Some Remarks on Elliptic Problems with Critical Growth in the Gradient

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This is a joint work with Andrea Dall'Aglio and Ireneo Peral.

In this work we analyze existence, nonexistence, multiplicity and regularity of solution to problem

$$\begin{cases} -\Delta u &= \beta(u)|\nabla u|^2 + \lambda f(x) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded open set in $I\!R^N$, β is a continuous nondecreasing positive function, λ is a positive constant and $f(x)$ is a positive measurable function. We will assume that Ω has a smooth enough boundary, as an example, the interior sphere condition is sufficient to do all the arguments below. Equations of the form (1) have been widely studied in the literature. For instance, in the case where $\beta \equiv$ constant and $f \equiv 0$, this equation may be reckoned as the stationary part of the equation

$$u_t - \varepsilon\Delta u = |\nabla u|^2,$$

which may be viewed as the viscosity approximation as $\varepsilon \to 0^+$ of Hamilton-Jacobi type equations from stochastic control theory (see [7]). The same parabolic equation appears in the physical theory of growth and roughening of surfaces, where it is known as the Kardar-Parisi-Zhang equation (see [2]). Existence results for problem (1) start from the classic references [6] and [5]. We will start from the study of the simpler case of equation (1), that is, the case where $\beta(s) \equiv$ constant. It is well known that in general there is no uniqueness of solutions of (1). For instance, if $N > 2$, the functions

$$u_m(x) = \log\left(\frac{|x|^{2-N} - m}{1-m}\right) \in W_0^{1,2}(B_1),\ 0 \le m < 1,$$

all solve the equation $-\Delta u = |\nabla u|^2$ in the unit ball $B_1 = \{x \in I\!R^N \ : \ |x| < 1\}$ (though only the zero function satisfies $e^u - 1 \in W_0^{1,2}(B_1)$). One of the main aims of this paper is to characterize this non-uniqueness phenomenon, and to show that *every* solution of problem (1) comes from a solution of a linear problem with measure data, after a suitable change of variable.

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