

# On the Space of Symmetric Operators with Multiple Ground States

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## Abstract

We study homological structure of the filtration of the space of self-adjoint operators by the multiplicity of the ground state. We consider only operators acting in a finite dimensional complex or real Hilbert space but infinite dimensional generalizations are easily guessed.

## 1 Introduction

This paper is dedicated to the memory of V. I. Arnold and is somehow inspired by his works [1], [2] (see also [5]). It opens a planned series of papers devoted to homological invariants of the families of quadratic forms and related geometric structures; Theorem 2 below forms a fundament of all further development.

In this paper we study the filtration of the space of self-adjoint operators by the multiplicity of the ground state. We restrict ourself to operators in a finite dimensional complex or real Hilbert space, but possible infinite dimensional generalizations are easily guessed.

Let  $\lambda_1(A) \leq \lambda_2(A) \leq \dots \leq \lambda_k(A) \leq \dots$  be the ordered eigenvalues of the operator  $A$ . The operators with the ground state of multiplicity at least  $k$  are characterized by the equation  $\lambda_1(A) = \lambda_k(A)$ . Theorem 1 describes the homotopy type of the space of nontrivial solutions to this equation, it appears to be the Thom space of certain vector bundle over a Grassmannian.

The growth of the multiplicity from  $k$  to  $k+1$  is realized by the intersection of the space of solutions to the equation  $\lambda_1(A) = \lambda_k(A)$  with the space of

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solutions to the equation  $\lambda_k(A) = \lambda_{k+1}(A)$ . We focus on the homological structure of this intersection procedure.

As often happens, it is more convenient to accept the dual viewpoint, i. e. to deal with the cohomology of the pair:  $(\mathbb{B}, \{A \in \mathbb{B} : \lambda_1(A) \neq \lambda_k(A)\})$  instead of the homology of the space  $\{A \in \mathbb{B} : \lambda_1(A) = \lambda_k(A)\}$ , where  $\mathbb{B}$  is the space of all self-adjoint operators. Then the intersections of cycles is substituted by the standard cohomological product.

The space of solutions to the equation  $\lambda_k(A) = \lambda_{k+1}(A)$  is a cycle of codimension 3 in the complex case and a cycle modulo 2 of codimension 2 in the real case. The dual object is a 3-dimensional cohomology class in the complex case and a 2-dimensional cohomology class modulo 2 in the real one; we mean the class of the pair  $(\mathbb{B}, \{A \in \mathbb{B} : \lambda_k(A) \neq \lambda_{k+1}(A)\})$ .

In both cases, we denote this cohomology class by  $\Gamma_k$  and study the map from the cohomology of the pair  $(\mathbb{B}, \{A \in \mathbb{B} : \lambda_1(A) \neq \lambda_k(A)\})$  to the cohomology of the pair  $(\mathbb{B}, \{A \in \mathbb{B} : \lambda_1(A) \neq \lambda_{k+1}(A)\})$  which sends any cohomology class to its product with  $\Gamma_k$ . The main result of the paper, Theorem 2 states that the sequence of these maps for  $k = 1, 2, \dots$  is an exact sequence.

Let us consider the simplest case of self-adjoint operators on  $\mathbb{C}^2$  or, in other words, of  $2 \times 2$  Hermitian matrices. The pair  $(\mathbb{B}, \{A \in \mathbb{B} : \lambda_1(A) \neq \lambda_k(A)\})$  equals  $(\mathbb{R}^4, \emptyset)$  for  $k = 1$  and  $(\mathbb{R}^4, \mathbb{R}^4 \setminus \mathbb{R})$  for  $k = 2$ . The exact sequence of Theorem 2 is reduced to the obvious sequence

$$0 \rightarrow H^*(\mathbb{R}^3) \rightarrow H^{*+3}(\mathbb{R}^3, \mathbb{R}^3 \setminus 0) \rightarrow 0.$$

The general multidimensional calculation is far from being trivial and has perhaps a fundamental nature as we hope to show in the forthcoming publications.

In the next section we introduce some notations and recall well-known facts on the spaces of self-adjoint operators. The Theorems 1 and 2 are formulated and proved in Section 3.

All pairs of topological spaces and their subspaces we deal with are homotopy equivalent to pairs of finite cell complexes and their subcomplexes. Let  $(M, X)$ ,  $(M, Y)$ ,  $(M, X \cup Y)$  be such pairs,  $\xi \in H^i(M, X)$ ,  $\eta \in H^j(M, Y)$ ; then  $\xi \smile \eta \in H^{i+j}(M, X \cup Y)$  is the cohomological product of  $\xi$  and  $\eta$ .

## 2 Preliminaries

We consider the spaces of self-adjoint operators on  $\mathbb{R}^n$  or  $\mathbb{C}^n$ . In both cases, given an operator  $A$  we denote by  $\lambda_1(A) \leq \dots \leq \lambda_n(A)$  its ordered eigenvalues. Let  $I$  be the unit operator and  $\alpha$  a positive real number. Obviously,  $\lambda_i(A \pm \alpha I) = \lambda_i(A) \pm \alpha$ ,  $\lambda_i(\alpha A) = \alpha \lambda_i(A)$ . Moreover,  $A \pm \alpha I$  and  $\alpha A$  have the same eigenvectors as  $A$ . It is convenient do not distinguish the operators obtained one from another by just described trivial transformations.

We denote by  $\mathbb{S}(\mathbb{R})$  (correspondingly by  $\mathbb{S}(\mathbb{C})$ ) the space of all non-scalar self-adjoint linear operators  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  (correspondingly  $A : \mathbb{C}^n \rightarrow \mathbb{C}^n$ ) factorized by the equivalence relation  $A \sim (\alpha A + \beta I)$ ,  $\forall \alpha > 0, \beta \in \mathbb{R}$ . Then  $\mathbb{S}(\mathbb{R})$  is homeomorphic to the sphere  $S^{\frac{n(n+1)}{2}-2}$  and  $\mathbb{S}(\mathbb{C})$  is homeomorphic to  $S^{n^2-2}$ . In what follows we often deal simultaneously with the real and Hermitian cases and simply omit the argument of  $\mathbb{S}$ . The join of  $\mathbb{S}$  and the origin is denoted by  $\mathbb{B}$ ; this is the ball of dimension  $\frac{n(n+1)}{2} - 1$  in the real case and the ball of dimension  $n^2 - 1$  in the Hermitian case.

We can substitute the factorization by the normalization and define  $\mathbb{S}$  as the space of self-adjoint operators  $A$  such that  $\sum_{i=1}^n \lambda_i(A) = 0$ ,  $\sum_{i=1}^n \lambda_i^2(A) = 1$ ; then  $\mathbb{B}$  is defined by the relations  $\sum_{i=1}^n \lambda_i(A) = 0$ ,  $\sum_{i=1}^n \lambda_i^2(A) \leq 1$ . Anyway, the normalization is sometimes less convenient than the factorization and we often use the same symbols for the equivalence classes and their representatives; this simplifies notations and does not lead to a confusion.

Now consider open subsets

$$\Sigma_{k,k+1} \doteq \{A \in \mathbb{S} : \lambda_k(A) \neq \lambda_{k+1}(A)\}.$$

The following facts are well-known:

**Proposition 1.**  $\mathbb{S} \setminus \Sigma_{k,k+1}$  is an algebraic subset of  $\mathbb{S}$  of codimension 2 in the real case and of codimension 3 in the Hermitian case. Singular locus of  $\mathbb{S} \setminus \Sigma_{k,k+1}$  consists of the operators with at least triple eigenvalue  $\lambda_k$ ; it is an algebraic subset of  $\mathbb{S}$  of codimension 5 in the real case and of codimension 8 in the Hermitian case. Moreover, regular part of  $\mathbb{S} \setminus \Sigma_{k,k+1}$  is orientable in the Hermitian case.

**Sketch of the proof.** Let  $A_0 \in \mathbb{S} \setminus \Sigma_{k,k+1}$  and  $J_{A_0}$  be the set of all  $j \in \{1, \dots, n\}$  such that  $\lambda_j(A_0) = \lambda_k(A_0)$ ; then  $\#J_{A_0} \geq 2$ . Given a self-adjoint operator  $A$ , we set  $K_A = \text{span}\{x \in X : Ax = \lambda_j(A)x, j \in J_{A_0}\}$ , where  $X$  is  $\mathbb{R}^n$  in the real case and  $X$  is  $\mathbb{C}^n$  in the Hermitian case.

Let  $\mathcal{O}_0$  be a neighborhood of  $A_0$  in  $\mathbb{S}$  such that  $K_A \cap K_{A_0}^\perp = 0$ ,  $\forall A \in \mathcal{O}_0$ . We denote by  $P_{A_0 A}^k : K_A \rightarrow K_{A_0}$  the restriction to  $K_A$  of the orthogonal projector  $X \rightarrow K_{A_0}$  and set  $\Phi(A) = P_{A_0 A} A P_{A_0 A}^{-1}$ ,  $A \in \mathcal{O}_0$ . Then  $\Phi$  is a well-defined rational map from  $\mathcal{O}_0$  to the space of self-adjoint operators on  $K_{A_0}$ . The differential of  $\Phi$  at  $A_0$  sends  $A$  to the composition of  $A|_{K_{A_0}}$  with the orthogonal projection  $X \rightarrow K_{A_0}$ . In particular,  $D_{A_0} \Phi$  is surjective; hence  $\Phi$  is a submersion on a neighborhood of  $A_0$ . We may assume that  $\Phi$  is a submersion on the whole  $\mathcal{O}_0$ . Moreover,  $\lambda_i(\Phi(A)) = \lambda_{i+j_0}(A)$ ,  $i = 1, \dots, \#J_{A_0}$ , where  $j_0 = \min J_{A_0}$ .

Let  $A \in \mathcal{O}_0$ ; we obtain that  $J_A = J_{A_0}$  if and only if  $\Phi(A)$  is a scalar operator. Hence  $\{A \in \mathcal{O}_0 : J_A = J_{A_0}\}$  is a regular algebraic subset of  $\mathcal{O}_0$  of codimension  $\frac{j_0(j_0+1)}{2} - 1$  in the real case and  $j_0^2 - 1$  in the Hermitian case.

It remains to prove the orientability in the Hermitian case. It is sufficient to show that the space of self-adjoint operators on  $K_A$ ,  $A \in \mathbb{S}$  has a canonical orientation. The orientation of the space of self-adjoint operators is induced by the orientation of the space  $K_A$  itself, and the orientation of  $K_A \subset \mathbb{C}^n$  is defined by the complex structure (any complex space has a canonical orientation).  $\square$

Proposition 1 implies that  $H_{\dim \mathbb{S}-2}(\mathbb{S} \setminus \Sigma_{k,k+1}; \mathbb{Z}_2) = \mathbb{Z}_2$  in the real case and  $H_{\dim \mathbb{S}-3}(\mathbb{S} \setminus \Sigma_{k,k+1}; \mathbb{Z}) = \mathbb{Z}$  in the Hermitian case. According to the Alexander duality,  $H^1(\Sigma_{k,k+1}; \mathbb{Z}_2) = \mathbb{Z}_2$  in the real case and the generator of  $H^1(\Sigma_{k,k+1}; \mathbb{Z}_2)$  applied to a closed curve in  $\Sigma_{k,k+1}$  equals the linking number of the curve and  $\mathbb{S} \setminus \Sigma_{k,k+1}$  modulo 2. Similarly,  $H^2(\Sigma_{k,k+1}; \mathbb{Z}) = \mathbb{Z}$  in the Hermitian case and the generator of  $H^1(\Sigma_{k,k+1}; \mathbb{Z})$  applied to a compact oriented surface in  $\Sigma_{k,k+1}$  equals the linking number of the surface and  $\mathbb{S} \setminus \Sigma_{k,k+1}$ .

Further in this paper we always consider homology and cohomology with coefficients in  $\mathbb{Z}_2$  in the real case and with coefficients in  $\mathbb{Z}$  in the Hermitian case, and we omit the indication of coefficients in order to simplify notations. We also denote by  $\varepsilon$  the codimension of  $\mathbb{S} \setminus \Sigma_{k,k+1}$  in  $\mathbb{S}$ ; then  $\varepsilon = 2$  in the real case and  $\varepsilon = 3$  in the Hermitian case.

Given  $A \in \Sigma_{k,k+1}$ , we set

$$E_A^k = \text{span}\{x \in X : Ax = \lambda_i(A)x, i = 1, \dots, k\},$$

where  $X = \mathbb{R}^n$  in the real case and  $X = \mathbb{C}^n$  in the complex case. Then  $\mathcal{E}^k = \{(x, A) : A \in \Sigma_{k,k+1}, x \in E_A^k\}$  is a  $k$ -dimensional vector subbundle of the trivial bundle  $X \times \Sigma_{k,k+1}$  over  $\Sigma_{k,k+1}$ . Let  $\gamma_k \in H^{\varepsilon-1}(\Sigma_{k,k+1})$  be the first

Stiefel–Whitney characteristic class of this bundle in the real case and the first Chern characteristic class in the Hermitian case.

**Proposition 2.**  $\gamma_k$  is a generator of  $H^{\varepsilon-1}(\Sigma_{k,k+1})$ .

**Proof.** We have to compute the characteristic classes of the restriction of the bundle  $\mathcal{E}^k \rightarrow \Sigma_{k,k+1}$  to a  $(\varepsilon - 1)$ -dimensional compact submanifold of  $\Sigma_{k,k+1}$  whose linking number with  $\mathbb{S} \setminus \Sigma_{k,k+1}$  equals  $\pm 1$ .

We denote by  $\mathbb{S}_2$  the space of self-adjoint operators  $B$  on  $\mathbb{R}^2$  (in the real case) or on  $\mathbb{C}^2$  (in the Hermitian case) such that  $\lambda_1(B) + \lambda_2(B) = 0$ ,  $\lambda_1^2(B) + \lambda_2^2(B) = 1$  and set  $\mathbb{B}_2 = \text{conv}(\mathbb{S}_2)$ . Then  $\mathbb{S}_2$  is a  $(\varepsilon - 1)$ -dimensional sphere and  $\mathbb{B}_2$  is a  $\varepsilon$ -dimensional ball,  $\mathbb{S}_2 = \partial\mathbb{B}_2$ . Let  $A_-$  be a self-adjoint operator on  $\mathbb{R}^{k-1}$  (or on  $\mathbb{C}^{k-1}$ ) with simple eigenvalues such that  $\lambda_{k-1}(A_-) < -1$  and  $A_+$  be a self-adjoint operator on  $\mathbb{R}^{n-k-1}$  (or on  $\mathbb{C}^{n-k-1}$ ) with simple eigenvalues such that  $\lambda_1(A_+) > 1$ . Then  $A_- \oplus \mathbb{S}_2 \oplus A_+$  is the required  $(\varepsilon - 1)$ -dimensional submanifold of  $\mathbb{S}$ . Indeed,  $A_- \oplus \mathbb{S}_2 \oplus A_+ = \partial(A_- \oplus \mathbb{B}_2 \oplus A_+)$  and  $(A_- \oplus \mathbb{B}_2 \oplus A_+) \cap (\mathbb{S} \setminus \Sigma_{k,k+1}) = (A_- \oplus 0 \oplus A_+)$ ; moreover, the intersection is transversal. Hence the linking number of  $A_- \oplus \mathbb{S}_2 \oplus A_+$  and  $\mathbb{S} \setminus \Sigma_{k,k+1}$  equals  $\pm 1$ .

The restriction of the bundle  $\mathcal{E}^k \rightarrow \Sigma_{k,k+1}$  to  $A_- \oplus \mathbb{S}_2 \oplus A_+$  splits in the sum of a trivial vector bundle and a linear bundle over  $\mathbb{S}_2$  whose fiber at  $B \in \mathbb{S}_2$  is the eigenspace of the eigenvalue  $\lambda_1(B)$ . It is easy to see that the map sending  $B \in \mathbb{S}_2$  to the eigenspace of the eigenvalue  $\lambda_1(B)$  is the diffeomorphism of  $\mathbb{S}_2$  and the projective line (real or complex). This diffeomorphism identifies our linear bundle with the tautological bundle of the projective line.  $\square$

Let  $\delta : H^i(\Sigma_{k,k+1}) \rightarrow H^{i+1}(\mathbb{B}, \Sigma_{k,k+1})$  be the isomorphism induced by the exact cohomological sequence of the pair  $(\mathbb{B}, \Sigma_{k,k+1})$ . We set  $\Gamma_k = \delta \circ \gamma_k \in H^\varepsilon(\mathbb{B}, \Sigma_{k,k+1})$ . The value of  $\Gamma_k$  on a relative cycle  $\xi \in C_\varepsilon(\mathbb{B}, \Sigma_{k,k+1})$  is the intersection number of  $\xi$  and  $\text{conv}(\mathbb{S} \setminus \Sigma_{k,k+1})$ .

We conclude this section with an explicit expression for a closed two-form representing the class  $\gamma_k$  in the Hermitian case.

Let  $A$  be a self-adjoint operator with simple eigenvalues and  $e_1, \dots, e_n$  an orthonormal basis of its eigenvectors such that  $Ae_i = \lambda_i(A)e_i$ ,  $i = 1, \dots, n$ . Then  $e_i$  are defined up to a complex multiplier of the absolute value 1. Let  $\langle \cdot, \cdot \rangle$  denotes the Hermitian product and  $B$  be another self-adjoint operator. It is easy to see that the wedge square over  $\mathbb{R}$  of the complex number  $\langle Be_i, e_j \rangle$  depends only on  $A, B, i, j$  and not on the choice of the eigenvectors. In particular,  $\bigwedge_{\mathbb{R}}^2 \langle dA e_i, e_j \rangle$  is a well-defined two-form on the space of self-adjoint

operators with simple eigenvalues. We have:  $\bigwedge_{\mathbb{R}}^2 \langle dA e_i, e_j \rangle \left( \frac{\partial}{\partial B_1}, \frac{\partial}{\partial B_2} \right) = \det_{\mathbb{R}} (\langle B_1 e_i, e_j \rangle, \langle B_2 e_i, e_j \rangle)$ , where complex numbers are treated as vectors in  $\mathbb{R}^2$ .

**Proposition 3.** *The form  $\Omega_k = \sum_{i=1}^k \sum_{j=k+1}^n \frac{1}{\pi(\lambda_i - \lambda_j)^2} \bigwedge_{\mathbb{R}}^2 \langle dA e_i, e_j \rangle$  represents the restriction of  $\gamma_k$  to the space of self-adjoint operators with simple eigenvalues<sup>1</sup>.*

**Sketch of the proof.** We have to demonstrate that  $\Omega_k$  represents the first Chern class of the vector bundle  $\mathcal{E}^k$  restricted to the space of self-adjoint operators with simple eigenvalues. This restriction of  $\mathcal{E}^k$  is the direct sum of line bundles  $\mathcal{L}^i$ ,  $i = 1, \dots, k$ , where the fiber of  $\mathcal{L}^i$  at  $A$  is the line  $L_A^i \doteq \{z \in \mathbb{C}^n : Az = \lambda_i(A)z\}$ . We have to show that  $\Omega_k$  represents the class  $\sum_{i=1}^k c_1(\mathcal{L}^i)$ .

Consider the associated with  $\mathcal{L}^i$  principal  $S^1$ -bundle  $\mathcal{C}^i$  whose fiber at  $A$  is  $C_A^i \doteq \{e_i \in \mathbb{C}^n : e_i \in L_A^i, |e_i| = 1\}$ . Given a smooth curve  $t \mapsto A(t)$  in the space of self-adjoint operators with simple eigenvalues and  $e_i(0) \in C_{A(0)}^i$ , the condition  $\langle \dot{e}_i(t), e_i(t) \rangle = 0$  defines a canonical lift  $t \mapsto e_i(t)$  of the curve  $A(\cdot)$  to the bundle  $\mathcal{C}^i$ . These lifts are the parallel translations for a connection on the bundle  $\mathcal{C}^i$  along curves in the base space of the bundle. The form of this connection equals  $\Im \langle de_i, e_i \rangle$ , where  $\Im$  denotes the imaginary part of a complex number.

The exterior differential of the form  $\Im \langle de_i, e_i \rangle$  is the pullback of the curvature form  $R_i$  of the connection. An immediate calculation shows that  $R_i \left( \frac{\partial}{\partial B_1}, \frac{\partial}{\partial B_2} \right) \Big|_A = 2\Im \left\langle \frac{\partial e_i(A)}{\partial B_2}, \frac{\partial e_i(A)}{\partial B_1} \right\rangle$ , where  $\frac{\partial e_i(A)}{\partial B} = \frac{d}{dt} e_i(A + tB) \Big|_{t=0}$  and  $t \mapsto e_i(A + tB)$  is parallel along the curve  $t \mapsto A + tB$ . The differentiation by  $t$  of the equation  $\langle (A + tB)e_i(A + tB), e_j(A) \rangle = \lambda_i(A + tB) \langle e_i(A + tB), e_j(A) \rangle$  gives:  $\left\langle \frac{\partial e_i(A)}{\partial B}, e_j(A) \right\rangle = \frac{1}{\lambda_i(A) - \lambda_j(A)} \langle B e_i(A), e_j(A) \rangle$ ,  $\forall j \neq i$ . Hence  $\frac{\partial e_i}{\partial B} = \sum_{i \neq j} \frac{1}{\lambda_i - \lambda_j} \langle B e_i, e_j \rangle e_j$  and  $R_i \left( \frac{\partial}{\partial B_1}, \frac{\partial}{\partial B_2} \right) = \sum_{j \neq i} \frac{2}{(\lambda_i - \lambda_j)^2} \Im (\langle B_2 e_i, e_j \rangle \langle B_1 e_i, e_j \rangle)$ .

On the other hand  $\Im(z^1 z^2) = \det_{\mathbb{R}}(z^2, z^1)$  for any complex numbers  $z^1, z^2$ . We obtain:  $R_i = \sum_{j \neq i} \frac{2}{(\lambda_i - \lambda_j)^2} \bigwedge_{\mathbb{R}}^2 \langle dA e_i, e_j \rangle$ . Summing up, we get the desired

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<sup>1</sup>The form  $\Omega_k$  is locally bounded in the topology of  $\Sigma_{k,k+1}$ . Moreover, any two-dimensional cycle in  $\Sigma_{k,k+1}$  is homotopic to a cycle in the space of self-adjoint operators with simple eigenvalues. Hence the form  $\Omega_k$  indeed represents  $\gamma_k$ .

expression for the form  $\Omega_k = \frac{1}{2\pi} \sum_{i=1}^k R_i$  representing the class  $\gamma_k$ .  $\square$

### 3 Main Results

We are going to study the filtrations

$$M_k = \{A \in \mathbb{S} : \lambda_1(A) = \lambda_{k+1}(A)\}, \quad M^k = \mathbb{S} \setminus M_k, \quad k = 0, \dots, n-1.$$

It is easy to see that

$$W_k \doteq \{A \in \mathbb{S} : \lambda_{k+1}(A) = \lambda_n(A)\} \subset M^k$$

is deformation retract of  $M^k$ . The retraction  $\phi_k : M^k \rightarrow W_k$  changes only eigenvalues of the operators while the eigenvectors are kept fixed.

The involution  $A \mapsto (-A)$ ,  $A \in \mathbb{S}$ , transforms  $W_k$  into  $M_{n-k-1}$ ; hence  $M^k$  is homotopy equivalent to  $M_{n-k-1}$ . Note also that the map  $A \mapsto A - \lambda_1(A)I$ ,  $A \in \mathbb{S}$ , induces homeomorphism of  $M_k$  and the space of nonzero nonnegative self-adjoint operators of rank  $< n - k$  factorized by the equivalence relation  $A \sim \alpha A$ ,  $\forall \alpha > 0$ .

In what follows,  $Gr_k(m)$  is the Grassmannian of  $k$ -dimensional subspaces of  $\mathbb{R}^m$  or  $\mathbb{C}^m$ .

**Theorem 1.**  *$M^k$  has homotopy type of the Thom space of a real vector bundle over the Grassmannian  $Gr_k(n-1)$ ; the dimension of the bundle equals  $\frac{k(k+1)}{2} + k - 1$  in the real case and  $k^2 + 2k - 1$  in the Hermitian case.*

**Proof.** Let  $e$  be a unit length vector (in  $\mathbb{R}^n$  or  $\mathbb{C}^n$ ). We set

$$G_k(e) = \{A \in M_{k-1} \cap W_k : Ae = \lambda_n(A)e\}.$$

Then  $G_k(e) \cong Gr_k(n-1)$ . Moreover, a neighborhood of  $G_k(e)$  in  $W_k$  is a smooth manifold and the normal bundle of  $G_k(e)$  in this manifold has dimension  $\frac{k(k+1)}{2} + k - 1$  in the real case and dimension  $k^2 + 2k - 1$  in the Hermitian case. In fact, the normal bundle splits in the sum of two subbundles. The first one is the normal bundle of  $G_k(e)$  in  $\{A \in W_k : Ae = \lambda_n(A)e\}$ ; it is isomorphic to the bundle of the self-adjoint endomorphisms with zero trace of the tautological bundle of  $Gr_k(n-1)$ . The second one is the normal bundle of  $G_k(e)$  in  $M_{k-1} \cap W_k$ ; it is isomorphic to the normal subbundle of  $Gr_k(n-1)$  in  $Gr_k(n)$ . The theorem is an immediate corollary of the following

**Lemma 1.**  $W_k \setminus G_k(e)$  is a contractible space.

**Proof.** We'll contract  $W_k \setminus G_k(e)$  to the point  $-e^* \otimes e \in W_k \setminus G_k(e)$ . The contraction sends  $(A, t) \in (W_k \setminus G_k(e)) \times [0, 1]$  to  $\phi_k(A_t)$ , where  $A_t = (1-t)A - te^* \otimes e$ .

It remains to prove that the contraction is correctly defined, i.e. that  $A_t \in M^k$  and  $\phi_k(A_t)$  does not belong to  $G_k(e)$ ,  $\forall t \in [0, 1]$ . We have:  $\phi_k(A_t) \in G_k(e)$  if and only if  $A_t \in M_{k-1}$  and  $e$  is orthogonal to  $\{x \in X : A_t x = \lambda_1(A_t)x\}$ , where  $X$  is  $\mathbb{R}^n$  (in the real case) or  $\mathbb{C}^n$  (in the Hermitian case).

Set  $A^t = A - \frac{t}{1-t}e^* \otimes e = \frac{1}{1-t}A_t$ ; the positive multiplier does not influence the multiplicity of the eigenvalues and we will work with  $A^t$  instead of  $A_t$ . We consider separately two cases.

1.  $e \perp \{x \in X : Ax = \lambda_i(A)x, i = 1, \dots, k\}$ . Then  $e$  is an eigenvector of  $A$ ,  $Ae = \lambda_n(A)e$ . Hence all  $A^t$  have common eigenvectors. Moreover,  $(n-1)$  eigenvalues of  $A^t$  (counted with the multiplicities) are equal to eigenvalues of  $A$  while the eigenvalue corresponding to the eigenvector  $e$  is monotone decreasing from  $\lambda_n(A)$  to  $-\infty$  as  $t$  runs from 0 to 1. Besides that,  $\lambda_1(A) \neq \lambda_k(A)$  since  $A \notin G_k(e)$ ; hence  $A^t \in M^k$ .

The equality  $\lambda_1(A^t) = \lambda_k(A^t)$  is valid for some  $t \in (0, 1]$  if and only if  $\lambda_1(A) = \lambda_{k-1}(A)$ ; then  $A^t e = \lambda_1(A^t)e$  and  $\phi_k(A^t) \notin G_k(e)$ .

2.  $e \not\perp \{x \in X : Ax = \lambda_i(A)x, i = 1, \dots, k\}$ . The restriction of the quadratic form  $x \mapsto \langle A^t x, x \rangle$  to the hyperplane  $e^\perp$  does not depend on  $t$ . Hence the minimax principle for the eigenvalues implies:

$$\lambda_1(A^t) \leq \lambda_1(A) \leq \lambda_2(A^t) \leq \dots \leq \lambda_k(A) \leq \lambda_{k+1}(A^t).$$

Assume that  $\lambda_1(A^t) = \lambda_{k+1}(A^t)$ . Hence

$$\lambda_1(A) = \lambda_k(A) = \lambda_1(A^t) = \min_{|x|=1} \langle A^t x, x \rangle.$$

At the same time,

$$\langle A^t x, x \rangle = \lambda_1(A) - \frac{t}{1-t} \langle e, x \rangle^2, \quad \forall x \in \{x \in X : Ax = \lambda_1(A)x\}.$$

We obtain the contradiction with the assumption

$$e \not\perp \{x \in X : Ax = \lambda_1(A)x\} = \{x \in X : Ax = \lambda_i(A)x, i = 1, \dots, k\}.$$

Hence  $A_t \in M^k$ .



Now assume that  $\lambda_1(A^t) = \lambda_k(A^t)$  and  $e \perp \{x \in X : A^t x = \lambda_1(A^t)x\}$ . Then

$$\begin{aligned}\lambda_1(A^t)|x|^2 &= \langle A^t x, x \rangle = \langle Ax, x \rangle = \lambda_1(A)|x|^2, \\ \forall x \in \{x \in X : A^t x &= \lambda_i(A^t)x, i = 1, \dots, k\}.\end{aligned}$$

Hence  $\lambda_1(A) = \lambda_k(A)$  and

$$\begin{aligned}\{x \in X : A^t x &= \lambda_i(A^t)x, i = 1, \dots, k\} = \\ \{x \in X : Ax &= \lambda_i(A)x, i = 1, \dots, k\}.\end{aligned}$$

We obtain the contradiction with the assumption **2**.  $\square$

Let  $u_k \in H^{\nu_k}(M^k)$  be the Thom class of the normal bundle of  $G_k(e)$  in  $W_k$ ,  $\nu_k = \frac{k(k+1)}{2} + k - 1$  in the real case and  $\nu_k = k^2 + 2k - 1$  in the Hermitian case. Let  $\mathcal{G}_k$  be the total space of this bundle and  $G_k(e) \subset \mathcal{G}_k$  its zero section. We have:

$$\tilde{H}^*(M^k) = H^*(\mathcal{G}_k, \mathcal{G}_k \setminus G_k(e)), \quad H^*(G_k(e)) = H^*(\mathcal{G}_k)$$

and the cohomology product of the classes from  $\tilde{H}^*(M^k)$  and  $H^*(G_k(e))$  is a well-defined class in  $\tilde{H}^*(M^k)$ . Then  $\xi \mapsto u_k \smile \xi$ ,  $\xi \in H^*(G_k(e))$ , is the Thom isomorphism of  $H^*(G_k(e))$  and  $\tilde{H}^*(M^k)$ . Recall that  $u_k|_{G_k(e)} \in H^{\nu_k}(G_k(e))$  is the Euler class of the bundle  $\mathcal{G}_k \rightarrow G_k(e)$ .

**Lemma 2.**  $u_k|_{G_k(e)} = 0$ .

**Proof.** The bundle  $\mathcal{G}_k$  splits in the sum of two subbundles as it was explained in the proof of Theorem 1. We'll prove that the first subbundle, i. e. the bundle of self-adjoint endomorphisms with zero trace of the tautological bundle of the Grassmannian has zero Euler class. It is sufficient to show that the induced bundle over the flag space has zero Euler class. This is easy. Indeed, the bundle over the flag space has natural non-vanishing sections: the value of such a section at a flag is the self-adjoint operator with prescribed simple eigenvalues whose eigenspaces are the elements of the flag.  $\square$

**Corollary 1.** *The cohomology product of any two elements of  $\tilde{H}^*(M^k)$  is zero.*

**Proof.** Due to the Thom isomorphism, it is sufficient to show that  $u_k \smile u_k = 0$ , but  $u_k \smile u_k$  is the image of  $u_k|_{Gr_k(e)} = 0$  under the Thom isomorphism.  $\square$

Obviously,  $M^k = M^{k-1} \cup \Sigma_{k,k+1}$ . We consider the homomorphisms

$$\mathbf{d}_k : H(\mathbb{B}, M^{k-1}) \rightarrow H(\mathbb{B}, M^k), \quad k = 1, \dots, n-1,$$

acting by multiplication with the class  $\Gamma_k \in H^\varepsilon(\mathbb{B}, \Sigma_{k,k+1})$  defined at the end of Section 2:

$$\mathbf{d}_k(\xi) = \Gamma_k \smile \xi, \quad \xi \in H(\mathbb{B}, M^{k-1}).$$

Recall that  $\varepsilon = 2$  in the real case and  $\varepsilon = 3$  in the Hermitian case.

**Theorem 2.**

$$0 \rightarrow H(\mathbb{B}) \xrightarrow{\mathbf{d}_1} H(\mathbb{B}, M^1) \xrightarrow{\mathbf{d}_2} \dots \xrightarrow{\mathbf{d}_{n-2}} H(\mathbb{B}, M^{n-2}) \xrightarrow{\mathbf{d}_{n-1}} H(\mathbb{B}, \mathbb{S}) \rightarrow 0 \quad (1)$$

is an exact sequence.

**Proof.** We make calculations only for the real case; the Hermitian version is obtained by obvious modifications.

First note that  $\Gamma_k \smile \Gamma_{k-1} = 0$ . Indeed,  $\Gamma_k \smile \Gamma_{k-1}$  is an element of  $H^4(\mathbb{B}, \Sigma_{k-1,k} \cup \Sigma_{k,k+1}) = H^4(\mathbb{S}, \Sigma_{k-1,k} \cup \Sigma_{k,k+1})$  but

$$\mathcal{S} \setminus (\Sigma_{k-1,k} \cup \Sigma_{k,k+1}) = \{A \in \mathcal{S} : \lambda_{k-1}(A) = \lambda_{k+1}(A)\}$$

is a codimension 5 algebraic subset of  $\mathcal{S}$  (see Proposition 1). Hence  $\mathbf{d}_k \circ \mathbf{d}_{k-1} = 0$  and (1) is a cochain complex. We have to prove that this complex has trivial cohomology.

Consider the spaces:

$$\Sigma_{1,k,k+1} \doteq \{A \in \mathbb{S} : \lambda_1(A) \neq \lambda_k(A) \neq \lambda_{k+1}(A)\} = M^{k-1} \cap \Sigma_{k,k+1}.$$

Then  $\phi_k(\Sigma_{k,k+1}) = \Sigma_{k,k+1} \cap W_k$  and  $\Sigma_{k,k+1} \cap W_k$  is a deformation retract of  $\Sigma_{k,k+1}$ . Similarly,  $\phi_k(\Sigma_{1,k,k+1}) = \Sigma_{1,k,k+1} \cap W_k$  and  $\Sigma_{1,k,k+1} \cap W_k$  is a deformation retract of  $\Sigma_{1,k,k+1}$ .

The map  $A \mapsto \text{span}\{x \in \mathbb{R}^n : Ax = \lambda_i(A)x, i = 1, \dots, k\}$  endows the space  $\Sigma_{k,k+1} \cap W_k$  with the structure of the fiber bundle over  $Gr_k(n)$ , where the fiber at  $E \in Gr_k(n)$  is the space of all self-adjoint operators  $A : E \rightarrow E$  such that  $\sum_{i=1}^k \lambda_i(A) \leq 0$ ,  $\sum_{i=1}^k \lambda_i^2(A) + \frac{1}{n-k} \left( \sum_{i=1}^k \lambda_i(A) \right)^2 = 1$ : such operators

are uniquely extended to (normalized) operators from  $W_k$ . The fiber is thus a ball of dimension  $\frac{k(k+1)}{2} - 1 = \nu_{k-1} + 1$ .

Moreover,  $(\Sigma_{k,k+1} \setminus \Sigma_{1,k,k+1}) \cap W_k$  is a section of the bundle  $\Sigma_{k,k+1} \cap W_k \rightarrow Gr_k(n)$ , where the value of the section at  $E \in Gr_k(n)$  is a scalar operator on  $E$ , “the center of the ball”. Hence “the spherical bundle”

$$\{A \in \Sigma_{k,k+1} \cap W_k : \sum_{i=1}^k \lambda_i(A) = 0\}$$

with a typical fiber  $S^{\nu_{k-1}}$  is a homotopy retract of  $\Sigma_{1,k,k+1}$ . Let

$$S_E^{\nu_{k-1}} \doteq \{A \in \Sigma_{k,k+1} \cap W_k : \sum_{i=1}^k \lambda_i(A) = 0, Ax_i = \lambda_i(A)x_i, \\ x_i \in E \setminus \{0\}, i = 1, \dots, k\} \quad (2)$$

be the fiber at  $E$  of this spherical bundle.

**Lemma 3.** *The restriction  $u_{k-1}|_{S_E^{\nu_{k-1}}}$  of the class  $u_{k-1} \in H^{\nu_{k-1}}(M^{k-1})$  induced by the inclusion*

$$M^{k-1} \supset \Sigma_{1,k,k+1} \supset S_E^{\nu_{k-1}}$$

*is the generator of  $H^{\nu_{k-1}}(S_E^{\nu_{k-1}})$ .*

**Proof.** The value of the Thom class  $u_{k-1}$  on the cycle  $S_E^{\nu_{k-1}}$  is the intersection number of the cycle  $\phi_{k-1}(S_E^{\nu_{k-1}})$  with  $G_{k-1}(e)$  in  $M^{k-1} \cap W_{k-1}$ . Obviously, this number does not depend on  $E$ . Take  $E$  such that  $e \notin E$ . Then the intersection of  $\phi_{k-1}(S_E^{\nu_{k-1}})$  and  $G_{k-1}(e)$  is transversal and consists of one point  $A_0$  characterized by the relations

$$A_0 \in G_{k-1}(e), \quad \{x \in \mathbb{R}^n : A_0 x = \lambda_1(A_0)x\} = e^\perp \cap E. \quad \square$$

**Corollary 2.** *Let  $v_{k-1} = u_{k-1}|_{\Sigma_{1,k,k+1}}$ . Then the ring  $H(\Sigma_{1,k,k+1})$  is a free module over the ring  $H(Gr_k(n))$  with the basis  $1, v_{k-1}$ . Moreover,  $v_{k-1} \smile v_{k-1} = 0$ .*

**Proof.** The module structure is induced by the bundle structure  $\Sigma_{1,k,k+1} \cap W_k \rightarrow Gr_k(n)$ . The fact that the module is free follows from Lemma 3 and the Leray–Hirsch theorem. The equality  $v_{k-1} \smile v_{k-1} = 0$  follows from the equality  $u_{k-1} \smile u_{k-1} = 0$  (see corollary 1).  $\square$

**Lemma 4.** *The inclusions  $M^{k-1} \subset M^k$  and  $\Sigma_{k,k+1} \subset M^k$  induce zero homomorphisms of the reduced cohomology groups.*

**Proof.** We have:  $\phi_k(M^{k-1}) \subset M^k \setminus G_k(e)$ . Hence  $M^{k-1}$  is contained in the contractible subset of  $M^k$  and the restriction to  $M^{k-1}$  makes trivial any cohomology class from  $\tilde{H}(M^k)$ . Now consider the inclusions

$$\Sigma_{1,k,k+1} \subset \Sigma_{k,k+1} \subset M^k.$$

Corollary 2 implies that the inclusion  $\Sigma_{1,k,k+1} \subset \Sigma_{k,k+1} \cong Gr_k(n)$  induces the injective homomorphism  $H^i(\Sigma_{k,k+1}) \rightarrow H^i(\Sigma_{1,k,k+1})$ . On the other hand,  $\phi_k(\Sigma_{1,k,k+1}) \subset M^k \setminus G_k(e)$ . Hence the composition of the induced by the inclusions homomorphisms

$$\tilde{H}^i(M^k) \rightarrow \tilde{H}^i(\Sigma_{k,k+1}) \rightarrow \tilde{H}^i(\Sigma_{1,k,k+1})$$

is zero. We obtain that the homomorphism  $\tilde{H}^i(M^k) \rightarrow \tilde{H}^i(\Sigma_{k,k+1})$  is zero.  $\square$

Let  $X \subset \mathbb{S}$  be an open subset of  $\mathbb{S}$  whose compliment is a neighborhood deformation retract, we denote by  $\hat{\delta} : \tilde{H}^i(X) \rightarrow H^{i+1}(\mathbb{B}, X)$  natural isomorphism induced by the exact sequence of the pair  $\mathbb{B}, X$ .

Now consider the Mayer–Vietoris exact sequence of the pair  $\Sigma_{k,k+1}, M^{k-1}$ :

$$\begin{aligned} \dots H^{i-1}(M^k) \rightarrow \\ H^{i-1}(\Sigma_{k,k+1}) \oplus H^{i-1}(M^{k-1}) \xrightarrow{\theta} H^{i-1}(\Sigma_{1,k,k+1}) \xrightarrow{d} H^i(M^k) \dots \end{aligned}$$

and its relative version:

$$\begin{aligned} \dots H^i(\mathbb{B}, M^k) \rightarrow \\ H^i(\mathbb{B}, \Sigma_{k,k+1}) \oplus H^i(\mathbb{B}, M^{k-1}) \xrightarrow{\theta} H^i(\mathbb{B}, \Sigma_{1,k,k+1}) \xrightarrow{d} H^{i+1}(\mathbb{B}, M^k) \dots \end{aligned}$$

Then  $\hat{\delta}$  establishes the isomorphism of these two exact sequences. Moreover, Lemma 4 implies that long exact sequences split in the short ones:

$$0 \rightarrow H^{i-1}(\Sigma_{k,k+1}) \oplus H^{i-1}(M^{k-1}) \xrightarrow{\theta} H^{i-1}(\Sigma_{1,k,k+1}) \xrightarrow{d} H^i(M^k) \rightarrow 0 \quad (3)$$

and similarly for the relative version.

**Lemma 5.** *Let  $\xi \in H^i(\Sigma_{k,k+1})$ ,  $\eta \in H^i(M^{k-1})$ . Then  $\hat{\delta}\xi \smile \hat{\delta}\eta = 0$  if and only if  $(\xi|_{\Sigma_{1,k,k+1}} \smile \eta|_{\Sigma_{1,k,k+1}}) \in \text{im } \theta$ .*

**Proof.** The Proposition from the Appendix A implies:

$$\hat{\delta}\xi \smile \hat{\delta}\gamma = \delta \circ d \left( \xi|_{\Sigma_{1,k,k+1}} \smile \eta|_{\Sigma_{1,k,k+1}} \right).$$

Now the statement of the Lemma follows from the fact that  $\hat{\delta}$  is an isomorphism and the sequence (3) is exact.  $\square$

The next step is to find  $\text{im } \theta$ . Given  $\xi \in H(\Sigma_{k,k+1})$ ,  $\eta \in H(M^{k-1})$ , we have:

$$\theta(\xi \oplus \eta) = \xi|_{\Sigma_{1,k,k+1}} - \eta|_{\Sigma_{1,k,k+1}}.$$

According to Corollary 2, the restriction  $H(\Sigma_{k,k+1}) \rightarrow H(\Sigma_{k,k+1})|_{\Sigma_{1,k,k+1}}$  is injective and

$$H(\Sigma_{1,k,k+1}) = H(\Sigma_{k,k+1})|_{\Sigma_{1,k,k+1}} \oplus \left( v_{k-1} \smile H(\Sigma_{k,k+1})|_{\Sigma_{1,k,k+1}} \right).$$

Recall that  $M^{k-1}$  has the homotopy type of the Thom space of a vector bundle over  $G_{k-1}(e) \subset M^{k-1}$  with the Thom class  $u_{k-1} \in H^{\nu_{k-1}}(M^{k-1})$ . We consider the map  $\varrho_k : H(G_{k-1}(e)) \rightarrow H(\Sigma_{k,k+1})$ , where

$$v_{k-1} \smile \varrho_k(\zeta)|_{\Sigma_{1,k,k+1}} = \pi_v(u_{k-1} \smile \zeta)|_{\Sigma_{1,k,k+1}}, \quad \forall \zeta \in H(G_{k-1}(e)). \quad (4)$$

The identity (4) uniquely defines  $\varrho_k$ . Moreover, the map  $\varrho_k$  is injective and

$$\text{im } \theta = H(\Sigma_{k,k+1})|_{\Sigma_{1,k,k+1}} \oplus \left( v_{k-1} \smile \text{im } \varrho_k|_{\Sigma_{1,k,k+1}} \right). \quad (5)$$

The space  $\Sigma_{k,k+1}$  has the homotopy type of the Grassmannian  $Gr_k(n)$  while  $G_{k-1}(e)$  is identified with the Grassmannian  $\{F \in Gr_{k-1}(n) : F \subset e^\perp\} = Gr_{k-1}(n-1)$ . We are going to explicitly compute the map  $\varrho_k$  in the bases provided by the Schubert cells in the Grassmannians.

In what follows, we identify the manifold

$$\Sigma_{k,k+1} \cap M_{k-1} \cap W_k = (\Sigma_{k,k+1} \setminus \Sigma_{1,k,k+1}) \cap W_k$$

with the Grassmannian  $Gr_k(n)$ , where  $A \in \Sigma_{k,k+1} \cap M_{k-1} \cap W_k$  is identified with the subspace  $\{x \in \mathbb{R}^n : Ax = \lambda_1(A)x\}$ . Obviously,  $\Sigma_{k,k+1} \cap M_{k-1} \cap W_k$  is a homotopy retract of  $\Sigma_{k,k+1}$ . In particular,  $H(\Sigma_{k,k+1}) = H(Gr_k(n))$ .

Let  $e_1 = e, e_2, \dots, e_n$  be an orthogonal basis of  $\mathbb{R}^n$ . The closed Schubert cells in  $Gr_k(n)$  associated to this basis are cycles which give an additive basis of  $H(Gr_k(n))$  (see Appendix B). We also consider the dual *Schubert basis*

of  $H(Gr_k(n))$ . The Schubert cells of dimension  $r \geq 0$  are in the one-to-one correspondence with the partitions of  $r$  in no more than  $k$  positive integral summands in such a way that each summand does not exceed  $n - k$ .

Similarly, Schubert cells associated to the basis  $e_2, \dots, e_n$  of  $e^\perp = \mathbb{R}^{n-1}$  give the Schubert basis of  $H(Gr_{k-1}(n-1)) = H(G_{k-1}(e))$ . The elements of dimension  $r$  of this basis are in the one-to-one correspondence with the partitions of  $r$  in less than  $k$  positive integral summands in such a way that each summand does not exceed  $n - k$ .

**Lemma 6.** *The map  $\varrho_k : H(G_{k-1}(e)) \rightarrow H(Gr_k(n))$  sends the element of the Schubert basis of  $H(G_{k-1}(e))$  associated to a partition in less than  $k$  summands to the element of the Schubert basis of  $H(Gr_k(n))$  associated to the same partition!*

**Proof.** We'll study the adjoint map  $\varrho_k^* : H(Gr_k(n)) \rightarrow H(G_{k-1}(e))$ . We have to prove that  $\varrho_k^*$  sends to zero the Schubert classes for  $H(Gr_k(n))$  associated to the partitions in exactly  $k$  summands, while the classes associated to the partitions in less than  $k$  summands are sent to the Schubert classes for  $H(G_{k-1}(e))$  associated to the same partitions.

Let  $C \subset Gr_k(n)$  be a Schubert cycle and  $[C]$  its homology class. We set  $S_C^{\nu_{k-1}} = \bigcup_{E \in C} S_E^{\nu_{k-1}}$ , c.f. (2). Then  $\varrho_k^*[C]$  is the homology class of the

intersection of  $\phi_{k-1}(S_C^{\nu_{k-1}})$  with  $G_{k-1}(e) = Gr_{k-1}(n-1)$ . In other words, the map  $\varrho_k^*$  is essentially determined by the set-valued map  $\mathfrak{r}_k : Gr_k(n) \dashrightarrow G_{k-1}(e)$ , where  $\mathfrak{r}_k(E) = \phi_{k-1}(S_E^{\nu_{k-1}}) \cap G_{k-1}(e)$ ,  $E \in Gr_k(n)$ .

It is easy to see that  $\mathfrak{r}_k(E) = \{F \in Gr_{k-1}(n-1) : F \subset E \cap e^\perp\}$ . In particular,  $\mathfrak{r}_k$  is one-valued on  $\{E \in Gr_k(n) : E \not\subset e\}$ ; if  $E \not\subset e$ , then  $\mathfrak{r}_k(E) = E \cap e^\perp$ . The (one-valued) map  $F \mapsto (F + \mathbb{R}e)$ ,  $F \in Gr_{k-1}(n-1)$ , is a right inverse of  $\mathfrak{r}_k$ .

Let  $\mathfrak{d}$  be a starting from the unit Schubert symbol for  $Gr_k(n)$ . Then  $\mathfrak{r}_k(Sc_k^{\mathfrak{d}}(n)) = Sc_{k-1}^{\mathfrak{d}'}(n-1)$ , where  $\mathfrak{d}'$  is obtained from  $\mathfrak{d}$  by removing the first unit. Indeed,  $e \in E$ ,  $\forall E \in Sc_k^{\mathfrak{d}}(n)$ , and the desired equality easily follows from the definitions.

Now assume that  $\mathfrak{d}$  is a starting from 0 Schubert symbol for  $Gr_k(n)$ . We'll show that  $\mathfrak{r}_k(Sc_k^{\mathfrak{d}}(n))$  is contained in the union of Schubert cells whose dimension is smaller than the dimension of  $Sc_k^{\mathfrak{d}}(n)$ . This fact completes the proof of Lemma 6.

Let  $F \in Sc_k^{\mathfrak{d}}(n)$  and  $\hat{F} \in \mathfrak{r}_k(Sc_{k-1}^{\hat{\mathfrak{d}}}(n-1))$ . Recall that

$$d_i^{\mathfrak{d}} = \min\{j : \dim(E_j \cap F) = i\}, \quad i = 1, \dots, k,$$

$$d_i^{\hat{\mathfrak{d}}} = \min\{j : \dim(E_{j+1} \cap \hat{F}) = i\}, \quad i = 1, \dots, k-1,$$

where  $E_j = \text{span}\{e_1, \dots, e_j\}$ ,  $j = 1, \dots, n$ . On the other hand,

$$\dim(E_j \cap F_j) - 1 \leq \dim(E_j \cap \hat{F}_j) \leq \dim(E_j \cap F_j);$$

hence  $d_i^{\mathfrak{d}} \leq d_i^{\hat{\mathfrak{d}}} + 1 \leq d_{i+1}^{\mathfrak{d}}$ . Moreover,  $d_1^{\mathfrak{d}} > 1$ , since the symbol  $\mathfrak{d}$  starts from 0. We obtain:

$$\begin{aligned} \dim Sc_{k-1}^{\hat{\mathfrak{d}}}(n-1) &= \\ &= \sum_{i=1}^{k-1} (d_i^{\hat{\mathfrak{d}}} - i) = \sum_{i=1}^{k-1} \left( (d_i^{\hat{\mathfrak{d}}} + 1) - (i+1) \right) \leq \sum_{i=2}^k (d_i^{\mathfrak{d}} - i) < \dim Sc_k^{\mathfrak{d}}(n). \end{aligned}$$

□

**Lemma 7.** *Let  $w \in H^1(Gr_k(n))$  be the first Stiefel–Whitney class of the tautological bundle and  $\xi \in \text{im } \varrho_k$ . Then  $(w \smile \xi) \in \text{im } \varrho_k$  if and only if  $\xi$  is a sum of Schubert classes associated to partitions in less than  $k-1$  summands.*

**Proof.** Let  $\Pi_j$  be the linear hull of the Schubert classes associated to the partitions in exactly  $j$  summands,  $j = 0, 1, \dots, k$ , and  $\xi$  the Schubert class associated to the partition  $a_1 + \dots + a_j$ . The Pieri formula (see Appendix B) implies that the difference of  $w \smile \xi$  and the Schubert class associated to the partition  $1 + a_1 + \dots + a_j$  belongs to  $\Pi_j$ . On the other hand, according to

Lemma 6,  $\text{im } \varrho_k = \bigoplus_{j=0}^{k-1} \Pi_j$ . □

Now we are ready to compute  $\ker \mathbf{d}_k$  and thus complete the proof of Theorem 2. Let  $\xi \in H^1(M^{k-1})$ ; then  $\xi = \hat{\delta}(u_{k-1} \smile \zeta)$  for a unique  $\zeta \in H^1(G_{k-1}(e))$ . We have:

$$\mathbf{d}_k(\xi) = \hat{\delta}(\gamma_k) \smile \hat{\delta}(u_{k-1} \smile \zeta),$$

where  $\gamma_k \in H^1(\Sigma_{k,k+1})$  was defined in Section 2. According to Lemma 5,  $\mathbf{d}_k(\xi) = 0$  if and only if

$$(u_{k-1} \smile \zeta)|_{\Sigma_{1,k,k+1}} \smile \gamma_k|_{\Sigma_{1,k,k+1}} = \left( v_{k-1} \smile (\varrho_k(\zeta) \smile \gamma_k)|_{\Sigma_{1,k,k+1}} \right) \in \text{im } \theta.$$

Further,  $Gr_k(n) = \Sigma_{k,k+1} \cap M_{k-1} \cap W_k$  is a homotopy retract of  $\Sigma_{k,k+1}$ , and  $\gamma_k|_{Gr_k(n)}$  is the first Stiefel–Whitney class of the tautological bundle of the

Grassmannian  $Gr_k(n)$ , i. e.  $\gamma_k|_{Gr_k(n)} = w$  (see Proposition 2). Now the equality (5) implies that  $\mathbf{d}_k(\xi) = 0$  if and only if

$$(\varrho_k(\zeta) \smile w) \in \text{im } \varrho_k.$$

It follows from Lemma 7 and the injectivity of  $\varrho_k$  that  $\dim \ker \mathbf{d}_k$  is equal to the number of partitions in no more than  $k - 2$  natural summands in such a way that each summand does not exceed  $n - k$ . In other words,  $\dim \ker \mathbf{d}_k = \binom{n-2}{k-2}$ . At the same time, the isomorphisms  $H^*(\mathbb{B}, M^{k-1}) \cong \tilde{H}^*(M^{k-1}) \cong H^*(Gr_{k-1}(n-1))$  imply that  $\dim H^*(\mathbb{B}, M^{k-1}) = \binom{n-1}{k-1}$ . The Pascal triangle identity  $\binom{n-1}{k-1} = \binom{n-2}{k-2} + \binom{n-2}{k-1}$  gives:

$$\dim H^*(\mathcal{B}, M^{k-1}) = \dim \ker \mathbf{d}_k + \dim \ker \mathbf{d}_{k+1}. \quad \square$$

## A A property of the cohomological product

Let  $M$  be a simplicial complex and  $X \subset M$  its subcomplex; we denote by  $\delta_X : H^*(X) \rightarrow H^{*+1}(M, X)$  the connecting homomorphism in the cohomological exact sequence of the pair  $M, X$ . Let  $Y \subset M$  be one more subcomplex and  $d : H^*(X \cap Y) \rightarrow H^{*+1}(X \cup Y)$  the connecting homomorphism in cohomological Mayer–Vietoris exact sequence of the pair  $X, Y$ .

**Proposition.** *Let  $\xi \in H^*(X)$ ,  $\eta \in H^*(Y)$ ; then*

$$\delta_X \xi \smile \delta_Y \eta = \delta_{X \cup Y} \circ d(\xi|_{X \cap Y} \smile \eta|_{X \cap Y}).$$

**Proof.** We set  $\zeta = d(\xi|_{X \cap Y} \smile \eta|_{X \cap Y})$ . Let  $x$  and  $y$  be cocycles representing cohomology classes  $\xi$  and  $\eta$ . Any cocycle  $z$  such that

$$z|_X = \delta u, \quad z|_Y = \delta v, \quad u|_{X \cap Y} - v|_{X \cap Y} = x|_{X \cap Y} \smile y|_{X \cap Y} \quad (A)$$

for some cochains  $u, v$  is a representative of  $\zeta$ . We do as follows: extend  $x$  and  $y$  to cochain  $\hat{x}$  and  $\hat{y}$  defined on  $X \cup Y$  and set  $z = \hat{x} \smile \delta \hat{y}$ . Then conditions (A) are satisfied for  $u = (-1)^{\dim x} x \smile \hat{y}|_X$ ,  $v = 0$  and we have:  $\delta z = \delta \hat{x} \smile \delta \hat{y}$ .  $\square$



## B Schubert cells

Schubert cells give cell complex structures of Grassmannians. They are indexed by *Schubert symbols*. A Schubert symbol  $\mathfrak{d}$  for  $Gr_k(n)$  is a sequence of zeros and units that contains exactly  $k$  units and  $n - k$  zeros. The total number of symbols for  $Gr_k(n)$  (i.e. the number of cells in the cell complex) is  $\binom{n}{k}$ . We denote by  $d_i^{\mathfrak{d}}$  the number of  $i$ th unit in the sequence; then  $1 \leq d_1^{\mathfrak{d}} < \dots < d_k^{\mathfrak{d}} \leq n$ .

We treat simultaneously the real and complex cases. Let  $e_1, \dots, e_n$  be a basis of  $\mathbb{R}^n$  in the real case and a basis of  $\mathbb{C}^n$  in the complex case. We set

$$E_i = \text{span}\{e_1, \dots, e_i\}, \quad i = 1, \dots, n.$$

The Schubert cell  $Sc_k^{\mathfrak{d}}(n)$  is defined as follows:

$$Sc_k^{\mathfrak{d}}(n) = \left\{ F \in Gr_k(n) : \dim(F \cap E_{d_i^{\mathfrak{d}}}) = i, \dim(F \cap E_{d_i^{\mathfrak{d}}-1}) = i - 1 \right\}.$$

There is a one-to-one correspondence between Schubert symbols for  $Gr_k(n)$  and partitions of nonnegative integers in no more than  $k$  positive integral summands in such a way that each summand does not exceed  $(n - k)$ . The summands associated to the symbol  $\mathfrak{d}$  are numbers of zeros to the left of each unit presented in the symbol. In other words, the summands are nonzero terms of the sequence  $(d_i^{\mathfrak{d}} - i)$ ,  $i = 1, \dots, k$ .

The dimension of the Schubert cell associated to a partition of the number  $r$  is equal to  $r$  in the real case and to  $2r$  in the complex case. We thus have:

$$\dim Sc_k^{\mathfrak{d}}(n) = \epsilon \sum_{i=1}^k (d_i^{\mathfrak{d}} - i),$$

where  $\epsilon = 1$  in the real case and  $\epsilon = 2$  in the complex case. The closure  $\overline{Sc_k^{\mathfrak{d}}(n)}$  is a cycle over  $\mathbb{Z}_2$  in the real case and a cycle over  $\mathbb{Z}$  in the complex case (a *Schubert cycle*). In both cases, the homology classes of the Schubert cycles form an additive basis of the total homology group of  $Gr_k(n)$  (over  $\mathbb{Z}_2$  in the real case and over  $\mathbb{Z}$  in the complex one). Moreover, in the complex case the homology groups are free.

The dual basis of the total cohomology group of  $Gr_k(n)$  is called the *Schubert basis*. Repeat that the elements of dimension  $r$  of this basis, the  $r$ -dimensional *Schubert classes*, are in the one-to-one correspondence with

partitions of  $r$  in no more than  $k$  natural summands, where each summand does not exceed  $n - k$ .

The Stiefel–Whitney (in the real case) and Chern (in the complex case) characteristic classes of the tautological bundle are Schubert classes associated to partitions in units. In particular, the Stiefel–Whitney class  $w_1$  in the real case and the Chern class  $c_1$  in the complex case are associated to the unique “partition” of 1.

There is a useful *Pieri formula* which computes the cohomological product of the Schubert class associated to a partition with one summand  $a$  and the Schubert class associated to any partition  $b_1 + \cdots + b_j$ , where  $b_1 \leq \cdots \leq b_j$ . The product equals the sum of all Schubert classes of dimension  $a + \sum_{i=1}^j b_i$  associated to the partitions  $c_0 + c_1 + \cdots + c_j$  such that  $b_{i-1} \leq c_i \leq b_i$ ,  $i = 1, \dots, j$ ,  $b_0 = 0$ .

See details in [4, Ch.5] and [3, Ch.1.5].

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