On geodesics in the metrics with singularities of the Klein type

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In this note, we announce some results for geodesic lines in the metric

$$ds^2 = \frac{\alpha(t,x)\,(dx^2 + \varepsilon\, dt^2)}{\omega^r(t,x)}, \quad \varepsilon = \pm 1, \tag{1}$$

where $\alpha(t,x) > 0$ and $\omega(t,x)$ are C^∞-smooth functions, $r > 0$ is a real number. Suppose that the equation $\omega(t,x) = 0$ defines a smooth regular curve A on the (t,x)-plane, i.e., at all points $(t,x) \in A$ the condition $\nabla\omega \neq 0$ holds. In the case $\varepsilon = -1$ we also assume that the curve A is not tangent to the radical of the form $dx^2 + \varepsilon\, dt^2$, i.e., at all points $(t,x) \in A$ the condition $|\omega'_t| \neq |\omega'_x|$ holds. The aim of this paper is to study singularities of the geodesic flow and geodesic lines which pass through the points of the curve A, in neighborhoods of these points. All results are local, and by means of isothermal coordinates they can be extended to the metrics of more general form $ds^2 = Q/\omega^r$, where Q is an arbitrary smooth non-degenerated metric (if Q is indefinite then we assume that the curve $A = \{\omega = 0\}$ is not tangent to $\mathrm{Rad}\, Q$).

The metric (1) is a natural generalization of the Klein metric used in the model of the Lobachevsky plane ($r = 2$). It is also related to Clairaut–Liouville metric ($r = 1$) which has some applications in optimal control theory. For instance, in [1] the authors deal with the metric

$$ds^2 = dx^2 + \frac{g(x^2,t)}{x^2}\, dt^2 = \frac{x^2\, dx^2 + g(x^2,t)\, dt^2}{x^2}, \tag{2}$$

where g is a positive smooth function (t and x are standard angular coordinates on the sphere, the curve $A = \{x = 0\}$ is the equator). In the case $g(x^2,t) \equiv 1$ it is well-known Grushin metric [2, 3]. After the change of variables $\tilde{x} = x^2$ the metric (2) becomes $ds^2 = Q/(4\tilde{x})$, where $Q = d\tilde{x}^2 + 4g(\tilde{x},t)\, dt^2$ is a positive definite quadratic form. By means of isothermal coordinates the last metric becomes (1) with $r = 1$ and $\varepsilon = +1$.

Geodesic lines generated by the metric (1) are extremals of the Euler–Lagrange equation with $L^2 = \alpha(p^2 + \varepsilon)/\omega^r$, where $p = dx/dt$. After some transformations this yields the equation

$$2\varepsilon\alpha\omega\, \frac{dp}{dt} = (\mu_1 p - \varepsilon\mu_0)(p^2 + \varepsilon), \quad p = \frac{dx}{dt}, \tag{3}$$

where $\mu_1 = r\alpha\omega'_t - \alpha'_t\omega$ and $\mu_0 = r\alpha\omega'_x - \alpha'_x\omega$. In the (t,x,p)-space the equation (3) generates the direction field

$$\dot{t} = 2\varepsilon\alpha\omega, \quad \dot{x} = 2p\varepsilon\alpha\omega, \quad \dot{p} = (\mu_1 p - \varepsilon\mu_0)(p^2 + \varepsilon). \tag{4}$$

The projections of integral curves of the field (4) onto the (t,x)-plane along the p-axis are geodesic lines. Direction field (4) has two key properties.

Firstly, for each point $q = (t,x) \in A$ and each p such that $(\mu_1 p - \varepsilon\mu_0)(p^2 + \varepsilon) \neq 0$ the field (4) has a unique integral curve that passes through the point (q,p). This integral curve is a straight line in the (t,x,p)-space parallel to the p-axis, and its projection on the (t,x)-plane is the point q, hence it does not correspond to geodesic. Thus geodesics that pass through the point $q \in A$ are projections of the integral curves of the field (4) that pass through the singular points (q,p) of this field. This yields the equality $(\mu_1 p - \varepsilon\mu_0)(p^2 + \varepsilon) = 0$.

Secondly, singular points of the field (4) are given by two equations $\omega = 0$ and $(\mu_1 p - \varepsilon\mu_0)(p^2 + \varepsilon) = 0$ which define a curve $W_0^c = \{q \in A,\ p = \varepsilon\mu_0/\mu_1\}$ if $\varepsilon = +1$. In the case $\varepsilon = -1$ the locus defined by these equations contains the curve W_0^c and two additional curves $W_\pm^c = \{q \in A,\ p = \pm 1\}$. The condition $|\omega'_t| \neq |\omega'_x|$ implies that the curves $W_0^c, W_\pm^c$ do not intersect in a neighborhood of A and form the central manifold of the field ($W^c =$

$W_0^c \cup W_+^c \cup W_-^c$). The spectrum of the linear part of the direction field (4) is defined uniquely up to a scalar factor; it is equal to $(2, r, 0)$ at the points of W_0^c and $(1, -r, 0)$ at the points of $W_\pm^c$. The eigenvalue r or $-r$ corresponds to the eigenvector proportional to ∂_p.

Combining the results of [4, 5], we get smooth orbital normal forms for the germ of the field (4) at the singular points. At any point $(p, q) \in W_0^c$, the germ (4) has the normal form

$$\dot{\xi} = r\xi, \quad \dot{\eta} = 2\eta, \quad \dot{\zeta} = 0, \quad \text{if} \quad \max\{r/2, 2/r\} \notin \mathbb{N}, \tag{5}$$

$$\dot{\xi} = n\xi + \varphi(\zeta)\eta^n, \quad \dot{\eta} = \eta, \quad \dot{\zeta} = 0, \quad \text{if} \quad \max\{r/2, 2/r\} = n \in \mathbb{N}. \tag{6}$$

If $\varphi(0) \neq 0$ then the coefficient $\varphi(\zeta)$ in the normal form (6) simplifies to 1. If $\varphi(\zeta)$ has a finite order s at the origin, then $\varphi(\zeta)$ simplifies to ζ^s. In the case $r = 1$ the coefficient $\varphi(\zeta) \equiv 0$. In the case $r = 2$ the identity $\varphi(\zeta) \equiv 0$ holds, in particular, if the function $\omega(t, x)$ is linear and the restriction of $\alpha(t, x)$ to the line A is identically constant.

At any point $(p, q) \in W_\pm^c$ the germ (4) has the normal form

$$\dot{\xi} = \xi, \quad \dot{\eta} = -r\eta, \quad \dot{\zeta} = 0, \quad \text{if} \quad r \notin \mathbb{Q}, \tag{7}$$

$$\dot{\xi} = n\xi, \quad \dot{\eta} = -m\eta\Phi(\rho, \zeta), \quad \dot{\zeta} = \rho\Psi(\rho, \zeta), \quad \text{if} \quad r \in \mathbb{Q}, \tag{8}$$

where $\rho = \xi^m\eta^n$ is the resonant monomial, $r = m/n$, $m, n \in \mathbb{N}$, $(m, n) = 1$, and $\Phi(0, 0) = 1$. If $\Psi(0, 0) \neq 0$ then the coefficients $\Phi(\rho, \zeta)$ and $\Psi(\rho, \zeta)$ in (8) simplify to 1. The condition $\Psi(0, 0) \neq 0$ for generic metric (1) holds, in the case $r = 1$ it always holds true. However both normal forms (7) and (8) are topologically equivalent to $\dot{\xi} = \xi$, $\dot{\eta} = -\eta$, $\dot{\zeta} = 0$.

To get the portrait of geodesic lines passing through the point $q \in A$ with tangent direction $p_0 = \varepsilon\mu_0/\mu_1$ we need to consider the invariant leaf ($\zeta = c$ (const) in the normal form (5) or (6)) that contains the given point (q, p_0). The projection of the integral curves of (4) from this leaf on the (t, x)-plane along the p-axis gives a pencil of geodesics γ_k, $k \in \mathbb{R}$, with the same 1-jet at the point q. The jets of higher orders depend on r and $\varphi(c)$. If $r < 2$ then all geodesics γ_k, except only one, have an infinite second derivative at the point q (in particular, in the case $r = 1$ the geodesics are semi-cubic parabolas having a cusp at q). Consider the case $r = 2n$, $n \in \mathbb{N}$, and the corresponding normal form (6). If $\varphi(c) \neq 0$ then all geodesics $\gamma_k \in C^n$ and have the same n-jet and infinite derivative of the order $n + 1$ at q. If $\varphi(c) = 0$ then all geodesics $\gamma_k \in C^\infty$ and have the same n-jet and different $(n + 1)$-jets.[1]

In the case $\varepsilon = +1$ all geodesics passing through the point $q = (t_*, x_*) \in A$ belong to the pencil γ_k, $k \in \mathbb{R}$. In the case $\varepsilon = -1$ apart from this pencil there are two more geodesics: isotropic lines $\gamma_\pm$: $(x - x_*) = \pm(t - t_*)$ with tangent directions $p = \pm 1$.[2]

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[1]For example, the standard Klein metric (1) with $r = 2$, $\varepsilon = +1$ and $\alpha(t, x) = 1$, $\omega(t, x) = t$ has the normal form (6) with $\varphi(\zeta) \equiv 0$. Geodesics in the Klein metric are the circles $(x - x_0)^2 + t^2 = R^2$ and the lines $x = c$. All these curves are C^∞-smooth near the absolute $A = \{t = 0\}$, and for the given point $q \in A$ they have different 2-jets.

[2]The Pseudo-Klein metric (1) with $r = 2$, $\varepsilon = -1$ and $\alpha(t, x) = 1$, $\omega(t, x) = t$ has the normal form (6) with $\varphi(\zeta) \equiv 0$. Geodesics in this metric are the hyperbolas $(x - x_0)^2 - t^2 = H$ and the lines $x = c$. For $H < 0$ the hyperbolas are timelike ($p^2 - 1 > 0$) and do not intersect A. For $H > 0$ the hyperbolas are spacelike ($p^2 - 1 < 0$) and intersect A with tangent direction $p_0 = \varepsilon\mu_0/\mu_1 = 0$ (also the lines $x = c$). At last, the value $H = 0$ gives the pairs of straight isotropic lines which intersect A with tangent directions $p = \pm 1$. For the given point $q = (t_*, x_*) \in A$ the pencil γ_k consists of the hyperbolas $(x - x_0)^2 - t^2 = k^2$ with various $k \neq 0$, where $x_0 = x_* + k$, and the line $x = x_*$. All these geodesics have different 2-jets at q.

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