domain of the semigroup. We can think of $t \mapsto w(t)$ as an approximate solution of (1.1), while $t \mapsto S_t w(0)$ is the exact solution having the same initial data. The distance $\big\| w(T) - S_T w(0)\big\|_{\mathbf{L}^1}$ is clearly bounded by the length of the curve $\gamma : [0,T] \mapsto \mathbf{L}^1$, with $\gamma(t) \doteq S_{T-t}w(t)$. According to (2.23), this length is estimated by the integral of an *instantaneous error rate*, amplified by the Lipschitz constant L of the semigroup.

Using (2.23), one can estimate the distance between a front tracking approximation and the corresponding exact solution. For approximate solutions constructed by the Glimm scheme, a direct application of this same formula is not possible because of the additional errors introduced by the restarting procedure at each time $t_k \doteq k\,\Delta t$. However, relying on a careful analysis of Liu [L3], one can construct a front tracking approximate solution having the same initial and terminal values as the Glimm solution. By this technique, in [BM] the authors proved the estimate

$$\lim_{\Delta x \to 0} \frac{\big\| u^{\text{Glimm}}(\tau,\cdot) - u^{\text{exact}}(\tau,\cdot)\big\|_{\mathbf{L}^1}}{\sqrt{\Delta x}\cdot |\ln \Delta x|} = 0 \tag{2.24}$$

for every $\tau > 0$. In other words, letting the mesh sizes $\Delta x, \Delta t \to 0$ while keeping their ratio $\Delta x/\Delta t$ constant, the $\mathbf{L}^1$ norm of the error in the Glimm approximate solution tends to zero at a rate slightly slower than $\sqrt{\Delta x}$.

2.6. Uniqueness and characterization of entropy weak solutions

The stability result stated in Theorem 1 refers to a special class of weak solutions: those obtained as limits of Glimm or front tracking approximations. It is desirable to have a uniqueness theorem valid for general weak solutions, without reference to any particular constructive procedure. Results in this direction were proved in, [BLF], [BG], [BLw]. They are all based on the error formula (2.23). In the proofs, one considers a weak solution $u = u(t,x)$ of the Cauchy problem

$$u_t + f(u)_x = 0\,, \qquad\qquad u(0,x) = \bar u(x)\,. \tag{2.25}$$

Assuming that u satisfies suitable entropy and regularity conditions, one shows that

$$\liminf_{h\to 0+} \frac{\big\|u(t+h) - S_h u(t)\big\|_{\mathbf{L}^1}}{h} = 0 \tag{2.26}$$

at almost every time t. By (2.23), u thus coincides with the semigroup trajectory $t\mapsto S_t u(0) = S_t\bar u$ obtained as limit of front tracking approximations. Of course, this implies uniqueness. We state here the main result in [BG]. A function $u = u(t,x)$ is said to have **tame oscillation** if there exist constants $C', \beta > 0$ such that the following holds. For every $\tau > 0$ and $a < b$, the oscillation of u over the triangle

$$\Delta^\tau_{a,b} \doteq \big\{(s,y)\,;\ s\geq t\,,\quad a+\beta(s-\tau) < y < b - \beta(s-\tau)\big\}$$

satisfies

$$\text{Osc.}\{u;\ \Delta^\tau_{a,b}\} \leq C'\cdot \text{Tot.Var.}\big\{u(\tau,\cdot);\ ]a,b[\,\big\}\,. \tag{2.27}$$

We recall that the oscillation of a u on a set Δ is defined as

$$\text{Osc.}\{u;\ \Delta\} \doteq \sup_{(s,y),\,(s',y')\in\Delta} \big|u(s,y) - u(s',y')\big|\,.$$

Theorem 3. *In the same setting as Theorem 1, let $u = u(t,x)$ be a weak solution of the Cauchy problem (2.20) satisfying the tame oscillation condition (2.27) and the Liu admissibility condition (2.15) at every point of approximate jump. Then u coincides with the corresponding semigroup trajectory, i.e.*

$$u(t,\cdot) = S_t\bar u \tag{2.28}$$

for all $t\geq 0$. In particular, the solution that satisfies the above conditions is unique.

An additional characterization of these admissible weak solutions, based on local integral estimates, was given in [B2]. To state this result, we need to introduce some notations. Given a function $u = u(t,x)$ and a point (τ,ξ), we denote by $U^\sharp_{(u;\tau,\xi)}$ the solution of the Riemann problem with initial data

$$u^- = \lim_{x\to\xi-} u(\tau,x), \qquad\qquad u^+ = \lim_{x\to\xi+} u(\tau,x). \tag{2.29}$$

In addition, we define $U^\flat_{(u;\tau,\xi)}$ as the solution of the linear hyperbolic Cauchy problem with constant coefficients

$$w_t + A^\flat w_x = 0, \qquad\qquad w(0,x) = u(\tau,x). \tag{2.30}$$

Here $A^\flat \doteq A\big(u(\tau,\xi)\big)$. Observe that (2.30) is obtained from the quasilinear system (1.2) by "freezing" the coefficients of the matrix $A(u)$ at the point (τ,ξ) and choosing $u(\tau)$ as initial data. A notion of "good solution" can now be introduced, by locally comparing a function u with the self-similar solution of a Riemann problem and with the solution of a linear hyperbolic system with constant coefficients. More precisely, we say that a function $u = u(t,x)$ is a **viscosity solution** of the system (1.1) if $t \mapsto u(t,\cdot)$ is continuous as a map with values into $\mathbf{L}^1_{loc}$, and moreover the following integral estimates hold.

(i) At every point (τ,ξ), for every $\beta' > 0$ one has

$$\lim_{h\to 0+} \frac{1}{h}\int_{-\beta' h}^{\beta' h} \Big|u(\tau+h,\ \xi+x) - U^\sharp_{(u;\tau,\xi)}(h,\ x)\Big|\, dx \ = \ 0. \tag{2.31}$$

(ii) There exist constants $C,\beta > 0$ such that, for every $\tau \geq 0$ and $a < \xi < b$, one has

$$\limsup_{h\to 0+} \frac{1}{h}\int_{a+\beta h}^{b-\beta h} \Big|u(\tau+h,\ x) - U^\flat_{(u;\tau,\xi)}(h,x)\Big|\, dx \leq C\cdot\Big(\mathrm{Tot.Var.}\big\{u(\tau);\ ]a,\,b[\ \big\}\Big)^2. \tag{2.32}$$

The above definition was introduced in [B2], with the expectation that the "viscosity solutions" in the above integral sense should precisely coincide with the limits of vanishing viscosity approximations (1.7). This conjecture was proved only many years later in [BiB]. The main result in [B2] shows that the integral estimates (2.31)-(2.32) completely characterize the semigroup trajectories.

Theorem 4. *Let $S : \mathcal{D}\times[0,\infty[\times\mathcal{D}$ be the semigroup generated by the system of conservation laws (1.1), whose trajectories are limits of front tracking approximations. A function $u : [0,T]\mapsto\mathcal{D}$ is a viscosity solution of (1.1) if and only if $u(t) = S_t u(0)$ for all $t\in[0,T]$.*

3 - The Vanishing Viscosity Approach

In view of the previous uniqueness and stability results, it is natural to expect that the entropy weak solutions of the hyperbolic system (1.1) should coincide with the unique limits of solutions to the parabolic system

$$u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon\, u^\varepsilon_{xx} \tag{3.1}$$

letting the viscosity coefficient $\varepsilon \to 0$. For smooth solutions, this convergence is easy to show. However, one should keep in mind that a weak solution of the hyperbolic system (1.1) in general is only a function with bounded variation, possibly with a countable number of discontinuities. In this case, as the smooth functions u^ε approach the discontinuous solution u, near points of jump their gradients u^ε_x must tend to infinity (fig. 11), while their second derivatives u^ε_{xx} become even more singular. Therefore, establishing the convergence $u^\varepsilon \to u$ is a highly nontrivial matter. In earlier literature, results in this direction relied on three different approaches:

1 - Comparison principles for parabolic equations. For a scalar conservation law, the existence, uniqueness and global stability of vanishing viscosity solutions was first established by Oleinik [O] in one space dimension. The famous paper by Kruzhkov [K] covers the more general class of $\mathbf{L}^\infty$ solutions and is also valid in several space dimensions.

2 - Singular perturbations. This technique was developed by Goodman and Xin [GX], and covers the case where the limit solution u is piecewise smooth, with a finite number of non-interacting, entropy admissible shocks. See also [Yu] and [Ro], for further results in this direction.

3 - Compensated compactness. With this approach, introduced by Tartar and DiPerna [DP2], one first considers a weakly convergent subsequence $u^\varepsilon \rightharpoonup u$. For a class of 2×2 systems, one can show that this weak limit u actually provides a distributional solution to the nonlinear system (1.1). The proof relies on a compensated compactness argument, based on the representation of the weak limit in terms of Young measures, which must reduce to a Dirac mass due to the presence of a large family of entropies.

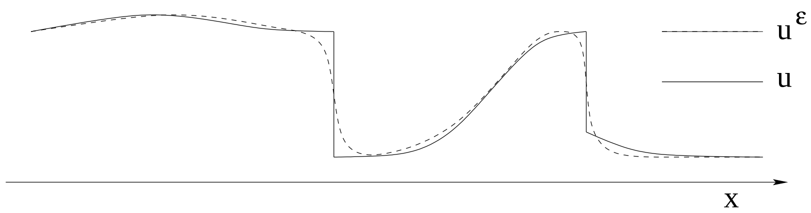

figure 11

Since the hyperbolic Cauchy problem is known to be well posed within a space of functions with small total variation, it seems natural to develop a theory of vanishing viscosity approximations within the same space BV. This was indeed accomplished in [BiB], in the more general framework of nonlinear hyperbolic systems not necessarily in conservation form. The only assumptions needed here are the strict hyperbolicity of the system and the small total variation of the initial data.

Theorem 5. *Consider the Cauchy problem for the hyperbolic system with viscosity*

$$u^\varepsilon_t + A(u^\varepsilon)u^\varepsilon_x = \varepsilon\, u^\varepsilon_{xx} \qquad\qquad u^\varepsilon(0,x) = \bar u(x)\,. \tag{3.2}$$

Assume that the matrices $A(u)$ are strictly hyperbolic, smoothly depending on u in a neighborhood of the origin. Then there exist constants C, L, L' and $\delta > 0$ such that the following holds. If

$$Tot.Var.\{\bar{u}\} < \delta\,, \qquad \|\bar{u}\|_{\mathbf{L}^\infty} < \delta, \tag{3.3}$$

then for each $\varepsilon > 0$ the Cauchy problem (3.2)$_\varepsilon$ has a unique solution u^ε, defined for all $t \geq 0$. Adopting a semigroup notation, this will be written as $t \mapsto u^\varepsilon(t,\cdot) \doteq S_t^\varepsilon \bar{u}$. In addition, one has

BV bounds :

$$Tot.Var.\big\{S_t^\varepsilon \bar{u}\big\} \leq C\ Tot.Var.\{\bar{u}\}\,. \tag{3.4}$$

$\mathbf{L}^1$ **stability** :

$$\big\|S_t^\varepsilon \bar{u} - S_t^\varepsilon \bar{v}\big\|_{\mathbf{L}^1} \leq L\,\big\|\bar{u} - \bar{v}\big\|_{\mathbf{L}^1}\,, \tag{3.5}$$

$$\big\|S_t^\varepsilon \bar{u} - S_s^\varepsilon \bar{u}\big\|_{\mathbf{L}^1} \leq L'\,\Big(|t-s| + \big|\sqrt{\varepsilon t} - \sqrt{\varepsilon s}\,\big|\Big)\,. \tag{3.6}$$

Convergence: *As $\varepsilon \to 0+$, the solutions u^ε converge to the trajectories of a semigroup S such that*

$$\big\|S_t \bar{u} - S_s \bar{v}\big\|_{\mathbf{L}^1} \leq L\,\|\bar{u} - \bar{v}\|_{\mathbf{L}^1} + L'\,|t-s|\,. \tag{3.7}$$

These vanishing viscosity limits can be regarded as the unique **vanishing viscosity solutions** *of the hyperbolic Cauchy problem*

$$u_t + A(u)u_x = 0, \qquad u(0,x) = \bar{u}(x)\,. \tag{3.8}$$

In the conservative case $A(u) = Df(u)$, every vanishing viscosity solution is a weak solution of

$$u_t + f(u)_x = 0, \qquad u(0,x) = \bar{u}(x)\,, \tag{3.9}$$

satisfying the Liu admissibility conditions.

Assuming, in addition, that each field is genuinely nonlinear or linearly degenerate, the vanishing viscosity solutions coincide with the unique limits of Glimm and front tracking approximations.

In the genuinely nonlinear case, an estimate on the rate of convergence of these viscous approximations was provided in [BY]:

Theorem 6. *For the strictly hyperbolic system of conservation laws (3.9), assume that every characteristic field is genuinely nonlinear. At any time $t > 0$, the difference between the corresponding solutions of (3.2) and (3.9) can be estimated as*

$$\big\|u^\varepsilon(t,\cdot) - u(t,\cdot)\big\|_{\mathbf{L}^1} = \mathcal{O}(1)\cdot(1+t)\sqrt{\varepsilon}|\ln\varepsilon|\ \text{Tot.Var.}\{\bar{u}\}\,.$$

The following remarks relate Theorem 5 with previous literature.

1. Concerning the global existence of weak solutions to the system of conservation laws (3.9), Theorem 5 contains the famous result of Glimm [G]. It also slightly extends the existence theorem of Liu [L4], which does notrequire the assumption (H). Its main novelty lies in the case where the system (3.8) is non-conservative. However, one should be aware that for non-conservative systems the limit solution might change if we choose a viscosity matrix different from the identity, say

$$u_t + A(u)u_x = \varepsilon \big(B(u)u_x\big)_x .$$

The analysis of viscous solutions, with a more general viscosity matrix $B(u)$, is still a wide open problem.

2. Concerning the uniform stability of entropy weak solutions, the results previously available for $n \times n$ hyperbolic systems [BC1], [BCP], [BLY] always required the assumption (H). For 2×2 systems, this condition was somewhat relaxed in [AM]. On the other hand, Theorem 5 established this uniform stability for completely general $n \times n$ strictly hyperbolic systems, without any reference to the assumption (H).

3. For the viscous system (1.10), previous results in [L5], [SX], [SZ], [Yu] have established the stability of special types of solutions, such as travelling viscous shocks or viscous rarefactions, w.r.t. suitably small perturbations. Taking $\varepsilon = 1$, our present theorem yields at once the uniform Lipschitz stability of all viscous solutions with sufficiently small total variation, w.r.t. the $\mathbf{L}^1$ distance. On the other hand, our approach has not yet been used to study the stability and convergence of viscous solutions with large total variation. For a result in this direction using singular perturbations, see [R].

4 - Linear systems in several space dimensions

Consider the linear system of P.D.E. with constant coefficients in several space variables

$$u_t + \sum_{\alpha=1}^{m} A^\alpha u_{x_\alpha} = 0 . \tag{4.1}$$

Here t is time, $x = (x_1, \ldots, x_m)$ is the space variable while $u = (u_1, \ldots, u_m)$ is the dependent variable. For each vector $\xi = (\xi_1, \ldots, \xi_m) \in I\!R^m$ we consider the linear combination

$$A(\xi) \doteq \sum_\alpha \xi_\alpha A^\alpha .$$

Definition 1. The system (4.1) is *hyperbolic* if there exists a constant C such that

$$\sup_{\xi \in I\!R^m} \big\| \exp iA(\xi) \big\| \leq C . \tag{4.2}$$

Of course, the matrix exponential means

$$\exp iA(\xi) \doteq \sum_{k=0}^{\infty} \frac{\big(iA(\xi)\big)^k}{k!}\,.$$

The above definition is motivated by

Theorem 1. *The system (4.1) is hyperbolic if and only if the corresponding Cauchy problem is well posed in* $\mathbf{L}^2$.

Proof. Given an initial condition $u(0,x)$, the solution can be computed by Fourier transform. The vector valued function

$$\hat u(t,\xi) = (2\pi)^{-m/2}\int_{I\!R^m} e^{-i\xi\cdot x}u(t,x)\,dx$$

satisfies the evolution equation

$$\frac{\partial}{\partial t}\hat u(t,\xi) = -iA(\xi)\hat u(t,\xi)\,.$$

Therefore

$$\hat u(t,\xi) = \exp\big(-itA(\xi)\big)\,\hat u(0,\xi)\,,$$

$$u(t,x) = (2\pi)^{-m/2}\int_{I\!R^m} e^{i\xi\cdot x}\,\hat u(t,\xi)\,dx\,.$$

Since the Fourier transform is an isometry on $\mathbf{L}^2$, we have

$$\big\|u(t)\big\|_{\mathbf{L}^2} = \big\|\hat u(t)\big\|_{\mathbf{L}^2} \le \sup_{\xi\in I\!R^m}\big\|\exp(-iA(\xi))\big\|\cdot\big\|\hat u(0)\big\|_{\mathbf{L}^2} = \sup_{\xi\in I\!R^m}\big\|\exp\,iA(\xi)\big\|\cdot\big\|u(0)\big\|_{\mathbf{L}^2}\,.$$

It is now clear that the linear flow map $u(0)\mapsto u(t)$ is bounded in $\mathbf{L}^2$ if and only if the system is hyperbolic. □

We now derive geometric conditions on the eigenvectors of the matrix A, related to the condition of hyperbolicity.

Lemma 2. *If the system (4.1) is hyperbolic, then for every* $\xi\in I\!R^m$ *the matrix* $A(\xi)$ *has a basis of eigenvectors* $r_1,\ldots,r_n$*, with real eigenvalues* $\lambda_1,\ldots,\lambda_n$ *(not necessarily distinct).*

Proof. Call $\rho(M)$ the spectral radius of a matrix M. The assumption of hyperbolicity implies

$$\sup_{\xi\in I\!R^m}\rho\big(\exp iA(\xi)\big) < \infty\,.$$

Observing that, for every integer k,

$$\exp\,iA(k\xi) = \big(\exp\,iA(\xi)\big)^k\,,$$

we deduce that the spectral radii of the matrices

$$\exp iA(\xi)\,, \qquad\qquad \exp\big(-iA(\xi)\big) = \big(\exp iA(\xi)\big)^{-1}$$

are both $=1$. Since the eigenvalues of $\exp M$ are the exponentials of the eigenvalues of the matrix M, we conclude that all eigenvalues of $A(\xi)$ must be real. To prove that $A(\xi)$ is actually diagonalisable let us assume, on the contrary, that its canonical form contains a nontrivial Jordan block. In other words, assume that there exist two nonzero vectors v, z such that

$$A(\xi)v = \lambda v\,, \qquad\qquad A(\xi)z = \lambda z + v\,.$$

In this case one computes

$$e^{iA(t\xi)}z = e^{itA(\xi)}z = e^{it\lambda}(z + itv),$$

against the boundedness assumption (4.2). □

Lemma 3. *Assume that, for $|\xi| = 1$, the matrices $A(\xi)$ can be diagonalized in terms of an invertible matrix $R(\xi)$ continuously depending on ξ. Then the system is hyperbolic.*

Proof. By assumption, the column vectors of $R(\xi)$ form a basis of right eigenvectors of $A(\xi)$, continuously depending on ξ :

$$R^{-1}(\xi)A(\xi)R(\xi) = D(\xi)\,, \tag{4.3}$$

where $D(\xi)$ is a diagonal matrix whose entries are the eigenvalues of $A(\xi)$. We can extend the definition of R by homogeneity to all $\mathbb{R}^m \setminus \{0\}$, by setting $R(\xi) \doteq R\big(\xi/|\xi|\big)$. This yields

$$R^{-1}(\xi)A(\xi)R(\xi) = |\xi|\, D\big(\xi/|\xi|\big) \doteq D(\xi)\,.$$

Therefore

$$\begin{aligned} \big\|\exp iA(\xi)\big\| &= \Big\|\exp\big(R(\xi)\, iD(\xi)\, R^{-1}(\xi)\big)\Big\| = \Big\|R(\xi)\, \exp\big(iD(\xi)\big)\, R^{-1}(\xi)\big)\Big\| \\ &\leq \big\|R(\xi)\big\|\, \big\|\exp\, iD(\xi)\big\|\, \big\|R^{-1}(\xi)\big\| = \big\|R(\xi)\big\|\, \big\|R^{-1}(\xi)\big\|\,. \end{aligned} \tag{4.4}$$

Since $R(\xi) = R\big(\xi/|\xi|\big)$ and the matrices R, R^{-1} are continuous on the surface of the unit ball, the right hand side of (4.4) is uniformly bounded for $\xi \in \mathbb{R}^m$. □

Another sufficient condition for hyperbolicity is given below.

Lemma 4. *Assume that there exists a positive definite symmetric matrix S such that all matrices $S^\alpha \doteq SA^\alpha$ are symmetric. Then the system is hyperbolic.*

Proof. By assumption, the matrices

$$SA(\xi) = \sum_\alpha \xi_\alpha SA^\alpha$$

are all symmetric. Call K the symmetric matrix which provides the positive definite square root of S^{-1}, so that $K^2 S = I$. Since $KSA(\xi)K$ is symmetric, it can be diagonalized by an orthogonal matrix $O(\xi)$, say

$$O(\xi)^{-1} KSA(\xi)KO(\xi) = D(\xi)$$

for some diagonal matrix $D(\xi)$. Choosing $P(\xi) \doteq KO(\xi)$ we obtain

$$P^{-1}(\xi)A(\xi)P(\xi) = O^{-1}(\xi)K^{-1}K^2 SA(\xi)KO(\xi) = O^{-1}KSA(\xi)KO(\xi) = D(\xi)\,.$$

Clearly,

$$\big\|P(\xi)\big\|\,\|P^{-1}(\xi)\big\| \leq \|K\|\,\|K^{-1}\|$$

is bounded uniformly w.r.t. ξ. □

We remark that in general, the Cauchy problem for a hyperbolic system is not well posed in $\mathbf{L}^p$ for $p \neq 2$. A special case is when the matrices A^α commute.

Lemma 5. *Two $n \times n$ hyperbolic matrices A, B commute if and only if they have a common basis of eigenvectors.*

Proof. Assume that A, B have a common basis of eigenvectors. Then they can be simultaneously diagonalised, say

$$R^{-1}AR = D\,, \qquad\qquad R^{-1}BR = D',$$

for some diagonal matrices D, D' and a suitable invertible matrix R whose columns are eigenvectors of both A and B. Since $DD' = D'D$, we have

$$AB = (RDR^{-1})\,(RD'R^{-1}) = RDD'R^{-1} = RD'DR^{-1} = (RD'R^{-1})\,(RDR^{-1}) = BA\,.$$

To prove the converse, assume that $AB = BA$. For each eigenvalue λ of A, call E_λ the space of all corresponding eigenvectors:

$$E_\lambda \doteq \{v \in I\!R^n\,;\quad Av = \lambda v\}\,.$$

If $v \in E_\lambda$ then also $Bv \in E_\lambda$. Indeed, the commutativity assumption yields

$$ABv = BAv = \lambda Bv.$$

Hence E_λ is invariant under the linear transformation B. For each eigenvalue λ_j of A we can thus construct a basis $\{r_{j,1}, \ldots, r_{j,n_j}\}$ of the corresponding eigenspace E_{λ_j} consisting of eigenvectors of B. Clearly, these are also eigenvectors of A. The set of all these eigenvectors is the desired basis. □

An Explicit Formula. If all matrices A^α in (4.1) commute, the solution of the Cauchy problem with initial data

$$u(0,x) = \bar u(x) \tag{4.5}$$

can be written explicitly. Indeed, by the previous lemma we can find a common basis of right eigenvectors $r_1, \ldots, r_n$, for all matrices A^α. Let $l_1, \ldots, l_n$ be the dual basis of left eigenvectors, so that

$$l_i \cdot r_j = \begin{cases} 1 & \text{if} \quad i = j\,, \\ 0 & \text{if} \quad i \neq j\,. \end{cases}$$

Call $\lambda_{i,\alpha}$ the eigenvalue of A^α corresponding to r_i, and define the velocity vectors $\mathbf{v}_i \in I\!R^m$ by setting $\mathbf{v}_i \doteq (\lambda_{i,1}, \ldots, \lambda_{i,m})$. The explicit solution of (4.1) with initial data (4.5) is now given by a linear superposition of travelling waves:

$$u(t,x) = \sum_i u_i(x - t\mathbf{v}_i) \qquad\qquad u_i(y) \doteq l_i \cdot \bar u(y)\,. \tag{4.6}$$

The following lemma clarifies the geometry of matrix symmetrizers.

Lemma 6. *Let H be a hyperbolic matrix, and let S be a strictly positive definite, symmetric matrix. Then SH is symmetric if and only if there exists a basis of eigenvectors for H which is orthogonal w.r.t. the inner product induced by S.*

Proof. Assume that there exists a matrix R whose columns are a basis of right eigenvectors of H, orthogonal w.r.t. S. In other words, assume

$$R^{-1}HR = D\,, \qquad\qquad R^*SR = \Lambda$$

with D and Λ diagonal matrices. Here * denotes transposition. Then

$$S = (R^*)^{-1}\Lambda R^{-1} \qquad\qquad H = RDR^{-1},$$

$$SH = (R^*)^{-1}\Lambda D R^{-1} = (R^*)^{-1} D\Lambda R^{-1} = (R^*)^{-1} D R^* \,(R^*)^{-1}\Lambda R^{-1} = H^* S\,.$$

Hence SH is symmetric. To prove the converse, it suffices to show that any two eigenvectors of H corresponding to distinct eigenvalues must be orthogonal w.r.t. S. Assume that $Hv = \lambda v$ and $Hw = \mu w$. Then

$$\lambda w^* S v = w^* S H v = v^* H^* S^* w = v^* S H w = \mu v^* S w = \mu w^* S v\,.$$

If $\lambda \neq \mu$, this implies $w^* S v = 0$, as claimed.

5 - Nonlinear systems

The quasilinear system

$$u_t + \sum_{\alpha=1}^{m} A^\alpha(u) u_{x_\alpha} = 0 \tag{5.1}$$

is called *hyperbolic* if, for every fixed value of $\bar{u}$, the corresponding system with constant coefficients

$$u_t + \sum_{\alpha=1}^{m} A^\alpha(\bar{u}) u_{x_\alpha} = 0 \tag{5.2}$$

is hyperbolic. More precisely, we shall ask that for every state u and every unit vector $\xi \in I\!R^m$ the matrix

$$A(u, \xi) \doteq \sum_\alpha \xi_\alpha A^\alpha(u)$$

can be diagonalized by an invertible matrix $R(u, \xi)$, so that

$$R^{-1}(u,\xi) A(u,\xi) R(u,\xi) = D(u,\xi)\,. \tag{5.3}$$

Moreover, we require that the norms of the matrices $R(u,\xi)$ and $R^{-1}(u,\xi)$ remain uniformly bounded as u ranges on compact sets.

Remark. The above definition of hyperbolic systems is invariant under smooth nonlinear transformations of coordinates. Indeed, if $v = \phi(u)$, then v satisfies the system of equations

$$v_t + \sum_{\alpha=1}^{m} B^\alpha(v) u_{x_\alpha} = 0\,, \tag{5.4}$$

where

$$B^\alpha = D\phi(u)\, A^\alpha(u)\, \big(D\phi(u)\big)^{-1}.$$

If (5.3) holds, then the matrices $B^\alpha\big(\xi, \phi(u)\big)$ are diagonalized by $P(v,\xi) = D\phi(u) R(u,\xi)$.

A system of conservation laws in several space dimensions takes the form

$$u_t + \sum_{\alpha=1}^{m} \frac{\partial}{\partial x_\alpha} f^\alpha(u) = 0\,. \tag{5.5}$$

Here $u = (u_1, \ldots, u_n)$ is the vector of *conserved quantities*, while the components of $f = (f_1, \ldots, f_n)$ are the (vector valued) fluxes. Equivalently, (5.3) can be written in divergence form as

$$u_{i,t} + \mathrm{div}_x f_i(u) = 0 \qquad\qquad i = 1, \ldots, n\,. \tag{5.6}$$

The system is called *hyperbolic* if the corresponding quasilinear system (5.1) is hyperbolic, where $A^\alpha(u) = Df^\alpha(u)$ is the corresponding $n \times n$ Jacobian matrix.

Definition. A function $\eta = \eta(u) : I\!R^n \mapsto I\!R$ is an entropy for the system (5.1), with entropy flux $q = (q^1, \ldots, q^m) : I\!R^n \mapsto I\!R^m$, if every smooth solution of (5.3) satisfies the additional conservation law

$$\eta(u)_t + \mathrm{div}_x\, q(u) = 0\,. \tag{5.7}$$

This holds if and only if

$$D\eta Df^\alpha = Dq^\alpha\,, \qquad\qquad \alpha = 1, \ldots, m\,. \tag{5.8}$$

Theorem 2. *If the system (5.3) admits a strictly convex entropy, then it is symmetric hyperbolic. Indeed, it is symmetrized by the Hessian matrix $D^2\eta$.*

Proof. In view of Lemma 6, it suffices to check that eigenvectors of any matrix A^α corresponding to distinct eigenvalues are orthogonal w.r.t. the inner product determined by the symmetric positive definite matrix $D^2\eta$.

From (5.8), for every vector v we deduce

$$D\eta\, Df^\alpha\, v = Dq^\alpha\, v\,. \tag{5.9}$$

Taking the directional derivative of (5.9) w.r.t. u, in the direction of another vector w, we find

$$D^2\eta\big((Df^\alpha\, v) \otimes w\big) + D\eta\big(D^2 f^\alpha (v \otimes w)\big) = D^2 q^\alpha (v \otimes w)\,.$$

Interchanging the roles of v and w,

$$D^2\eta\big((Df^\alpha\, w) \otimes v\big) + D\eta\big(D^2 f^\alpha (w \otimes v)\big) = D^2 q^\alpha (w \otimes v)\,.$$

Recalling that second order derivatives are symmetric, from the two above inequalities one obtains

$$D^2\eta\big((Df^\alpha v) \otimes w\big) - D^2\eta\big(v \otimes (Df^\alpha w)\big) = 0. \tag{5.10}$$

The above relation is valid for every couple of vectors v, w. In particular, let v, w be eigenvectors

$$Df^\alpha v = \lambda v\,, \qquad\qquad Df^\alpha w = \mu w\,.$$

If $\lambda \neq \mu$, from (5.10) we conclude $D^2\eta(v \otimes w) = 0$, proving our claim. □

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