A STRATEGY FOR NON-STRICTLY CONVEX DISTANCE TRANSPORT COST AND THE OBSTACLE PROBLEM

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JULY 6, 2010
PREPRINT SISSA 50/2010/M

ABSTRACT. We address the Monge problem in metric spaces with a geodesic distance: (X, d) is a Polish space and d_N is a geodesic Borel distance which makes (X, d_N) a possibly branching geodesic space. We show that under some assumptions on the transference plan we can reduce the transport problem to transport problems along family of geodesics.

We introduce two assumptions on the transference plan π which imply that the conditional probabilities of the first marginal on each family of geodesics are continuous and that each family of geodesics is a hourglass-like set. We show that this regularity is sufficient for the construction of a transport map.

We apply these results to the Monge problem in $\mathbb{R}^d$ with smooth, convex and compact obstacle obtaining the existence of an optimal map provided the first marginal is absolutely continuous w.r.t. the d-dimensional Lebesgue measure.

CONTENTS

1. INTRODUCTION

This paper concerns the Monge minimization problem in metric spaces with geodesic structure: given two Borel probability measure $\mu, \nu \in \mathcal{P}(X)$, where (X, d) is a Polish space, i.e. complete and separable metric space, we study the minimization of the functional

$$\mathcal{I}(T) = \int d_N(x, T(x))\mu(dy)$$

where T varies over all Borel maps $T : X \to X$ such that $T_\sharp \mu = \nu$ and d_N is a Borel distance that makes (X, d_N) a possibly branching geodesic space.

Before describing our investigation, we present a little bit of the existing literature referring to [15] and [16] for a deeper insight into optimal transportation.

In the original formulation given by Monge in 1781 the problem was settled in $\mathbb{R}^n$, with the cost given by the Euclidean norm and the measures μ, ν were supposed to be absolutely continuous and supported on two disjoint compact sets. The original problem remained unsolved for a long time. In 1978 Sudakov [13] claimed to have a solution for any distance cost function induced by a norm: an essential ingredient in the proof was that if $\mu \ll \mathcal{L}^d$ and $\mathcal{L}^d$-a.e. $\mathbb{R}^d$ can be decomposed into convex sets of dimension k, then then the conditional probabilities are absolutely continuous with respect to the $\mathcal{H}^k$ measure of the correct dimension. But it turns out that when $d > 2$, $0 < k < d-1$ the property claimed by Sudakov is not true. An example with $d = 3$, $k = 1$ can be found in [11].

The Euclidean case has been correctly solved only during the last decade. L. C. Evans and W. Gangbo in [8] solved the problem under the assumptions that $\operatorname{spt}\mu \cap \operatorname{spt}\nu = \emptyset$, $\mu, \nu \ll \mathcal{L}^d$ and their densities are Lipschitz function with compact support. The first existence results for general absolutely continuous measures μ, ν with compact support have been independently obtained by L. Caffarelli, M. Feldman and R.J. McCann in [6] and by N. Trudinger and X.J. Wang in [14]. Afterwards M. Feldman and R.J. McCann [9] extended the results to manifolds with geodesic cost. The case of a general norm as cost function on $\mathbb{R}^d$, including also the case with non strictly convex unitary ball, has been solved first in the particular case of crystalline norm by L. Ambrosio, B. Kirchheim and A. Pratelli in [2], and then in fully generality by T. Champion and L. De Pascale in [7]. The case with (X, d_N) non-branching geodesic space has been studied by S. Bianchini and the author in [4].

1.1. **Overview of the paper.** We introduce the setting considered in this paper: (X, d_N) is a geodesic space, not necessarily Polish. To assure that standard measure theory can be used, there exists a second distance d on X which makes (X, d) Polish and d_N is a Borel function on $X \times X$ with the metric $d \times d$. We do not require d_N to be l.s.c., hence the existence of an optimal transference plan is not guaranteed. We will consider a d_N-cyclically monotone transference plan $\pi \in \Pi(\mu, \nu)$ and we will show that, under appropriate assumptions on the first marginal and on the plan π, there exists an admissible map $T : X \to X$ that lowers the transference cost of π. Since d_N is not smooth enough, it is worth notice that we will not use the existence of optimal potentials (ϕ, ψ).

The strategy to cope with the Monge problem with branching distance cost is the one presented in [4]:

(1) reduce the problem, via Disintegration Theorem, to transportation problems in sets where, under a regularity assumption on the first marginal and on π, we know how to produce an optimal map;
(2) show that the disintegration of the first marginal μ on each of this sets verifies this regularity assumption;
(3) find a transport map on each sets and piece them together.

In the case d_N non-branching, the natural set where we can reduce the problem is a single geodesic and it do not depend on the choice of the d_N-cyclically monotone transference plan considered. Indeed the reduced problem becomes essentially one dimensional and the right regularity assumption is that the first marginal has no atoms (is continuous).

If d_N is a general geodesic distance this reduction can't be done anymore and there is not another reference set where the existence of Monge minimizer is known. The reduction set will be a concatenation of more geodesics and to produce an optimal map we will need a regularity assumption also on the shape of this set.

As in the non-branching case, the reduction sets come from the class of geodesics used by a d_N-monotone plan π. This class can be obtained from a d_N-cyclical monotone set Γ on which π is concentrated: one can construct the set of transport rays R, the transport set $\mathcal{T}_e$, i.e. the set of geodesics used by π, and from them construct

- the set $\mathcal{T}$ made of inner points of geodesics,
- the set $a \cup b := \mathcal{T}_e \setminus \mathcal{T}$ of initial points a and end points b.

Since branching of geodesics is admitted, R is not a partition on $\mathcal{T}$. To obtain an equivalence relation we have to consider the set H of chain of transport rays: it is the set of couples (x, y) such that we can go from x to y with a finite number of transport rays such that their common points are not final or initial points.

Hence H will provide the partition of the transport set $\mathcal{T}$ and each equivalence class, $H(y)$ for y in the quotient space, will be a reduction set.

Even if a partition is given, the reduction to transport problems on the equivalence classes is not straightforward: a necessary and sufficient condition is that the disintegration of the measure μ w.r.t. the partition H is strongly consistent. This is equivalent to the fact that there exists a μ-measurable quotient map $f : \mathcal{T} \to \mathcal{T}$ of the equivalence relation induced by the partition.
Since this partition is closely related to the geodesics of d_N, the strong consistency will follow from a topological property of the geodesic as set in (X, d) and from a metric property of d_N as a function:

(a) each set of geodesics $H(y)$ restricted to a d_N closed ball is d-closed;
(b) $d_N(x, \cdot)$ restricted to $H(x)$ is bounded on d-bounded sets.

Observe that these conditions on H and d_N are the direct generalization of the ones on geodesics in [4] (continuity and local compactness) and they depend on the particular choice of the transference plan.

Under this assumption, it is possible to disintegrate μ restricted to $\mathcal{T}$. Hence one can write

$$\mu = \int \mu_y m(dy), \quad m := f_\sharp \mu, \quad \mu_y(f^{-1}(y)) = 1,$$

i.e. the conditional probabilities μ_y are concentrated on the counterimages $f^{-1}(y)$ (which is an equivalence class). Then we can obtain the reduced problems by disintegrating π w.r.t. the partition $H \times (X \times X)$,

$$\pi = \int \pi_y m(dy), \quad \nu = \int \nu_y m(dy) \quad \nu_y := (P_2)_\sharp \pi_y,$$

and consider the problems on the sets $H(y)$ with marginals μ_y, ν_y and cost d_N.

At this level of generality we don't know how to obtain a d_N-monotone admissible map for the restricted problem even if the marginal μ_y satisfies some regularity assumptions. Therefore we need to assume that $H(y)$ has a particular structure: $H(y)$ is contained, up to set of μ_y-measure zero, in the uncountable "increasing" family of disjoint measurable sets. Then if the quotient measure and the marginal measures of μ_y are continuous, we prove the existence of an optimal map between μ_y and ν_y.

We can then study the problem of when the conditional probabilities μ_y are continuous. A natural operation on sets can be considered: the translation along geodesics. If A is a subset of $\mathcal{T}$, we denote by A_t the set translated by t in the direction determined by π. It turns out that the fact that $\mu(a \cup b) = 0$ and the measures μ_y are continuous depends on how the function $t \mapsto \mu(A_t)$ behaves.

Theorem 1.1 (Proposition 5.2 and Proposition 5.3)**.** *If for all A Borel such that $\mu(A) > 0$ there exists a sequence $\{t_n\} \subset \mathbb{R}$ and $C > 0$ such that $\mu(A_{t_n}) \geq C\mu(A)$ as $t_n \to 0$, then $\mu(a \cup b) = 0$ and the conditional probabilities μ_y are continuous.*

Once we have this result we can solve the Monge problem.

Theorem 1.2 (Proposition 6.2 and Theorem 6.1)**.** *Assume that the hypothesis of Theorem 1.1 are verified and the plan π is concentrated on a set Γ satisfying the assumption on the shape of the sets $H(y)$. Then there exists an admissible d_N-monotone map that lowers the transference cost of π.*

It follows immediately that in the hypothesis of Theorem 1.2, if π is also optimal, the Monge minimization problem admits a solution.

In the last part of the paper we show an application of Theorem 1.2. Consider $X = \mathbb{R}^d$ and a smooth hyper-surface M connected that is the boundary of a convex and compact set. Take as cost function d_M: the minimum of the euclidean length among all Lipschitz curves that do not cross M. Hence we have a Monge minimization problem with M as obstacle. We show that if μ is absolutely continuous w.r.t. $\mathcal{L}^d$, the Monge minimization problem with cost d_M admits a solution.

1.2. **Structure of the paper.** The paper is organized as follows.

In Section 2, we recall the mathematical tools we use in this paper. In Section 2.1 the fundamental results of projective set theory are listed. In Section 2.2 we recall the Disintegration Theorem. Next, the basic results of selection principles are listed in Section 2.3, and in Section 2.4 we define the geodesic structure (X, d, d_N) which is studied in this paper. Finally, Section 2.5 recalls some fundamental results in optimal transportation.

Section 3 shows how using only the d_N-cyclical monotonicity of a set Γ we can obtain a partial order relation $G \subset X \times X$ as follows (Lemma 3.3 and Proposition 3.9): xGy iff there exists $(w, z) \in \Gamma$ and a geodesic γ, passing trough w and z and with direction $w \to z$, such that x, y belongs to γ and $\gamma^{-1}(x) \leq \gamma^{-1}(y)$. This set G is analytic, and allows to define

- the transport rays set R (3.3),
- the transport sets $\mathcal{T}_e$, $\mathcal{T}$ (with and without and points) (3.4),
- the set of initial points a and final points b (3.7).

Even if this part of Section 3 contains the same results of the first part of Section 3 of [4], for being as self contained as possible, we state this results and show their proofs again.

The main difference with the strictly convex case [4] is that here R is not an equivalence relation. To obtain an equivalence relation $H \subset X \times X$ we have to consider the set of couples (x, y) for $x, y \in \mathcal{T}$ such that there is a continuous path from x to y, union of a finite number of transport rays never passing through $a \cup b$, Definition 3.8. In Proposition 3.9 we prove that H is an equivalence relation.

Section 4 proves that the compatibility conditions (a) and (b) between d_N and d, imply that the disintegration induced by H on $\mathcal{T}$ is strongly consistent (Proposition 4.4). Using this fact we can reduce the analysis on $H(y)$ for y in the quotient set.

In Section 5 we prove Theorem 1.1. We first introduce the operation $A \mapsto A_t$, the translation along geodesics (5.1), and show that $t \mapsto \mu(A_t)$ is a $\mathcal{A}$-measurable function if A is analytic (Lemma 5.1). Next, we show that under the assumption

$$\mu(A) > 0 \quad \Longrightarrow \quad \mu(A_{t_n}) \geq C\mu(A)$$

for an infinitesimal sequence t_n and $C > 0$, the set of initial points a is μ-negligible (Proposition 5.2) and the conditional probabilities μ_y are continuous.

In Section 6 we prove Theorem 1.2. First in Theorem 6.1 we prove that gluing all the d_N-cyclically monotone maps defined on $H(y)$ we obtain a measurable transference map T from μ to ν d_N-cyclically monotone. Then the assumption on the structure of Γ is stated (Assumption 2) and in Proposition 6.2 we show that on the equivalence class $H(y)$ satisfying Assumption 2 there exists a d_N-cyclically monotone transference map T_y from μ_y to ν_y, provided the quotient measure and the marginal probabilities of μ_y induced by the partition given by Assumption 2 are continuous.

Section 7 gives an application of Theorem 1.2: $X = \mathbb{R}^d$, M is a smooth hyper-surface, connected that is the boundary of a convex and compact set. The distance d_M is the minimum of the euclidean length among all the Lipschitz curves that do not cross M (7.1). Hence M is to be intended as an obstacle for euclidean geodesics. The geodesic space (X, d_M) fits into the setting of Theorem 1.2 (Lemma 7.1). If $\mu \ll \mathcal{L}^d$ then the μ-measure of the set of initial points is zero and the marginal μ_y are continuous (Lemma 7.2). Finally we show in Proposition 7.5 and Proposition 7.7 that any d_M-cyclically monotone set and μ satisfy the hypothesis of Proposition 6.2. It follows the existence of a solution for the Monge minimization problem.

2. Preliminaries

In this section we recall some general facts about projective classes, the Disintegration Theorem for measure, measurable selection principles, geodesic spaces and optimal transportation problems.

2.1. **Borel, projective and universally measurable sets.** The *projective class* $\Sigma^1_1(X)$ is the family of subsets A of the Polish space X for which there exists Y Polish and $B \in \mathcal{B}(X \times Y)$ such that $A = P_1(B)$. The *coprojective class* $\Pi^1_1(X)$ is the complement in X of the class $\Sigma^1_1(X)$. The class Σ^1_1 is called *the class of analytic sets*, and Π^1_1 are the *coanalytic sets*.

The *projective class* $\Sigma^1_{n+1}(X)$ is the family of subsets A of the Polish space X for which there exists Y Polish and $B \in \Pi^1_n(X \times Y)$ such that $A = P_1(B)$. The *coprojective class* $\Pi^1_{n+1}(X)$ is the complement in X of the class Σ^1_{n+1}.

If Σ^1_n, Π^1_n are the projective, coprojective pointclasses, then the following holds (Chapter 4 of [12]):

(1) Σ^1_n, Π^1_n are closed under countable unions, intersections (in particular they are monotone classes);
(2) Σ^1_n is closed w.r.t. projections, Π^1_n is closed w.r.t. coprojections;
(3) if $A \in \Sigma^1_n$, then $X \setminus A \in \Pi^1_n$;
(4) the *ambiguous class* $\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$ is a σ-algebra and $\Sigma^1_n \cup \Pi^1_n \subset \Delta^1_{n+1}$.

We will denote by $\mathcal{A}$ the σ-algebra generated by Σ^1_1: clearly $\mathcal{B} = \Delta^1_1 \subset \mathcal{A} \subset \Delta^1_2$.

We recall that a subset of X Polish is *universally measurable* if it belongs to all completed σ-algebras of all Borel measures on X: it can be proved that every set in $\mathcal{A}$ is universally measurable. We say that $f : X \to \mathbb{R} \cup \{\pm\infty\}$ is a *Souslin function* if $f^{-1}(t, +\infty] \in \Sigma^1_1$.

Lemma 2.1. *If $f : X \to Y$ is universally measurable, then $f^{-1}(U)$ is universally measurable if U is.*

See [4] for the proof.

2.2. **Disintegration of measures.** We follow the approach of [3].

Given a measurable space $(R, \mathscr{R})$ and a function $r : R \to S$, with S generic set, we can endow S with the *push forward σ-algebra $\mathscr{S}$* of $\mathscr{R}$:

$$Q \in \mathscr{S} \iff r^{-1}(Q) \in \mathscr{R},$$

which could be also defined as the biggest σ-algebra on S such that r is measurable. Moreover given a measure space $(R, \mathscr{R}, \rho)$, the *push forward measure* η is then defined as $\eta := (r_\sharp \rho)$.

Consider a probability space $(R, \mathscr{R}, \rho)$ and its push forward measure space $(S, \mathscr{S}, \eta)$ induced by a map r. From the above definition the map r is clearly measurable and inverse measure preserving.

Definition 2.2. A *disintegration* of ρ *consistent with* r is a map $\rho : \mathscr{R} \times S \to [0, 1]$ such that

(1) $\rho_s(\cdot)$ is a probability measure on $(R, \mathscr{R})$, for all $s \in S$,
(2) $\rho_\cdot(B)$ is η-measurable for all $B \in \mathscr{R}$,

and satisfies for all $B \in \mathscr{R}, C \in \mathscr{S}$ the consistency condition

$$\rho\left(B \cap r^{-1}(C)\right) = \int_C \rho_s(B)\eta(ds).$$

A disintegration is *strongly consistent with* r if for all s we have $\rho_s(r^{-1}(s)) = 1$.

We say that a σ-algebra $\mathcal{A}$ is *essentially countably generated* with respect to a measure m, if there exists a countably generated σ-algebra $\hat{\mathcal{A}}$ such that for all $A \in \mathcal{A}$ there exists $\hat{A} \in \hat{\mathcal{A}}$ such that $m(A \triangle \hat{A}) = 0$.

We recall the following version of the theorem of disintegration of measure that can be found on [10], Section 452.

Theorem 2.3 (Disintegration of measure)**.** *Assume that $(R, \mathscr{R}, \rho)$ is a countably generated probability space, $R = \cup_{s \in S} R_s$ a decomposition of R, $r : R \to S$ the quotient map and $(S, \mathscr{S}, \eta)$ the quotient measure space. Then $\mathscr{S}$ is essentially countably generated w.r.t. η and there exists a unique disintegration $s \to \rho_s$ in the following sense: if ρ_1, ρ_2 are two consistent disintegration then $\rho_{1,s}(\cdot) = \rho_{2,s}(\cdot)$ for $\eta-$a.e. s.*

If $\{S_n\}_{n \in \mathbb{N}}$ is a family essentially generating $\mathscr{S}$ define the equivalence relation:

$$s \sim s' \iff \{s \in S_n \iff s' \in S_n,\ \forall\, n \in \mathbb{N}\}.$$

Denoting with p the quotient map associated to the above equivalence relation and with $(L, \mathscr{L}, \lambda)$ the quotient measure space, the following properties hold:

- *$R_l := \cup_{s \in p^{-1}(l)} R_s = (p \circ r)^{-1}(l)$ is ρ-measurable and $R = \cup_{l \in L} R_l$;*
- *the disintegration $\rho = \int_L \rho_l \lambda(dl)$ satisfies $\rho_l(R_l) = 1$, for λ-a.e. l. In particular there exists a strongly consistent disintegration w.r.t. $p \circ r$;*
- *the disintegration $\rho = \int_S \rho_s \eta(ds)$ satisfies $\rho_s = \rho_{p(s)}$, for η-a.e. s.*

In particular we will use the following corollary.

Corollary 2.4. *If $(S, \mathscr{S}) = (X, \mathcal{B}(X))$ with X Polish space, then the disintegration is strongly consistent.*

2.3. **Selection principles.** Given a multivalued function $F : X \to Y$, X, Y metric spaces, the *graph* of F is the set

$$\operatorname{graph}(F) := \big\{(x, y) : y \in F(x)\big\}. \tag{2.1}$$

The *inverse image* of a set $S \subset Y$ is defined as:

$$F^{-1}(S) := \big\{x \in X \ :\ F(x) \cap S \neq \emptyset\big\}. \tag{2.2}$$

For $F \subset X \times Y$, we denote also the sets

$$F_x := F \cap \{x\} \times Y, \quad F^y := F \cap X \times \{y\}. \tag{2.3}$$

In particular, $F(x) = P_2(\mathrm{graph}(F)_x)$, $F^{-1}(y) = P_1(\mathrm{graph}(F)^y)$. We denote by F^{-1} the graph of the inverse function

$$F^{-1} := \{(x,y) : (y,x) \in F\}. \tag{2.4}$$

We say that F is $\mathcal{R}$*-measurable* if $F^{-1}(B) \in \mathcal{R}$ for all B open. We say that F is *strongly Borel measurable* if inverse images of closed sets are Borel. A multivalued function is called *upper-semicontinuous* if the preimage of every closed set is closed: in particular u.s.c. maps are strongly Borel measurable.

In the following we will not distinguish between a multifunction and its graph. Note that the *domain of* F (i.e. the set $P_1(F)$) is in general a subset of X. The same convention will be used for functions, in the sense that their domain may be a subset of X.

Given $F \subset X \times Y$, a *section* u *of* F is a function from $P_1(F)$ to Y such that $\mathrm{graph}(u) \subset F$. We recall the following selection principle, Theorem 5.5.2 of [12], page 198.

Theorem 2.5 (Von Neumann)**.** *Let X and Y be Polish spaces, $A \subset X \times Y$ analytic, and $\mathcal{A}$ the σ-algebra generated by the analytic subsets of X. Then there is an $\mathcal{A}$-measurable section $u : P_1(A) \to Y$ of A.*

A *cross-section of the equivalence relation* E is a set $S \subset E$ such that the intersection of S with each equivalence class is a singleton. We recall that a set $A \subset X$ is saturated for the equivalence relation $E \subset X \times X$ if $A = \cup_{x \in A} E(x)$.

The next result is taken from [12], Theorem 5.2.1.

Theorem 2.6. *Let Y be a Polish space, X a nonempty set, and $\mathcal{L}$ a σ-algebra of subset of X. Every $\mathcal{L}$-measurable, closed value multifunction $F : X \to Y$ admits an $\mathcal{L}$-measurable section.*

A standard corollary of the above selection principle is that if the disintegration is strongly consistent in a Polish space, then up to a saturated set of negligible measure there exists a Borel cross-section.

In particular, we will use the following corollary.

Corollary 2.7. *Let $F \subset X \times X$ be $\mathcal{A}$-measurable, X Polish, such that F_x is closed and define the equivalence relation $x \sim y \Leftrightarrow F(x) = F(y)$. Then there exists a $\mathcal{A}$-section $f : P_1(F) \to X$ such that $(x, f(x)) \in F$ and $f(x) = f(y)$ if $x \sim y$.*

Proof. For all open sets $G \subset X$, consider the sets $F^{-1}(G) = P_1(F \cap X \times G) \in \mathcal{A}$, and let $\mathcal{R}$ be the σ-algebra generated by $F^{-1}(G)$. Clearly $\mathcal{R} \subset \mathcal{A}$.

If $x \sim y$, then

$$x \in F^{-1}(G) \iff y \in F^{-1}(G),$$

so that each equivalence class is contained in an atom of $\mathcal{R}$, and moreover by construction $x \mapsto F(x)$ is $\mathcal{R}$-measurable.

We thus conclude by using Theorem 2.6 that there exists an $\mathcal{R}$-measurable section f: this measurability condition implies that f is constant on atoms, in particular on equivalence classes. $\square$

2.4. **Metric setting.** In this section we refer to [5].

Definition 2.8. A *length structure* on a topological space X is a class $\mathtt{A}$ of admissible paths, which is a subset of all continuous paths in X, together with a map $L : \mathtt{A} \to [0, +\infty]$: the map L is called *length of path*. The class $\mathtt{A}$ satisfies the following assumptions:

- **closure under restrictions:** if $\gamma : [a,b] \to X$ is admissible and $a \leq c \leq d \leq b$, then $\gamma\llcorner_{[c,d]}$ is also admissible.
- **closure under concatenations of paths:** if $\gamma : [a,b] \to X$ is such that its restrictions γ_1, γ_2 to $[a,c]$ and $[c,b]$ are both admissible, then so is γ.
- **closure under admissible reparametrizations:** for an admissible path $\gamma : [a,b] \to X$ and a for $\varphi : [c,d] \to [a,b]$, $\varphi \in B$, with B class of admissible homeomorphisms that includes the linear one, the composition $\gamma(\varphi(t))$ is also admissible.

The map L satisfies the following properties:

- **additivity:** $L(\gamma\llcorner_{[a,b]}) = L(\gamma\llcorner_{[a,c]}) + L(\gamma\llcorner_{[c,b]})$ for any $c \in [a,b]$.
- **continuity:** $L(\gamma\llcorner_{[a,t]})$ is a continuous function of t.
- **invariance:** The length is invariant under admissible reparametrizations.

topology: Length structure agrees with the topology of X in the following sense: for a neighborhood U_x of a point $x \in X$, the length of paths connecting x with points of the complement of U_x is separated from zero:

$$\inf \big\{ L(\gamma) : \gamma(a) = x, \gamma(b) \in X \setminus U_x \big\} > 0.$$

Given a length structure, we can define a distance

$$d_N(x,y) = \inf \Big\{ L(\gamma) : \gamma : [a,b] \to X, \gamma \in \mathtt{A}, \gamma(a) = x, \gamma(b) = y \Big\},$$

that makes (X, d_N) a metric space (allowing d_N to be $+\infty$). The metric d_N is called *intrinsic*. It follows from Proposition 2.5.9 of [5] that every admissible curve of finite length admits a constant speed parametrization, i.e. γ defined on $[0,1]$ and $L(\gamma \llcorner [t,t']) = v(t'-t)$, with v velocity.

Definition 2.9. A length structure is said to be *complete* if for every two points x, y there exists an admissible path joining them whose length $L(\gamma)$ is equal to $d_N(x,y)$.

Observe that in the previous definition we do no require $d_N(x,y) < +\infty$.

Intrinsic metrics associated with complete length structure are said to be *strictly intrinsic*. The metric space (X, d_N) with d_N strictly intrinsic is called a *geodesic space*. A curve whose length equals the distance between its end points is called *geodesic*.

From now on we assume the following:

(1) (X, d) Polish space;
(2) $d_N : X \times X \to [0, +\infty]$ is a Borel distance;
(3) (X, d_N) is a geodesic space;

Since we have two metric structures on X, we denote the quantities relating to d_N with the subscript N: for example

$$B_r(x) = \big\{ y : d(x,y) < r \big\}, \quad B_{r,N}(x) = \big\{ y : d_N(x,y) < r \big\}.$$

In particular we will use the notation

$$D_N(x) = \big\{ y : d_N(x,y) < +\infty \big\},$$

$(\mathcal{K}, d_H)$ for the compact sets of (X, d) with the Hausdorff distance d_H and $(\mathcal{K}_N, d_{H,N})$ for the compact sets of (X, d_N) with the Hausdorff distance $d_{H,N}$. We recall that $(\mathcal{K}, d_H)$ is Polish.

Lemma 2.10. *If A is analytic in (X, d), then $\{x : d_N(A, x) < \varepsilon\}$ is analytic for all $\varepsilon > 0$.*

Proof. Observe that

$$\big\{ x : d_N(A,x) < \varepsilon \big\} = P_1 \Big(X \times A \cap \big\{ (x,y) : d_N(x,y) < \varepsilon \big\} \Big),$$

so that the conclusion follows from the invariance of the class Σ^1_1 w.r.t. projections. □

In particular, $\overline{A}^{d_N}$, the closure of A w.r.t. d_N, is analytic if A is analytic.

2.5. **General facts about optimal transportation.** Let (X, Ω, μ) and (Y, Σ, ν) be two probability spaces and $c : X \times Y \to \mathbb{R}^+$ be a $\Omega \times \Sigma$ measurable function. Consider the set of transference plans

$$\Pi(\mu, \nu) := \Big\{ \pi \in \mathcal{P}(X \times Y) : (P_1)_\sharp \pi = \mu, (P_2)_\sharp \pi = \nu \Big\},$$

where $P_i(x_1, x_2) = x_i, i = 1, 2$. Define the functional

$$\begin{array}{rcccl} \mathcal{I} & : & \Pi(\mu,\nu) & \longrightarrow & \mathbb{R}^+ \\ & & \pi & \longmapsto & \mathcal{I}(\pi) := \displaystyle\int_{X \times Y} c\pi. \end{array} \tag{2.5}$$

The *Monge-Kantorovich minimization problem* is the minimization of $\mathcal{I}$ over all transference plans.

If we consider a map $T : X \to Y$ such that $T_\sharp \mu = \nu$, the functional (2.5) becomes

$$\mathcal{I}(T) := \mathcal{I}((Id \times T)_\sharp \mu) = \int_X c(x, T(x)) \mu(dx).$$

The minimization problem over all T is called *Monge minimization problem.*

The Kantorovich problem admits a (pre) dual formulation: before stating it, we introduce two definitions.

Definition 2.11. A map $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is said to be *c-concave* if it is not identically $-\infty$ and there exists $\psi : Y \to \mathbb{R} \cup \{-\infty\}$, $\psi \not\equiv -\infty$, such that

$$\varphi(x) = \inf_{y \in Y} \{c(x, y) - \psi(y)\}.$$

The *c-transform* of φ is the function

$$\varphi^c(y) := \inf_{x \in X} \{c(x, y) - \varphi(x)\}. \tag{2.6}$$

The *c-superdifferential* $\partial^c\varphi$ of φ is the subset of $X \times Y$ defined by

$$(x, y) \in \partial^c\varphi \iff c(x, y) - \varphi(x) \leq c(z, y) - \varphi(z) \quad \forall z \in X. \tag{2.7}$$

Definition 2.12. A set $\Gamma \subset X \times Y$ is said to be *c-cyclically monotone* if, for any $n \in \mathbb{N}$ and for any family $(x_1, y_1), \ldots, (x_n, y_n)$ of points of Γ, the following inequality holds

$$\sum_{i=0}^{n} c(x_i, y_i) \leq \sum_{i=0}^{n} c(x_{i+1}, y_i),$$

with $x_{n+1} = x_1$. A transference plan is said to be *c-cyclically monotone* (or just *c-monotone*) if it is concentrated on a c-cyclically monotone set.

Consider the set

$$\Phi_c := \Big\{(\varphi, \psi) \in L^1(\mu) \times L^1(\nu) : \varphi(x) + \psi(y) \leq c(x, y)\Big\}. \tag{2.8}$$

Define for all $(\varphi, \psi) \in \Phi_c$ the functional

$$J(\varphi, \psi) := \int \varphi\mu + \int \psi\nu. \tag{2.9}$$

The following is a well known result (see Theorem 5.10 of [16]).

Theorem 2.13 (Kantorovich Duality)**.** *Let X and Y be Polish spaces, let $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$, and let $c : X \times Y \to \mathbb{R}^+ \cup \{+\infty\}$ be lower semicontinuous. Then the following holds:*

(1) *Kantorovich duality:*

$$\inf_{\pi \in \Pi[\mu,\nu]} \mathcal{I}(\pi) = \sup_{(\varphi,\psi) \in \Phi_c} J(\varphi, \psi).$$

Moreover, the infimum on the left-hand side is attained and the right-hand side is also equal to

$$\sup_{(\varphi,\psi) \in \Phi_c \cap C_b} J(\varphi, \psi),$$

where $C_b = C_b(X, \mathbb{R}) \times C_b(Y, \mathbb{R})$.

(2) *If c is real valued and the optimal cost*

$$C(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} I(\pi)$$

is finite, then there is a measurable c-cyclically monotone set $\Gamma \subset X \times Y$, closed if c is continuous, such that for any $\pi \in \Pi(\mu, \nu)$ the following statements are equivalent:

(a) *π is optimal;*

(b) *π is c-cyclically monotone;*

(c) *π is concentrated on Γ;*

(d) *there exists a c-concave function φ such that π-a.s. $\varphi(x) + \varphi^c(y) = c(x, y)$.*

(3) *If moreover*

$$c(x, y) \leq c_X(x) + c_Y(y), \qquad c_X \ \mu\text{-measurable}, \ c_Y \ \nu\text{-measurable},$$

then there exist a couple of potentials and the optimal transference plan π is concentrated on the set

$$\big\{(x, y) \in X \times Y \,|\, \varphi(x) + \psi(y) = c(x, y)\big\}.$$

Finally if $(c_X, c_Y) \in \mathcal{L}^1(\mu) \times \mathcal{L}^1(\nu)$ then the supremum is attained

$$\sup_{\Phi_c} J = J(\varphi, \varphi^c).$$

We recall also that if $-c$ is Souslin, then every optimal transference plan π is concentrated on a c-cyclically monotone set [3].

3. Optimal transportation in geodesic spaces

Let $\mu, \nu \in \mathcal{P}(X)$ and consider the transportation problem with cost $c(x,y) = d_N(x,y)$, and let $\pi \in \Pi(\mu,\nu)$ be a d_N-cyclically monotone transference plan with finite cost. By inner regularity, we can assume that the optimal transference plan is concentrated on a σ-compact d_N-cyclically monotone set $\Gamma \subset \{d_N(x,y) < +\infty\}$.

Consider the set

$$\Gamma' := \bigg\{ (x,y) : \exists I \in \mathbb{N}_0, (w_i, z_i) \in \Gamma \text{ for } i = 0, \dots, I, \ z_I = y$$

$$w_{I+1} = w_0 = x, \ \sum_{i=0}^{I} d_N(w_{i+1}, z_i) - d_N(w_i, z_i) = 0 \bigg\}. \tag{3.1}$$

In other words, we concatenate points $(x,z), (w,y) \in \Gamma$ if they are initial and final point of a cycle with total cost 0.

Lemma 3.1. *The following holds:*

(1) $\Gamma \subset \Gamma' \subset \{d_N(x,y) < +\infty\}$;
(2) if Γ is analytic, so is Γ';
(3) if Γ is d_N-cyclically monotone, so is Γ'.

Proof. For the first point, set $I = 0$ and $(w_{n,0}, z_{n,0}) = (x,y)$ for the first inclusion. If $d_N(x,y) = +\infty$, then $(x,y) \notin \Gamma$ and all finite set of points in Γ are bounded.

For the second point, observe that

$$\begin{aligned}
\Gamma' &= \bigcup_{I \in \mathbb{N}_0} P_{1,2I+1}(A_I) \\
&= \bigcup_{I \in \mathbb{N}_0} P_{1,2I+1}\bigg(\prod_{i=0}^{I} \Gamma \cap \bigg\{ \prod_{i=0}^{I} (w_i, z_i) : \sum_{i=0}^{I} d_N(w_{i+1}, z_i) - d_N(w_i, z_i) = 0, w_{I+1} = w_0 \bigg\}\bigg).
\end{aligned}$$

For each $I \in \mathbb{N}_0$, since d_N is Borel, it follows that

$$\bigg\{ \prod_{i=0}^{I} (w_i, z_i) : \sum_{i=0}^{I} d_N(w_{i+1}, z_i) - d_N(w_i, z_i) = 0, w_{I+1} = w_0 \bigg\}$$

is Borel in $\prod_{i=0}^{I}(X \times X)$, so that for Γ analytic each set $A_{n,I}$ is analytic. Hence $P_{1,2I+1}(A_I)$ is analytic, and since the class Σ^1_1 is closed under countable unions and intersections it follows that Γ' is analytic.

For the third point, observe that for all $(x_j, y_j) \in \Gamma'$, $j = 0, \dots, J$, there are $(w_{j,i}, z_{j,i}) \in \Gamma$, $i = 0, \dots, I_j$, such that

$$d_N(x_j, y_j) + \sum_{i=0}^{I_j - 1} d_N(w_{j,i+1}, z_{j,i}) - \sum_{i=0}^{I_j} d_N(w_{j,i}, z_{j,i}) = 0.$$

Hence we can write for $x_{J+1} = x_0$, $w_{j,I_j+1} = w_{j+1,0}$, $w_{J+1,0} = w_{0,0}$

$$\sum_{j=0}^{J} d_N(x_{j+1}, y_j) - d_N(x_j, y_j) = \sum_{j=0}^{J} \sum_{i=0}^{I_j} d_N(w_{j,i+1}, z_{j,i}) - d_N(w_{j,i}, z_{j,i}) \geq 0,$$

using the d_N-cyclical monotonicity of Γ. □

Definition 3.2 (Transport rays)**.** Define the *set of oriented transport rays*

$$G := \Big\{(x,y) : \exists (w,z) \in \Gamma', d_N(w,x) + d_N(x,y) + d_N(y,z) = d_N(w,z)\Big\}. \tag{3.2}$$

For $x \in X$, the *outgoing transport rays from* x is the set $G(x)$ and the *incoming transport rays in* x is the set $G^{-1}(x)$. Define the *set of transport rays* as the set

$$R := G \cup G^{-1}. \tag{3.3}$$

The set G is the set of all couples of points on oriented geodesics with endpoints in Γ'. In R the couples are non oriented.

Lemma 3.3. *The following holds:*

(1) G is d_N-cyclically monotone;
(2) $\Gamma' \subset G \subset \{d_N(x,y) < +\infty\}$;
(3) the sets G, $R := G \cup G^{-1}$ are analytic.

Proof. The second point follows by the definition: if $(x,y) \in \Gamma'$, just take $(w,z) = (x,y)$ in the r.h.s. of (3.2).

The third point is consequence of the fact that

$$G = P_{34}\Big(\big(\Gamma' \times X \times X\big) \cap \Big\{(w,z,x,y) : d_N(w,x) + d_N(x,y) + d_N(y,z) = d_N(w,z)\Big\}\Big),$$

and the result follows from the properties of analytic sets.

The first point follows from the following observation: if $(x_i,y_i) \in \gamma_{[w_i,z_i]}$, then from triangle inequality

$$\begin{aligned} d_N(x_{i+1},y_i) - d_N(x_i,y_i) + d_N(x_i,y_{i-1}) &\geq d_N(x_{i+1},z_i) - d_N(z_i,y_i) - d_N(x_i,y_i) + d_N(x_i,y_{i-1}) \\ &= d_N(x_{i+1},z_i) - d_N(x_i,z_i) + d_N(x_i,y_{i-1}) \\ &\geq d_N(x_{i+1},z_i) - d_N(x_i,z_i) + d_N(w_i,y_{i-1}) - d_N(w_i,x_i) \\ &= d_N(x_{i+1},z_i) - d_N(w_i,z_i) + d_N(w_i,y_{i-1}). \end{aligned}$$

Repeating the above inequality finitely many times one obtain

$$\sum_i d_N(x_{i+1},y_i) - d_N(x_i,y_i) \geq \sum_i d_N(w_{i+1},z_i) - d_N(w_i,z_i) \geq 0.$$

Hence the set G is d_N-cyclically monotone. □

Definition 3.4. Define the *transport sets*

$$\mathcal{T} := P_1\big(G^{-1} \setminus \{x=y\}\big) \cap P_1\big(G \setminus \{x=y\}\big), \tag{3.4a}$$

$$\mathcal{T}_e := P_1\big(G^{-1} \setminus \{x=y\}\big) \cup P_1\big(G \setminus \{x=y\}\big). \tag{3.4b}$$

Since G and G^{-1} are analytic sets, $\mathcal{T}$, $\mathcal{T}_e$ are analytic. The subscript e refers to the endpoints of the geodesics: clearly we have

$$\mathcal{T}_e = P_1(R \setminus \{x=y\}). \tag{3.5}$$

The following lemma shows that we have only to study the Monge problem in $\mathcal{T}_e$.

Lemma 3.5. *It holds $\pi(\mathcal{T}_e \times \mathcal{T}_e \cup \{x=y\}) = 1$.*

Proof. If $x \in P_1(\Gamma \setminus \{x=y\})$, then $x \in G^{-1}(y) \setminus \{y\}$. Similarly, $y \in P_2(\Gamma \setminus \{x=y\})$ implies that $y \in G(x) \setminus \{x\}$. Hence $\Gamma \setminus \mathcal{T}_e \times \mathcal{T}_e \subset \{x=y\}$. □

As a consequence, $\mu(\mathcal{T}_e) = \nu(\mathcal{T}_e)$ and any maps T such that for $\nu \llcorner_{\mathcal{T}_e} = T_\sharp \mu \llcorner_{\mathcal{T}_e}$ can be extended to a map T' such that $\nu = T_\sharp \mu$ with the same cost by setting

$$T'(x) = \begin{cases} T(x) & x \in \mathcal{T}_e \\ x & x \notin \mathcal{T}_e \end{cases} \tag{3.6}$$

Definition 3.6. Define the multivalued *endpoint graphs* by:

$$a := \big\{(x,y) \in G^{-1} : G^{-1}(y) \setminus \{y\} = \emptyset\big\}, \tag{3.7a}$$

$$b := \big\{(x,y) \in G : G(y) \setminus \{y\} = \emptyset\big\}. \tag{3.7b}$$

We call $P_2(a)$ the set of *initial points* and $P_2(b)$ the set of *final points*.

Proposition 3.7. *The following holds:*

1. *the sets*
$$a, b \subset X \times X, \quad a(A), b(A) \subset X,$$
belong to the $\mathcal{A}$-class if A analytic;
2. $a \cap b \cap \mathcal{T}_e \times X = \emptyset$*;*
3. $a(\mathcal{T}) = a(\mathcal{T}_e)$, $b(\mathcal{T}) = b(\mathcal{T}_e)$*;*
4. $\mathcal{T}_e = \mathcal{T} \cup a(\mathcal{T}) \cup b(\mathcal{T})$, $\mathcal{T} \cap (a(\mathcal{T}) \cup b(\mathcal{T})) = \emptyset$.

Proof. Define

$$C := \big\{(x,y,z) \in \mathcal{T}_e \times \mathcal{T}_e \times \mathcal{T}_e : y \in G(x), z \in G(y)\big\} = (G \times X) \cap (X \times G) \cap \mathcal{T}_e \times \mathcal{T}_e \times \mathcal{T}_e,$$

that is clearly analytic. Then

$$b = \big\{(x,y) \in G : y \in G(x), G(y) \setminus \{y\} = \emptyset\big\} = G \setminus P_{12}(C \setminus X \times \{y = z\}),$$

$$b(A) = \big\{y : y \in G(x), G(y) \setminus \{y\} = \emptyset, x \in A\big\} = P_2(G \cap A \times X) \setminus P_2(C \setminus X \times \{y = z\}).$$

A similar computation holds for a:

$$a = G^{-1} \setminus P_{23}(C \setminus \{x = y\} \times X), \quad a(A) = P_1(G_S \cap X \times A) \setminus P_1(C \setminus \{x = y\} \times X).$$

Hence $a, b \in \mathcal{A}(X \times X)$, $a(A), b(A) \in \mathcal{A}(X)$, being the intersection of an analytic set with a coanalytic one. If $x \in \mathcal{T}_e \setminus \mathcal{T}$, then it follows that $G(x) = \{x\}$ or $G^{-1}(x) = \{x\}$ hence $x \in a(x) \cup b(x)$.

The other points follow easily. □

Definition 3.8 (Chain of transport rays)**.** Define the set of *chain of transport rays*

$$H := \bigg\{(x,y) \in \mathcal{T}_e \times \mathcal{T}_e : \exists I \in \mathbb{N}_0, z_i \in \mathcal{T} \text{ for } 1 \leq i \leq I, \\ (z_i, z_{i+1}) \in R,\ 0 \leq i \leq I+1,\ z_0 = x,\ z_{I+1} = y\bigg\}. \tag{3.8}$$

Using similar techniques of Lemma 3.1 it can be shown that H is analytic.

Proposition 3.9. *The set $H \cap \mathcal{T} \times \mathcal{T}$ is an equivalence relation on $\mathcal{T}$. The set G is a partial order relation on $\mathcal{T}_e$.*

Proof. Using the definition of H, one has in $\mathcal{T}$:

1. $x \in \mathcal{T}$ clearly implies that $(x,x) \in H$;
2. since R is symmetric, if $y \in H(x)$ then $x \in H(y)$;
3. if $y \in H(x)$, $z \in H(y)$, $x, y, z \in \mathcal{T}$. Glue the path from x to y to the one from y to z. Since $y \in \mathcal{T}$, $z \in H(x)$.

The second part follows similarly:

1. $x \in \mathcal{T}_e$ implies that
$$\exists (x,y) \in \big(G \setminus \{x = y\}\big) \cup \big(G^{-1} \setminus \{x = y\}\big),$$
so that in both cases $(x,x) \in G$;
2. $(x,y), (y,z) \in G \setminus \{x = y\}$ implies by d_N-cyclical monotonicity that $(x,z) \in G$.

□

We finally show that we can assume that the μ-measure of final points and the ν-measure of the initial points are 0.

Lemma 3.10. *The sets $G \cap b(\mathcal{T}) \times X$, $G \cap X \times a(\mathcal{T})$ is a subset of the graph of the identity map.*

Proof. From the definition of b one has that

$$x \in b(\mathcal{T}) \quad \Longrightarrow \quad G(x) \setminus \{x\} = \emptyset,$$

A similar computation holds for a. □

Hence we conclude that

$$\pi(b(\mathcal{T}) \times X) = \pi(G \cap b(\mathcal{T}) \times X) = \pi(\{x = y\})$$

and following (3.6) we can assume that

$$\mu(b(\mathcal{T})) = \nu(a(\mathcal{T})) = 0.$$

4. Partition of the transport set

To perform a disintegration we have to assume some regularity of the support Γ of the transport plan $\pi \in \Pi(\mu, \nu)$. From now on we will assume the following:

(1) for all $x \in \mathcal{T}$ and for all $r > 0$ the set $H(x) \cap \overline{B_{r,N}(x)}^{d_N}$ is d-closed;
(2) for all $x \in \mathcal{T}$ there exists $r > 0$ such that $d_N(x, \cdot)_{\llcorner H(x) \cap \overline{B_r(x)}}$ is bounded.

Let $\{x_i\}_{i \in \mathbb{N}}$ be a dense sequence in (X, d).

Lemma 4.1. *The sets*

$$W_{ijk} := \Big\{ x \in \mathcal{T} \cap \bar{B}_{2^{-j}}(x_i) : d_N(x, \cdot)_{\llcorner H(x) \cap \bar{B}_{2^{-j}}(x_i)} \leq k \Big\}$$

form a countable covering of $\mathcal{T}$ of class $\mathcal{A}$.

Proof. We first prove the measurability. We consider separately the conditions defining W_{ijk}.

Point 1. The set

$$A_{ij} := \mathcal{T} \cap \bar{B}_{2^{-j}}(x_i)$$

is clearly analytic.

Point 2. The set

$$D_{ijk} := \Big\{ (x, y) \in H : d(x_i, y) \leq 2^{-j}, d_N(x, y) > k \Big\}$$

is again analytic. We finally can write

$$W_{ijk} = A_{ij} \cap P_1(D_{ijk})^c,$$

and the fact that $\mathcal{A}$ is a σ-algebra proves that $W_{ijk} \in \mathcal{A}$.

To show that it is a covering, notice that for all $x \in \mathcal{T}$ there exists $r > 0$ such that, on the set $H(x) \cap \bar{B}_r(x)$, $d_N(x, \cdot)$ is bounded. Choose j and i such that $2^{-j-1} \leq r$ and $d(x_i, x) \leq 2^{-j-1}$, hence

$$\bar{B}_{2^{-j}}(x_i) \subset \bar{B}_r(x)$$

and therefore for some $\bar{k} \in \mathbb{N}$ we obtain that $x \in W_{ijk}$. □

Remark 4.2. Observe that $\bar{B}_{2^{-j}}(x_i) \cap H(x)$ is closed for all $x \in W_{ijk}$.

Indeed take $\{y_n\}_{n \in \mathbb{N}} \subset \bar{B}_{2^{-j}}(x_i) \cap H(x)$ with $d(y_n, y) \to 0$ as $n \to +\infty$, then since $x \in W_{ijk}$ it holds $d_N(x, y_n) \leq k$. By Assumption (1) above, $d_N(x, y) \leq k$ and $y \in \bar{B}_{2^{-j}}(x_i) \cap H(x)$.

Lemma 4.3. *There exist μ-negligible sets $N_{ijk} \subset W_{ijk}$ such that the family of sets*

$$\mathcal{T}_{ijk} = H^{-1}(W_{ijk} \setminus N_{ijk}) \cap \mathcal{T}$$

is a countable covering of $\mathcal{T} \setminus \cup_{ijk} N_{ijk}$ into saturated analytic sets.

Proof. First of all, since $W_{ijk} \in \mathcal{A}$, then there exists μ-negligible set $N_{ijk} \subset W_{ijk}$ such that $W_{ijk} \setminus N_{ijk} \in \mathcal{B}(X)$. Hence $\{W_{ijk} \setminus N_{ijk}\}_{i,j,k \in \mathbb{N}}$ is a countable covering of $\mathcal{T} \setminus \cup_{ijk} N_{ijk}$. It follows immediately that $\{\mathcal{T}_{ijk}\}_{i,j,k \in \mathbb{N}}$ satisfies the lemma. □

From any analytic countable covering, we can find a countable partition into $\mathcal{A}$-class saturated sets by defining

$$\mathcal{Z}_m := \mathcal{T}_{i_m j_m k_m} \setminus \bigcup_{m'=1}^{m-1} \mathcal{T}_{i_{m'} j_{m'} k_{m'}}, \tag{4.1}$$

where

$$\mathbb{N} \ni m \mapsto (i_m, j_m, k_m) \in \mathbb{N}^3$$

is a bijective map. Since H is an equivalence relation on $\mathcal{T}$, we use this partition to prove the strong consistency.

On $\mathcal{Z}_m$, $m > 0$, we define the closed valued map

$$\mathcal{Z}_m \ni x \mapsto F(x) := H(x) \cap \bar{B}_{2^{-j_m}}(x_{i_m}). \tag{4.2}$$

Proposition 4.4. *There exists a μ-measurable cross section $f : \mathcal{T} \to \mathcal{T}$ for the equivalence relation H.*

Proof. First we show that F is $\mathcal{A}$-measurable: for $\delta > 0$,

$$\begin{aligned} F^{-1}(B_\delta(y)) &= \Big\{x \in \mathcal{Z}_m : H(x) \cap B_\delta(y) \cap \bar{B}_{2^{-j_m}}(x_{i_m}) \neq \emptyset\Big\} \\ &= \mathcal{Z}_m \cap P_1\Big(H \cap \big(X \times B_\delta(y) \cap \bar{B}_{2^{-j_m}}(x_{i_m})\big)\Big). \end{aligned}$$

Being the intersection of two $\mathcal{A}$-class sets, $F^{-1}(B_\delta(y))$ is in $\mathcal{A}$. In Remark 4.2 we have observed that F is a closed-valued map, hence, from Lemma 5.1.4 of [12], graph(F) is $\mathcal{A}$-measurable.

By Corollary 2.7 there exists a $\mathcal{A}$-class section $f_m : \mathcal{Z}_m \to \bar{B}_{2^{-j_m}}(x_{i_m})$. The proposition follows by setting $f \llcorner_{\mathcal{Z}_m} = f_m$ on $\cup_m \mathcal{Z}_m$, and defining it arbitrarily on $\mathcal{T} \setminus \cup_m \mathcal{Z}_m$: the latter being μ-negligible, f is μ-measurable. □

Up to a μ-negligible saturated set $\mathcal{T}_N$, we can assume it to have σ-compact range: just let $S \subset f(\mathcal{T})$ be a σ-compact set where $f_\sharp \mu$ is concentrated, and set

$$\mathcal{T}_S := H^{-1}(S) \cap \mathcal{T}, \quad \mathcal{T}_N := \mathcal{T} \setminus \mathcal{T}_S, \quad \mu(\mathcal{T}_N) = 0. \tag{4.3}$$

Hence we have a measurable cross-section

$$\mathcal{S} := S \cup f(\mathcal{T}_N) = (\text{Borel}) \cup (f(\mu\text{-negligible})).$$

Hence Disintegration Theorem 2.3 yields

$$\mu \llcorner_{\mathcal{T}} = \int_S \mu_y m(dy), \quad m = f_\sharp \mu \llcorner_{\mathcal{T}}, \ \mu_y \in \mathcal{P}(H(y)) \tag{4.4}$$

and the disintegration is strongly consistent since the quotient map $f : \mathcal{T} \to \mathcal{T}$ is μ-measurable and $(\mathcal{T}, \mathcal{B}(\mathcal{T}))$ is countably generated.

Observe that H induces an equivalence relation also on $\mathcal{T} \times X \cap \Gamma$ where the equivalence classes are $H(y) \cap \mathcal{T} \times X$ and the quotient map is the f of Proposition 4.4. Hence

$$\pi \llcorner_{\mathcal{T} \times X \cap \Gamma} = \int_S \pi_y m_\pi(dy), \quad m_\pi = f_\sharp \pi \llcorner_{\mathcal{T} \times X \cap \Gamma}, \ \pi_y \in \mathcal{P}(H(y) \cap \mathcal{T} \times X). \tag{4.5}$$

Observe that $m = m_\pi$.

5. Regularity of the disintegration

In this Section we consider the translation of Borel sets by the optimal geodesic flow, we introduce the fundamental regularity assumption (Assumption 1) on the measure μ and we show that an immediate consequence is that the set of initial points is negligible and consequently we obtain a disintegration of μ on the whole space. A second consequence is that the disintegration of μ w.r.t. the H has continuous conditions probabilities.

5.1. **Evolution of Borel sets.** Let $A \subset \mathcal{T}_e$ be an analytic set and define for $t \in \mathbb{R}$ the *t-evolution A_t of A* by:

$$A_t := \begin{cases} P_2\big\{(x,y) \in G \cap A \times X : d_N(x,y) = t\big\} & t \geq 0 \\ P_2\big\{(x,y) \in G^{-1} \cap A \times X : d_N(x,y) = t\big\} & t < 0. \end{cases} \tag{5.1}$$

It is clear from the definition that if A is analytic, also A_t is analytic . We can show that $t \mapsto \mu(A_t)$ is measurable.

Lemma 5.1. *Let A be analytic. The function $t \mapsto \mu(A_t)$ is $\mathcal{A}$-measurable for $t \in \mathbb{R}$.*

Proof. We divide the proof in three steps.

Step 1. Define the subset of $X \times \mathbb{R}$

$$\hat{A} := \big\{(x,t) : x \in A_t\big\}.$$

Note that

$$\hat{A} = P_{13}\Big\{(x,y,t) \in X \times X \times \mathbb{R}^+ : (x,y) \in G \cap A \times X, d_N(x,y) = t\Big\}$$
$$\cup P_{13}\Big\{(x,y,t) \in X \times X \times \mathbb{R}^- : (x,y) \in G^{-1} \cap A \times X, d_N(x,y) = -t\Big\},$$

hence it is analytic. Clearly $A_t = \hat{A}(t)$.

Step 2. Define the closed set in $\mathcal{P}(X \times [0,1])$

$$\Pi(\mu) := \big\{\pi \in \mathcal{P}(X \times [0,1]) : (P_1)_\sharp(\pi) = \mu\big\}$$

and let $B \subset X \times \mathbb{R} \times [0,1]$ be a Borel set such that $P_{12}(B) = \hat{A}$.

Consider the function

$$\mathbb{R} \times \Pi(\mu) \ni (t,\pi) \mapsto \pi(B(t)).$$

A slight modification of Lemma 4.12 in [3] shows that this function is Borel.

Step 3. Since supremum of Borel function are $\mathcal{A}$-measurable, pag. 134 of [12], the proof is concluded once we show that

$$\mu(A_t) = \mu(\hat{A}(t)) = \sup_{\pi \in \Pi(\mu)} \pi(B(t)).$$

Since $\hat{A}(t) \times [0,1] \supset B(t)$

$$\mu(\hat{A}(t)) = \pi(\hat{A}(t) \times [0,1]) \geq \pi(B(t)).$$

On the other hand from Theorem 2.5, there exists an $\mathcal{A}$-measurable section of the analytic set $B(t)$, so we have $u : \hat{A}(t) \to B(t)$. Clearly for $\pi_u = (\mathbb{I}, u)_\sharp(\mu)$ it holds $\pi_u(B(t)) = \mu(\hat{A}(t))$. $\square$

The next assumption is the fundamental assumption of the paper.

Assumption 1 (Non-degeneracy assumption)**.** For each analytic set $A \subset \mathcal{T}_e$ such that $\mu(A) > 0$ there exists a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and a strictly positive constant C such that $t_n \to 0$ as $n \to +\infty$ and $\mu(A_{t_n}) \geq C\mu(A)$.

Clearly it is enough to verify Assumption 1 for A compact set. An immediate consequence of the Assumption 1 is that the measure μ is concentrated on $\mathcal{T}$.

Proposition 5.2. *If μ satisfies Assumption 1 then*

$$\mu(\mathcal{T}_e \setminus \mathcal{T}) = 0.$$

Proof. Let $A = \mathcal{T}_e \setminus \mathcal{T}$. Suppose by contradiction $\mu(A) > 0$. By the inner regularity there exists $\hat{A} \subset A$ closed with $\mu(\hat{A}) > 0$. By Assumption 1 there exist $C > 0$ and $\{t_n\}_{n\in\mathbb{N}}$ converging to 0 such that $\mu(\hat{A}_{t_n}) \geq C\mu(\hat{A})$.

Define $\hat{A}^\varepsilon := \big\{x : d_N(\hat{A}, x) < \varepsilon\big\}$. Since $\hat{A} \subset A$, for all $n \in \mathbb{N}$ it holds $\hat{A}_{t_n} \cap A = \emptyset$. Moreover for $t_n \leq \varepsilon$ we have $\hat{A}^\varepsilon \supset \hat{A}_{t_n}$. So we have

$$\mu(\hat{A}) = \lim_{\varepsilon \to 0} \mu(\hat{A}^\varepsilon) \geq \mu(\hat{A}) + \mu(\hat{A}_{t_n}) \geq (1 + C)\mu(\hat{A}),$$

that gives the contradiction. $\square$

Once we know that $\mu(\mathcal{T}) = 1$, we can use the Disintegration Theorem 2.3 to write

$$\mu = \int_S \mu_y m(dy), \quad m = f_\sharp \mu, \ \mu_y \in \mathcal{P}(H(y)). \tag{5.2}$$

The disintegration is strongly consistent since the quotient map $f : \mathcal{T} \to \mathcal{T}$ is μ-measurable and $(\mathcal{T}, \mathcal{B}(\mathcal{T}))$ is countably generated.

The second consequence of Assumption 1 is that μ_y is continuous, i.e. $\mu_y(\{x\}) = 0$ for all $x \in X$.

Proposition 5.3. *If μ satisfies Assumption 1 then the conditional probabilities μ_y are continuous for m-a.e. $y \in S$.*

Proof. From the regularity of the disintegration and the fact that $m(S) = 1$, we can assume that the map $y \mapsto \mu_y$ is weakly continuous on a compact set $K \subset S$ of comeasure $< \varepsilon$. It is enough to prove the proposition on K.

Step 1. From the continuity of $K \ni y \mapsto \mu_y \in \mathcal{P}(X)$ w.r.t. the weak topology, it follows that the map

$$y \mapsto A(y) := \big\{x \in H(y) : \mu_y(\{x\}) > 0\big\} = \cup_n \big\{x \in H(y) : \mu_y(\{x\}) \geq 2^{-n}\big\}$$

is σ-closed: in fact, if $(y_m, x_m) \to (y, x)$ and $\mu_{y_m}(\{x_m\}) \geq 2^{-n}$, then $\mu_y(\{x\}) \geq 2^{-n}$ by u.s.c. on compact sets. Hence A is Borel.

Step 2. The claim is equivalent to $\mu(P_2(A)) = 0$. Suppose by contradiction $\mu(P_2(A)) > 0$. By Lusin Theorem (Theorem 5.8.11 of [12]) A is the countable union of Borel graphs. Therefore we can take a Borel selection of A just considering one of the Borel graphs, say $\hat{A}$. Clearly $m(P_1(\hat{A})) > 0$ and therefore by (5.2) $\mu(P_2(\hat{A})) > 0$. By Assumption 1 $\mu((P_2(\hat{A}))_{t_n}) \geq C\mu(P_2(\hat{A}))$ for some $C > 0$ and $t_n \to 0$. From $(P_2(\hat{A}))_{t_n} \cap (P_2(\hat{A}))_{t_m} = \emptyset$ we have a contradiction with the fact that the measure is finite. □

6. Solution to the Monge problem

Throughout the section we assume μ to satisfies Assumption 1. It follows from Disintegration Theorem 2.3, Proposition 5.2 and Proposition 5.3 that

$$\mu = \int \mu_y m(dy), \ \pi = \int \pi_y m(dy), \quad \mu_y \text{ continuous}, \ (P_1)_\sharp \pi_y = \mu_y,$$

where $m = f_\sharp \mu$ and $\mu_y \in \mathcal{P}(H(y))$. We write moreover

$$\nu = \int \nu_y m(dy) = \int (P_2)_\sharp \pi_y m(dy).$$

Note that $\pi_y \in \Pi(\mu_y, \nu_y)$ is d_N-cyclically monotone (and since $d_N \llcorner_{H(y)\times H(y)}$ is finite, also optimal) for m-a.e. y. If $\nu(\mathcal{T}) = 1$, then the above formula is the disintegration of ν w.r.t. H.

Theorem 6.1. *Let $\pi \in \Pi(\mu, \nu)$ be d_N-monotone such that for all $y \in S$ there exists an optimal map T_y from μ_y to ν_y. Then there exists a μ-measurable map $T : X \to X$ with the same transference cost of π.*

Proof. The idea is to use Theorem 2.5. Recall $S \subset \mathcal{T}$ introduced in (4.3). *Step 1.* Let $\mathbf{T} \subset S \times \mathcal{P}(X^2)$ be the set: for $y \in S$, $\mathbf{T}_y$ is the family of optimal transference plans in $\Pi(\tilde{\mu}_y, \tilde{\nu}_y)$ concentrated on a graph,

$$\mathbf{T} = \Big\{(y, \pi) \in S \times \mathcal{P}(X^2) : \pi \in \Pi(\mu_y, \nu_y) \text{ optimal}, \exists T : X \to X, \pi(\text{graph}(T)) = 1\Big\}.$$

where for optimal in $\Pi(\mu_y, \nu_y)$ we mean

$$\int d_N \pi = \min_{\pi \in \Pi(\mu_y, \nu_y)} \int d_N \pi.$$

Note that, since π is a Borel measure, in the definition of $\mathbf{T}$, T can be taken Borel. Moreover the y section $\mathbf{T}_y = \mathbf{T} \cap \{y\} \times \mathcal{P}(X^2)$ is not empty.

Step 2. Since the projection is a continuous map, then the set

$$\tilde{\Pi} = \Big\{(y, \pi) : (P_1)_\sharp \pi = \mu_y, (P_2)_\sharp \pi = \nu_y\Big\}$$

is a Borel subsets of $S \times \mathcal{P}(X^2)$: in fact it is the counter-image of the Borel set graph$((\mu_y, \nu_y)) \subset S \times \mathcal{P}(X)^2$ w.r.t. the weakly continuous map $(y, \pi) \mapsto (y, (P_1)_\sharp \pi, (P_2)_\sharp \pi)$.

Define the Borel function

$$S \times \mathcal{P}(X^2) \ni (y,\pi) \mapsto f(y,\pi) := \begin{cases} \int d_N \pi & \pi \in \Pi(\mu_y,\nu_y) \\ +\infty & \text{otherwise} \end{cases}$$

It follows that $y \mapsto g(y) := \inf_\pi f(y,\pi)$ is an $\mathcal{A}$-function: we can redefine it on a m-negligible set to make it Borel, where m is the quotient measure of μ. Hence the set

$$\tilde{\Pi}^{\mathrm{opt}} = \left\{ (y,\pi) : \pi \in \tilde{\Pi}(\mu_y,\nu_y), \int d_N \pi \leq g(y) \right\} = \tilde{\Pi} \cap \left\{ (y,\pi) : \int d_N \pi \leq g(y) \right\}$$

is Borel.concentrated

Step 3. Now we show that the set of $\pi \in \mathcal{P}(X^2)$ concentrated on a graph is analytic. By Borel Isomorphism Theorem, see [12] page 99, it is enough to prove the same statement for $\pi \in \mathcal{P}([0,1])^2$. Consider the function

$$\mathcal{P}([0,1]^2) \times C_b([0,1],[0,1]) \ni (\pi,\phi) \mapsto h(\pi,\phi) := \pi(\mathrm{graph}(\phi)) \in [0,1].$$

Since $\mathrm{graph}(\phi)$ is compact, h is u.s.c.. Hence the set $B^n = h^{-1}([1-2^{-n},1])$ is closed, so that

$$\mathscr{T} = \bigcap_n P_1(B^n) = \left\{ \pi : \forall \varepsilon > 0 \ \exists \phi_\varepsilon, \pi(\phi_\varepsilon) > 1 - \varepsilon \right\}$$

is an analytic set. It is easy to prove that $\pi \in \mathscr{T}$ iff π is concentrated on a graph.

Step 4. It follows that

$$\mathbf{T} = S \times \mathscr{T} \cap \tilde{\Pi}^{\mathrm{opt}}$$

is analytic and by Theorem 2.5 there exists a m-measurable selection $y \mapsto \pi_y \in \mathbf{T}_y$. It is fairly easy to prove that $\int \pi_y m(dy)$ is an optimal map concentrated on a graph. □

It follows from Theorem 6.1 that it is enough to solve for each $y \in S$ the Monge minimization problem with marginal μ_y and ν_y on the set $H(y)$. In order to solve it, we introduce an assumption on the geometry of the set $H(y)$.

Assumption 2. For a given $y \in S$, $H(y)$ satisfies Assumption 2 if there exist two families of disjoint $\mathcal{A}$-measurable sets $\{K_t\}_{t\in[0,1]}$ and $\{Q_s\}_{s\in[0,1]}$ such that

- $\mu_y(H(y) \setminus \cup_{t\in[0,1]} K_t) = \nu_y(H(y) \setminus \cup_{s\in[0,1]} Q_s) = 0$;
- the associated quotient maps φ_K and φ_Q are respectively μ_y-measurable and ν_y-measurable;
- for $t \leq s$, $K_t \times Q_s$ is d_N-cyclically monotone.

In the measurability condition of Assumption 2 the set $[0,1]$ is equipped with the Borel σ-algebra $\mathcal{B}([0,1])$. If $H(y)$ satisfies Assumption 2 we can disintegrate the marginal measures μ_y and ν_y respectively w.r.t. the family $\{K_t\}$ and $\{Q_s\}$:

$$\mu_y = \int \mu_{y,t} m_{\mu_y}(dt), \quad \nu_y = \int \nu_{y,t} m_{\nu_y}(dt)$$

where $m_{\mu_y} = \varphi_{K\,\sharp}\mu_y$, $m_{\nu_y} = \varphi_{Q\,\sharp}\nu_y$ and the disintegrations are strongly consistent.

Proposition 6.2. *Suppose that $H(y)$ satisfies Assumption 2 and that the following conditions hold true:*

- m_{μ_y} *is continuous;*
- $\mu_{y,t}$ *is continuous for* m_{μ_y}*-a.e.* $t \in [0,1]$*;*
- $m_{\mu_y}([0,t]) \geq m_{\nu_y}([0,t])$ *for* m_{μ_y}*-a.e.* $t \in [0,1]$*.*

Then there exists a d_N-cyclically monotone μ_y-measurable map T_y such that $T_{y\,\sharp}\mu_y = \nu_y$.

Proof. Step 1. Since m_{μ_y} is continuous and $m_{\mu_y}([0,t]) \geq m_{\nu_y}([0,t])$, there exists an increasing map $\psi : [0,1] \to [0,1]$ such that $\psi_\sharp m_{\mu_y} = m_{\nu_y}$.

Moreover, since for m_{μ_y}-a.e. $t \in [0,1]$ $\mu_{y,t}$ is continuous, there exists a Borel map $T_t : K_t \to Q_{\psi(t)}$ such that $T_{t\,\sharp}\mu_{y,t} = \nu_{y,\psi(t)}$ for m_{μ_y}-a.e. $t \in [0,1]$. Since $\psi(t) \geq t$ the map T_t is d_N-cyclically monotone, hence optimal between $\mu_{y,t}$ and $\nu_{y,t}$.

Step 2. Reasoning as in the proof of Theorem 6.1, one can prove the existence of a μ_y-measurable map $T : H(y) \to H(y)$ that is the gluing of all the maps T_t constructed in *Step 1.*. Hence there exists a μ_y-measurable map $T : H(y) \to H(y)$ such that $T_\sharp \mu_{y,t} = \nu_{y,\psi(t)}$. It follows that T is d_N-cyclically monotone and

$$T_\sharp \mu_y = \int T_\sharp \mu_{y,t} m_{\mu_y}(dt) = \int \nu_{y,\psi(t)} m_{\mu_y}(dt) = \int \nu_{y,t} (\psi_\sharp m_{\mu_y})(dt) = \nu_y.$$

□

Corollary 6.3. *Let $\pi \in \Pi(\mu,\nu)$ be d_N-monotone plan. Assume that for m-a.e. $y \in S$ the set $H(y)$ satisfies Assumption 2 and for m-a.e. $y \in S$ the hypothesis of Proposition 6.2 are verified. Then there exists an admissible Borel map T with d_N-cyclically monotone graph and same transference cost of π.*

The hypothesis of Proposition 6.2 and Assumption 2 are justified by the last part of this Section that is devoted to an example.

6.1. **Example.** We conclude this Section with the analysis of a particular case in which the set $H(y)$ satisfies Assumption 2. Fix the following notation: a continuous curve $\gamma : [0,1] \to X$ is *increasing* if for $t, s \in [0,1]$

$$t \le s \Longrightarrow (\gamma(t), \gamma(s)) \in G$$

Definition 6.4 (Hourglass sets)**.** For $z \in X$ define the *hourglass set*

$$K(z) := \Big\{ (x,y) \in X \times X : (x,z),(z,y) \in G \Big\}.$$

Assume that there exists an increasing curve γ such that

$$H(y) \times X \cap \Gamma \subset \bigcup_{t\in[0,1]} K(\gamma(t)) \cap \Gamma,$$

then $H(y)$ satisfies Assumption 2. Indeed first notice that $K(z)$ is analytic, then define the family of sets

$$K_t := G^{-1}(\gamma(t)) \setminus \bigcap_{s<t} G^{-1}(\gamma(s)), \qquad Q_t := G(\gamma(t)) \setminus \bigcap_{t<s} G(\gamma(s)).$$

Since γ is increasing, K_t and Q_s are $\mathcal{A}$-measurable and the quotient maps are $\mathcal{A}$-measurable: let $[a,b] \subset [0,1]$

$$\varphi_K^{-1}([a,b]) = \bigcup_{t\in[a,b]} K_t = G^{-1}(\gamma(b)) \setminus G^{-1}(\gamma(a)) \cup K_a \; \in \mathcal{A}$$

and the same calculation holds true for φ_Q. The d_N-cyclical monotonicity follows directly from the increasing property of γ.

Observe moreover that in this framework it is also verified that $m_{\mu_y}([0,t]) \ge m_{\nu_y}([0,t])$.

7. An application

Throughout this section $|\cdot|$ will be the euclidean distance of $\mathbb{R}^d$.

Let $C \subset \mathbb{R}^d$ be an open convex set such that $M := \partial C$ is a smooth compact sub-manifold of $\mathbb{R}^d$ of dimension $d-1$. Hence $\mathbb{R}^d \setminus M$ is disconnected with two open connected components, say A_1 and C. Let $C_1 := \mathrm{cl}\, A_1$.

Consider the following geodesic distance: $d_M : \mathbb{R}^d \times \mathbb{R}^d \to [0,+\infty]$

$$d_M(x,y) := \begin{cases} \inf\{L(\gamma) : \gamma \in \mathrm{Lip}([0,1],C_1), \gamma(0)=x, \gamma(1)=y\}, & x,y \in C_1 \\ d(x,y), & x,y \in C \\ +\infty, & \text{elsewhere} \end{cases} \tag{7.1}$$

where L is the standard euclidean arc-length: $L(\gamma) = \int |\dot\gamma|$. Hence M can be seen as an obstacle for geodesics connecting points in C_1.

Since any minimizing sequence has uniformly bounded Lipschitz constant, in (7.1) we can substitute inf with min, hence d_M is a geodesic distance.

We will show that the Monge minimization problem with marginals $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, $\mu \ll \mathcal{L}^d$ and geodesic cost d_M admits a solution. From now on $\mu \ll \mathcal{L}^d$. Clearly we have to assume that μ and ν are concentrated on the same connected component otherwise all transference plans have infinite transference cost. Hence $\mu(C_1) = \nu(C_1) = 1$. Since $\mathcal{L}^d(M) = 0$, it follows that $\mu(A_1) = \mu(C_1) = 1$. From now on all the sets and structures introduced during the paper, will be referred to this Monge problem.

The strategy to solve the Monge minimization problem is the one used in Section 6: build a d_M-monotone map on each equivalence class $H(y)$ and then use Theorem 6.1.

Lemma 7.1. *The distance $d_M \llcorner C_1 \times C_1$ is a continuous map.*

Proof. Step 1. Let $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \in C_1$ such that $|x_n - x| \to 0$ $|y_n - y| \to 0$. Since the boundary of C_1 is a smooth manifold, for every $n \in \mathbb{N}$ there exist curves $\gamma_{1,n}, \gamma_{2,n} \in \mathrm{Lip}([0,1], C_1)$ such that

- $\gamma_{1,n}(0) = x, \gamma_{1,n}(1) = x_n$;
- $\gamma_{2,n}(0) = y, \gamma_{2,n}(1) = y_n$;
- $L(\gamma_{i,n}) \to 0$ as $n \to +\infty$, for $i = 1, 2$.

Consider $\gamma_n \in \mathrm{Lip}([0,1], C_1)$ such that $\gamma_n(0) = x_n, \gamma_n(1) = y_n$ and $L(\gamma_n) \le d_M(x_n, y_n) + 2^{-n}$. Gluing $\gamma_{1,n}$ and $\gamma_{2,n}$ to γ_n it follows

$$d_M(x,y) \le d_M(x_n,y_n) + 2^{-n} + L(\gamma_{1,n}) + L(\gamma_{2,n}).$$

Hence d_M is l.s.c..

Step 2. Taking a minimizing sequence of admissible curves for $d_M(x,y)$ and gluing them with $\gamma_{i,n}$ as in *Step 1.*, it is fairly easy to prove that d_M is u.s.c. and therefore continuous. □

As a corollary we have the existence of an optimal transference plan π. Hence from now on π will be an optimal transference plan and all the structures defined during the paper starting from a generic d_N-monotone plan, are referred to it. Moreover there exists $\varphi \in \mathrm{Lip}_{d_M}(C_1, \mathbb{R})$ such that $\Gamma = \Gamma' = G = \{(x,y) \in C_1 \times C_1 : \varphi(x) - \varphi(y) = d_M(x,y)\}$. Note that Γ is closed.

In the following Lemma we prove that the problem can be reduced to the equivalence classes $H(y)$. We use the following notation: the quotient map induced by H will be denoted by f^y and the corresponding quotient measure $f^y_\sharp \mu$ by m_H.

Lemma 7.2. *The μ-measure of the set of initial points is zero, hence*

$$\mu = \int \mu_y m_H(dy).$$

Moreover μ_y is continuous for m_H-a.e. y.

Proof. Since $\mu \ll \mathcal{L}^d$, w.l.o.g. we can prove the statement for $\mathcal{L}^d$ restricted to any set of finite measure.

The property requested in Assumption 1 is straightforward for $\mathcal{L}^d$ w.r.t. the flow induced by a $|\cdot|$-cyclically monotone set, see for example [4]. Moreover it is clear that Assumption 1 prescribes just a local regularity for the map $t \mapsto \mu(A_t)$. Hence if we show that for subsets of A_1 the evolution by a d_M-cyclically monotone set is locally induced by a $|\cdot|$-monotone set, then the claim follows.

Let $K \subset A_1$ compact set with $\mathcal{L}^d(K) > 0$. Possibly intersecting K with $B_r(x)$ for some $x \in A_1$ and $r > 0$, we can assume that $K \subset B_\varepsilon(x)$ and $B_{2\varepsilon}(x) \cap M = \emptyset$. Since $d_M \ge d$, $K_t \subset B_{2\varepsilon}(x)$ for all $t \le \varepsilon$. Hence the set $G \cap K \times K_\varepsilon$ is d-cyclically monotone, therefore Assumption 1 holds for $\mathcal{L}^d$ restricted to any set of finite d-dimensional Lebesgue measure. □

Hence we can assume w.l.o.g. that $\mu(G^{-1}(M)) = \nu(G(M)) = 1$: if $H(y)$ do not intersect the obstacle, it is a straight line and the marginal μ_y is continuous. Since the existence of an optimal transport map on a straight line with first marginal continuous is a standard fact in optimal transportation, the reduction follows.

The next result shows that, due to smoothness and convexity of the obstacle, the sets $H(y)$ have the structure of Example 6.1.

Lemma 7.3. *For all $y \in S$, $H(y)$ has the geometry of Example 6.1: there exists an increasing curve $\gamma_y : [0,1] \to X$ such that*

$$H(y) \times X \cap \Gamma \subset \bigcup_{t\in[0,1]} K(\gamma_y(t)) \cap \Gamma.$$

Proof. Since due to convexity and smoothness of the obstacle, the geodesics of d_M are smooth and composed by a first straight line, a geodesic of the manifold and a final straight line, a branching structure can appear only on the manifold M. If $H(y) \neq R(y)$, consider the following sets:

$$Z := \bigcap_{z\in H(y)\cap M} G^{-1}(z) \cap M, \quad W := \bigcap_{z\in H(y)\cap M} G(z) \cap M.$$

By d_M-monotonicity, smoothness and convexity of M, for all $z \in H(y) \cap M$ the set $G^{-1}(z) \cap M$ is always contained in the same geodesic of M. Using the compactness of M, $Z = \{z\}$ and $W = \{w\}$ and $(z,w) \in G$. Consider the unique increasing geodesic $\gamma_y \in \gamma_{[z,w]}$ such that $\gamma_y = G(z) \cap G^{-1}(w)$. Hence

$$H(y) \times X \cap \Gamma \subset \bigcup_{t\in[0,1]} K(\gamma_y(t)) \cap \Gamma.$$

□

Recall the two family of sets introduced in Example 6.1

$$K_{y,t} := G^{-1}(\gamma_y(t)) \setminus \bigcap_{s<t} G^{-1}(\gamma_y(s)), \qquad Q_{y,t} := G(\gamma_y(t)) \setminus \bigcap_{t<s} G(\gamma_y(s)).$$

It follows from Lemma 7.3 and Example 6.1 that

$$\mu_y = \int \mu_{y,t} m_{\mu_y}(dt), \quad \nu_y = \int \nu_{y,t} m_{\nu_y}(dt).$$

with $\mu_{y,t}(K_{y,t}) = \nu_{y,t}(Q_{y,t}) = 1$. Moreover using the increasing curve γ_y, we can assume that $m_{\mu_y} \in \mathcal{P}(M)$, indeed

$$\mu_y = \int_{[0,1]} \mu_{y,t} m_{\mu_y}(dt) = \int_{\gamma_y([0,1])} \mu_{y,\gamma_y^{-1}(z)} (\gamma_{y\,\sharp} m_{\mu_y})(dz). \tag{7.2}$$

And the same calculation holds true for ν_y and m_{ν_y}. Therefore in the following

$$\mu_y = \int_M \mu_{y,z} m_{\mu_y}(dz), \quad \nu_y = \int_M \nu_{y,z} m_{\nu_y}(dz) \tag{7.3}$$

with $\mu_{y,z}(K_{y,\gamma_y^{-1}(z)}) = \nu_{y,z}(Q_{y,\gamma_y^{-1}(z)}) = 1$ and $m_{\mu_y}(\gamma_y([0,1])) = m_{\nu_y}(\gamma_y([0,1])) = 1$.

Moreover w.l.o.g. we can assume that $S = f^y(\mathbb{R}^d) \subset M$, in particular we can assume that for all $y \in S$ there exists $t(y) \in [0,1]$ such that $y = \gamma_y(t(y))$.

According to Proposition 6.2, to obtain the existence of an optimal map on $H(y)$ it is enough to prove that m_{μ_y} is continuous and $\mu_{y,z}$ is continuous for m_{μ_y}-a.e. $z \in M$. Recall that $m_{\mu_y}(\gamma_y([0,t])) \geq m_{\nu_y}(\gamma_y([0,t]))$ is a straightforward consequence of the increasing property of γ_y.

Remark 7.4. Consider the following $\mathcal{A}$-measurable map:

$$G^{-1}(M) \setminus (a(M) \cap M) \ni w \mapsto f^M(w) := \mathrm{Argmin}\{d(z,w) : z \in M \cap G(w)\} \in M.$$

Consider the measure $m := f^M_\sharp \mu \in \mathcal{P}(M)$. Observing that $f^M(H(y)) = \gamma_y([0,1])$, it follows that the support of m is partitioned by a d_M-cyclically monotone equivalence relation:

$$m\Big(\bigcup_{y\in S} \gamma_y([0,1])\Big) = 1, \qquad \bigcup_{y\in S} \gamma_y([0,1]) \times \bigcup_{y\in S} \gamma_y([0,1]) \cap G \ \text{ is } d_M\text{-cyclically monotone}$$

Moreover f^y is a quotient map also for this equivalence relation. Note that $f^y_\sharp m = m_H$: consider $I \subset S$

$$\begin{aligned}(f^y_\sharp m)(I) &= m\Big(\bigcup_{y\in I} \gamma_y([0,1])\Big) = \mu\Big(G^{-1}\big(\bigcup_{y\in I} \gamma_y([0,1])\big)\Big) \\ &= \mu\Big(\bigcup_{y\in I} H(y)\Big) = (f^y_\sharp \mu)(I) = m_H(I).\end{aligned}$$

It follows that

$$m = \int_S (f^M_\sharp \mu_y) m_H(dy)$$

and from (7.3) $f^M_\sharp \mu_y = m_{\mu_y}$. Hence the final disintegration formula for m is the following one:

$$m = \int_S m_{\mu_y} m_H(dy) \tag{7.4}$$

Proposition 7.5. *The measure m is absolutely continuous w.r.t. the Hausdorff measure $\mathcal{H}^{d-1}$ restricted to M.*

Proof. Recall that $\varphi \in \mathrm{Lip}_{d_M}(\mathbb{R}^d)$ is the potential associated to Γ and consider the following set

$$M_2 := P_1\Big(\{(x,y) \in M \times M : |\varphi(x) - \varphi(y)| = d_M(x,y)\} \setminus \{x = y\}\Big).$$

Step 1. The function φ is a potential for any Monge minimization problem on M with cost the geodesic distance, that coincides with d_M, with first marginal m and as second marginal any probability measure supported on M_2. It follows from Proposition 15 of [9] that $\nabla\varphi$ is a Lipschitz function: for all $x, y \in M_2$

$$|\nabla\varphi(x) - \nabla\varphi(y)| \le L d_M(x,y).$$

In [9] the Lipschitz constant L is uniform for x, y belonging to sets uniformly far from the starting and ending points of the geodesics on M of the transport set. Since in our setting the geodesics on M do not intersect, L is uniform on the whole M. Moreover note that if $z = \gamma_y(t)$, then

$$\nabla\varphi(z) = -\frac{\dot\gamma_y(t)}{|\dot\gamma_y(t)|}.$$

Step 2. For $t \ge 0$, define the following map

$$M_2 \ni x \mapsto \psi_t(x) := x + \nabla\varphi(x)t.$$

Possibly restricting ψ_t to a subset of M of points coming from transport rays of uniformly positive length, since $t \mapsto \psi_t(x)$ is a parametrization of the transport ray touching M in x, by d_M-cyclical monotonicity of Γ, we can assume that ψ_t is injective. Moreover ψ_t is bi-lipschitz, provided t is small enough: indeed

$$|x + \nabla\varphi(x)t - y - \nabla\varphi(y)t| \ge |x - y|(1 - Lt).$$

It follows that

$$M_2 \times [-\delta, \delta] \ni (x,s) \mapsto \psi(x,s) := x + \nabla\varphi(x)(t+s)$$

is bi-Lipschitz and injective provided $\delta \le 1/L + t$. Hence the Jacobian determinant of φ, $Jd\varphi$, is uniformly positive.

Step 3. Consider the following set

$$B := \{x \in \mathbb{R}^d : t - \delta \le d(M,x) \le t + \delta\} \cap G^{-1}(M)$$

where d is the euclidean distance. Clearly B is the range of ψ and $\mathcal{L}^d(B) > 0$. Since M is a smooth manifold, we can pass to local charts: let $U_\alpha \subset \mathbb{R}^{d-1}$ be an open set and $h_\alpha : U_\alpha \to M$ the corresponding parametrization map. The map

$$U_\alpha \times [-\delta, \delta] \ni (x,s) \mapsto \psi_\alpha(x,s) := \psi(h_\alpha(x), s)$$

is a bi-Lipschitz parametrization of the set $B_\alpha := B \cap G^{-1}(h_\alpha(U_\alpha))$.

It follows directly from the Area Formula, see for example [1], that

$$\mathcal{L}^d \llcorner_{B_\alpha} = \psi_{\alpha\,\sharp}\Big(Jd\psi_\alpha(\mathcal{L}^{d-1} \times dt)\llcorner_{U_\alpha \times [-\delta,\delta]}\Big),$$

hence $f^M_\sharp \mathcal{L}^d \llcorner_{B_\alpha} \ll \mathcal{H}^{d-1}\llcorner_M$. Since B can be covered with a finite number of B_α and $\mathcal{L}^d\llcorner_{B_\alpha}$ is equivalent to m, the claim follows. □

Recall the following result. Let (M, g) be a n-dimensional compact Riemannian manifold, let d_M be the geodesic distance induced by g and η the volume measure. Then the disintegration of η w.r.t. any d_M-cyclically monotone set is strongly consistent and the conditional probabilities are continuous. This result is proved in [4], Theorem 9.5, in the more general setting of metric measure space satisfying the measure contraction property.

Corollary 7.6. *For m_H-a.s. $y \in S$, the quotient measure m_{μ_y} is continuous.*

Proof. We have proved in Remark 7.4 that the measures m_{μ_y} are the conditional probabilities of the disintegration of m w.r.t. the equivalence relation given by the membership to geodesics γ_y and m_H is the corresponding quotient measure. Hence the claim follows directly from Theorem 9.5 of [4] and Proposition 7.5. □

Proposition 7.7. *For m_H-a.e. $y \in S$, the measures $\mu_{y,z}$ are continuous for m_{μ_y}-a.e. $z \in M$.*

Proof. Recall that $f^M_\sharp \mu = m$.

Step 1. The measure μ can be disintegrated w.r.t. the partition given by the family of pre-images of the $\mathcal{A}$-measurable map f^M: $\{(f^M)^{-1}(p)\}_{p\in f^M(\mathbb{R}^d)}$. Clearly f^M is a possible quotient map, hence

$$\mu = \int \mu_z m(dz), \tag{7.5}$$

The set $G^{-1}(M)\setminus a(M) \times G^{-1}(M)\setminus a(M) \cap G$ is $|\cdot|$-cyclically monotone and $\mu \ll \mathcal{L}^d$, hence it follows that for m-a.e. $z \in f^M(\mathbb{R}^d)$, μ_z is continuous.

Step 2. From Lemma 7.2 $\mu = \int \mu_y m_H(dy)$, therefore

$$m = f^M_\sharp \mu = \int (f^M_\sharp \mu_y) m_H(dy),$$

hence using (7.5) and the uniqueness of the disintegration

$$\mu = \int \bigg(\int \mu_z (f^M_\sharp \mu_y)(dz) \bigg) m_H(dy), \qquad \mu_y = \int \mu_z (f^M_\sharp \mu_y)(dz),$$

where the last equality holds true for m_H-a.e. $y \in S$. Hence for m_H-a.e. $y \in S$ the measures $\mu_{y,z}$ are continuous for m_{μ_y}-a.e. $z \in M$. □

Finally we can prove the existence of an optimal map for the Monge problem with obstacle.

Theorem 7.8. *There exists a solution for the Monge minimization problem with cost d_M and marginal μ, ν with $\mu \ll \mathcal{L}^d$.*

Proof. From Lemma 7.2 it follows that μ can be disintegrated w.r.t. the equivalence relation H. From Theorem 6.1 it follows that to prove the claim it is enough to prove the existence of an optimal map on each equivalence class $H(y)$. Hence we restrict the analysis to the classes $H(y)$ such that $H(y) \neq R(y)$ and for them we proved in Lemma 7.3 that Assumption 2 holds true. In Proposition 7.5, Corollary 7.6 and Proposition 7.7 we proved that for m_H-a.e. $y \in S$ the measures m_{μ_y} and $\mu_{y,z}$ verify the hypothesis of Proposition 6.2. Therefore the claim follows. □

Acknowledgements

I would like to express my gratitude to Stefano Bianchini for his unfailing guidance and support throughout the preparation of this paper. I wish to thank Luigi De Pascale for drawing my attention on the obstacle problem.

Appendix A. Notation

$P_{i_1\dots i_I}$	projection of $x \in \Pi_{k=1,\dots,K} X_k$ into its $(i_1, \dots, i_I)$ coordinates, keeping order
$\mathcal{P}(X)$ or $\mathcal{P}(X,\Omega)$	probability measures on a measurable space (X,Ω)
$\mathcal{M}(X)$ or $\mathcal{M}(X,\Omega)$	signed measures on a measurable space (X,Ω)
$f\llcorner_A$	the restriction of the function f to A
$\mu\llcorner_A$	the restriction of the measure μ to the σ-algebra $A\cap\Sigma$
$\mathcal{L}^d$	Lebesgue measure on $\mathbb{R}^d$
$\mathcal{H}^k$	k-dimensional Hausdorff measure
$\Pi(\mu_1,\dots,\mu_I)$	$\pi \in \mathcal{P}(\Pi_{i=1}^I X_i, \otimes_{i=1}^I \Sigma_i)$ with marginals $(P_i)_\sharp \pi = \mu_i \in \mathcal{P}(X_i)$
$\mathcal{I}(\pi)$	cost functional (2.5)
c	cost function : $X \times Y \mapsto [0,+\infty]$
$\mathcal{I}$	transportation cost (2.5)
ϕ^c	c-transform of a function ϕ (2.6)
$\partial^c\varphi$	d-subdifferential of φ (2.7)
Φ_c	subset of $L^1(\mu)\times L^1(\nu)$ defined in (2.8)
$J(\phi,\psi)$	functional defined in (2.9)
C_b or $C_b(X,\mathbb{R})$	continuous bounded functions on a topological space X
(X,d)	Polish space
(X,d_L)	non-branching geodesic separable metric space
$D_N(x)$	the set $\{y : d_N(x,y) < +\infty\}$

$L(\gamma)$	length of the Lipschitz curve γ, Definition 2.8
$B_r(x)$	open ball of center x and radius r in (X, d)
$B_{r,L}(x)$	open ball of center x and radius r in (X, d_L)
$\mathcal{K}(X)$	space of compact subsets of X
$d_H(A, B)$	Hausdorff distance of A, B w.r.t. the distance d
A_x, A^y	x, y section of $A \subset X \times Y$ (2.3)
$\mathcal{B}$, $\mathcal{B}(X)$	Borel σ-algebra of X Polish
Σ^1_1, $\Sigma^1_1(X)$	the pointclass of analytic subsets of Polish space X, i.e. projection of Borel sets
Π^1_1	the pointclass of coanalytic sets, i.e. complementary of Σ^1_1
Σ^1_n, Π^1_n	the pointclass of projections of Π^1_{n-1}-sets, its complementary
Δ^1_n	the ambiguous class $\Sigma^1_n \cap \Pi^1_n$
$\mathcal{A}$	σ-algebra generated by Σ^1_1
$\mathcal{A}$-function	$f : X \to \mathbb{R}$ such that $f^{-1}((t, +\infty])$ belongs to $\mathcal{A}$
$h_\sharp\mu$	push forward of the measure μ through h, $h_\sharp\mu(A) = \mu(h^{-1}(A))$
$\mathrm{graph}(F)$	graph of a multifunction F (2.1)
F^{-1}	inverse image of multifunction F (2.2)
F_x, F^y	sections of the multifunction F (2.3)
$\mathrm{Lip}_1(X)$	Lipschitz functions with Lipschitz constant 1
Γ'	transport set (3.1)
G, G^{-1}	outgoing, incoming transport ray, Definition 3.2
R	set of transport rays (3.3)
$\mathcal{T}$, $\mathcal{T}_e$	transport sets (3.4)
$a, b : \mathcal{T}_e \to \mathcal{T}_e$	endpoint maps (3.7)
$\mathcal{Z}_{m,e}$, $\mathcal{Z}_m$	partition of the transport set Γ (4.1), (4.2)
$\mathcal{S}$	cross-section of $R \llcorner_{\mathcal{T}\times\mathcal{T}}$
A_t	evolution of $A \subset \mathcal{Z}_{k,i,j}$ along geodesics (5.1)

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