Singularities of a geodesic flow on surfaces with a cuspidal edge

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Abstract

This paper is a study of singularities of geodesic flows on surfaces with non-isolated singular points, which form a smooth curve (like cuspidal edge). The main results of the paper are normal forms of the corresponding direction field on the tangent bundle of the local coordinates plane and projection of its trajectories to the surface.

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Introduction

In 3-dimensional Euclidean space E with the Cartesian coordinates $r = (r_1, r_2, r_3)$ and the standard metric $ds^2 = \sum dr_i^2$ consider the surface S given parametrically by the formula

$$r = F\left(x, \frac{t^2}{2}\varphi(t,x), \frac{t^2}{2}\psi(t,x)\right), \quad F = (F_1, F_2, F_3), \tag{1}$$

where $F(x, y, z)$ is a generic diffeomorphism,[1] and $\varphi(t, x), \psi(t, x)$ are C^∞-smooth functions. The surface S is smooth at all regular points, i.e., points where the mapping $\sigma : (t, x) \to r$ given by the formula (1) is an immersion. Non-regular points are given by the equation $t = 0$, which defines a curve on the surface S. This curve is called the *cuspidal edge* (briefly, *edge*) of the surface. Surfaces of this kind occur in various problems connected with Lagrangian and Legendrian singularities, for instance, in studying caustics and wave fronts [1, 2].

The aim of this paper is to study singularities of the geodesic flow and geodesic lines on the surface S which pass through the non-regular points, in neighborhoods of these points. Here we consider only continuously differentiable curves, passing through non-regular points with definite tangent directions.[2] Non-regular points of the surface S are singular points of the geodesic flow, that is, points where the coefficient of higher derivative in the corresponding second-order differential equation vanishes. At such points this equation can not be resolved with respect to the higher derivative, and the standard existence and uniqueness theorem is

[1] This means that F belongs to an open and everywhere dense set in the space of smooth mappings with the fine Whitney topology.

[2] In the qualitative theory of differential equation such integral curves are often called *regular*.

not valid. Notice that singularities of a geodesic flow were considered before for the surfaces of another kind [3, 4].

It is natural to consider the projectivized tangent bundle of the (t, x)-plane with the coordinates (t, x, p), where $p = dx/dt$, as the phase space of the differential equation of the geodesic flow. Let π be the projection $\pi : (t, x, p) \to (t, x)$ along the axis p. The differential equation of the geodesic flow defines in the (t, x, p)-space a direction field χ. The singular points of the field χ correspond to non-regular points of the surface S.

The first aim of the paper is to get normal forms of the field χ in neighborhoods of its singular points. Then by means of the mappings π and σ (see the left side of the diagram (2)) we get the phase portraits of the geodesics passing through the edge of S. Notice that the similar scheme was used in [5] for studying geodesics on a 2-dimensional smooth surface with pseudo-Riemannian metric passing through the point where the metrics changes the signature, i.e., is degenerated. But unlike the present problem the mapping σ in [5] was an immersion and did not have any singularities.

In the first part of the paper we study singular points of the geodesic flow in the (t, x, p)-space. Here we define geodesics from the variational principle: as extremals of the Euler–Lagrange equation for the length functional corresponding to the metric of the ambient Euclidean space E. (The same equation of the geodesic flow can be also obtained from the geometrical definition of geodesic: as curves having identically zero geodesic curvature.) The singular points of the corresponding direction field χ in the (t, x, p)-space are not isolated, they form curves which are (possibly, apart from some discrete subset of the points) the central manifolds (W^c) of the field. Such fields were studied in [6, 7]. The results yield the local normal forms of the field χ at the singular points.

Then using the projection $\pi : (t, x, p) \to (t, x)$ along the p-direction, which we call vertical, we get the pre-images of the geodesic lines on the (t, x)-plane. In most cases in a neighborhood of the singular point the (t, x, p)-space has a smooth foliation invariant with respect to the field χ. This foliation is transversal to W^c, and each leaf intersects W^c by a unique point. Hence one can consider the projection of each leaf Σ separately (see the right side of the diagram (2)). The main difficulty is that the leafs are tangent to the kernel of the projection (the p-direction), and moreover, each leaf Σ contains a vertical straight line, which projects to a point. Notice that the projection with similar properties occurs in studying the net of curvature lines in neighborhoods of umbilic points [8, 9].

In the second part of the paper we study the phase portraits of geodesic lines on the surface S itself. The mapping $\sigma : (t, x) \to r$ sends the pre-images of the geodesics on the (t, x)-plane to the geodesics on the surface S itself:

$$
\begin{tikzcd}
(t,x,p) \arrow[d, "\pi"'] \arrow[dr] & & & \Sigma \arrow[d, "\pi"'] \arrow[dr] & \\
{} & S \arrow[rr, "\text{restriction to the leaf } \Sigma"] & & {} & S \\
(t,x) \arrow[ur, "\sigma"'] & & & (t,x) \arrow[ur, "\sigma"'] &
\end{tikzcd}
\tag{2}
$$

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1 The geodesic flow

The geodesic lines on the surface S equipped by the metric induced from the ambient Euclidean space E are the extremals of the Euler–Lagrange equation

$$\frac{d}{dt}L'_p - L'_x = 0, \quad p = \frac{dx}{dt}, \quad L = \sqrt{ap^2 + 2bp + c}, \tag{3}$$

with the smooth coefficients

$$\begin{aligned} a = \langle r'_x, r'_x\rangle = \langle F'_x, F'_x\rangle + t^2(\varphi'_x\langle F'_x, F'_y\rangle + \psi'_x\langle F'_x, F'_z\rangle) + O(t^4), \\ b = \langle r'_x, r'_t\rangle = t(\varphi\langle F'_x, F'_y\rangle + \psi\langle F'_x, F'_z\rangle) + \frac{t^2}{2}(\varphi'_t\langle F'_x, F'_y\rangle + \psi'_t\langle F'_x, F'_z\rangle) + O(t^3), \\ c = \langle r'_t, r'_t\rangle = t^2(\varphi^2\langle F'_y, F'_y\rangle + 2\varphi\psi\langle F'_y, F'_z\rangle + \psi^2\langle F'_z, F'_z\rangle) + O(t^3), \end{aligned} \tag{4}$$

where triangle brackets mean the scalar product corresponding to the metric of the Euclidean space E. The equation (3) reads

$$\frac{\frac{d}{dt}(ap+b)}{L} - \frac{(ap+b)\frac{d}{dt}(ap^2+2bp+c)}{2L^3} - \frac{a'_xp^2 + 2b'_xp + c'_x}{2L} = 0. \tag{5}$$

It is not defined at the points of the (t, x, p)-space, where $L = 0$.

We claim that if the diffeomorphism F is generic then the implicit differential equation $L(t, x, p) = 0$ does not have solutions. Indeed, from the formulas (4) it follows that the discriminant $b^2 - ac$ of the quadratic trinomial $ap^2 + 2bp + c$ is not positive, and $b^2 - ac = 0$ if and only if $t = 0$. At at points of the edge ($t = 0$) we have the inequality $a \neq 0$ and the equalities $b = c = 0$, hence the equation $L = 0$ implies $p = 0$. On the other hand, the tangent direction to the edge ($t = 0$) is $dx : dt = 1 : 0$, i.e., $p = \infty$. Thus the case $L = 0$ does not correspond to any solution of (5), and we can multiply the equation (5) by $2L^3$. After that we get the equation

$$2(ac - b^2)\frac{dp}{dt} + M(t, x, p) = 0, \quad p = \frac{dx}{dt}, \quad M(t, x, p) = \sum_{i=0}^{3}\mu_i(t, x)p^i, \tag{6}$$

with the coefficients

$$\begin{aligned} \mu_3 = a(a'_t - 2b'_x) + ba'_x, \quad \mu_2 = b(3a'_t - 2b'_x) + ca'_x - 2ac'_x, \\ \mu_1 = b(2b'_t - 3c'_x) + 2ca'_t - ac'_t, \quad \mu_0 = c(2b'_t - c'_x) - bc'_t. \end{aligned} \tag{7}$$

The coefficient $2(ac-b^2)$ of higher derivative in (6) depending only on the variables (t, x) is proportional to the discriminant of the Riemannian metric on S, hence it vanishes only at the points of the edge ($t = 0$).

The equation (6) defines in the (t, x, p)-space the direction field χ:

$$\dot t = 2(b^2 - ac), \quad \dot x = 2p(b^2 - ac), \quad \dot p = M(t, x, p). \tag{8}$$

The projections of the integral curves of the field (8) onto the (t, x)-plane along the p-direction are geodesics (more precisely, the pre-images of geodesics on the plane of parameters).

Substituting the expressions (4) for a, b, c in (7), we see that all functions $\mu_i(t, x)$ are divisible by t. Then we can write the coefficients $\mu_i(t, x)$ in the form $\mu_i(t, x) = 2t\widetilde{\mu}_i(t, x) + O(t^2)$, where the symbol $O(t^n)$ means a term containing the factor t^n (note that here the order of $O(t^n)$ can be greater that n). The direct calculation taking account of (4) and (7) gives $\widetilde{\mu}_0 \equiv \widetilde{\mu}_2 \equiv 0$,

$$\widetilde{\mu}_1 = (\langle F'_x, F'_y\rangle^2 - \langle F'_x, F'_x\rangle\langle F'_y, F'_y\rangle)\varphi^2 + 2(\langle F'_x, F'_y\rangle\langle F'_x, F'_z\rangle - \langle F'_x, F'_x\rangle\langle F'_y, F'_z\rangle)\varphi\psi +$$
$$+ (\langle F'_x, F'_z\rangle^2 - \langle F'_x, F'_x\rangle\langle F'_z, F'_z\rangle)\psi^2 =$$
$$= -[(F_1)'_x G_2 - (F_2)'_x G_1]^2 - [(F_1)'_x G_3 - (F_3)'_x G_1]^2 - [(F_2)'_x G_3 - (F_3)'_x G_2]^2 \le 0, \quad (9)$$

$$\widetilde{\mu}_3 = (\langle F'_x, F'_y\rangle\langle F'_x, F''_{xx}\rangle - \langle F'_x, F'_x\rangle\langle F'_y, F''_{xx}\rangle)\varphi + (\langle F'_x, F'_z\rangle\langle F'_x, F''_{xx}\rangle - \langle F'_x, F'_x\rangle\langle F'_z, F''_{xx}\rangle)\psi, \quad (10)$$

where $G_i = (F_i)'_y\varphi + (F_i)'_z\psi$, and also $\mu_0(t,x) = O(t^3)$.

Thus we can divide the components of the field (8) by $2t$. Since $\mu_0 = O(t^3)$, we get the field

$$\dot{t} = t(\widetilde{\mu}_1 + O(t)), \quad \dot{x} = tp(\widetilde{\mu}_1 + O(t)), \quad \dot{p} = \widetilde{M}(t,x,p) = \sum_{i=1}^{3}(\widetilde{\mu}_i + O(t))p^i + O(t^2). \quad (11)$$

Each non-regular point $(0, x^*)$ of the surface S corresponds to the vertical straight line in the (t,x,p)-space, which consists of the points $(0, x^*, p)$. It is readily seen that such line is an integral curve of the field (11). Thus the geodesic lines on the surface S passing through the point $(0, x^*)$ correspond to the integral curves of the field (11) passing through the singular points $(0, x^*, p)$ of this field with various p, that is, $\widetilde{M}(0, x^*, p) = 0$. The last equation is cubic with respect to the variable p and reads

$$\widetilde{M}(0, x^*, p) = p\,(\widetilde{\mu}_3 p^2 + \widetilde{\mu}_1) = 0. \quad (12)$$

According to the previous assumption that the diffeomorphism F in (1) is generic, we will always assume that $\widetilde{\mu}_1 \neq 0$ (from (9) it follows that $\widetilde{\mu}_1 < 0$) and $\widetilde{\mu}_3 \neq 0$. Then the equation (12) has the prime root $p_0 = 0$. If $\widetilde{\mu}_3 < 0$ then $p_0 = 0$ is a unique real root, if $\widetilde{\mu}_3 > 0$ then the equation (12) has two more real roots: $p_\pm = \pm\sqrt{-\widetilde{\mu}_1/\widetilde{\mu}_3}$. We conclude that if $\widetilde{\mu}_3 < 0$ then non-regular point $(0, x^*)$ corresponds to the unique singular point $P_0^* = (0, x^*, 0)$ of the field (11), and if $\widetilde{\mu}_3 > 0$ then it corresponds to three singular points: P_0^* and $P_\pm^* = (0, x^*, p_\pm)$.

Theorem 1 *The germ of the field* (11) *at the point* P_0^* *is smoothly orbitally equivalent to the germ*

$$\dot{\xi} = \xi, \quad \dot{\eta} = \eta, \quad \dot{\zeta} = 0 \quad (13)$$

at the origin.

Proof. Direct calculation of the linear part Λ of the field (11) at the singular point $P_0^* = (0, x^*, 0)$ gives the spectrum $(\lambda, \lambda, 0)$, where $\lambda = \widetilde{\mu}_1(0, x^*)$. Here the zero eigenvalue corresponds to the central manifold W^c, which consists of the singular points of (11) and coincides with the x-axis. The double non-zero eigenvalue λ corresponds to 2-dimensional invariant manifold having the vertical tangent plane at P_0^*. The germ of such field is orbitally equivalent to the germ

$$\dot{\xi} = \xi + \varphi(\zeta)\eta, \quad \dot{\eta} = \eta, \quad \dot{\zeta} = 0 \quad (14)$$

at the origin with some smooth function $\varphi(\zeta)$. In the finite-smooth (C^k-smooth with any arbitrary natural k) category this statement follows from the Theorem 3 [7], which is a consequence of more general results [10]. In the C^∞-smooth category it follows from the Theorem 20 [6].

Let us show that in the normal form (14) the coefficient $\varphi(\zeta) \equiv 0$. Consider the matrix $\Lambda - \lambda I$ at the singular point P_0^*. Clearly, the rang of this matrix is equal 1 or 2 and it is an invariant of the field, i.e., $\varphi(0) = 0$ if $\mathrm{rg}\,(\Lambda - \lambda I) = 1$ and $\varphi(0) \neq 0$ if $\mathrm{rg}\,(\Lambda - \lambda I) = 2$. Direct calculation shows that

$$\Lambda - \lambda I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\lambda & 0 \\ \widetilde{M}'_t & \widetilde{M}'_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

hence we get $\mathrm{rg}\,(\Lambda - \lambda I) = 1$, i.e., $\varphi(0) = 0$. Arguing as above, we see that the same reasoning is valid for any singular point $P_0 = (0, x, 0)$ sufficiently close to $P_0^* = (0, x^*, 0)$, and we get the equality $\varphi(\zeta) \equiv 0$ for all ζ sufficiently close to zero. □

The statement of the Theorem 1 is valid also in the analytic case, see [11, 12]. Moreover, in the analytic case it directly follows from general results by A.D. Bryuno [13, 14]. Indeed, it is clear that the condition A for the formal normal form (13) holds. The condition ω also holds, since the pair of non-zero eigenvalues of the linear part of the germ (11) at the singular point P_0^* lies in the Poincaré domain.

Theorem 2 *Suppose $\widetilde{\mu}_3 > 0$. The germ of the field* (11) *at the point $P_\pm^*$ is smoothly orbitally equivalent to the germ*

$$\dot{\xi} = \xi, \quad \dot{\eta} = -2\eta, \quad \dot{\zeta} = \xi^2 \eta \tag{15}$$

at the origin.

Proof. Direct calculation of the linear part Λ of the field (11) at the singular point $P_\pm^* = (0, x^*, p_\pm)$ gives the spectrum $(\lambda, -2\lambda, 0)$, where $\lambda = \widetilde{\mu}_1(0, x^*)$. Like in the previous theorem the zero eigenvalue corresponds to the central manifold W^c, which consists of the singular points of (11). Here W^c is a smooth curve in the (t, x, p)-space passing through the point $P_\pm^*$. Each non-zero eigenvalue corresponds to 1-dimensional invariant manifold of the field. Let W_1 and W_2 be these invariant manifolds, T_1 and T_2 be the corresponding tangent direction at $P_\pm^*$. Notice that here we distinguish modules, but not signs of the eigenvalues, since we deal with direction fields (orbital equivalence).

Since the diffeomorphism $F(x, y, z)$ is generic, the germ of this field is orbitally equivalent to the normal form (15). In the finite-smooth category it follows from the Theorem 6 [7]. In the C^∞-smooth category this statement can be proved with help of the method used for the Theorem 20 [6]. □

The statement of the Theorem 2 is not valid in the analytic case. The analytic normal form comes out from the smooth normal form (15) after addition of some module, see e.g. [11, 12]. From the viewpoint of the general theory developed by A.D. Bryuno, it can be explained in the following way. The condition A for the formal normal form (15) does not hold, since the third equation in (15) has the form $\dot{\zeta} = \xi^2 \eta$ instead of $\dot{\zeta} = 0$. Besides, the pair of non-zero eigenvalues of the linear part of the germ (11) at the singular point $P_\pm^*$ lies in the Siegel domain, where formal normalizing series generally diverge.

Theorems 1 and 2 allow to get the phase portraits of pre-images of the geodesics (we will briefly call them geodesics) on the (t, x)-plane, i.e., on the plane of parameters from the formula (1). For getting these phase portraits we need to project the integral curves of the field (11) onto the (t, x)-plane along the p-axis.

Theorem 3 *Suppose $\widetilde{\mu}_1 \neq 0$. Then on the (t, x)-plane there exists a pencil of geodesics γ_α passing through the point $(0, x^*)$ with various 2-jets $x(t) = x^* + \frac{\alpha}{2} t^2$, $\alpha \in \mathbb{R}$. If $\widetilde{\mu}_3 < 0$ then* *there are no others geodesics passing through* $(0, x^*)$*. If* $\widetilde{\mu}_3 > 0$ *then there exist also two geodesics* $\gamma_\pm$ *passing through* $(0, x^*)$ *with 1-jets* $x(t) = x^* + p_\pm t$.

Proof. Consider the projection of the (t, x, p)-space onto the (t, x)-plane along the p-axis in a neighborhood of the point P_0^*. The normal form (13) has the invariant foliation $\zeta =$ const, hence the field (11) has a smooth 2-dimensional invariant foliation with coordinates (t, p) on the leafs (the base is the x-axis). At the points of intersection with the x-axis all leafs have vertical tangent planes, and the restriction of the field (11) to each leaf is a bicritical node. Thus each non-regular point $(0, x^*)$ of the surface S corresponds to the invariant leaf Σ passing through the point P_0^*. The leaf Σ contains an infinite set of the integral curves of (11) passing through the point P_0^* with various tangent directions $dp : dt$. The integral curve with the tangent direction $dp : dt = 1 : 0$ is a straight vertical line, its projection is the point $(0, x^*)$. The projections of others integral curves from the leaf Σ onto the (t, x)-plane form a pencil of geodesics γ_α passing through the point $(0, x^*)$ with the common tangent direction $dx : dt = 0 : 1$ and various 2-jets $x(t) = x^* + \frac{\alpha}{2}t^2$, $\alpha \in \mathbb{R}$.

Now consider the projection of the (t, x, p)-space onto the (t, x)-plane along the p-axis in a neighborhood of the point $P_\pm^*$. The normal form (15) has the first integral $\xi^2\eta$ and the corresponding invariant foliation $\xi^2\eta =$ const with singularities at the points of the ζ-axis. The invariant manifolds W_1 and W_2 of the germ (15) coincide with the ξ-axis and η-axis, respectively. Direct calculation shows that for the initial field (11) the direction T_2 is vertical (i.e., it coincides with the p-axis), and the direction T_1 is not vertical. Hence the invariant manifold W_2 of the germ (11) at $P_\pm^*$ coincides with the vertical straight line passing through $P_\pm^*$, and its projection onto the (t, x)-plane is the point $(0, x^*)$. This means that there is a unique geodesic passing through the point $(0, x^*)$, it is the projection of W_1.

These arguments show that the singular points $P_\pm^*$ of the field (11) correspond to two geodesics $\gamma_\pm$ passing through the point $(0, x^*)$ with 1-jets $x(t) = x^* + p_\pm t$. $\square$

The geodesic lines on the (t, x)-plane are shown in Fig. 1. Here the solid lines are geodesics γ_α and the dashed lines are geodesics $\gamma_\pm$. The double line is the edge ($t = 0$).

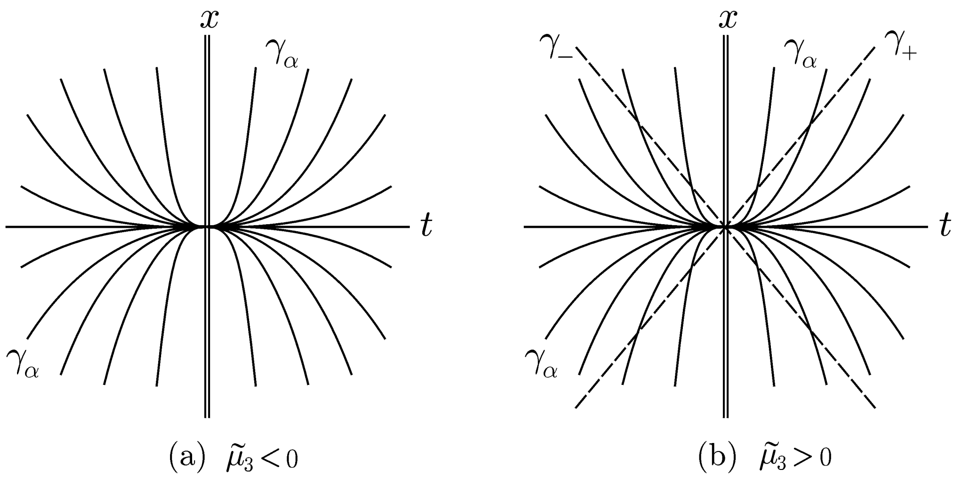

Fig. 1

Notice that in the initial formula (1) the parameters (t, x) have equal status, but when we write the equation (3) we consider one of them (t) as an independent variable, and consider another one (x) as a dependent variable. Moreover, we suppose that the derivative $p = dx/dt$ is finite. Geometrically, this means that we use only one affine chart ($p \neq \infty$)

on the projectivized tangent plane and hence we ignore the possibility of geodesics passing through the point $(0, x^*)$ with tangent direction $dx : dt = 1 : 0$, i.e., tangent to the edge. However it is not hard to show that if the conditions $\widetilde{\mu}_1 \neq 0$, $\widetilde{\mu}_3 \neq 0$ hold then such geodesics do not exist.

Indeed, after interchanging t and x in the formula (1) one can write the Euler–Lagrange equation (3) with new variables and get the direction field similar to (8), where the first and second components are the same as in the case of old variables (since the discriminant $b^2 - ac$ of the Riemannian metric on S does not change by interchanging t and x), and the third component has the form $\dot{p} = N(t, x, p)$, where $N = \sum\limits_{i=0}^{3} \nu_i(t, x)p^i$. From the formulas (4) and (7) it follows that the coefficients of the cubic monomials $N(t, x, p)$ and $M(t, x, p)$ are connected with the relations $\nu_i + \mu_{3-i} = 0$, $i = 0, \ldots, 3$.

Hence all coefficients ν_i are divisible by x, i.e., they have the form $\nu_i = 2x\widetilde{\nu}_i + O(x^2)$. Substituting $x = 0$, we get $\widetilde{\nu}_3 = -\widetilde{\mu}_0 = 0$, $\widetilde{\nu}_1 = -\widetilde{\mu}_2 = 0$, $\widetilde{\nu}_2 = -\widetilde{\mu}_1 \neq 0$, $\widetilde{\nu}_0 = -\widetilde{\mu}_3 \neq 0$. The tangent direction to the edge corresponds to the value $p = \infty$ in old variables, and the value $p = 0$ in new variables. Thus for existence of geodesics with such tangent direction it is necessary that the polynomial $N(t, x, p)$ has the root $p = 0$ (in new variables). But the last condition does not hold since $\widetilde{\nu}_0 \neq 0$.

Example 1. The simplest examples of the surfaces (1) are the semi-cubic caspidal edge and the folded Whitney umbrella. These surfaces are given by the formula (1) with the functions $\varphi(t, x) = 1$, $\psi(t, x) = t$ and $\varphi(t, x) = 1$, $\psi(t, x) = tx$, respectively. In both cases from the formula (9) with $t = 0$ it follows:

$$\widetilde{\mu}_1 = - \begin{vmatrix} (F_1)'_x & (F_1)'_y \\ (F_2)'_x & (F_2)'_y \end{vmatrix}^2 - \begin{vmatrix} (F_1)'_x & (F_1)'_y \\ (F_3)'_x & (F_3)'_y \end{vmatrix}^2 - \begin{vmatrix} (F_2)'_x & (F_2)'_y \\ (F_3)'_x & (F_3)'_y \end{vmatrix}^2 < 0, \tag{16}$$

since the vectors F'_x and F'_y are linearly independent. Thus for the semi-cubic caspidal edge and the folded Whitney umbrella the geodesics γ_α, $\alpha \in \mathbb{R}$, always exist. The condition $\widetilde{\mu}_3 \neq 0$ holds for any diffeomorphism $(x, y, z) \to F(x, y, z)$ having the generic 2-jet at the considering point.

Example 2. Consider the surface S obtained from the algebraic surface $r_1^{2n} r_2^m = r_3^2$ with integer $n \geq 0$ and $m \geq 3$ by means of smooth generic deformation $F(x, y, z)$. This surface can be given in the form (1) with $\varphi(t, x) = 1$ and $\psi(t, x) = t^{m-2}x^n$ (in the case $m = 3$, $n = 0$ it is semi-cubic caspidal edge, and in the case $m = 3$, $n = 1$ it is folded Whitney umbrella). Like in the previous example, the coefficient $\widetilde{\mu}_1$ at points of the edge is given by the formula (16), hence it is not equal to zero. The formula (10) implies that for generic deformation F the condition $\widetilde{\mu}_3 \neq 0$ also holds.

Example 3. Let S be the surface obtained from the helicoid

$$r_1 = \theta, \quad r_2 = \rho \cos \theta, \quad r_3 = \rho \sin \theta, \quad \rho > 0. \tag{17}$$

by means of smooth generic deformation $F(x, y, z)$. This surface can be given in the form (1), where $\theta = x$, $\rho = \frac{t^2}{2}$, and the functions $\varphi(t, x) = \cos x$, $\psi(t, x) = \sin x$. Let A be the Jacobi matrix of the diffeomorphism F:

$$A = \frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)},$$

and A_{ij} be the cofactor corresponding to the element a_{ij}. From the formula (9) we get

$$\widetilde{\mu}_1 = -\varphi^2 \sum_{i=1}^{3} A_{i3}^2 - 2\varphi\psi \sum_{i=1}^{3} A_{i2}A_{i3} - \psi^2 \sum_{i=1}^{3} A_{i2}^2. \tag{18}$$

The right side of (18) is a quadratic form of φ, ψ. It is not hard to prove that its discriminant D is given by the formula similar to (16), where the derivatives $(F_i)'_x$ and $(F_i)'_y$ are be replaced by the cofactors A_{i2} and A_{i3}, respectively. Thus we get $D \leq 0$, and the equality $D = 0$ holds if and only if $A_{i2}A_{j3} - A_{i3}A_{j2} = 0$ for all $i \neq j$. The expressions $A_{i2}A_{j3} - A_{i3}A_{j2}$ coincide with the minors standing in the second and third columns of the matrix A' with the elements $a'_{ij} = A_{ij}$. Since the matrix A is invertible, the matrix A' is also invertible, and three equalities $A_{i2}A_{j3} - A_{i3}A_{j2} = 0$ can not be valid simultaniously. Thus we get $D < 0$, and the equality $\widetilde{\mu}_1 = 0$ holds only if $\varphi = \psi = 0$. But the condition $\varphi = \psi = 0$ is not possible, since $\varphi = \cos x$ and $\psi = \sin x$.

Thus the condition $\widetilde{\mu}_1 < 0$ always holds, and for each non-regular point ($\rho = 0$) of the surface S there exists the infinite set of geodesics γ_α, $\alpha \in \mathbb{R}$, passing through this point and being not tangent to the edge ($\rho = 0$), which corresponds to the axis of the helicoid (17). For any generic non-linear deformation F the equality $\widetilde{\mu}_3 = 0$ holds only at isolated points of the edge, and between these points one of two inequalities $\widetilde{\mu}_3 < 0$ or $\widetilde{\mu}_3 > 0$ holds. In the first case there are no geodesics passing through non-regular point apart from γ_α. In the second case we have two more geodesics $\gamma_\pm$ tangent to the edge ($\rho = 0$).

2 Geometry of the surface

At first find the geometrical meaning of the sign of the coefficient $\widetilde{\mu}_3$. Recall that the sign of $\widetilde{\mu}_3$ defines existence or non-existence of the pair of geodesics $\gamma_\pm$, which have the pre-images on the (t, x)-plane transversal to the edge ($t = 0$) and the direction $p = 0$.

At any regular point the surface S has two linearly independent tangent vectors

$$r'_t = t\left(F'_y\Big(\varphi + \frac{t}{2}\varphi'_t\Big) + F'_z\Big(\psi + \frac{t}{2}\psi'_t\Big)\right), \qquad r'_x = F'_x + O(t^2).$$

Dividing the vector r'_t by t and tending to limit as $t \to 0$, we get the vectors $v = (G_1, G_2, G_3)$ and $w = ((F_1)'_x, (F_2)'_x, (F_3)'_x)$, where $G_i = (F_i)'_y\varphi + (F_i)'_z\psi$. Since $F(x, y, z)$ is a diffeomorphism, the vectors v and w are linearly independent if the functions φ and ψ do not vanish simultaniously. From the formulas (9) and (10) we see that it is also necessary condition for the inequalities $\widetilde{\mu}_1 \neq 0$ and $\widetilde{\mu}_3 \neq 0$, and we will assume that this condition always holds.

The vectors v and w define the plane Π, which is the limit of a tangent plane to S at a regular point (t, x) as $t \to 0$. The plane Π is said to be the *tangent plane* to the surface S at the corresponding non-regular point.

Let $\Pi^\perp$ be the plane passing through the non-regular point of the surface S tangent to the edge ($t = 0$) and orthogonal to the tangent plane Π. Without loss of generality we can assume that the non-regular point of interest corresponds to the origin of the (t, x)-plane. The plane $\Pi^\perp$ is spanned by the vector F'_x and the vector $v \times w$, which is normal to the plane Π. Hence the normal vector to the plane $\Pi^\perp$ has the components:

$$n_i = [(F_i)'_x\langle F'_x, F'_y\rangle - (F_i)'_y\langle F'_x, F'_x\rangle]\varphi + [(F_i)'_x\langle F'_x, F'_z\rangle - (F_i)'_z\langle F'_x, F'_x\rangle]\psi, \quad i = 1, 2, 3. \tag{19}$$

The intersection $\Pi^\perp \cap S$ is given by the equation

$$\sum_{i=1}^{3} n_i F_i\left(x, \frac{t^2}{2}\varphi(t,x), \frac{t^2}{2}\psi(t,x)\right) = 0. \tag{20}$$

The left side of the equation (20) is a function of two variables t and x with zero linear part at the origin. It is not hard to see that the quadratic part of this function is $\alpha x^2 + \beta t^2$, where

$$\alpha = \sum_{i=1}^{3} n_i (F_i)''_{xx}, \quad \beta = \sum_{i=1}^{3} n_i [(F_i)'_y \varphi + (F_i)'_z \psi]. \tag{21}$$

Substituting the expressions (19) for n_i in (21), after some simple transformations we get $\alpha = \widetilde{\mu}_3$ and $\beta = \widetilde{\mu}_1$.

This shows that the sign of $\widetilde{\mu}_1\widetilde{\mu}_3$ at non-regular points of the surface has the same geometrical sense as the sign of a Gaussian curvature at regular points. Namely, if $\widetilde{\mu}_1\widetilde{\mu}_3 > 0$ (that is, $\widetilde{\mu}_3 < 0$) there is a neighborhood of the non-regular point throughout which the surface S lies on the one side of the plane $\Pi^\perp$. The intersection $\Pi^\perp \cap S$ consists of a unique point, similarly to a smooth surface with positive Gaussian curvature (it is shown in Fig. 2 (a) for the semi-cubic cuspidal edge). If $\widetilde{\mu}_1\widetilde{\mu}_3 < 0$ (that is, $\widetilde{\mu}_3 > 0$) then the surface S intersects the plane $\Pi^\perp$ and lies on the both sides of $\Pi^\perp$. The intersection $\Pi^\perp \cap S$ consists of two curves, similarly to a smooth surface with negative Gaussian curvature (it is shown in Fig. 2 (b) for the semi-cubic cuspidal edge).

In the first case (we call it elliptic) there is only family of geodesics γ_α, $\alpha \in \mathbb{R}$, passing through the given non-regular point. In the second case (we call it hyperbolic) there are also two geodesics $\gamma_\pm$ tangent to the lines $\Pi^\perp \cap S$.

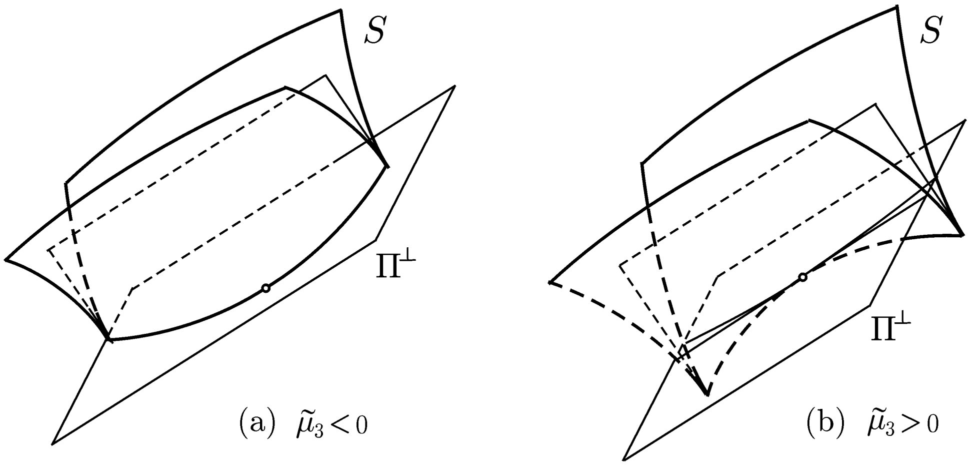

Fig. 2

Now consider the construction, which allows to get the phase portraits of geodesics not only on the (t,x)-plane, but on the surface S itself. The condition $\widetilde{\mu}_1 \neq 0$ implies (see the formula (9)) that there exist two indexes $i, j \in \{1,2,3\}$ such that $(F_i)'_x G_j - (F_j)'_x G_i \neq 0$. Consider projection of geodesics onto the (r_i, r_j)-plane with such indexes. Geodesics of the family γ_α passing through the point $(0, x^*)$ have various 2-jets $x(t) = x^* + \frac{\alpha}{2}t^2$, $\alpha \in \mathbb{R}$, hence the projection of γ_α onto the (r_i, r_j)-plane has the tangent direction

$$\frac{dr_i}{dr_j} = \lim_{t\to 0} \frac{\frac{d}{dt}F_i\big(x, \frac{t^2}{2}\varphi(t,x), \frac{t^2}{2}\psi(t,x)\big)}{\frac{d}{dt}F_j\big(x, \frac{t^2}{2}\varphi(t,x), \frac{t^2}{2}\psi(t,x)\big)} = \lim_{t\to 0} \frac{(\alpha(F_i)'_x + G_i)t + O(t^2)}{(\alpha(F_j)'_x + G_j)t + O(t^2)} = \frac{\alpha(F_i)'_x + G_i}{\alpha(F_j)'_x + G_j}. \tag{22}$$

at the point $(0, x^*)$.

The formula (22) defines the mapping $f : \mathbb{R} \to \mathbb{R}P^1$, which associates every real α with the corresponding tangent direction $dr_i : dr_j$. It is clear that if α takes all values from $\mathbb{R}$, then $f(\alpha)$ takes all values from $\mathbb{R}P^1$ except only $(F_i)'_x : (F_j)'_x$, which is tangent direction to the edge ($t = 0$). Indeed, if some real α corresponds to the direction $f(\alpha) = (F_i)'_x : (F_j)'_x$ then from the formula (22) we get the equality

$$\frac{\alpha(F_i)'_x + G_i}{\alpha(F_j)'_x + G_j} = \frac{(F_i)'_x}{(F_j)'_x},$$

which implies $(F_i)'_x G_j - (F_j)'_x G_i = 0$. Thus the formula (22) shows that the projections of geodesics γ_α, $\alpha \in \mathbb{R}$, on the (r_i, r_j)-plane pass through the given non-regular point with all possible tangent directions except the direction of the edge ($t = 0$).

In the hyperbolic case ($\widetilde{\mu}_3 > 0$) we have two more geodesics $\gamma_\pm$ passing through the point $(0, x^*)$ with 1-jets $x(t) = x^* + p_\pm t$, where $p_\pm = \pm\sqrt{-\widetilde{\mu}_1/\widetilde{\mu}_3}$. Their projections onto the (r_i, r_j)-plane have the common tangent direction

$$\frac{dr_i}{dr_j} = \lim_{t\to 0} \frac{\frac{d}{dt} F_i\big(x, \frac{t^2}{2}\varphi(t,x), \frac{t^2}{2}\psi(t,x)\big)}{\frac{d}{dt} F_j\big(x, \frac{t^2}{2}\varphi(t,x), \frac{t^2}{2}\psi(t,x)\big)} = \lim_{t\to 0} \frac{(F_i)'_x p_\pm + O(t)}{(F_j)'_x p_\pm + O(t)} = \frac{(F_i)'_x}{(F_j)'_x},$$

at the point $(0, x^*)$. This direction is tangent to the edge ($t = 0$).

The phase portraits of geodesics obtained above are also the same for any plane not orthogonal to the (r_i, r_j)-plane, in particular, for the tangent plane Π at the given non-regular point. The geodesic lines on the semi-cubic caspidal edge (for one of its leafs) are shown in Fig. 3 (a) and (b) in elliptic and hyperbolic cases, respectively. Here the solid lines are geodesics γ_α and the dashed lines are geodesics $\gamma_\pm$ as before.

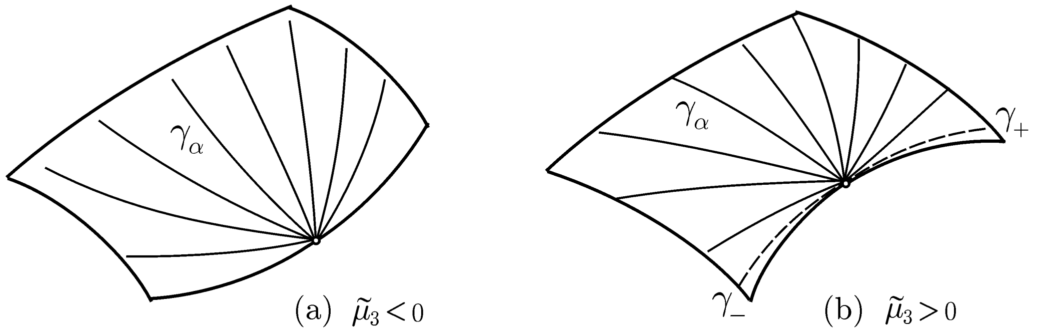

Fig. 3

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