CONSERVATION OF GEOMETRIC STRUCTURES FOR NON-HOMOGENEOUS INVISCID INCOMPRESSIBLE FLUIDS

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Abstract

In this paper we obtain a result about propagation of geometric properties for solutions of non-homogeneous incompressible Euler system in any dimension $N \geq 2$. In particular, we investigate conservation of striated and conormal regularity, which is a natural way of generalising the 2-D structure of vortex patches. The results we get are only local in time, even in the dimension $N = 2$: in contrast with the homogeneous case, the global existence issue is still an open problem, because the vorticity is not preserved during the time evolution. Moreover we will be able to give an explicit lower bound for the lifespan of the solution, in terms of the norms of initial data only. In the case of physical dimension $N = 2$ or 3, we will investigate also propagation of Hölder regularity in the interior of a bounded domain.

1 Introduction

In this paper we are interested in studying conservation of geometric properties for solutions of the density-dependent incompressible Euler system

eq:ddeuler

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 \\ \rho \left(\partial_t u + u \cdot \nabla u \right) + \nabla \Pi = 0 \\ \operatorname{div} u = 0 \,, \end{cases} \tag{1}$$

which describes the evolution of a non-homogeneous inviscid fluid with no body force acting on it, an assumption we will make throughout all this paper to simplify the presentation. Here, $\rho(t,x) \in \mathbb{R}_+$ represents the density of the fluid, $u(t,x) \in \mathbb{R}^N$ its velocity field and $\Pi(t,x) \in \mathbb{R}$ its pressure. The term $\nabla \Pi$ can be also seen as the Lagrangian multiplier associated to the divergence-free constraint over the velocity.

We will always suppose that the variable x belongs to the whole space $\mathbb{R}^N$.

The problem of preserving geometric structures came out already in the homogeneous case, for which $\rho \equiv 1$ and system (1) becomes

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla \Pi = 0 \\ \operatorname{div} u = 0 \,, \end{cases} \tag{E}$$

in studying 2-dimensional vortex patches, that is to say the initial vorticity Ω_0 is the characteristic function of a bounded domain D_0. As we will explain below, in the case of higher dimension $N \geq 3$ this notion was generalized by the properties of striated and conormal regularity.

The vorticity of the fluid is defined as the skew-symmetric matrix

def:vort

$$\Omega := \nabla u - {}^t\nabla u \tag{2}$$

and in the homogenous case it satisfies the equation

$$\partial_t \Omega + u \cdot \nabla \Omega + \Omega \cdot \nabla u + {}^t\nabla u \cdot \Omega = 0 .$$

In dimension $N = 2$ it can be identified with the scalar function $\omega = \partial_1 u^2 - \partial_2 u^1$, while for $N = 3$ with the vector-field $\omega = \nabla \times u$. Let us recall also that in the bidimensional case this quantity is transported by the velocity field:

$$\partial_t \omega + u \cdot \nabla \omega = 0 .$$

The notion of vortex patches was introduced in [24] and gained new interest after the survey paper [22] of Majda. In the case $N = 2$ Yudovich's theorem ensures the existence of a unique global solution of the homogeneous Euler system, which preserves the geometric structure: the vorticity remains the characteristic function of the evolution (by the flow associated to this solution) of the domain D_0. Vortex patches in bounded domains of $\mathbb{R}^2$ were also studied by Depauw (see [15]), while Dutrifoy in [16] focused on the case of domains in $\mathbb{R}^3$. Moreover, in [6] Chemin proved that, if the initial domain has boundary ∂D_0 of class $\mathcal{C}^{1+\varepsilon}$ for some $\varepsilon > 0$, then this regularity is preserved during the evolution for small times; in [7] he also showed a global-in-time persistence issue. In [11] Danchin considered instead the case in which initial data of the Euler system are vortex patches with singular boundary: he proved that if ∂D_0 is regular apart from a closed subset, then it remains regular for all times, apart from the closed subset transported by the flow associated to the solution.

In the case $N \geq 3$ one can't expect to have global results anymore, nor to preserve the initial vortex patch structure, because of the presence of the stretching term in the vorticity equation. Nevertheless, it's possible to introduce the definition of *striated regularity*, which generalizes in a quite natural way the previous one of vortex patch: it means that the vorticity is more regular along some fixed directions, given by a nondegenerate family of vector-fields (see definition 2.1 below). This notion was introduced first by Bony in [3] in studying hyperbolic equations, and then adapted by Alinhac (see [1]) and Chemin (see [5]) for nonlinear partial differential equations.

In [17], Gamblin and Saint-Raymond proved that striated regularity is preserved during the evolution in any dimension $N \geq 3$, but, as already remarked, only locally in time (see also [23]). They also obtained global results if initial data have other nice properties (e.g., if the initial velocity is axisymmetric).

As Euler system is, in a certain sense, a limit case of the Navier-Stokes system as the viscosity of the fluid goes to 0, it's interesting to study if there is also "convergence" of the geometric properties of the solutions. Recently Danchin proved results on striated regularity for the solutions of the Navier-Stokes system

$$(NS_\nu) \qquad \begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla \Pi = 0 \\ \mathrm{div}\, u = 0 , \end{cases}$$

in [10] for the 2-dimensional case, in [12] for the general one. Already in the former paper, he had to dismiss the vortex patch structure "stricto sensu" due to the presence of the viscous term, which comes out also in the vorticity equation and has a smoothing effect; however, he still got global in time results. Moreover, in both his works he had to handle with spaces of type $B^{1+\varepsilon}_{p,\infty}$ (with $p \in]2, +\infty[$ and $\varepsilon \in]2/p, 1[$) due to technical reasons which come out with a viscous fluid.

Let us immediately clarify that these problems have been recently solved by Hmidi in [20] (see also [2]), and this fact allows us to consider again the Hölder spaces framework. In the above mentioned works Danchin proved also a priori estimates for solutions of (NS_ν) independent of the viscosity ν, therefore preservation of the geometric structures in passing from solutions of (NS_ν) to solutions of (E) in the limit $\nu \to 0$.

In this paper we will come back to the inviscid case and we will study the non-homogeneous incompressible Euler system (1). We want to investigate if preservation of geometric properties of initial data, such as striated and conormal regularity, still holds in this setting, as in the classical (homogeneous) one. Let us note that in the 2-dimensional case the equation for the vorticity reads

$$\partial_t\omega \,+\, u\cdot\nabla\omega \,+\, \nabla\left(\frac{1}{\rho}\right)\wedge\nabla\Pi \,=\, 0\,,$$

so it's not better than in higher dimension due to the presence of the density term, which doesn't allow us to get conservation of Lebesgue norms. This is also the reason why it's not clear if Yudovich's theorem still holds true for non-homogeneous fluids: having $\omega_0 \in L^q \cap L^\infty$, combined with suitable hypothesis on ρ_0, doesn't give rise to a local solution.
So, we will immediately focus on the general case $N \geq 2$. We will assume the initial velocity u_0 and the initial vorticity Ω_0 to be in some Lebesgue spaces, in order to assure the pressure term to belong to L^2, a requirement we could not bypass. As a matter of fact, $\nabla\Pi$ satisfies an elliptic equation with low regularity coefficient,

$$-\operatorname{div}\,(a\,\nabla\Pi) \,=\, \operatorname{div}F\,,$$

and it can be solved independently of a only in the energy space L^2. Moreover, we will suppose Ω_0 to have regularity properties of geometric type. Obviously, we will require some natural but quite general hypothesis also on the initial density ρ_0 of the fluid: we suppose ρ_0 to be bounded with its gradient and that it satisfies geometric assumptions analogous to those for Ω_0. Let us point out that proving the velocity field to be Lipschitz, which was the key part in the homogeneous case, works as in this setting: it relies on Biot-Savart law and it requires no further hypothesis on the density term. Let us also remark that no smallness condition over the density are needed. Of course, we will get only local in time results. Moreover, we will see that geometric structures propagate also to the velocity field and to the pressure term.

Our paper is organized in the following way.

In the first part, we will recall basic facts about Euler system: some properties of the vorticity and how to associate a flow to the velocity field. In this section we will also give the definition of the geometric properties we are studying and we will state the main results we got about striated and conormal regularity.

Then, we will explain the mathematical tools, from Fourier Analysis, we need to prove our claims: so, we will introduce the Littlewood-Paley decomposition and some techniques coming from paradifferential calculus. In particular, we will introduce the notion of paravector-field, as defined in [12]: it will play a fundamental role in our analysis, because it is, in a certain sense, the principal part of the derivation operator along a fixed vector-field. Moreover, we will also quote some results about transport equations in Hölder spaces and about elliptic equations in divergence form with low regularity coefficients.

This having been done, we will finally be able to tackle the proof of our result about striated regularity. First of all, we will state a priori estimates for suitable smooth solutions of the Euler system (1). Then from them we will get, in a quite classical way, the existence of a solution with the required properties: we will construct a sequence of regular solutions of system (1) with approximated data, and, using a compactness argument, we will show the convergence of this sequence to a "real" solution. Proving preservation of the geometric structure requires instead strong convergence in rough spaces of type $\mathcal{C}^{-\alpha}$ (for some $\alpha > 0$). Uniqueness of the solution will follow from a stability result for our equations. In the following section, we will also give an estimate from below for the lifespan of the solution.

Finally, we will spend a few words about conormal regularity: proving its propagation from the previous result is standard and can be done as in the homogenous setting. As a consequence, inspired by what done in Huang's paper [21], in the physical case of space dimension $N = 2$ or 3 we can improve our result: we will also show that, if the initial data are Hölder continuous in the interior of a suitably smooth bounded domain, the solution conserves this property during the time evolution, i.e. it is still Hölder continuous in the interior of the domain transported by the flow.

2 Basic definitions and main results

Let $(\rho, u, \nabla\Pi)$ be a solution of the density-dependent incompressible Euler system (1) over $[0,T] \times \mathbb{R}^N$ and let us denote the vorticity of the fluid by Ω. As in the homogeneous case, it will play a fundamental role throughout all this paper, so let us spend a few words about it.

From the definition (2), it is obvious that, for all $q \in [1,\infty]$, if $\nabla u \in L^q$, then also $\Omega \in L^q$. Conversely, if u is divergence-free then for all $1 \leq i \leq N$ we have $\Delta u^i = \sum_{j=1}^N \partial_j \Omega_{ij}$, and so, formally,

eq:BS-law

$$u^i \,=\, -\,(-\Delta)^{-1} \sum_{j=1}^{N} \partial_j \Omega_{ij}\,. \tag{3}$$

This is the Biot-Savart law, and it says that a divergence free vector-field u is completely determined by its vorticity. From (3) we immediately get

eq:BS_grad

$$\nabla u^i \,=\, -\nabla\,(-\Delta)^{-1} \sum_{j=1}^{N} \partial_j \Omega_{ij}\,. \tag{4}$$

Now, as the symbol of the operator $-\partial_i\,(-\Delta)^{-1}\,\partial_j$ is $\sigma(\xi) = \xi_i \xi_j / |\xi|^2$, the classical Calderon-Zygmund theorem ensures that[1] for all $q \in]1,\infty[$, if $\Omega \in L^q$ then $\nabla u \in L^q$ and

est:CZ

$$\|\nabla u\|_{L^q} \leq C\,\frac{q^2}{q-1}\,\|\Omega\|_{L^q}\,. \tag{5}$$

In dimension $N = 2$ the vorticity equation is simpler than in the general case due to the absence of the stretching term. Nevertheless, as remarked above, the exterior product involving density and pressure terms makes it impossible to get conservation of Lebesgue norms, which was the fundamental issue to get global existence. So, we immediately focus on the case $N \geq 2$ whatever, in which the vorticity equation reads

eq:vort

$$\partial_t \Omega \,+\, u \cdot \nabla\Omega \,+\, \Omega \cdot \nabla u \,+\, {}^t\nabla u \cdot \Omega \,+\, \nabla\left(\frac{1}{\rho}\right) \wedge \nabla\Pi \,=\, 0\,, \tag{6}$$

where, for two vector-fields v and w, we have set $v \wedge w$ to be the skew-symmetric matrix with components

$$(v \wedge w)_{ij} \,=\, v^j w^i \,-\, v^i w^j\,.$$

Finally, recall that we can associate a flow ψ to the velocity field u of the fluid: it is defined by the relation

$$\psi(t,x) \,\equiv\, \psi_t(x) \,:=\, x \,+\, \int_0^t u(\tau, \psi_\tau(x))\, d\tau$$

[1] This time the extreme values of q are not included.

for all $(t,x)\in[0,T]\times\mathbb{R}^N$ and it is, for all fixed $t\in[0,T]$, a diffeomorphism over $\mathbb{R}^N$.

Let us now introduce the geometric properties we are handling throughout this paper. The first notion we are interested in is the *striated regularity*, that is to say initial data are more regular along some given directions.

So, let us take a family $X=(X_\lambda)_{1\leq\lambda\leq m}$ of m vector-fields with components and divergence of class $\mathcal{C}^\varepsilon$ for some fixed $\varepsilon\in]0,1[$. We also suppose this family to be non-degenerate, i.e.

$$I(X)\;:=\;\inf_{x\in\mathbb{R}^N}\;\sup_{\Lambda\in\Lambda^m_{N-1}}\left|\overset{N-1}{\wedge}\,X_\Lambda(x)\right|^{\frac{1}{N-1}}\;>\;0\,.$$

Here $\Lambda\in\Lambda^m_{N-1}$ means that $\Lambda=(\lambda_1,\ldots,\lambda_{N-1})$, with each $\lambda_i\in\{1,\ldots,m\}$ and $\lambda_i<\lambda_j$ for $i<j$, while the symbol $\overset{N-1}{\wedge}\,X_\Lambda$ stands for the element of $\mathbb{R}^N$ such that

$$\forall\,Y\in\mathbb{R}^N\,,\qquad\left(\overset{N-1}{\wedge}\,X_\Lambda\right)\cdot Y\;=\;\det\left(X_{\lambda_1}\ldots X_{\lambda_{N-1}},Y\right)\,.$$

For each vector-field of this family we put

$$\widetilde{\|}X_\lambda\|_{\mathcal{C}^\varepsilon}\;:=\;\|X_\lambda\|_{\mathcal{C}^\varepsilon}\;+\;\|\mathrm{div}\,X_\lambda\|_{\mathcal{C}^\varepsilon}\,,$$

while we will use the symbol $|||\cdot|||$ in considering the supremum over all indices $\lambda\in\Lambda^m_1=\{1\ldots m\}$.

d:stri **Definition 2.1.** Take a vector-field Y with components and divergence in $\mathcal{C}^\varepsilon$ and fix a $\eta\in[\varepsilon,1+\varepsilon]$. A function $f\in L^\infty$ is said to be of class $\mathcal{C}^\eta$ along Y, and we write $f\in\mathcal{C}^\eta_Y$, if $\mathrm{div}\,(f\,Y)\in\mathcal{C}^{\eta-1}\left(\mathbb{R}^N\right)$.

If $X=(X_\lambda)_{1\leq\lambda\leq m}$ is a non-degenerate family of vector-fields as above, we define

$$\mathcal{C}^\eta_X\;:=\;\bigcap_{1\leq\lambda\leq m}\mathcal{C}^\eta_{X_\lambda}\qquad\text{and}\qquad\|f\|_{\mathcal{C}^\eta_X}\;:=\;\frac{1}{I(X)}\left(\|f\|_{L^\infty}\,\widetilde{|||}X|||_{\mathcal{C}^\varepsilon}\;+\;|||\,\mathrm{div}\,(f\,X)\,|||_{\mathcal{C}^{\eta-1}}\right)\,.$$

r:div **Remark 2.2.** Our aim is to investigate Hölder regularity of the derivation of f along the fixed vector-field (say) Y, i.e. the quantity

$$\partial_Y f\;:=\;\sum_{i=1}^N Y^i\,\partial_i f\,.$$

If f is only bounded, however, this expression has no meaning: this is why we decided to focus on $\mathrm{div}\,(f\,Y)$, as done in the literature about this topic (see also [12]). Lemma 4.5 below will clarify the relation between these two quantities.

Now, let us take a vector-field X_0 and define its time evolution $X(t)$:

def:X
$$X(t,x)\;\equiv\;X_t(x)\;:=\;\partial_{X_0(x)}\psi_t\left(\psi_t^{-1}(x)\right)\,,\tag{7}$$

that is $X(t)$ is the vector-field X_0 transported by the flow associated to u. From this definition, it immediately follows that $[X(t)\,,\,\partial_t+u\cdot\nabla]=0$, i.e. $X(t)$ satisfies the following system:

eq:X
$$\begin{cases}(\partial_t\,+\,u\cdot\nabla)\,X\;=\;\partial_X u\\ X_{|t=0}\;=\;X_0\,.\end{cases}\tag{8}$$

We are now ready for stating our first result, on striated regularity.

t:stri-N **Theorem 2.3.** *Fix $\varepsilon \in]0,1[$ and take a non-degenerate family of vector-fields $X_0 = (X_{0,\lambda})_{1\leq\lambda\leq m}$ over $\mathbb{R}^N$, whose components and divergence are in $\mathcal{C}^\varepsilon$.*
Let the initial velocity field $u_0 \in L^p$, with $p \in]2,+\infty]$, and its vorticity $\Omega_0 \in L^\infty \cap L^q$, with $q \in [2,+\infty[$ such that $1/p + 1/q \geq 1/2$. Let us suppose $\Omega_0 \in \mathcal{C}^\varepsilon_{X_0}$.
Finally, let the initial density $\rho_0 \in W^{1,\infty}$ be such that $0 < \rho_* \leq \rho_0 \leq \rho^*$ and $\nabla\rho_0 \in \mathcal{C}^\varepsilon_{X_0}$.

Then there exist a time $T > 0$ and a unique solution $(\rho, u, \nabla\Pi)$ of system (1), such that:

- *$\rho \in L^\infty([0,T];W^{1,\infty}) \cap \mathcal{C}_b([0,T]\times\mathbb{R}^N)$, such that $0 < \rho_* \leq \rho \leq \rho^*$ at every time;*
- *$u \in \mathcal{C}([0,T];L^p) \cap L^\infty([0,T];\mathcal{C}^{0,1})$, with $\partial_t u \in \mathcal{C}([0,T];L^2)$ and vorticity $\Omega \in \mathcal{C}([0,T];L^q)$;*
- *$\nabla\Pi \in \mathcal{C}([0,T];L^2)$, with $\nabla^2\Pi \in L^\infty([0,T];L^\infty)$.*

Moreover, the family of vector-fields transported by the flow still remains, at every time, non-degenerate and with components and divergence in $\mathcal{C}^\varepsilon$, and striated regularity is preserved: at every time $t \in [0,T]$, one has that

- *$\nabla\rho(t)$ and $\Omega(t) \in \mathcal{C}^\varepsilon_{X(t)}$,*
- *$u(t)$ and $\nabla\Pi(t) \in \mathcal{C}^{1+\varepsilon}_{X(t)}$*

uniformly on $[0,T]$.

Another interesting notion, strictly related to the previous one, is that of *conormal regularity.* First of all, we have to recall a definition (see also [12]).

Definition 2.4. Let $\Sigma \subset \mathbb{R}^N$ be a compact hypersurface of class $\mathcal{C}^{1+\varepsilon}$. Let us denote by $\mathcal{T}^\varepsilon_\Sigma$ the set of all vector-fields X with components and divergence in $\mathcal{C}^\varepsilon$, which are tangent to Σ, i.e. $\partial_X H_{|\Sigma} \equiv 0$ for all local equations H of Σ.

Given a $\eta \in [\varepsilon, 1+\varepsilon]$, we say that a function $f \in L^\infty$ belongs to the space $\mathcal{C}^\eta_\Sigma$ if

$$\forall\, X \in \mathcal{T}^\varepsilon_\Sigma \ , \qquad \mathrm{div}\,(f\,X) \in \mathcal{C}^{\eta-1}\,.$$

Similarly to what happens for striated regularity, also conormal structure propagates during the time evolution.

t:conorm-N **Theorem 2.5.** *Fix $\varepsilon \in]0,1[$ and take a compact hypersurface $\Sigma_0 \subset \mathbb{R}^N$ of class $\mathcal{C}^{1+\varepsilon}$.*
Let us suppose the initial velocity field $u_0 \in L^p$, with $p \in]2,+\infty]$, and its vorticity $\Omega_0 \in L^\infty \cap L^q$, with $q \in [2,+\infty[$ such that $1/p + 1/q \geq 1/2$. Moreover, let us suppose $\Omega_0 \in \mathcal{C}^\varepsilon_{\Sigma_0}$.
Finally, let the initial density $\rho_0 \in W^{1,\infty}$ be such that $0 < \rho_* \leq \rho_0 \leq \rho^*$ and $\nabla\rho_0 \in \mathcal{C}^\varepsilon_{\Sigma_0}$.

Then there exist a time $T > 0$ and a unique solution $(\rho, u, \nabla\Pi)$ of system (1), which verifies the same properties of theorem 2.3.
Moreover, if we define

$$\Sigma(t) := \psi_t(\Sigma_0) \ ,$$

$\Sigma(t)$ is, at every time $t \in [0,T]$, a hypersurface of class $\mathcal{C}^{1+\varepsilon}$ of $\mathbb{R}^N$, and conormal regularity is preserved: at every time $t \in [0,T]$, one has

- *$\nabla\rho(t)$ and $\Omega(t) \in \mathcal{C}^\varepsilon_{\Sigma(t)}$,*
- *$u(t)$ and $\nabla\Pi(t) \in \mathcal{C}^{1+\varepsilon}_{\Sigma(t)}$*

uniformly on $[0,T]$.

3 Tools

In this section we will introduce the main tools used to prove our results; they are mostly based on Fourier analysis techniques. Unless otherwise specified, one can find the proof of all the results quoted here in [2].

3.1 Littlewood-Paley decomposition and Besov spaces

Let us first define the so called "Littlewood-Paley decomposition", based on a non-homogeneous dyadic partition of unity with respect to the Fourier variable. So, fix a smooth radial function χ supported in (say) the ball $B(0,\frac{4}{3})$, equal to 1 in a neighborhood of $B(0,\frac{3}{4})$ and such that $r\mapsto\chi(r\,e_r)$ is nonincreasing over $\mathbb{R}_+$, and set $\varphi\left(\xi\right)=\chi\left(\xi/2\right)-\chi\left(\xi\right)$.

The dyadic blocks $(\Delta_j)_{j\in\mathbb{Z}}$ are defined by[2]

$$\Delta_j:=0\ \text{ if }\ j\leq-2,\quad \Delta_{-1}:=\chi(D)\quad\text{and}\quad \Delta_j:=\varphi(2^{-j}D)\ \text{ if }\ j\geq0.$$

We also introduce the following low frequency cut-off:

$$S_ju:=\chi(2^{-j}D)=\sum_{k\leq j-1}\Delta_k\quad\text{for}\quad j\geq0.$$

The following classical properties will be used freely throughout the paper:

- for any $u\in\mathcal{S}'$, the equality $u=\sum_j\Delta_ju$ holds true in $\mathcal{S}'$;
- for all u and v in $\mathcal{S}'$, the sequence $(S_{j-1}u\ \Delta_jv)_{j\in\mathbb{N}}$ is spectrally supported in dyadic annuli.

One can now define what a Besov space $B^s_{p,r}$ is.

d:besov **Definition 3.1.** Let u be a tempered distribution, s a real number, and $1\leq p,r\leq\infty$. We set

$$\|u\|_{B^s_{p,r}}:=\left(\sum_j 2^{rjs}\|\Delta_ju\|^r_{L^p}\right)^{\frac{1}{r}}\quad\text{if}\quad r<\infty\quad\text{and}\quad \|u\|_{B^s_{p,\infty}}:=\sup_{j\geq-1}\left(2^{js}\|\Delta_ju\|_{L^p}\right).$$

We then define the space $B^s_{p,r}$ as the subset of distributions $u\in\mathcal{S}'$ such that $\|u\|_{B^s_{p,r}}$ is finite.

From the above definition, it is easy to show that for all $s\in\mathbb{R}$, the Besov space $B^s_{2,2}$ coincides with the non-homogeneous Sobolev space H^s, while for all $s\in\mathbb{R}_+\backslash\mathbb{N}$, the space $B^s_{\infty,\infty}$ is actually the Hölder space $\mathcal{C}^s$.

If $s\in\mathbb{N}$, instead, we set $\mathcal{C}^s_*:=B^s_{\infty,\infty}$, to distinguish it from the space $\mathcal{C}^s$ of the differentiable functions with continuous partial derivatives up to the order s. Moreover, the strict inclusion $\mathcal{C}^s_b\hookrightarrow\mathcal{C}^s_*$ holds, where $\mathcal{C}^s_b$ denotes the subset of $\mathcal{C}^s$ functions bounded with all their derivatives up to the order s.

If $s<0$, we define the "negative Hölder space" $\mathcal{C}^s$ as the Besov space $B^s_{\infty,\infty}$.

Finally, let us also point out that for any $k\in\mathbb{N}$ and $p\in[1,+\infty]$, we have the following chain of continuous embeddings:

$$B^k_{p,1}\hookrightarrow W^{k,p}\hookrightarrow B^k_{p,\infty}\,,$$

where $W^{k,p}$ denotes the set of L^p functions with derivatives up to order k in L^p.

Besov spaces have many nice properties which will be recalled throughout the paper whenever they are needed. For the time being, let us just mention that if the condition

$$s\,>\,1+\frac{N}{p}\qquad\qquad\text{or}\qquad\qquad s\,=\,1+\frac{N}{p}\quad\text{and}\quad r=1$$

holds true, then $B^s_{p,r}$ is an algebra continuously embedded in the set $\mathcal{C}^{0,1}$ of bounded Lipschitz functions, and that the gradient operator maps $B^s_{p,r}$ in $B^{s-1}_{p,r}$.

[2]Throughout we agree that $f(D)$ stands for the pseudo-differential operator $u\mapsto\mathcal{F}^{-1}(f\,\mathcal{F}u)$.

The following result will be also needed.

p:CZ **Proposition 3.2.** *Let $F : \mathbb{R}^N \to \mathbb{R}$ be a smooth homogeneous function of degree m away from a neighborhood of the origin.*

Then for all $(p,r) \in [1,\infty]^2$ and all $s \in \mathbb{R}$, the operator $F(D)$ maps $B^s_{p,r}$ in $B^{s-m}_{p,r}$.

The following fundamental lemma describes, by the so-called *Bernstein's inequalities*, the way derivatives act on spectrally localized functions.

l:bern **Lemma 3.3.** *Let $0 < r < R$. A constant C exists so that, for any nonnegative integer k, any couple (p,q) in $[1,\infty]^2$ with $1 \leq p \leq q$ and any function $u \in L^p$, we have, for all $\lambda > 0$,*

$$\operatorname{Supp}\widehat{u} \subset B(0,\lambda R) \Longrightarrow \|\nabla^k u\|_{L^q} \leq C^{k+1}\lambda^{k+N\left(\frac{1}{p}-\frac{1}{q}\right)}\|u\|_{L^p}\,;$$
$$\operatorname{Supp}\widehat{u} \subset \{\xi \in \mathbb{R}^N \,/\, r\lambda \leq |\xi| \leq R\lambda\} \Longrightarrow C^{-k-1}\lambda^k\|u\|_{L^p} \leq \|\nabla^k u\|_{L^p} \leq C^{k+1}\lambda^k\|u\|_{L^p}\,.$$

As an immediate consequence of the first Bernstein inequality, one gets the following embedding result.

c:embed **Corollary 3.4.** *The space $B^{s_1}_{p_1,r_1}$ is continuously embedded in the space $B^{s_2}_{p_2,r_2}$ for all indices satisfying $1 \leq p_1 \leq p_2 \leq +\infty$ and*

$$s_2 \,<\, s_1 - N\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \qquad \text{or} \qquad s_2 \,=\, s_1 - N\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \quad \text{and} \quad 1 \leq r_1 \leq r_2 \leq +\infty\,.$$

As we are interested in the class of Hölder spaces, from now on we will focus on this particular case.

3.2 Paradifferential calculus

Let us now introduce Bony's decomposition of the product of two tempered distrubutions u and v: we will define the paraproduct operator and recall a few nonlinear estimates in Hölder spaces. Constructing the paraproduct operator relies on the observation that, formally, the product $u\,v$, may be decomposed into

eq:bony (9)
$$u\,v \,=\, T_u v \,+\, T_v u \,+\, R(u,v)\,,$$

with

$$T_u v \,:=\, \sum_j S_{j-1}u\,\Delta_j v \qquad \text{and} \qquad R(u,v) \,:=\, \sum_j \sum_{|k-j|\leq 1} \Delta_j u\,\Delta_k v\,.$$

The above operator T is called "paraproduct", whereas R is called "remainder".

The paraproduct and remainder operators have many nice continuity properties. The following ones will be of constant use in this paper.

p:op **Proposition 3.5.** *For any $s \in \mathbb{R}$ and $t > 0$, the paraproduct operator T maps $L^\infty \times \mathcal{C}^s$ in $\mathcal{C}^s$ and $\mathcal{C}^{-t} \times \mathcal{C}^s$ in $\mathcal{C}^{s-t}$, and the following estimates hold:*

$$\|T_u v\|_{\mathcal{C}^s} \,\leq\, C\,\|u\|_{L^\infty}\,\|\nabla v\|_{\mathcal{C}^{s-1}} \qquad \text{and} \qquad \|T_u v\|_{\mathcal{C}^{s-t}} \,\leq\, C\,\|u\|_{\mathcal{C}^{-t}}\,\|\nabla v\|_{\mathcal{C}^{s-1}}\,.$$

For any s_1 and s_2 in $\mathbb{R}$ such that $s_1 + s_2 > 0$, the remainder operator R maps $\mathcal{C}^{s_1} \times \mathcal{C}^{s_2}$ in $\mathcal{C}^{s_1+s_2}$ continuously.

Combining the above proposition with Bony's decomposition (9), we easily get the following "tame estimate":

c:op **Corollary 3.6.** *Let u be a bounded function such that $\nabla u \in \mathcal{C}^{s-1}$ for some $s > 0$.*

Then for any $v \in \mathcal{C}^s$ we have $u\,v \in \mathcal{C}^s$ and there exists a constant C, depending only on N and s, such that

$$\|u\,v\|_{\mathcal{C}^s} \leq C\Big(\|u\|_{L^\infty}\,\|v\|_{\mathcal{C}^s} + \|v\|_{L^\infty}\,\|\nabla u\|_{\mathcal{C}^{s-1}}\Big).$$

In our computations we will often have to handle compositions between a paraproduct operator and a Fourier multiplier. The following lemma (see the proof e.g. in [10]) provides us with estimates for the commutator operator.

l:comm **Lemma 3.7.** *Let $m \in \mathbb{R}$, $R > 0$ and $f \in \mathcal{C}^\infty(\mathbb{R}^N)$ be a homogeneous smooth function of degree m out of the ball $B(0,R)$.*

Then, there exists a constant C, depending only on R, such that, for all $s \in \mathbb{R}$ and all $\sigma < 1$, one has:

est:comm
$$\|[T_u, f(D)]\,v\|_{\mathcal{C}^{s-m+\sigma}} \leq \frac{C}{1-\sigma}\,\|\nabla u\|_{\mathcal{C}^{\sigma-1}}\,\|v\|_{\mathcal{C}^s}. \tag{10}$$

Let us now quote another result (see [13] for the proof of the former part, [10] for the proof of the latter), pertaining to the composition of functions in Besov spaces, which will be of great importance in the sequel.

p:comp **Proposition 3.8.** *(i) Let I be an open interval of $\mathbb{R}$ and $F : I \to \mathbb{R}$ a smooth function.*

Then for all compact subset $J \subset I$ and all $s > 0$, there exists a constant C such that, for all function u valued in J and with gradient in $\mathcal{C}^{s-1}$, we have $\nabla(F \circ u) \in \mathcal{C}^{s-1}$ and

$$\|\nabla(F \circ u)\|_{\mathcal{C}^{s-1}} \leq C\,\|\nabla u\|_{\mathcal{C}^{s-1}}.$$

(ii) Let $s > 0$ and $m \in \mathbb{N}$ be such that $m > s$. Let $u \in \mathcal{C}^s$ and $\psi \in \mathcal{C}^m_b$ such that the Jacobian of ψ^{-1} is bounded.

Then $u \circ \psi \in \mathcal{C}^s$. Moreover, if $s \in]0,1[$ the following estimate holds:

$$\|u \circ \psi\|_{\mathcal{C}^s} \leq C\,(1 + \|\nabla\psi\|_{L^\infty})\,\|u\|_{\mathcal{C}^s}.$$

Finally, let us introduce the notion of paravector-field.

d:pvec-f **Definition 3.9.** Let X be a vector-field with coefficients in $\mathcal{S}'$. We can formally define the paravector-field operator T_X in the following way:

$$T_X u := \sum_{i=1}^N T_{X^i}\partial_i u$$

for all $u \in \mathcal{S}'$.

The following result (see [12] for the proof) says that the paravector-field operator is, in a certain sense, the principal part of the derivation ∂_X: the derivative along X is more regular if and only if the "paraderivation" along X is.

l:T_X **Lemma 3.10.** *For all vector field $X \in \mathcal{C}^s$ and all $u \in \mathcal{C}^t$, we have:*

- *if $t < 1$ and $s + t > 1$, then*

$$\|\partial_X u - T_X u\|_{\mathcal{C}^{s+t-1}} \leq \frac{C}{(1-t)\,(s+t-1)}\,\|X\|_{\mathcal{C}^s}\,\|\nabla u\|_{\mathcal{C}^{t-1}};$$

- *if* $t<0$, $s<1$ *and* $s+t>0$, *then*

$$\|T_X u\,-\,\mathrm{div}\,(u\,X)\|_{\mathcal{C}^{s+t-1}}\,\leq\,\frac{C}{t\,(s+t)\,(s-1)}\,\|X\|_{\mathcal{C}^s}\,\|u\|_{\mathcal{C}^t}\,;$$

- *if* $t<1$ *and* $s+t>0$, *then*

$$\|\partial_X u\,-\,T_X u\|_{\mathcal{C}^{s+t-1}}\,\leq\,\frac{C}{(s+t)\,(1-t)}\,\widetilde{\|}X\|_{\mathcal{C}^s}\,\|\nabla u\|_{\mathcal{C}^{t-1}}\,.$$

Moreover, first and last inequalities are still true even in the case $t=1$, *provided that one replaces* $\|\nabla u\|_{\mathcal{C}^0_*}$ *with* $\|\nabla u\|_{L^\infty}$, *while the second is still true even if* $t=0$, *with* $\|u\|_{L^\infty}$ *instead of* $\|u\|_{\mathcal{C}^0_*}$.

We will heavily use also the following statement about composition of paravector-field and paraproduct operators (see again [12] for its proof).

l:pvec-pprod **Lemma 3.11.** *Fix* $s\in]0,1[$. *There exist constants* C, *depending only on* s, *such that, for all* $t_1<0$ *and* $t_2\in\mathbb{R}$,

$$\begin{aligned}\|T_X\,T_u\,v\|_{\mathcal{C}^{s-1+t_1+t_2}}\quad&\leq\quad C\,(\|X\|_{\mathcal{C}^s}\,\|u\|_{\mathcal{C}^{t_1}}\,\|v\|_{\mathcal{C}^{t_2}}\,+\\&\qquad+\|v\|_{\mathcal{C}^{t_2}}\,\|T_X u\|_{\mathcal{C}^{s-1+t_1}}\,+\,\|u\|_{\mathcal{C}^{t_1}}\,\|T_X v\|_{\mathcal{C}^{s-1+t_2}})\,,\end{aligned}$$

and this is true still in the case $t_1=0$ *with* $\|u\|_{L^\infty}$ *instead of* $\|u\|_{\mathcal{C}^0_*}$.

Moreover, if $s-1+t_1+t_2\,>\,0$, *then we have also*

$$\begin{aligned}\|T_X\,R(u,v)\|_{\mathcal{C}^{s-1+t_1+t_2}}\quad&\leq\quad C\,(\|X\|_{\mathcal{C}^s}\,\|u\|_{\mathcal{C}^{t_1}}\,\|v\|_{\mathcal{C}^{t_2}}\,+\\&\qquad+\|v\|_{\mathcal{C}^{t_2}}\,\|T_X u\|_{\mathcal{C}^{s-1+t_1}}\,+\,\|u\|_{\mathcal{C}^{t_1}}\,\|T_X v\|_{\mathcal{C}^{s-1+t_2}})\,.\end{aligned}$$

3.3 Transport and elliptic equations

System (1) is basically a coupling of transport equations of the type

$$(T)\qquad\qquad\begin{cases}\partial_t f\,+\,v\cdot\nabla f\;=\;g\,,\\ f_{|t=0}\;=\;f_0\,.\end{cases}$$

So, we often need to use the following result, which enables us to solve (T) in the Hölder spaces framework.

p:transport **Proposition 3.12.** *Let* $\sigma>0$ *(*$\sigma>-1$ *if* $\mathrm{div}\,v=0$*).*
Let $f_0\in\mathcal{C}^\sigma$, $g\in L^1([0,T];\mathcal{C}^\sigma)$ *and* v *be a time dependent vector field in* $\mathcal{C}_b([0,T]\times\mathbb{R}^N)$ *such that*

$$\begin{aligned}\nabla v\quad&\in\quad L^1([0,T];L^\infty)\quad\text{ if }\quad\sigma<1\,,\\ \nabla v\quad&\in\quad L^1([0,T];\mathcal{C}^{\sigma-1})\quad\text{if}\quad\sigma>1\,.\end{aligned}$$

Then equation (T) *has a unique solution* f *in the space* $\left(\bigcap_{\sigma'<\sigma}\mathcal{C}([0,T];\mathcal{C}^{\sigma'})\right)\bigcap\mathcal{C}_w([0,T];\mathcal{C}^\sigma)$.

Moreover, for all $t\in[0,T]$ *we have*

est:no-loss-1 (11) $$e^{-CV(t)}\,\|f(t)\|_{\mathcal{C}^\sigma}\,\leq\,\|f_0\|_{\mathcal{C}^\sigma}\,+\,\int_0^t e^{-CV(\tau)}\,\|g(\tau)\|_{\mathcal{C}^\sigma}\,d\tau$$

$$\textit{with}\quad V'(t):=\begin{cases}\|\nabla v(t)\|_{L^\infty}&\textit{if }\sigma<1\,,\\ \|\nabla v(t)\|_{\mathcal{C}^{\sigma-1}}&\textit{if }\sigma>1\,.\end{cases}$$

If $f\equiv v$ *then, for all* $\sigma>0$ *(*$\sigma>-1$ *if* $\mathrm{div}\,v=0$*), estimate* (11) *holds with* $V'(t):=\|\nabla f(t)\|_{L^\infty}$.

Finally, we shall make an extensive use of energy estimates for the following elliptic equation:

eq:elliptic
$$-\operatorname{div}(a\,\nabla\Pi) = \operatorname{div} F \quad \text{in} \quad \mathbb{R}^N\,, \tag{12}$$

where $a = a(x)$ is a given suitably smooth bounded function satisfying

q:ellipticity
$$a_* \,:=\, \inf_{x\in\mathbb{R}^N} a(x) \,>\, 0\,. \tag{13}$$

We shall use the following result based on Lax-Milgram's theorem (see the proof in e.g. [13]).

l:laxmilgram
Lemma 3.13. *For all vector field F with coefficients in L^2, there exists a tempered distribution Π, unique up to constant functions, such that $\nabla\Pi \in L^2$ and equation (12) is satisfied. In addition, we have*

$$a_*\,\|\nabla\Pi\|_{L^2} \,\leq\, \|F\|_{L^2}\,.$$

4 Propagation of striated regularity

Now we are ready to tackle the proof of theorem 2.3. We will carry out it in a standard way: first of all we will prove a priori estimates for solutions of the non-homogeneous Euler equations. Then, we will construct a sequence of regular approximated solutions. Finally, thanks to upper bounds proved in the first part, we will get convergence of this sequence to a solution of our initial system, with the required properties.

4.1 A priori estimates

First of all, we will prove a priori estimates for a smooth solution $(\rho, u, \nabla\Pi)$ of system (1).

4.1.1 Estimates for density and velocity field

From first equation of (1), it follows that

$$\rho(t,x) \,=\, \rho_0\left(\psi_t^{-1}(x)\right)\,,$$

so, as the flow ψ_t is a diffeomorphism over $\mathbb{R}^N$ at all fixed time, we have that

est:rho_L^inf
$$0 \,<\, \rho_* \,\leq\, \rho(t) \,\leq\, \rho^*\,. \tag{14}$$

Applying the operator ∂_i to the same equation, using classical L^p estimates for the transport equation and Gronwall's lemma, we get

t:Drho_L^inf
$$\|\nabla\rho(t)\|_{L^\infty} \,\leq\, \|\nabla\rho_0\|_{L^\infty}\,\exp\left(C\int_0^t \|\nabla u\|_{L^\infty}\,d\tau\right)\,. \tag{15}$$

From the equation for the velocity, instead, we get, in a classical way,

$$\|u(t)\|_{L^p} \,\leq\, \|u_0\|_{L^p} \,+\, \int_0^t \left\|\frac{\nabla\Pi}{\rho}\right\|_{L^p}\,d\tau\,;$$

so, using (14) and Hölder inequalities, for a certain $\theta \in\,]0,1[$, the following estimate holds:

est:u_L^p
$$\|u(t)\|_{L^p} \,\leq\, \|u_0\|_{L^p} \,+\, \frac{C}{\rho_*}\int_0^t \|\nabla\Pi\|_{L^2}^{\theta}\,\|\nabla\Pi\|_{L^\infty}^{1-\theta}\,d\tau\,. \tag{16}$$

Remark 4.1. Let us observe that, as regularity of the pressure goes like that of the velocity field, one can try to estimate directly the L^p norm of the pressure term. Unfortunately, we can't solve its (elliptic) equation in this space without assuming a smallness condition on the gradient of the density. So, we will prove that $\nabla\Pi$ is in $L^2\cap L^\infty$, which is actually stronger than previous property and requires no other hypothesis on the density term.

Already from (15) it's clear that we need an estimate for the L^∞ norm for the gradient of the velocity. As remarked before, we can't expect to get it from the hypothesis $\Omega \in L^\infty$; the key will be the further assumption, i.e. the hypothesis of more regularity of the vorticity along some fixed directions, that is to say along the family X_0.
Here we quote also a fundamental lemma, whose proof can be found in [2] for the 2-dimensional case, in [12] and [17] for the general one. It is the key point to get the velocity field to be Lipschitz and it turns out to be immediately useful in the sequel.

l:Du_L^inf **Lemma 4.2.** *Fix $\varepsilon \in]0,1[$ and an integer $m \geq N-1$ and take a non-degenerate family $Y = (Y_\lambda)_{1\leq\lambda\leq m}$ of $\mathcal{C}^\varepsilon$ vector-fields over $\mathbb{R}^N$ such that also their divergences are in $\mathcal{C}^\varepsilon$.*

Then, for all indices $1 \leq i,j \leq N$, there exist $\mathcal{C}^\varepsilon$ functions a_{ij}, $b_{ij}^{k\lambda}$ (with $1 \leq k \leq N$, $1 \leq \lambda \leq m$) such that, for all $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N$, the following equality holds:

$$\xi_i\,\xi_j \;=\; a_{ij}(x)|\xi|^2 \,+\, \sum_{k,\lambda} b_{ij}^{k\lambda}(x)\,(Y_\lambda(x)\cdot\xi)\,\xi_k\,.$$

Moreover, the functions in the previous relation could be chosen such that

$$\begin{aligned}\|a_{ij}\|_{L^\infty} &\leq 1\\ \left\|b_{ij}^{k\lambda}\right\|_{\mathcal{C}^\varepsilon} &\leq C\,\frac{m^{2N-2}}{I(Y)}\,|||Y|||_{\mathcal{C}^\varepsilon}^{9N-10}\,.\end{aligned}$$

Now, we can state the stationary estimate which says that the velocity field u is Lipschitz. This can be done as in the classical case, because it's based only on the Biot-Savart law, or better on it's gradient version (4).

p:Du_L^inf **Proposition 4.3.** *Fix $\varepsilon \in]0,1[$ and $q \in]1,+\infty[$ and take a non-degenerate family $Y = (Y_\lambda)_{1\leq\lambda\leq m}$ of $\mathcal{C}^\varepsilon$ vector-fields over $\mathbb{R}^N$ such that also their divergences are still in $\mathcal{C}^\varepsilon$.*

Then there exists a constant C, depending only on the space dimension N and on the number of vector-fields m, such that, for all skew-symmetric matrices Ω with coefficients in $L^q \cap \mathcal{C}^\varepsilon_Y$, the corresponding (by (3)) divergence-free vector-field u satisfies

est:Du_L^inf
$$\|\nabla u\|_{L^\infty} \,\leq\, C\left(\frac{q^2}{q-1}\,\|\Omega\|_{L^q} \,+\, \frac{1}{\varepsilon\,(1-\varepsilon)}\,\|\Omega\|_{L^\infty}\log\left(e \,+\, \frac{\|\Omega\|_{\mathcal{C}^\varepsilon_Y}}{\|\Omega\|_{L^\infty}}\right)\right). \tag{17}$$

4.1.2 Estimates for the vorticity

As in [14], using the well-known L^q estimates for transport equation and taking advantage of Gronwall's lemma and Hölder inequality in Lebesgue spaces, from (6) we obtain, for a certain $\gamma \in]0,1[$,

est:Om_L^q
$$\begin{aligned}\|\Omega(t)\|_{L^q} \;&\leq\; C\,\exp\left(\int_0^t\|\nabla u\|_{L^\infty}d\tau\right)\times\\ &\times\left(\|\Omega_0\|_{L^q} \,+\, \frac{1}{(\rho_*)^2}\int_0^t e^{-\int_0^\tau\|\nabla u\|_{L^\infty}d\tau'}\,\|\nabla\rho\|_{L^\infty}\,\|\nabla\Pi\|_{L^2}^\gamma\,\|\nabla\Pi\|_{L^\infty}^{1-\gamma}\,d\tau\right).\end{aligned} \tag{18}$$

Moreover, of course an analogue estimate holds also for the L^∞ norm:

est:Om_L^inf
$$\begin{aligned}\|\Omega(t)\|_{L^\infty} \;&\leq\; C\,\exp\left(\int_0^t\|\nabla u\|_{L^\infty}d\tau\right)\times\\ &\times\left(\|\Omega_0\|_{L^\infty} \,+\, \frac{1}{(\rho_*)^2}\int_0^t e^{-\int_0^\tau\|\nabla u\|_{L^\infty}d\tau'}\,\|\nabla\rho\|_{L^\infty}\,\|\nabla\Pi\|_{L^\infty}\,d\tau\right).\end{aligned} \tag{19}$$

r:q **Remark 4.4.** Let us fix the index p pertaining to u and let us call $\overline{q}$ the real number in $[2,+\infty[$ such that $1/p + 1/\overline{q} = 1/2$. From our hypothesis, it's clear that $q \leq \overline{q}$; therefore, thanks to Hölder and Young inequalities, we have

$$\|\Omega\|_{L^{\overline{q}}} \,\leq\, \|\Omega\|_{L^q}^\eta\,\|\Omega\|_{L^\infty}^{1-\eta} \,\leq\, \|\Omega\|_{L^q\cap L^\infty}\,.$$

4.1.3 Estimates for the pressure term

sss:est-p

Now, let us focus on the pressure term: taking the divergence of the second equation of system (1), we discover that it solves the elliptic equation

eq:Pi

$$-\operatorname{div}\left(\frac{\nabla\Pi}{\rho}\right)\,=\,\operatorname{div}\,(u\cdot\nabla u)\,. \tag{20}$$

From this, remembering our hypothesis and remark 4.4, estimate (5) and lemma 3.13, the control of L^2 norm immeditely follows:

est:Pi_L^2

$$\frac{1}{\rho^*}\left\|\nabla\Pi\right\|_{L^2}\,\leq\,C\left\|u\right\|_{L^p}\left\|\Omega\right\|_{L^q\cap L^\infty}\,. \tag{21}$$

Moreover, we have that $\nabla\Pi$ belongs also to L^∞, and so, by interpolation, $\nabla\Pi\in L^a$ for all $a\in[2,+\infty]$. As a matter of fact, now we are going to show a stronger claim, that is to say $\nabla\Pi\in\mathcal{C}^1_*$. Cutting in low and high frequencies, we have that

$$\left\|\nabla\Pi\right\|_{\mathcal{C}^1_*}\,\leq\,\left\|\Delta_{-1}\nabla\Pi\right\|_{\mathcal{C}^1_*}\,+\,\left\|(\mathrm{Id}\,-\,\Delta_{-1})\nabla\Pi\right\|_{\mathcal{C}^1_*}\,\leq\,C\left(\left\|\nabla\Pi\right\|_{L^2}\,+\,\left\|\Delta\Pi\right\|_{\mathcal{C}^0_*}\right)\,.$$

Now, from (20) we get

eq:Lapl-Pi

$$-\,\Delta\Pi\,=\,\nabla\left(\log\rho\right)\cdot\nabla\Pi\,+\,\rho\,\operatorname{div}\,(u\cdot\nabla u)\,. \tag{22}$$

From this last relation, from the fact that $\operatorname{div}(u\cdot\nabla u)=\nabla u:\nabla u$ and the immersion $L^\infty\hookrightarrow\mathcal{C}^0_*$, we obtain

$$\begin{aligned}\left\|\Delta\Pi\right\|_{\mathcal{C}^0_*}\,\leq\,\left\|\Delta\Pi\right\|_{L^\infty}\,&\leq\,\left\|\nabla\left(\log\rho\right)\cdot\nabla\Pi\right\|_{L^\infty}\,+\,\left\|\rho\,\operatorname{div}\,(u\cdot\nabla u)\right\|_{L^\infty}\\&\leq\,C\left(\left\|\nabla\rho\right\|_{L^\infty}\left\|\nabla\Pi\right\|_{L^\infty}\,+\,\rho^*\left\|\nabla u\right\|^2_{L^\infty}\right)\,.\end{aligned}$$

Now, $\mathcal{C}^1_*\hookrightarrow\mathcal{C}^\eta\hookrightarrow L^\infty$ for all $\eta\in\,]0,1[$; taking for instance $\eta=1/2$ and using interpolation inequalities between Besov spaces, we thus have, for a certain $\beta\in\,]0,1[\,$,

$$\left\|\nabla\Pi\right\|_{L^\infty}\,\leq\,\left\|\nabla\Pi\right\|_{\mathcal{C}^{1/2}}\,\leq\,C\left\|\nabla\Pi\right\|^\beta_{\mathcal{C}^{-N/2}}\left\|\nabla\Pi\right\|^{1-\beta}_{\mathcal{C}^1_*}\,\leq\,C\left\|\nabla\Pi\right\|^\beta_{L^2}\left\|\nabla\Pi\right\|^{1-\beta}_{\mathcal{C}^1_*}\,.$$

Thanks to Young's inequality, from this relation and (21) one finally gets

est:Pi_C^1_*

$$\left\|\nabla\Pi\right\|_{\mathcal{C}^1_*}\,\leq\,C\,\left(\left(1+\left\|\nabla\rho\right\|^\delta_{L^\infty}\right)\left\|u\right\|_{L^p}\left\|\Omega\right\|_{L^q\cap L^\infty}\,+\,\rho^*\left\|\nabla u\right\|^2_{L^\infty}\right)\,, \tag{23}$$

for some δ depending only on the space dimension N. So we have proved our claim, i.e. $\nabla\Pi\in\mathcal{C}^1_*$, and so it belongs also to L^∞.

Finally, we want to prove boundedness of second derivatives of the pressure term. This property is a consequence of striated regularity for $\nabla\Pi$ we will show in next section: for the time being, let us admit this fact. So, passing in Fourier variables and using lemma 4.2, for all $1\leq i\,,j\leq N$ we can write

$$\xi_i\,\xi_j\,\widehat{\Pi}(\xi)\,=\,a_{ij}(x)|\xi|^2\,\widehat{\Pi}(\xi)\,+\,\sum_{k,\lambda}b^{k\lambda}_{ij}(x)\left(X_\lambda(x)\cdot\xi\right)\xi_k\,\widehat{\Pi}(\xi)\,.$$

Applying the inverse Fourier transform $\mathcal{F}^{-1}_\xi$ and passing to L^∞ norms, we get

$$\left\|\nabla^2\Pi\right\|_{L^\infty}\,\leq\,C\left(\left\|\Delta\Pi\right\|_{L^\infty}\,+\,\left\|\partial_X\nabla\Pi\right\|_{L^\infty}\right)\,.$$

As we will see later, $\partial_X\nabla\Pi\in\mathcal{C}^\varepsilon$; nevertheless, due to technical reasons it's convenient for us to estimate its L^∞ norm in an intermediate space $\mathcal{C}^\varepsilon\hookrightarrow\mathcal{C}^\eta\hookrightarrow\mathcal{C}^0_*$ and then use interpolation inequalities. For instance, the choice $\eta=\varepsilon/4$ will be suitable for our purposes:

$$\left\|\partial_X\nabla\Pi\right\|_{L^\infty}\,\leq\,\left\|\partial_X\nabla\Pi\right\|_{\mathcal{C}^{\varepsilon/4}}\,\leq\,\left\|\partial_X\nabla\Pi\right\|^{3/4}_{\mathcal{C}^0_*}\left\|\partial_X\nabla\Pi\right\|^{1/4}_{\mathcal{C}^\varepsilon}\,.$$

Using Bony's paraproduct decomposition to handle the norm in $\mathcal{C}^0_*$ finally leads us to the following control:

$$\|\partial_X \nabla\Pi\|_{L^\infty} \,\leq\, \|X\|^{3/4}_{\mathcal{C}^\varepsilon}\,\|\nabla\Pi\|^{3/4}_{\mathcal{C}^1_*}\,\|\partial_X\nabla\Pi\|^{1/4}_{\mathcal{C}^\varepsilon}\,.$$

Therefore, from previous inequality and from the control for $\Delta\Pi$, we get

est:D^2-Pi

$$\left\|\nabla^2\Pi\right\|_{L^\infty} \,\leq\, C\left(\|\nabla\rho\|_{L^\infty}\,\|\nabla\Pi\|_{\mathcal{C}^1_*} \,+\, \rho^*\,\|\nabla u\|^2_{L^\infty} \,+\, \|X\|^{3/4}_{\mathcal{C}^\varepsilon}\,\|\nabla\Pi\|^{3/4}_{\mathcal{C}^1_*}\,\|\partial_X\nabla\Pi\|^{1/4}_{\mathcal{C}^\varepsilon}\right)\,, \tag{24}$$

and this finally proves our claim, once one admits striated regularity for $\nabla\Pi$.

4.2 A priori estimates for striated regularity

After having established the "classical" estimates, let us now focus on the conservation of striated regularity. The most important step lies in finding a priori estimates for the derivations along the vector-field X. So, let us now state a lemma which explains the relation between the operators ∂_X and $\mathrm{div}\,(\,\cdot\,X)$ (see also remark 2.2).

l:div

Lemma 4.5. *For every vector-field X with components and divergence in $\mathcal{C}^\varepsilon$, and every function $f\in\mathcal{C}^\eta$ for some $\eta\in\,]0,1]$, we have*

$$\|\mathrm{div}\,(f\,X)\,-\,\partial_X f\|_{\mathcal{C}^{\min\{\varepsilon,\eta\}-1}} \,\leq\, C\,\widetilde{\|X\|}_{\mathcal{C}^\varepsilon}\,\|f\|_{\mathcal{C}^\eta}\,.$$

Moreover, the previous inequality is still true in the limit case $\eta=0$, with $\|f\|_{L^\infty}$ instead of $\|f\|_{\mathcal{C}^0_*}$.

Proof. The thesis immediately follows from the identity $\mathrm{div}\,(f\,X)\,-\,\partial_X f \,=\, f\,\mathrm{div}X$ and from Bony's paraproduct decomposition. □

4.2.1 The evolution of the family of vector-fields

First of all, we want to prove that the family of vector-fields $X(t)=(X_\lambda(t))_{1\leq\lambda\leq m}$, where each $X_\lambda(t)$ is defined by (7), still remains non-degenerate for all t, and that each $X_\lambda(t)$ still has components and divergence in $\mathcal{C}^\varepsilon$. Throughout this paragraph we will denote by $Y(t)$ a generic element of the family $X(t)$.

Applying the divergence operator to (8), an easy computation shows us that $\mathrm{div}\,Y$ satisfies

$$(\partial_t\,+\,u\cdot\nabla)\;\mathrm{div}\,Y \,=\, 0\,,$$

which immediately implies $\mathrm{div}\,Y(t)\in\mathcal{C}^\varepsilon$ for all t and

est:div-X

$$\|\mathrm{div}\,Y(t)\|_{\mathcal{C}^\varepsilon} \,\leq\, C\,\|\mathrm{div}\,Y_0\|_{\mathcal{C}^\varepsilon}\,\exp\left(c\int_0^t\|\nabla u\|_{L^\infty}\,d\tau\right)\,. \tag{25}$$

Moreover, starting again from (8), we get (for the details, see proposition 4.1 of [12])

$$(\partial_t\,+\,u\cdot\nabla)\left(\overset{N-1}{\wedge}\,X_\lambda\right) \,=\, {}^t\nabla u\cdot\left(\overset{N-1}{\wedge}\,X_\lambda\right)\,,$$

from which it follows

$$\left(\overset{N-1}{\wedge}\,X_\lambda\right)(t,x) \,=\, \left(\overset{N-1}{\wedge}\,X_\lambda\right)(0,\psi_t^{-1}(x)) \,-\, \int_0^t {}^t\nabla u\cdot\left(\overset{N-1}{\wedge}\,X_\lambda\right)(\tau,\psi_t^{-1}(\psi_\tau(x)))\,d\tau\,.$$

This relation gives us

$$\left|\left(\overset{N-1}{\wedge}\,X_\lambda\right)(0,\psi_t^{-1}(x))\right| \;\leq\; \left|\left(\overset{N-1}{\wedge}\,X_\lambda\right)(t,x)\right| \,+$$
$$+\,\int_0^t\|\nabla u(t-\tau)\|_{L^\infty}\,\left|\left(\overset{N-1}{\wedge}\,X_\lambda\right)(t-\tau,\psi_\tau^{-1}(x))\right|\,d\tau\,,$$

and by Gronwall's lemma one gets

$$\left|\left(\overset{N-1}{\wedge} X_\lambda\right)(t,x)\right| \geq \left|\left(\overset{N-1}{\wedge} X_{0,\lambda}\right)(\psi_t^{-1}(x))\right| e^{-c\int_0^t \|\nabla u\|_{L^\infty}\,d\tau}.$$

From this inequality we immediately have that the family still remains non-degenerate at every time t:

est:I (26)
$$I(X(t)) \geq I(X_0)\,\exp\left(-c\int_0^t \|\nabla u\|_{L^\infty}\,d\tau\right). \tag{26}$$

Finally, again from the evolution equation (8), it's clear that, to prove that $Y(t)$ is of class $\mathcal{C}^\varepsilon$, we need a control on the norm in this space of the term $\partial_Y u$. To get this, we use, as very often in the sequel, the following decomposition:

$$\partial_Y u = T_Y u + (\partial_Y - T_Y)\,u,$$

with (by lemma 3.10)

$$\|(\partial_Y - T_Y)\,u\|_{\mathcal{C}^\varepsilon} \leq C\,\widetilde{\|}Y\|_{\mathcal{C}^\varepsilon}\,\|\nabla u\|_{L^\infty}.$$

Moreover, for all $1 \leq i \leq N$ thanks to (3) we can write

$$T_Y u^i = -\sum_{k,j}\left(\partial_k(-\Delta)^{-1}T_{Y^j}\partial_j\Omega_{ik} - \left[\partial_k(-\Delta)^{-1}, T_{Y^j}\partial_j\right]\Omega_{ik}\right).$$

Obviously, from lemma 3.10 we have

$$\left\|\partial_k(-\Delta)^{-1}\sum_j T_{Y^j}\partial_j\Omega_{ik}\right\|_{\mathcal{C}^\varepsilon} \leq \|T_Y\Omega\|_{\mathcal{C}^{\varepsilon-1}} \leq \|\partial_Y\Omega\|_{\mathcal{C}^{\varepsilon-1}} + C\,\widetilde{\|}Y\|_{\mathcal{C}^\varepsilon}\,\|\Omega\|_{L^\infty},$$

while for the commutator term we use lemma 3.7, which gives us the following control:

$$\left\|\left[\partial_k(-\Delta)^{-1}, T_{Y^j}\partial_j\right]\Omega_{ik}\right\|_{\mathcal{C}^\varepsilon} \leq C\,\|Y\|_{\mathcal{C}^\varepsilon}\,\|\Omega\|_{L^\infty}.$$

So, in the end, from the hypothesis of striated regularity for the vorticity we get that also the velocity field u is more regular along the fixed directions and

est:d_X-u (27)
$$\|\partial_Y u\|_{\mathcal{C}^\varepsilon} \leq C\left(\|\partial_Y\Omega\|_{\mathcal{C}^{\varepsilon-1}} + \widetilde{\|}Y\|_{\mathcal{C}^\varepsilon}\,\|\nabla u\|_{L^\infty}\right). \tag{27}$$

Moreover, applying proposition 3.12 to (8) and using (27), (25) and Gronwall's inequality finally give us

est:X_C^e (28)
$$\widetilde{\|}Y(t)\|_{\mathcal{C}^\varepsilon} \leq C\,\exp\left(c\int_0^t\|\nabla u\|_{L^\infty}\,d\tau\right)\left(\widetilde{\|}Y_0\|_{\mathcal{C}^\varepsilon} + \int_0^t e^{-c\int_0^\tau\|\nabla u\|_{L^\infty}\,d\tau'}\,\|\partial_Y\Omega\|_{\mathcal{C}^{\varepsilon-1}}\,d\tau\right). \tag{28}$$

These estimates having being established, from now on for simplicity we will consider the case of only one vector-field $X(t)$: the generalization to the case of a finite family is quite obvious, and where the difference is substantial, we will suggest references for the details.

4.2.2 Striated regularity for the density

Now, we want to investigate propagation of striated regularity for the density. First of all, let us state a stationary lemma.

l:f->Df **Lemma 4.6.** *Let f be a function in $\mathcal{C}^1_*$.*

(i) If $\partial_X f \in \mathcal{C}^\varepsilon$ and $\nabla f \in L^\infty$, then one has $\partial_X \nabla f \in \mathcal{C}^{\varepsilon-1}$ and the following inequality holds:

est:f->Df (29)
$$\|\partial_X \nabla f\|_{\mathcal{C}^{\varepsilon-1}} \leq C \left(\|\partial_X f\|_{\mathcal{C}^\varepsilon} + \widetilde{\|} X\|_{\mathcal{C}^\varepsilon} \left(\|f\|_{\mathcal{C}^1_*} + \|\nabla f\|_{L^\infty} \right) \right) .$$

(ii) Conversely, if $\partial_X \nabla f \in \mathcal{C}^{\varepsilon-1}$, then $\partial_X f \in \mathcal{C}^\varepsilon$ and one has

est:Df->f (30)
$$\|\partial_X f\|_{\mathcal{C}^\varepsilon} \leq C \left(\widetilde{\|} X\|_{\mathcal{C}^\varepsilon} \left(\|f\|_{\mathcal{C}^1_*} + \|\nabla f\|_{L^\infty} \right) + \|\partial_X \nabla f\|_{\mathcal{C}^{\varepsilon-1}} \right) .$$

Proof. (i) Using the paravector-field operator (remember definition 3.9), we can write:

$$\partial_X \nabla f = (\partial_X - T_X) \nabla f + T_X \nabla f .$$

From lemma 3.10, we have that the first term of the previous equality is in $\mathcal{C}^{\varepsilon-1}$ and

est:d-T_D (31)
$$\|(\partial_X - T_X) \nabla f\|_{\mathcal{C}^{\varepsilon-1}} \leq C \widetilde{\|} X\|_{\mathcal{C}^\varepsilon} \|\nabla f\|_{L^\infty} .$$

Now, we have to estimate the paravector-field term: note that

$$T_X \nabla f = \nabla (T_X f) + [T_X, \nabla] f .$$

From hypothesis of the lemma, it's obvious that $\nabla (T_X f) \in \mathcal{C}^{\varepsilon-1}$. For the last term, remembering that ∇ and T_X are operators of order 1, we can use lemma 3.7 and get

est:T_D (32)
$$\|[T_X, \nabla] f\|_{\mathcal{C}^{\varepsilon-1}} \leq C \|X\|_{\mathcal{C}^\varepsilon} \|f\|_{\mathcal{C}^1_*} .$$

Putting together (31), (32) and the control for $\|\nabla (T_X f)\|_{\mathcal{C}^{\varepsilon-1}}$ gives us the first part of the lemma.

(ii) For the second part, we can write:

$$\partial_X f = T_X f + (\partial_X - T_X) f .$$

By definition of the space $\mathcal{C}^\varepsilon_X$, we know that ∇f is bounded: so, second term can be easily controlled in $\mathcal{C}^\varepsilon$ thanks to lemma 3.10. Now let us define the operator Ψ such that, in Fourier variables, for all vector-fields v we have

$$\mathcal{F}_x (\Psi v)(\xi) = -i \frac{1}{|\xi|^2} \xi \cdot \widehat{v}(\xi) .$$

So, noting that the paravector term involves only high frequencies of f, we can write

$$T_X f = T_X (\Psi \nabla f) = \Psi T_X \nabla f + [T_X, \Psi] \nabla f .$$

Now, applying lemmas 3.10 and 3.7 completes the proof.

□

r:lem_f->Df **Remark 4.7.** Let us note that, if $f \in L^b$ (for some $b \in [1, +\infty]$) is such that $\nabla f \in L^\infty$, then $f \in \mathcal{C}^1_*$ (indeed $f \in \mathcal{C}^{0,1}$) and (separating low and high frequencies)

$$\|f\|_{\mathcal{C}^1_*} \leq C \left(\|f\|_{L^b} + \|\nabla f\|_{L^\infty} \right) .$$

Both u and ρ satisfy such an estimate, respectively with $b = p$ and $b = +\infty$.

Thanks to lemma 4.6, we can equally deal with ρ or $\nabla\rho$: as the equation for ρ is very simple, we choose to work with it. Keeping in mind that $[X(t)\,,\,\partial_t + u\cdot\nabla] = 0$, we have

$$\partial_t\left(\partial_X\rho\right) \,+\, u\cdot\nabla\left(\partial_X\rho\right) \,=\, 0\,,$$

from which (remember also (30)) it immediately follows that

est:d_X-rho
$$\left\|\partial_{X(t)}\rho(t)\right\|_{\mathcal{C}^\varepsilon} \,\leq\, C\left(\widetilde{\|}X_0\|_{\mathcal{C}^\varepsilon}\left(\rho^* + \|\nabla\rho_0\|_{L^\infty}\right) \,+\, \|\partial_{X_0}\nabla\rho_0\|_{\mathcal{C}^{\varepsilon-1}}\right)\exp\left(c\int_0^t\|\nabla u\|_{L^\infty}\,d\tau\right)\,. \tag{33}$$

Therefore, one gets also

est:d_X-Drho
$$\begin{aligned}\left\|\partial_{X(t)}\nabla\rho(t)\right\|_{\mathcal{C}^{\varepsilon-1}} \quad\leq\quad & C\,\exp\left(\int_0^t\|\nabla u\|_{L^\infty}\,d\tau\right)\,\times \\ & \times\left(\left(\rho^* + \|\nabla\rho_0\|_{L^\infty}\right)\widetilde{\|}X_0\|_{\mathcal{C}^\varepsilon} \,+\, \|\partial_{X_0}\nabla\rho_0\|_{\mathcal{C}^{\varepsilon-1}} \,+\right. \\ & \left. +\,\int_0^t e^{-\int_0^\tau\|\nabla u\|_{L^\infty}d\tau'}\,\|\partial_X\Omega\|_{\mathcal{C}^{\varepsilon-1}}\,d\tau\right)\,.\end{aligned} \tag{34}$$

4.2.3 Striated regularity for the pressure term

In this paragraph we want to show geometric properties for the pressure term we have pointed out in section 4.1.3, i.e. we want to prove $\partial_X\nabla\Pi \in \mathcal{C}^\varepsilon$. As a matter of fact, as regularity of the gradient of the pressure goes like that of the velocity field, it seems quite natural to expect such a property.

Again, we use the decomposition

$$\partial_X\nabla\Pi \,=\, T_X\left(\nabla\Pi\right) \,+\, \left(\partial_X - T_X\right)\nabla\Pi\,.$$

Let us consider the second term: from lemma 3.10 we have

$$\|\left(\partial_X \,-\, T_X\right)\nabla\Pi\|_{\mathcal{C}^\varepsilon} \,\leq\, C\,\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\left\|\nabla^2\Pi\right\|_{L^\infty}\,.$$

Hence, using estimate (24) and applying Young's inequality to isolate the term $\|\partial_X\nabla\Pi\|_{\mathcal{C}^\varepsilon}$, we see that it can be controlled by the quantity

est:d_X-T_X-Pi
$$C\,\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\left(\left(\|\nabla\rho\|_{L^\infty} + \widetilde{\|}X\|^{4/3}_{\mathcal{C}^\varepsilon}\right)\|\nabla\Pi\|_{\mathcal{C}^1_*} \,+\, \rho^*\,\|\nabla u\|^2_{L^\infty}\right) \,+\, \frac{1}{4}\,\|\partial_X\nabla\Pi\|_{\mathcal{C}^\varepsilon}\,. \tag{35}$$

To deal with the paravector term, we keep in mind that $\nabla\Pi \,=\, \nabla\left(-\Delta\right)^{-1}\left(g_1 \,+\, g_2\right)$, where we have set

$$\begin{cases} g_1 \;=\; -\,\nabla\left(\log\rho\right)\cdot\nabla\Pi \\ g_2 \;=\; \rho\;\mathrm{div}\left(u\cdot\nabla u\right)\,, \end{cases}$$

so it's enough to prove that both $T_X\nabla\left(-\Delta\right)^{-1}g_1$ and $T_X\nabla\left(-\Delta\right)^{-1}g_2$ belong to $\mathcal{C}^\varepsilon$. Let us consider first the term

eq:T_X-Op
$$T_X\nabla\left(-\Delta\right)^{-1}g_2 \,=\, \nabla\left(-\Delta\right)^{-1}T_Xg_2 \,+\, \left[T_X, \nabla\left(-\Delta\right)^{-1}\right]g_2\,. \tag{36}$$

From lemma 3.7 one immediately gets that

$$\left\|\left[T_X, \nabla\left(-\Delta\right)^{-1}\right]g_2\right\|_{\mathcal{C}^\varepsilon} \,\leq\, C\,\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\|g_2\|_{\mathcal{C}^0_*} \,\leq\, C\,\rho^*\,\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\|\nabla u\|^2_{L^\infty}\,,$$

while it's obvious that

$$\left\|\nabla\left(-\Delta\right)^{-1}T_Xg_2\right\|_{\mathcal{C}^\varepsilon} \,\leq\, C\,\|T_Xg_2\|_{\mathcal{C}^{\varepsilon-1}}\,.$$

Now we use Bony's paraproduct decomposition and write

$$T_X g_2 \,=\, T_X T_\rho \mathrm{div}\,(u\cdot\nabla u) \,+\, T_X T_{\mathrm{div}\,(u\cdot\nabla u)}\rho \,+\, T_X R\,(\rho, \mathrm{div}\,(u\cdot\nabla u))\,.$$

Lemma 3.11 and the fact that $\mathrm{div}\,(u\cdot\nabla u)=\nabla u:\nabla u$ tell us that the the first two terms of the previous relation can be bounded in $\mathcal{C}^{\varepsilon-1}$ by

$$\|X\|_{\mathcal{C}^\varepsilon}\,\|\rho\|_{\mathcal{C}^1_*}\,\|\nabla u\|^2_{L^\infty} \,+\, \|\nabla u\|^2_{L^\infty}\,\|T_X\rho\|_{\mathcal{C}^\varepsilon} \,+\, \|\rho\|_{\mathcal{C}^1_*}\,\|T_X \mathrm{div}\,(u\cdot\nabla u)\|_{\mathcal{C}^{\varepsilon-1}}\,;$$

the remainder term is actually more regular, and it can be estimated in $\mathcal{C}^\varepsilon$ by the same quantity as before. Now the problem is the control of the $\mathcal{C}^{\varepsilon-1}$ norm of $T_X\mathrm{div}\,(u\cdot\nabla u)$. We write

$$\begin{aligned} T_X \mathrm{div}\,(u\cdot\nabla u) &= \sum_{i,j} 2\,T_X T_{\partial_i u^j}\partial_j u^i \,+\, T_X\partial_i R(u^j,\partial_j u^i) \\ &= \sum_{i,j,k} 2\,T_{X^k}\partial_k T_{\partial_i u^j}\partial_j u^i \,+\partial_i T_{X^k}\partial_k R(u^j,\partial_j u^i) \,-\, T_{\partial_i X^k}\partial_k R(u^j,\partial_j u^i)\,. \end{aligned}$$

Again from lemma 3.11 we can easily see that the quantity

$$\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\left(\|u\|_{L^p}+\|\nabla u\|_{L^\infty}\right)\|\nabla u\|_{L^\infty} \,+\, \left(\|u\|_{L^p}+\|\nabla u\|_{L^\infty}\right)\|T_X\nabla u\|_{\mathcal{C}^{\varepsilon-1}} \,+\, \|\nabla u\|_{L^\infty}\,\|T_X u\|_{\mathcal{C}^\varepsilon}$$

controls the $\mathcal{C}^{\varepsilon-1}$ norm of all the previous terms; so, keeping in mind lemmas 3.10 and 4.6, it follows that

$$\|T_X\mathrm{div}\,(u\cdot\nabla u)\|_{\mathcal{C}^{\varepsilon-1}} \,\leq\, C\left(\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\left(\|u\|_{L^p}+\|\nabla u\|_{L^\infty}\right)^2 \,+\, \|\partial_X u\|_{\mathcal{C}^\varepsilon}\left(\|u\|_{L^p}+\|\nabla u\|_{L^\infty}\right)\right)\,.$$

Putting together all these estimates, applying again lemma 3.10 and remembering remark 4.7, we finally get

est:g_2

$$\begin{aligned} \left\|T_X\nabla\,(-\Delta)^{-1}g_2\right\|_{\mathcal{C}^\varepsilon} \;\leq\; C\Big(& (\rho^* \,+\, \|\nabla\rho\|_{L^\infty})\,\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\left(\|u\|_{L^p}+\|\nabla u\|_{L^\infty}\right)^2 \,+ \\ & +\,(\rho^* \,+\, \|\nabla\rho\|_{L^\infty})\,\|\partial_X u\|_{\mathcal{C}^\varepsilon}\left(\|u\|_{L^p}+\|\nabla u\|_{L^\infty}\right) \,+ \\ & +\,\|\partial_X\rho\|_{\mathcal{C}^\varepsilon}\,\|\nabla u\|^2_{L^\infty}\Big)\,. \end{aligned} \tag{37}$$

Before going on, let us state a simple lemma.

l:d_X-F

Lemma 4.8. *Fix a $\varepsilon\in\,]0,1[$ and an open interval $I\subset\mathbb{R}$.*
Let X be a $\mathcal{C}^\varepsilon$ vector-field with divergence in $\mathcal{C}^\varepsilon$ and $F:I\to\mathbb{R}$ be a smooth function.

Then for all compact set $J\subset I$ and all $\rho\in W^{1,\infty}$ valued in J and such that $\partial_X\rho\in\mathcal{C}^\varepsilon$, one has that $\partial_X(F\circ\rho)\in\mathcal{C}^\varepsilon$ and $\partial_X\nabla(F\circ\rho)\in\mathcal{C}^{\varepsilon-1}$. Moreover, the following estimates hold:

$$\begin{aligned} \|\partial_X(F\circ\rho)\|_{\mathcal{C}^\varepsilon} &\leq C\,\|\rho\|_{W^{1,\infty}}\,\|\partial_X\rho\|_{\mathcal{C}^\varepsilon} \\ \|\partial_X\nabla(F\circ\rho)\|_{\mathcal{C}^{\varepsilon-1}} &\leq C\,\|\rho\|_{W^{1,\infty}}\left(\|\partial_X\rho\|_{\mathcal{C}^\varepsilon} \,+\, \widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\|\rho\|_{W^{1,\infty}}\right)\,, \end{aligned}$$

for a constant C depending only on F and on the fixed subset J.

Proof. The first inequality is immediate keeping in mind identity $\partial_X(F\circ\rho)\,=\,F'(\rho)\,\partial_X\rho$ and estimate

$$\left\|F'(\rho)\right\|_{\mathcal{C}^\varepsilon} \,\leq\, C\,\left\|F''\right\|_{L^\infty(J)}\,\|\rho\|_{\mathcal{C}^\varepsilon} \,\leq\, C\,\left\|F''\right\|_{L^\infty(J)}\,\|\rho\|_{W^{1,\infty}}\,.$$

For the second one, we write:

$$\partial_X\nabla(F\circ\rho) \,=\, \partial_X\left(F'(\rho)\,\nabla\rho\right) \,=\, F'(\rho)\,\partial_X\nabla\rho \,+\, F''(\rho)\,\nabla\rho\;\partial_X\rho\,.$$

Let us observe that the first term is well-defined in $\mathcal{C}^{\varepsilon-1}$, and using decomposition in paraproducts and remainder operators, we have

$$\left\|F'(\rho)\,\partial_X\nabla\rho\right\|_{\mathcal{C}^{\varepsilon-1}} \,\leq\, C\,\left\|F'(\rho)\right\|_{W^{1,\infty}}\,\|\partial_X\nabla\rho\|_{\mathcal{C}^{\varepsilon-1}}\,.$$

Now, the thesis immediately follows from lemma 4.6. $\square$

Let us come back to g_1: we use the same trick as (36). Again, the control of the commutator term follows from lemma 3.7:

$$\left\|\left[T_X, \nabla\,(-\Delta)^{-1}\right] g_1\right\|_{\mathcal{C}^\varepsilon} \leq C\,\|X\|_{\mathcal{C}^\varepsilon}\,\|g_1\|_{\mathcal{C}^0_*} \leq \frac{C}{\rho_*}\,\|X\|_{\mathcal{C}^\varepsilon}\,\|\nabla\rho\|_{L^\infty}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\,.$$

For the second term, we use again Bony's paraproduct decomposition. From lemma 3.11, we see that both $T_X T_{\nabla(\log\rho)}\nabla\Pi$ and $T_X T_{\nabla\Pi}\nabla(\log\rho)$ belong to $\mathcal{C}^{\varepsilon-1}$, while the remainder term is in $\mathcal{C}^{\varepsilon/2}$, and the quantity

$$\|X\|_{\mathcal{C}^\varepsilon}\,\|\nabla\rho\|_{L^\infty}\,\|\nabla\Pi\|_{\mathcal{C}^1_*} + \|\nabla\Pi\|_{\mathcal{C}^1_*}\,\|T_X\nabla(\log\rho)\|_{\mathcal{C}^{\varepsilon-1}} + \|\nabla\rho\|_{L^\infty}\,\|T_X\nabla\Pi\|_{\mathcal{C}^{\varepsilon/2}}$$

bounds their norms in the respective spaces.
Now, from lemmas 3.10 and 4.8 we get

$$\|T_X\nabla(\log\rho)\|_{\mathcal{C}^{\varepsilon-1}} \leq C\,(\rho^* + \|\nabla\rho\|_{L^\infty})\left(\|\partial_X\rho\|_{\mathcal{C}^\varepsilon} + (\rho^* + \|\nabla\rho\|_{L^\infty})\,\widetilde{\|X\|}_{\mathcal{C}^\varepsilon}\right)\,,$$

while interpolation inequalities for the inclusions $\mathcal{C}^\varepsilon \hookrightarrow \mathcal{C}^{\varepsilon/2} \hookrightarrow \mathcal{C}^0_*$ and proposition 3.5 (just as done in section 4.1.3 for bounding the L^∞ norm of $\partial_X\nabla\Pi$) give us

$$\|T_X\nabla\Pi\|_{\mathcal{C}^{\varepsilon/2}} \leq C\,\|X\|^{1/2}_{\mathcal{C}^\varepsilon}\,\|\nabla\Pi\|^{1/2}_{\mathcal{C}^1_*}\,\|T_X\nabla\Pi\|^{1/2}_{\mathcal{C}^\varepsilon}\,.$$

Note that lemma 3.10 and inequality (35) imply

$$\|T_X\nabla\Pi\|_{\mathcal{C}^\varepsilon} \leq \frac{5}{4}\,\|\partial_X\nabla\Pi\|_{\mathcal{C}^\varepsilon} + C\,\widetilde{\|X\|}_{\mathcal{C}^\varepsilon}\left(\left(\|\nabla\rho\|_{L^\infty} + \widetilde{\|X\|}^{4/3}_{\mathcal{C}^\varepsilon}\right)\|\nabla\Pi\|_{\mathcal{C}^1_*} + \rho^*\,\|\nabla u\|^2_{L^\infty}\right).$$

So, putting all these inequalities together, applying Young's inequality and performing some manipulations we get the estimate for the term g_1:

est:g_1

$$\begin{aligned}(38)\qquad \left\|T_X\nabla\,(-\Delta)^{-1}\,g_1\right\|_{\mathcal{C}^\varepsilon} \leq\;& C\,\Big(\,(\rho^* + \|\nabla\rho\|_{L^\infty})\,\|\partial_X\rho\|_{\mathcal{C}^\varepsilon}\,\|\nabla\Pi\|_{\mathcal{C}^1_*} + \rho^*\,\|\nabla u\|^2_{L^\infty} + \\ & +\widetilde{\|X\|}_{\mathcal{C}^\varepsilon}\left(1 + \widetilde{\|X\|}^{4/3}_{\mathcal{C}^\varepsilon}\right)(\rho^* + \|\nabla\rho\|_{L^\infty})^2\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\Big) + \\ & + \frac{5}{8}\,\|\partial_X\nabla\Pi\|_{\mathcal{C}^\varepsilon}\,.\end{aligned}$$

Therefore, from (35), (37) and (38), we finally obtain the estimate of the $\mathcal{C}^\varepsilon$ norm of the pressure term along the fixed direction:

est:d_X-Pi

$$\begin{aligned}(39)\qquad \|\partial_X\nabla\Pi\|_{\mathcal{C}^\varepsilon} \leq\;& C\,\Big((\rho^* + \|\nabla\rho\|_{L^\infty})\,\|\partial_X\rho\|_{\mathcal{C}^\varepsilon}\,\|\nabla\Pi\|_{\mathcal{C}^1_*} + (\rho^* + \|\partial_X\rho\|_{\mathcal{C}^\varepsilon})\,\|\nabla u\|^2_{L^\infty} + \\ & + \widetilde{\|X\|}_{\mathcal{C}^\varepsilon}\left(1 + \widetilde{\|X\|}^{4/3}_{\mathcal{C}^\varepsilon}\right)(\rho^* + \|\nabla\rho\|_{L^\infty})^2\,\|\nabla\Pi\|_{\mathcal{C}^1_*} + \\ & + (\rho^* + \|\nabla\rho\|_{L^\infty})\,\widetilde{\|X\|}_{\mathcal{C}^\varepsilon}\,(\|u\|_{L^p} + \|\nabla u\|_{L^\infty})^2 + \\ & + (\rho^* + \|\nabla\rho\|_{L^\infty})\,\|\partial_X u\|_{\mathcal{C}^\varepsilon}\,(\|u\|_{L^p} + \|\nabla u\|_{L^\infty})\Big).\end{aligned}$$

4.2.4 Conservation of striated regularity for the vorticity

Let us now establish a control on the regularity of Ω along the vector-fields $(X_\lambda)_{1\leq\lambda\leq m}$. Applying the operator ∂_X to (6), we obtain the evolution equation for $\partial_X\Omega$:

eq:vort_str

$$(40)\qquad \partial_t\,(\partial_X\Omega) + u\cdot\nabla\,(\partial_X\Omega) = \partial_X\left(\frac{1}{\rho^2}\,\nabla\rho\wedge\nabla\Pi\right) - \partial_X\,(\Omega\cdot\nabla u) - \partial_X\left({}^t\nabla u\cdot\Omega\right)\,.$$

Second and third terms of the right-hand side of (40) can be treated taking advantage once again of the following decomposition:

$$\partial_X\left(\Omega\cdot\nabla u + {}^t\nabla u\cdot\Omega\right) = (\partial_X - T_X)\left(\Omega\cdot\nabla u + {}^t\nabla u\cdot\Omega\right) + T_X\left(\Omega\cdot\nabla u + {}^t\nabla u\cdot\Omega\right)\,.$$

Lemma 3.10 says that the operator $\partial_X - T_X$ maps $\mathcal{C}^0_*$ in $\mathcal{C}^{\varepsilon-1}$ continuously: as $L^\infty \hookrightarrow \mathcal{C}^0_*$, one has

$$\left\|(\partial_X \,-\, T_X)\left(\Omega\cdot\nabla u \,+\, {}^t\nabla u\cdot\Omega\right)\right\|_{\mathcal{C}^{\varepsilon-1}} \,\leq\, C\,\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\|\Omega\|_{L^\infty}\,\|\nabla u\|_{L^\infty} \,\leq\, C\,\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\|\nabla u\|^2_{L^\infty}\,.$$

To handle the paravector term, we proceed in the following way. First of all, we note that, as $\operatorname{div} u = 0$, we can write

$$\left(\Omega\cdot\nabla u \,+\, {}^t\nabla u\cdot\Omega\right)_{ij} \,=\, \sum_k\left(\partial_i u^k\,\partial_k u^j \,-\, \partial_j u^k\,\partial_k u^i\right) \,=\, \sum_k\left(\partial_k\left(u^j\,\partial_i u^k\right) \,-\, \partial_k\left(u^i\,\partial_j u^k\right)\right)\,.$$

So, we have to estimate the $\mathcal{C}^{\varepsilon-1}$ norm of terms of the type $T_X T_{\nabla u}\nabla u$ and $T_X\nabla R(u,\nabla u)$. Using the same trick as in (36) for the remainder terms and applying lemmas 3.11 and 3.7 give us the control of $T_X\left(\Omega\cdot\nabla u \,+\, {}^t\nabla u\cdot\Omega\right)$ in $\mathcal{C}^{\varepsilon-1}$ by the quantity

$$\|X\|_{\mathcal{C}^\varepsilon}\left(\|u\|_{L^p}+\|\nabla u\|_{L^\infty}\right)^2 \,+\, \left(\|u\|_{L^p}+\|\nabla u\|_{L^\infty}\right)^2\|T_X\nabla u\|_{\mathcal{C}^{\varepsilon-1}} \,+\, \|\nabla u\|_{L^\infty}\,\|T_X u\|_{\mathcal{C}^\varepsilon}\,.$$

So, from lemmas 3.10 and 4.6 it easily follows

est:Om-Du

$$\left\|\partial_X\left(\Omega\cdot\nabla u \,+\, {}^t\nabla u\cdot\Omega\right)\right\|_{\mathcal{C}^{\varepsilon-1}} \;\leq\; C\left(\,\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\left(\|u\|_{L^p}+\|\nabla u\|_{L^\infty}\right)^2 + \right. \\ \left. +\,\|\partial_X u\|_{\mathcal{C}^\varepsilon}\left(\|u\|_{L^p}+\|\nabla u\|_{L^\infty}\right)\right)\,. \tag{41}$$

Now, let us analyse the first term of (40). It can be written as the sum of three terms:

$$\partial_X\left(\frac{1}{\rho^2}\,\nabla\rho\wedge\nabla\Pi\right) \,=\, -\,\frac{2}{\rho^3}\,(\partial_X\rho)\,(\nabla\rho\wedge\nabla\Pi) \,+\, \frac{1}{\rho^2}\,(\partial_X\nabla\rho)\wedge\nabla\Pi \,+\, \frac{1}{\rho^2}\,\nabla\rho\wedge(\partial_X\nabla\Pi)\,.$$

So, let us consider each one separately and prove that it belongs to the space $\mathcal{C}^{\varepsilon-1}$.

Obviously, from previous estimates we have that first and third terms are in $L^\infty \hookrightarrow \mathcal{C}^{\varepsilon-1}$ and satisfy

$$\begin{aligned}
\left\|\frac{1}{\rho^3}\,(\partial_X\rho)\,(\nabla\rho\wedge\nabla\Pi)\right\|_{\mathcal{C}^{\varepsilon-1}} \;&\leq\; \frac{C}{(\rho_*)^3}\,\|\partial_X\rho\|_{\mathcal{C}^\varepsilon}\,\|\nabla\rho\|_{L^\infty}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\\
\left\|\frac{1}{\rho^2}\,\nabla\rho\wedge(\partial_X\nabla\Pi)\right\|_{\mathcal{C}^{\varepsilon-1}} \;&\leq\; \frac{C}{(\rho_*)^2}\,\|\nabla\rho\|_{L^\infty}\,\|\partial_X\nabla\Pi\|_{\mathcal{C}^\varepsilon}\,.
\end{aligned}$$

Now, let us find a $\mathcal{C}^{\varepsilon-1}$ control for the second term.
From lemma 4.6, we know that $\partial_X\nabla\rho$ belongs already to this space, but the product of a $\mathcal{C}^\sigma$ function, $\sigma<0$, with a L^∞ one is not in $\mathcal{C}^\sigma$: in general, we can't even define it. Nevertheless, we know that the pressure term is more regular, i.e. $\nabla\Pi\in\mathcal{C}^1_*$, so $(\partial_X\nabla\rho)\wedge\nabla\Pi$ still belongs (using Bony's paraproduct decomposition) to the Hölder space $\mathcal{C}^{\varepsilon-1}$.
Moreover, as already pointed out in remark 4.7, also $\rho\in\mathcal{C}^1_*$, and so does $a:=1/\rho$ (it satisfies the same hypothesis and the same equation of ρ) and also a^2 (because $\mathcal{C}^1_*$ is an algebra), so the term is well-defined and we can control it in $\mathcal{C}^{\varepsilon-1}$.

Let us make rigorous what we have just said. With a little abuse of notation (at the end, we would have to deal with the sum of products of components of the two vector-fields), we write

$$(\partial_X\nabla\rho)\,\nabla\Pi \,=\, T_{(\partial_X\nabla\rho)}\nabla\Pi \,+\, T_{\nabla\Pi}\,(\partial_X\nabla\rho) \,+\, R\,(\partial_X\nabla\rho,\nabla\Pi)\;;$$

remembering the continuity properties of the paraproduct and remainder operators and that $\mathcal{C}^1_*\hookrightarrow L^\infty\hookrightarrow\mathcal{C}^0_*$, we get:

$$\|(\partial_X\nabla\rho)\wedge\nabla\Pi\|_{\mathcal{C}^{\varepsilon-1}} \,\leq\, C\,\|\partial_X\nabla\rho\|_{\mathcal{C}^{\varepsilon-1}}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\,.$$

In the same way, we can control also the previous term multiplied by $1/\rho^2$, and from remark 4.7 applied to $1/\rho^2$ we get

$$\left\|\frac{1}{\rho^2}\left(\partial_X\nabla\rho\right)\wedge\nabla\Pi\right\|_{\mathcal{C}^{\varepsilon-1}}\,\leq\,\frac{C}{(\rho_*)^2}\left(1+\frac{\|\nabla\rho\|_{L^\infty}}{\rho_*}\right)^2\,\|\partial_X\nabla\rho\|_{\mathcal{C}^{\varepsilon-1}}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\,.$$

So, finally we obtain, for a constant C depending also on ρ_* and ρ^*,

est:wedge

$$\left\|\partial_X\left(\frac{1}{\rho^2}\,\nabla\rho\wedge\nabla\Pi\right)\right\|_{\mathcal{C}^{\varepsilon-1}}\;\leq\;C\Big(\|\partial_X\rho\|_{\mathcal{C}^\varepsilon}\,\|\nabla\rho\|_{L^\infty}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\,+\,\|\nabla\rho\|_{L^\infty}\,\|\partial_X\nabla\Pi\|_{\mathcal{C}^\varepsilon}\,+\,\|\nabla\rho\|^2_{L^\infty}\,\|\partial_X\nabla\rho\|_{\mathcal{C}^{\varepsilon-1}}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\Big)\,.\tag{42}$$

Therefore, from equation (40), classical estimates for transport equation in the Hölder spaces framework and inequalities (41) and (42) (in which we apply also (30)), we obtain

est:d_X-Om

$$\begin{aligned}\|\partial_X\Omega(t)\|_{\mathcal{C}^{\varepsilon-1}}\;&\leq\;C\,\exp\left(c\int_0^t\|\nabla u\|_{L^\infty}\,d\tau\right)\left(\|\partial_{X_0}\Omega_0\|_{\mathcal{C}^{\varepsilon-1}}\,+\,\int_0^t e^{-\int_0^t\|\nabla u\|_{L^\infty}d\tau'}\,\times\right.\\&\times\Big(\,\widetilde{\|X\|}_{\mathcal{C}^\varepsilon}\left(\|u\|_{L^p}+\|\nabla u\|_{L^\infty}\right)^2\,+\,\|\partial_X u\|_{\mathcal{C}^\varepsilon}\left(\|u\|_{L^p}+\|\nabla u\|_{L^\infty}\right)\,+\\&+\widetilde{\|X\|}_{\mathcal{C}^\varepsilon}\,\|\nabla\rho\|^2_{L^\infty}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\,+\,\|\nabla\rho\|_{L^\infty}\,\|\partial_X\nabla\Pi\|_{\mathcal{C}^\varepsilon}\,+\\&\left.+\,\|\nabla\rho\|^2_{L^\infty}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\,\|\partial_X\nabla\rho\|_{\mathcal{C}^{\varepsilon-1}}\Big)\,d\tau\right)\,,\end{aligned}\tag{43}$$

and this relation allows us to close estimates: let us see how.

4.3 Final estimates

ss:fin-est

First of all, we note that for all $\eta\in\,[0,1]$, thanks to Young's inequality and estimates (21) and (23), we have

f-est:Pi

$$\|\nabla\Pi\|^\eta_{L^2}\,\|\nabla\Pi\|^{1-\eta}_{L^\infty}\,\leq\,\|\nabla\Pi\|_{L^2\cap\mathcal{C}^1_*}\,\leq\,C\left(\left(1+\|\nabla\rho\|^\delta_{L^\infty}\right)\|u\|_{L^p}\,\|\Omega\|_{L^q\cap L^\infty}\,+\,\rho^*\,\|\nabla u\|^2_{L^\infty}\right).\tag{44}$$

So, setting

$$L(t)\,:=\,\|u(t)\|_{L^p}\,+\,\|\Omega(t)\|_{L^q\cap L^\infty}\,,$$

putting (15) into (16), (18) and (19), for all fixed $T>0$ we obtain, in the time interval $[0,T]$, an inequality of the form

$$L(t)\,\leq\,C\,\exp\left(c\int_0^T\|\nabla u\|_{L^\infty}d\tau\right)\left(L(0)\,+\,\int_0^t\|\nabla u\|^2_{L^\infty}\,d\tau\,+\,\int_0^t L^2(\tau)\,d\tau\right)\,,$$

with a constant C depending only on initial data. Now, if we define

cond-T_1

$$T\,:=\,\sup\left\{t>0\,\middle|\,\int_0^t\left(e^{-\int_0^\tau L(\tau')d\tau'}L(\tau)+\|\nabla u(\tau)\|^2_{L^\infty}\right)d\tau\,\leq\,2L(0)\right\}\,,\tag{45}$$

from previous inequality and Gronwall's lemma and applying a standard bootstrap procedure, we manage to estimate the norms of the solution on $[0,T]$ in terms of initial data only:

$$\|u(t)\|_{L^p}\,+\,\|\Omega(t)\|_{L^q\cap L^\infty}\,\leq\,C\left(\|u_0\|_{L^p}\,+\,\|\Omega_0\|_{L^q\cap L^\infty}\right)\,.$$

From this, keeping in mind (21) and (23), we also have

$$\|\nabla\Pi(t)\|_{L^2\cap\mathcal{C}^1_*}\,\leq\,C\left(1\,+\,\|\nabla\rho_0\|^\delta_{L^\infty}\right)\left(\|u_0\|_{L^p}\,+\,\|\Omega_0\|_{L^q\cap L^\infty}\right)^2\,.$$

Now, let us focus on estimates about striated regularity. For notation convenience, let us set $S(t) := \left\|\partial_{X(t)}\Omega(t)\right\|_{\mathcal{C}^{\varepsilon-1}}$.

First of all, from (26) we get that the family $X(t)$ remains non-degenerate in time evolution. Moreover, as $\log(e+y) \leq y+1$ for $y \geq 0$, from (17) we get

$$\|\nabla u\|_{L^\infty} \leq C\,L\,\log\left(e+\frac{S}{L}\right) \leq C\,(C_0 + S)$$

in $[0,T]$, for some constants C, C_0 which depend only on initial data. Hence, keeping in mind (25), (27) and (28) we obtain also (for different constants we will keep to call C, C_0)

$$\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,,\ \|\partial_X u\|_{\mathcal{C}^\varepsilon} \leq C\left(C_0 + S + \int_0^t e^{-\int_0^\tau S(\tau')d\tau'}S(\tau)\,d\tau\right),$$

while (34) and (33) tell us that

$$\|\partial_X \nabla\rho\|_{\mathcal{C}^\varepsilon} \leq C\,e^{\int_0^t S(\tau)d\tau}\left(C_0 + \int_0^t e^{-\int_0^\tau S(\tau')d\tau'}S(\tau)\,d\tau\right)$$

and that the same holds for $\|\partial_X\rho\|_{\mathcal{C}^\varepsilon}$. These estimates together with (39) give us

$$\begin{aligned}
\|\partial_X\nabla\Pi\|_{\mathcal{C}^\varepsilon} \ &\leq\ C\left(C_0 + \int_0^t e^{-\int S}S\,d\tau + S + S\int_0^t e^{-\int S}S\,d\tau + S^2 + S^2\int_0^t e^{-\int S}S\,d\tau + \right.\\
&\qquad\qquad \left. + S^3 + S^{7/3} + \left(\int_0^t e^{-\int S}S\,d\tau\right)^{7/3}\right)\\
&\leq\ C\left(C_0 + \int_0^t e^{-\int S}S\,d\tau + S^2\int_0^t e^{-\int S}S\,d\tau + S^3 + \left(\int_0^t e^{-\int S}S\,d\tau\right)^{7/3}\right),
\end{aligned}$$

where in second inequality we have kept only terms to the smallest and biggest powers. Now, putting all these relations in (43) and absorbing intermediate powers as just done, we get an inequality of the type

$$\begin{aligned}
S(t) \ &\leq\ C\,e^{\int_0^t S(\tau)d\tau}\left(C_0 + \int_0^t e^{\int_0^\tau S}\left(C_0 + \int_0^\tau e^{-\int S}S\,d\tau' + S^3 + \right.\right.\\
&\qquad\qquad \left.\left. + S^2\int_0^\tau e^{-\int S}S\,d\tau' + \left(\int_0^\tau e^{-\int S}S\,d\tau'\right)^{7/3}\right)d\tau\right).
\end{aligned}$$

Therefore, if we pass to $Z(t) := \sup_{[0,t]} S$, we have

$$Z(t) \leq C\,e^{\int_0^t Z(\tau)d\tau}\left(C_0 + \int_0^t e^{\int_0^\tau Z}\left(C_0 + \tau Z + Z^3 + \tau Z^3 + \tau^{7/3}Z^{7/3}\right)d\tau\right).$$

So, let us suppose also T to have been chosen so small that, for all $t \in [0,T]$, one has

cond-T_2 (46)
$$\int_0^t e^{\int_0^\tau Z}\left(C_0 + \tau Z + Z^3 + \tau Z^3 + \tau^{7/3}Z^{7/3}\right)d\tau \leq 2C_0\,;$$

hence a bootstrap argument again allows us to get the bound $\left\|\partial_{X(t)}\Omega(t)\right\|_{\mathcal{C}^{\varepsilon-1}} \leq C\,K_0$ uniformly on $[0,T]$, for a universal constant C (depending only on N, q, ε, ρ_* and ρ^*) and a K_0 depending only on the norms of initial data in the relative functional spaces.

r:T **Remark 4.9.** The lifespan T of the solution is essentially determined by conditions (45) and (46), and it's quite clear that $T > 0$ because of continuity of the function $t \mapsto \int_0^t$. In addition, in section 5 we will establish a lower bound for T in terms of the norms of initial data only and we will compare it with the classical result in the case of constant density.

4.4 Proof of the existence of a solution

After establishing a priori estimates, we want to give the proof of the existence of a solution for system (1) under our assumptions.

We will get it in a classical way: first of all, we will construct a sequence of approximate solutions of our problem, for which a priori estimates of the previous section hold uniformly, and then we will show the convergence of such a sequence to a solution of (1).

Now, we will work only for positive times, but it goes without saying that the same argument holds also for negative times evolution.

4.4.1 Construction of a sequence of approximate solutions

For each $n \in \mathbb{N}$, let us define $u_0^n := S_n u_0$; obviously $u_0^n \in L^p$, and an easy computation shows that it belongs also to the space $B^\sigma_{p,r}$ for all $\sigma \in \mathbb{R}$ and all $r \in [1, +\infty]$. Let us notice that $\bigcap_\sigma B^\sigma_{p,r} \subset \mathcal{C}^\infty_b$, so in particular we have that $u_0^n \in L^p \cap B^s_{\infty,r}$, for some fixed $s > 1$ and $r \in [1, +\infty]$ such that $B^s_{\infty,r} \hookrightarrow \mathcal{C}^{0,1}$.

Keeping in mind that $[S_n, \nabla] = 0$, we have that $\Omega_0^n = S_n \Omega_0 \in L^q \cap B^{s-1}_{\infty,r}$; in particular, from (5) we get $\nabla u_0^n \in L^q$.

Now let us take an even radial function $\theta \in \mathcal{C}^\infty_0(\mathbb{R}^N)$, supported in the unitary ball, such that $0 \leq \theta \leq 1$ and $\int_{\mathbb{R}^N} \theta(x)\, dx = 1$, and set $\theta_n(x) = n^N \theta(nx)$ for all $n \in \mathbb{N}$. We define $\rho_0^n := \theta_n * \rho_0$: it belongs to $B^s_{\infty,r}$ and it satisfies the bounds $0 < \rho_* \leq \rho_0^n \leq \rho^*$.

Moreover, by properties of localisation operators S_n and of θ_n, we also have:

- $\rho_0^n \rightharpoonup \rho_0$ in $W^{1,\infty}$ and $\|\nabla \rho_0^n\|_{L^\infty} \leq c\, \|\nabla \rho_0\|_{L^\infty}$;
- $u_0^n \to u_0$ in the space L^p and $\|u_0^n\|_{L^p} \leq c\, \|u_0\|_{L^p}$;
- $\Omega_0^n \to \Omega_0$ in L^q and $\|\Omega_0^n\|_{L^q} \leq c\, \|\Omega_0\|_{L^q}$, $\|\Omega_0^n\|_{L^\infty} \leq c\, \|\Omega_0\|_{L^\infty}$.

So, for each n, theorem 3 and remark 4 of [14] give us a unique solution of (1) such that:

(i) $\rho^n \in \mathcal{C}([0, T^n]; B^s_{\infty,r})$, with $0 < \rho_* \leq \rho^n \leq \rho^*$;

(ii) $u^n \in \mathcal{C}([0, T^n]; L^p \cap B^s_{\infty,r})$, with $\Omega^n \in \mathcal{C}([0, T^n]; L^q \cap B^{s-1}_{\infty,r})$;

(iii) $\nabla \Pi^n \in \mathcal{C}([0, T^n]; L^2) \cap L^1([0, T^n]; B^s_{\infty,r})$.

For such a solution, a priori estimates of the previous section hold at every step n. Moreover, remembering previous properties about approximated initial data and that the function $y \mapsto y \log\left(e + \frac{c}{y}\right)$ is nondecreasing, we can find a control independent of $n \in \mathbb{N}$. So, we can find a positive time $T \leq T^n$ for all $n \in \mathbb{N}$, such that in $[0, T]$ approximate solutions are all defined for every n and satisfy uniform bounds.

4.4.2 Convergence of the sequence of approximate solutions

To prove convergence of the obtained sequence, we appeal to a compactness argument. Actually, we weren't able to apply the classical method used for the homogeneous case, i.e. proving estimates in rough spaces as $\mathcal{C}^{-\alpha}$ ($\alpha > 0$): we couldn't solve the elliptic equation for the pressure term in this framework.

We know that $(\rho^n)_{n\in\mathbb{N}} \subset L^\infty([0, T]; W^{1,\infty})$, $(u^n)_{n\in\mathbb{N}} \subset L^\infty([0, T]; L^p)$ and $(\nabla \Pi^n)_{n\in\mathbb{N}} \subset L^\infty([0, T]; L^2)$ and, thanks to a priori estimates, all these sequences are bounded in the respective functional spaces.

Due to the reflexivity of L^2 and L^p and seeing L^∞ as the dual of L^1, up to a subsequence, we obtain the existence of functions ρ, u and $\nabla \Pi$ such that:

- $\rho^n \overset{*}{\rightharpoonup} \rho$ in the space $L^\infty([0,T];W^{1,\infty})$,
- $u^n \rightharpoonup u$ in $L^\infty([0,T];L^p)$ and
- $\nabla\Pi^n \rightharpoonup \nabla\Pi$ in $L^\infty([0,T];L^2)$.

Nevertheless, we are not able to prove that $(\rho, u, \nabla\Pi)$ is indeed a solution of system (1): passing to the limit in nonlinear terms requires strong convergence in (even rough) suitable functional spaces. So let us argue in a different way and establish strong convergence properties, which will be useful also to prove preservation of striated regularity.

First of all, let us recall that, by construction, $u_0^n \to u_0$ in L^p and $\Omega_0^n \to \Omega_0$ in L^q, and $(\rho_0^n)_n$ is bounded in $W^{1,\infty}$. So, for $\alpha > 0$ big enough (for instance, take $\alpha = \max\{N/p\,, N/q\}$), we have that $(\rho_0^n)_n$, $(u_0^n)_n$, $(\Omega_0^n)_n$ are all bounded in the space $\mathcal{C}^{-\alpha}$.

r:conv-data **Remark 4.10.** It goes without saying that the sequences of u_0^n and Ω_0^n still converge in $\mathcal{C}^{-\alpha}$; moreover, also $\rho_0^n \to \rho_0$ in this space. Remember that ρ_0 belongs to the space $\mathcal{C}^1_*$, which coincides (see [8] for the proof) with the Zygmund space, i.e. the set of bounded functions f for which there exists a constant Z_f such that

$$|f(x+y) \,+\, f(x-y) \,-\, 2\,f(x)| \,\leq\, Z_f\,|y|$$

for all x, $y \in \mathbb{R}^N$. So, using the symmetry of θ, we can write

$$\rho_0^n(x) \,-\, \rho_0(x) \,=\, \frac{1}{2}\,n^N \int_{\mathbb{R}^N} \theta(ny)\,(\rho_0(x+y) \,+\, \rho_0(x-y) \,-\, 2\,\rho_0(x))\,dy\,;$$

from this identity we get that $\rho_0^n \to \rho_0$ in L^∞, and so also in $\mathcal{C}^{-\alpha}$.

Now, let us consider the equation for ρ^n:

$$\partial_t\rho^n \,=\, -\,u^n\cdot\nabla\rho^n\,.$$

From a priori estimates we get that $(u^n)_n$ is bounded in $L^\infty([0,T];\mathcal{C}^1_*)$ and $(\nabla\rho^n)_n$ is bounded in the space $L^\infty([0,T];L^\infty)$; so, from the properties of paraproduct and remainder operators, one has that the sequence $(\partial_t\rho^n)_n$ is bounded in $L^\infty([0,T];\mathcal{C}^{-\alpha})$. Therefore $(\rho^n)_n$ is bounded in $\mathcal{C}^{0,1}([0,T];\mathcal{C}^{-\alpha})$, and in particular uniformly equicontinuous in the time variable.

Now, up to multiply by a $\varphi \in \mathcal{D}(\mathbb{R}^N)$ (recall theorem 2.94 of [2]) and extract a subsequence, Ascoli-Arzelà theorem and Cantor diagonal process ensure us that $\rho^n \to \rho$ in the space $\mathcal{C}([0,T];\mathcal{C}^{-\alpha}_{loc})$.

Exactly in the same way, one can show that $(\rho^n)_n$ is bounded in $\mathcal{C}_b([0,T]\times\mathbb{R}^N)$ and it converges to ρ in this space.

Finally, remembering that $\rho \in L^\infty([0,T];W^{1,\infty})$ (recall the compactness argument), by interpolation we have convergence also in $L^\infty([0,T];\mathcal{C}^{1-\eta}_{loc})$ for all $\eta > 0$.

We repeat the same argument for the velocity field. For all n, we have

$$\partial_t u^n \,=\, -\,u^n\cdot\nabla u^n \,-\, a^n\,\nabla\Pi^n\,,$$

where we have set $a^n \,:=\, (\rho^n)^{-1}$. Let us notice that, as ρ_0, $a_0 \,:=\, (\rho_0)^{-1}$ satisfy the same hypothesis and a^n, ρ^n satisfy the same equations, they have also the same properties.

Keeping this fact in mind, let us consider each term separately.

- Thanks to what we have just said, $(a^n)_n \subset \mathcal{C}_b([0,T]\times\mathbb{R}^N) \cap L^\infty([0,T];\mathcal{C}^1_*)$ is bounded; moreover, from a priori estimates, we have that also $(\nabla\Pi^n)_n$ is bounded in the space $L^1([0,T];\mathcal{C}^1_*)$. Therefore, it follows that $(a^n\,\nabla\Pi^n)_n$ is a bounded sequence in $L^\kappa([0,T];\mathcal{C}^{-\alpha})$ for all $\kappa \in [1,+\infty[$.

- In the same way, as $(u^n)_n \subset L^\infty([0,T];\mathcal{C}^1_*)$ and $(\nabla u^n)_n \subset L^\infty([0,T];L^\infty)$ are both bounded sequences, one has that the sequence $(u^n \cdot \nabla u^n)_n$ is bounded in $L^\infty([0,T];\mathcal{C}^{-\alpha})$.

Therefore, exactly as done for the density, we get that $(u^n)_n$ is bounded in $\mathcal{C}^\beta([0,T];\mathcal{C}^{-\alpha})$ for some $0 < \beta < 1$, so uniformly equicontinuous in the time variable, and this fact implies that $u^n \to u$ in $\mathcal{C}([0,T];\mathcal{C}^{-\alpha}_{loc})$.

Finally, thanks to uniform bounds and Fatou's property of Besov spaces, we have that $u \in L^\infty([0,T];\mathcal{C}^1_*)$ and, by interpolation, that $u^n \to u$ in $\mathcal{C}([0,T];\mathcal{C}^{1-\eta}_{loc})$ for all $\eta > 0$.

So, thanks to strong convergence properties if we test the equations on a $\varphi \in \mathcal{C}^1([0,T];\mathcal{S}(\mathbb{R}^N))$ (here we have set $\mathcal{S}$ to be the Schwartz class), we can pass to the limit and get that $(\rho, u, \nabla\Pi)$ is indeed a solution of the Euler system (1).

Before going on with the striated regularity, let us establish continuity properties of the solutions with respect to the time variable.

First of all, from

$$\partial_t\rho \,=\, -\,u\cdot\nabla\rho\,,$$

as $u \in \mathcal{C}([0,T];L^\infty)$ (from the properties of convergence stated before) and $\nabla\rho \in L^\infty([0,T];L^\infty)$, we obtain that $\rho \in \mathcal{C}^{0,1}([0,T];L^\infty)$, and the same holds for $a := \rho^{-1}$.

Remember that $u \in L^\infty([0,T];L^p)$, ∇u and $a \in L^\infty([0,T];L^\infty)$. Moreover, as $\nabla\Pi \in L^\infty([0,T];L^2)\cap L^1([0,T];L^\infty)$, it belongs also to $L^1([0,T];L^p)$ (in fact, it belongs to $L^\kappa([0,T];L^p)$, where $\kappa = (1-\theta)^{-1}$, θ being the interpolation exponent between L^2 and L^∞, see also (16)). So, from the equation

$$\partial_t u \,=\, -\,u\cdot\nabla u \,-\, a\,\nabla\Pi\,,$$

we get that $\partial_t u \in L^1([0,T];L^p)$, therefore $u \in \mathcal{C}([0,T];L^p)$.

In the same way, from (6) we get that $\Omega \in \mathcal{C}([0,T];L^q)$, and therefore the same holds true for ∇u.

Now, using elliptic equation (20) and keeping in mind properties just proved for ρ and a, one can see that $\nabla\Pi \in \mathcal{C}([0,T];L^2)$. So, coming back to the previous equation, we discover that also $\partial_t u$ belongs to the same space.

4.4.3 Final checking about striated regularity

It remains us to prove that also properties of striated regularity are preserved in passing to the limit. For doing this, we will follow the outline of the proof in [10].

1. *Convergence of the flow*

   Let ψ^n and ψ be the flows associated respectively to u^n and u; for all fixed $\varphi \in \mathcal{D}(\mathbb{R}^N)$, by definition we have:

$$\begin{aligned}
|\varphi(x)\,(\psi^n(t,x) - \psi(t,x))| \;&\leq\; \int_0^t |\varphi(x)\,(u^n(\tau,\psi^n(\tau,x)) - u(\tau,\psi(\tau,x)))|\,d\tau \\
&\leq\; \int_0^t |\varphi(x)\,(u^n - u)\,(\tau,\psi^n(\tau,x))| \,+ \\
&\qquad\qquad + |\varphi(x)u^n(\tau,\psi^n(\tau,x)) - \varphi(x)u^n(\tau,\psi(\tau,x))|\,d\tau \\
&\leq\; \int_0^t \|\nabla u^n\|_{L^\infty}\,|\varphi(x)\,(\psi^n - \psi)\,(\tau,x)|\,d\tau \,+ \\
&\qquad\qquad + \int_0^t \|\varphi u^n - \varphi u\|_{L^\infty}\,d\tau\,.
\end{aligned}$$

   So, from convergence properties stated in previous part, we have that $\psi^n \to \psi$ in the space $L^\infty([0,T];Id + L^\infty_{loc})$. Moreover, it's easy to see that

$$\|\nabla\psi^n(t)\|_{L^\infty} \,\leq\, c\,\exp\left(\int_0^t \|\nabla u^n\|_{L^\infty}\,d\tau\right),$$

which tells us that the sequence $(\psi^n)_n$ is bounded in $L^\infty([0,T]; Id + \mathcal{C}^{0,1})$. Hence, finally we discover that $\psi^n \to \psi$ also in the spaces $L^\infty([0,T]; Id + \mathcal{C}^{1-\eta}_{loc})$ for all $\eta > 0$.

2. *Regularity of* $\partial_{X_0}\psi$

First of all, let us notice that, by definition,

$$\partial_{X_0(x)}\psi^n(t,x) \;=\; X^n_t(\psi^n(t,x))\,;$$

applying proposition 3.8, we get

est:X-o-psi

$$\|\partial_{X_0}\psi^n_t\|_{\mathcal{C}^\varepsilon} \;=\; \|X^n_t \circ \psi^n_t\|_{\mathcal{C}^\varepsilon} \;\leq\; c\,\|\nabla\psi^n_t\|_{L^\infty}\,\|X^n_t\|_{\mathcal{C}^\varepsilon}\,, \tag{47}$$

which implies that $(\partial_{X_0}\psi^n)_n$ is bounded in the space $L^\infty([0,T]; \mathcal{C}^\varepsilon)$. Now we note that, for every fixed $\varphi \in \mathcal{D}(\mathbb{R}^N)$, we have

$$\varphi\,\partial_{X_0}\psi^n \;-\; \varphi\,\partial_{X_0}\psi \;=\; \partial_{X_0}(\varphi\psi^n - \varphi\psi) \;-\; (\partial_{X_0}\varphi)\,(\psi^n - \psi)\;;$$

the second term is compactly supported, hence it converges in L^∞ because of what we have already proved. So let us focus on the first one and consider the difference

$$\partial_{X_0}(\varphi\,\psi^n) - \partial_{X_0}(\varphi\,\psi) \;=\; \mathrm{div}\,(X_0 \otimes \varphi(\psi^n - \psi)) \;-\; \varphi(\psi^n - \psi)\,\mathrm{div}\,X_0\,;$$

decomposing both terms in paraproduct and remainder and remembering hypothesis over X_0, it's easy to see that

$$\|\partial_{X_0}(\varphi\,\psi^n) - \partial_{X_0}(\varphi\,\psi)\|_{\mathcal{C}^{\varepsilon-1}} \;\leq\; c\,\|\varphi\,\psi^n - \varphi\,\psi\|_{\mathcal{C}^\varepsilon}\;\widetilde{\|}X_0\|_{\mathcal{C}^\varepsilon}\,.$$

Therefore, from what we have just proved, $\partial_{X_0}\psi^n \to \partial_{X_0}\psi$ in $L^\infty([0,T]; \mathcal{C}^{\varepsilon-1}_{loc})$; moreover, by Fatou's property, one gets that $\partial_{X_0}\psi \in L^\infty([0,T]; \mathcal{C}^\varepsilon)$ and it verifies estimate (47). So, by interpolation, convergence occurs also in $L^\infty([0,T]; \mathcal{C}^{\varepsilon-\eta}_{loc})$ for all $\eta > 0$.

3. *Regularity of* X_t

Remembering the definitions

$$\begin{aligned} X_t(x) \;&:=\; \left(\partial_{X_0(x)}\psi\right)(t, \psi_t^{-1}(x)) \\ \mathrm{div}\,X_t \;&=\; \mathrm{div}\,X_0 \circ \psi_t^{-1}\,, \end{aligned}$$

from proposition 3.8 it immediately follows that X_t and $\mathrm{div}\,X_t$ both belong to $\mathcal{C}^\varepsilon$. Moreover, the same proposition implies that $X^n \to X$ in the space $L^\infty([0,T]; \mathcal{C}^{\varepsilon-\eta}_{loc})$ for all $\eta > 0$, and the same holds for the divergence. In particular, we have convergence also in $L^\infty([0,T]; L^\infty_{loc})$, which finally tells us that X_t remains non-degenerate for all $t \in [0,T]$, i.e. $I(X_t) \geq c\,I(X_0)$.

4. *Striated regularity for the density and the vorticity*

Let us first prove that regularity of the density with respect to the vector field X_t is preserved during the time evolution. To simplify the presentation, we will omit the localisation by $\varphi \in \mathcal{D}(\mathbb{R}^N)$: formally, we should repeat the same reasoning applied to prove regularity of $\partial_{X_0}\psi$. So, let us consider

$$\partial_{X^n}\rho^n - \partial_X\rho \;=\; \mathrm{div}\,(\rho^n\,(X^n - X)) - \rho^n\,\mathrm{div}\,(X^n - X) + \mathrm{div}\,((\rho^n - \rho)\,X) - (\rho^n - \rho)\,\mathrm{div}\,X$$

and prove the convergence in $L^\infty([0,T]; \mathcal{C}^{-1}_{loc})$. Using Bony's paraproduct decomposition, it's not difficult to see that first and third terms can be bounded by $\|\rho^n\|_{L^\infty}\,\|X^n - X\|_{L^\infty} + \|\rho^n - \rho\|_{L^\infty}\,\|X\|_{L^\infty}$, while second and last terms can be controlled by $\|\rho^n\|_{L^\infty}\,\|\mathrm{div}\,(X^n - X)\|_{\mathcal{C}^{\varepsilon/2}} + \|\rho^n - \rho\|_{L^\infty}\,\|\mathrm{div}\,X\|_{\mathcal{C}^{\varepsilon/2}}$, for instance. So, from the convergence properties stated for $(\rho^n)_n$ and $(X^n)_n$, we get that $\partial_{X^n}\rho^n \to \partial_X\rho$ in the space $L^\infty([0,T]; \mathcal{C}^{-1}_{loc})$, as claimed.

Moreover, from a priori bounds and Fatou's property of Besov spaces, we have that $\partial_X \rho \in L^\infty([0,T];\mathcal{C}^\varepsilon)$ and so, by interpolation, convergence occurs also in $L^\infty([0,T];\mathcal{C}^{\varepsilon-\eta}_{loc})$ for all $\eta > 0$.

Now we consider the vorticity term (again, we omit the multiplication by a $\mathcal{D}(\mathbb{R}^N)$ function):

$$\begin{aligned}\partial_{X^n}\Omega^n \,-\, \partial_X\Omega \;&=\; \mathrm{div}\,\left((X^n - X)\otimes\Omega^n\right) \,-\, \Omega^n\,\mathrm{div}\,(X^n - X) \,+ \\ &\qquad + \mathrm{div}\,\left(X\otimes(\Omega^n - \Omega)\right) \,-\, (\Omega^n - \Omega)\,\mathrm{div}\,X\,.\end{aligned}$$

From the convergence properties of $(u^n)_n$, we get that $\Omega^n \,\rightarrow\, \Omega$ in $L^\infty([0,T];\mathcal{C}^{-\eta}_{loc})$ for all $\eta > 0$, so for instance also for $\eta = \varepsilon/2$. From this, using again paraproduct decomposition as done before, one can prove that $\partial_{X^n}\Omega^n \,\rightarrow\, \partial_X\Omega$ in $L^\infty([0,T];\mathcal{C}^{-1-\varepsilon/2}_{loc})$. Therefore, as usual from a priori estimates and Fatou's property of Besov spaces, we have that $\partial_X\Omega \in L^\infty([0,T];\mathcal{C}^{\varepsilon-1})$, and moreover convergence remains true (by interpolation) in spaces $L^\infty([0,T];\mathcal{C}^{\varepsilon-1-\eta}_{loc})$ for all $\eta > 0$.

So, all the properties linked to striated regularity are now verified, and this concludes the proof of the existence part of theorem 2.3.

4.5 Uniqueness

Let us spend a few words on proof of uniqueness: it is an immediate consequence of the following stability result.

p:stab **Proposition 4.11.** *Let* $\left(\rho^1, u^1, \nabla\Pi^1\right)$ *and* $\left(\rho^2, u^2, \nabla\Pi^2\right)$ *be solutions of system* (1) *with*

$$0 \,<\, \rho_* \,\leq\, \rho^1\,,\,\rho^2 \,\leq\, \rho^*\,.$$

Let us suppose that $\delta\rho := \rho^1 - \rho^2 \,\in\, \mathcal{C}([0,T];L^2)$ *and that* $\delta u := u^1 - u^2 \,\in\, \mathcal{C}^1([0,T];L^2)$. *Finally, assume that* $\nabla\rho^2$, ∇u^1, ∇u^2 *and* $\nabla\Pi^2$ *all belong to* $L^1([0,T];L^\infty)$.

Then, for all $t \in [0,T]$, *we have the following estimate:*

$$\|\delta\rho(t)\|_{L^2} \,+\, \|\delta u(t)\|_{L^2} \,\leq\, C\,e^{c\,I(t)}\,\left(\|\delta\rho(0)\|_{L^2} \,+\, \|\delta u(0)\|_{L^2}\right)\,,$$

where we have defined

$$I(t) \,:=\, \int_0^t \left(\left\|\nabla\rho^2\right\|_{L^\infty} + \left\|\nabla u^1\right\|_{L^\infty} + \left\|\nabla u^2\right\|_{L^\infty} + \left\|\nabla\Pi^2\right\|_{L^\infty}\right)d\tau\,.$$

Proof. From $\partial_t\delta\rho \,+\, u^1\cdot\nabla\delta\rho \,=\, -\,\delta u\cdot\nabla\rho^2$, we immediately get

$$\|\delta\rho(t)\|_{L^2} \,\leq\, \|\delta\rho(0)\|_{L^2} \,+\, \int_0^t \|\delta u\|_{L^2}\,\left\|\nabla\rho^2\right\|_{L^\infty}\,d\tau\,.$$

Moreover, the equation for δu reads as follows:

$$\partial_t\delta u \,+\, u^1\cdot\nabla\delta u \,=\, -\,\delta u\cdot\nabla u^2 \,-\, \frac{\nabla\delta\Pi}{\rho^1} \,+\, \frac{\nabla\Pi^2}{\rho^1\,\rho^2}\,\delta\rho\,,$$

where we have set $\delta\Pi \,=\, \Pi^1 - \Pi^2$. So, from standard L^p estimates for transport equations, one has:

$$\|\delta u(t)\|_{L^2} \,\leq\, \|\delta u(0)\|_{L^2} + C\int_0^t \left(\|\delta u\|_{L^2}\left\|\nabla u^2\right\|_{L^\infty} + \|\nabla\delta\Pi\|_{L^2} + \left\|\nabla\Pi^2\right\|_{L^\infty}\|\delta\rho\|_{L^2}\right)d\tau\,.$$

Now, to get bounds for $\nabla\delta\Pi$, we analyse its equation:

$$
\begin{aligned}
-\operatorname{div}\left(\frac{\nabla\delta\Pi}{\rho^1}\right) &= \operatorname{div}\left(-\frac{\nabla\Pi^2}{\rho^1\,\rho^2}\,\delta\rho\,+\,u^1\cdot\nabla\delta u\,+\,\delta u\cdot\nabla u^2\right) \\
&= \operatorname{div}\left(-\frac{\nabla\Pi^2}{\rho^1\,\rho^2}\,\delta\rho\,+\,\delta u\cdot(\nabla u^1+\nabla u^2)\right)\,,
\end{aligned}
$$

where, to get the second equality, we have used the algebraic identity

$$
\operatorname{div}(v\cdot\nabla w)\;=\;\operatorname{div}(w\cdot\nabla v)\,+\,\operatorname{div}(v\,\operatorname{div}w)\,-\,\operatorname{div}(w\,\operatorname{div}v)\,.
$$

So, from lemma 3.13 we obtain

$$
\|\nabla\delta\Pi\|_{L^2}\,\leq\,C\left(\left\|\nabla\Pi^2\right\|_{L^\infty}\,\|\delta\rho\|_{L^2}\,+\,\|\delta u\|_{L^2}\left(\left\|\nabla u^1\right\|_{L^\infty}+\left\|\nabla u^2\right\|_{L^\infty}\right)\right)\,,
$$

and Gronwall's inequality completes the proof of the proposition. $\square$

Now, let us prove uniqueness: let $\left(\rho^1,u^1,\nabla\Pi^1\right)$ and $\left(\rho^2,u^2,\nabla\Pi^2\right)$ satisfy system (1) with same initial data (ρ_0,u_0), under hypothesis of theorem 2.3.

As $\delta u(0)=0$ and $u\in\mathcal{C}([0,T];L^p)$, $\nabla u\in\mathcal{C}([0,T];L^q)$, one easily gets that $\delta u\in\mathcal{C}^1([0,T];L^2)$. Moreover, from this fact, observing that also $\delta\rho(0)=0$, the equation for $\delta\rho$ tells us that $\delta\rho\in\mathcal{C}([0,T];L^2)$. Hence proposition 4.11 can be applied and uniqueness immediately follows.

5 On the lifespan of the solution

s:lifespan

The aim of this section is to establish, in the most accurate way, an explicit lower bound for the lifespan of the solution of system (1) in terms of initial data only. Our starting point is subsection 4.3: with the same notations, we define moreover

$$
\begin{aligned}
&A(t):=\|\nabla\rho(t)\|_{L^\infty}\,,\quad U(t):=\|\nabla u(t)\|_{L^\infty}\,,\quad V(t):=\int_0^t\left(L(\tau)+Z(\tau)\right)d\tau\,,\\
&\Gamma(t):=\widetilde{\|}X(t)\|_{\mathcal{C}^\varepsilon}\,,\quad R(t):=\left\|\partial_{X(t)}\nabla\rho(t)\right\|_{\mathcal{C}^{\varepsilon-1}}\,.
\end{aligned}
$$

It's only matter of repeating previous computations in a more accurate way. As

est:U (48)
$$
U(t)\,\leq\,C\,\left(L(t)\,+\,Z(t)\right)\,=\,C\,V'(t)
$$

and the exponent $\delta>1$, we can write

life:L (49)
$$
L(t)\,\leq\,C\,e^{cV(t)}\left(L_0+A_0\,(1+A_0)^\delta\int_0^t\left(e^{cV}L^2+Z^2\right)d\tau\right)\,.
$$

Concerning the "striated norms", first of all, from (28) and (27), we have

$$
\begin{aligned}
\Gamma(t)\;&\leq\;C\,e^{cV(t)}\left(\Gamma_0\,+\int_0^t e^{-cV(\tau)}S(\tau)\,d\tau\right)\;\leq\;C\,e^{cV(t)}\left(\Gamma_0\,+\,Z\,t\right)\\
\left\|\partial_{X(t)}u(t)\right\|_{\mathcal{C}^\varepsilon}\;&\leq\;C\left(Z\,+\,\Gamma_0\,e^{cV(t)}(L+Z)\,+\,e^{cV(t)}(L+Z)\,Z\,t\right)
\end{aligned}
$$

while (33) and (34) imply

$$
\|\partial_X\rho\|_{\mathcal{C}^\varepsilon}\,\leq\,C\,e^{cV(t)}\left(\Gamma_0(1+A_0)+R_0\right)\,,\quad R(t)\,\leq\,C\,e^{cV(t)}\left(\Gamma_0(1+A_0)+R_0+Z\,t\right).
$$

Now, we analyse carefully the terms in (39) one by one: keeping in mind also (44), we get that the first of them can be bounded by

$$
e^{cV(t)}\left(L^2+Z^2\right)(1+A_0)^{2+\delta}\left(\Gamma_0+R_0\right)
$$

and the same holds for the second one. The third term, instead, can be decomposed into two parts, one in which we have $\widetilde{\|X\|}_{\mathcal{C}^\varepsilon}$ and the other one with $\widetilde{\|X\|}^{7/3}_{\mathcal{C}^\varepsilon}$. It's easy to see that we can control the former with

$$e^{cV(t)}\left(L^2+Z^2\right)(1+A_0)^{2+\delta}\,\Gamma_0\;+\;e^{cV(t)}\left(L^2+Z^2\right)(1+A_0)^{2+\delta}\,Z\,t$$

and the latter with

$$e^{cV(t)}\left(L^2+Z^2\right)(1+A_0)^{2+\delta}\,\Gamma_0^{7/3}\;+\;e^{cV(t)}\left(L^2+Z^2\right)(1+A_0)^{2+\delta}\,Z^{7/3}\,t^{7/3}\,.$$

Finally, fourth and last terms of (39) can be bounded by

$$e^{cV(t)}\left(L^2+Z^2\right)(1+A_0)\,(1+\Gamma_0)\;+\;e^{cV(t)}\left(L^2+Z^2\right)(1+A_0)^{2+\delta}\,Z\,t\,,$$

and all these inequalities lead us to control $\|\partial_X\nabla\Pi\|_{\mathcal{C}^\varepsilon}$ with

life:d_X-P

$$\begin{aligned}e^{cV(t)}\,(1+A_0)^{2+\delta}\,(V')^{13/3}\,t^{7/3}\;+\;e^{cV(t)}\,(1+A_0)^{2+\delta}\,(V')^3\,t\;+&\\ +\,e^{cV(t)}\,(1+A_0)^{2+\delta}\left(1+R_0+\Gamma_0^{7/3}\right)(V')^2\,.&\end{aligned}\tag{50}$$

Now let us find a bound for $Z(t)$. Again, let us proceed carefully, starting from estimate (43). First and second terms under the integral are actually smaller than the last two ones we have just analysed in considering the gradient of the pressure. Moreover, $\|\nabla\rho\|_{L^\infty}\,\|\partial_X\nabla\Pi\|_{\mathcal{C}^\varepsilon}$, like the last term, can be bounded by the same quantity in (50) multiplied by $1+A_0$, while

$$\widetilde{\|X\|}_{\mathcal{C}^\varepsilon}\,\|\nabla\rho\|^2_{L^\infty}\;\|\nabla\Pi\|_{\mathcal{C}^1_*}\;\leq\;C\,\Gamma_0\,(1+A_0)^{2+\delta}\;e^{cV(t)}\,(V')^2$$

which is controlled again by (50).

So, keeping in mind (49), we finally get

life:V

$$\begin{aligned}V'(t)\;\leq\;C_1\,e^{C_2V(t)}\Bigg(V'(0)+\int_0^t e^{C_2V}\Big((1+A_0)^{3+\delta}\left(1+R_0+\Gamma_0^{7/3}\right)(V')^2+&\\ +\,(1+A_0)^{3+\delta}\,(V')^{13/3}\tau^{7/3}+(1+A_0)^{3+\delta}\,(V')^3\tau\Big)\,d\tau\Bigg)\,.&\end{aligned}\tag{51}$$

Let us set T^* the supremum of the positive times for which the integral in the right-hand side of the (51) is less than or equal to $2\,V'(0)$. Hence on $[0,T^*]$ we have

$$V'(t)\;\leq\;C_3\,V'(0)\,e^{C_2\,V(t)}\qquad\Longrightarrow\qquad 1\,-\,e^{-C_2\,V(t)}\;\leq\;C_4\,V'(0)\,t\,.$$

Now let us define

life:T

$$T\;:=\;\frac{\widetilde{K}}{C_4}\left(V'(0)\,(1+A_0)^{3+\delta}\left(1+R_0+\Gamma_0^{7/3}\right)\right)^{-1}\,,\tag{52}$$

for a constant $0<\widetilde{K}<1$ small enough (we will get later an estimate for it). For the sequel it is convenient to define $K:=\left(1-\widetilde{K}\right)^{-1}$.

We claim that $T^*\geq T$.

First of all, in $[0,T]\cap[0,T^*]$, remembering last two inequalities we have $e^{C_2\,V(t)}\;\leq\;K$ and $V'(t)\;\leq\;C_3\,K\,V'(0)$. Therefore, in this time interval the integral in (51) can be controlled by

$$\begin{aligned}K\Bigg((1+A_0)^{3+\delta}\left(1+R_0+\Gamma_0^{7/3}\right)(C_3\,K)^2\;(V'(0))^2\,T\,+&\\ +\,\frac{3}{10}\,(1+A_0)^{3+\delta}\,(C_3\,K)^{13/3}\;(V'(0))^{13/3}\,T^{10/3}\;+&\\ +\,\frac{1}{2}\,(1+A_0)^{3+\delta}\,(C_3\,K)^3\;(V'(0))^3\,T^2\Bigg)\,.&\end{aligned}$$

Now we want this expression to be less than or equal to $2\,V'(0)$; hence, from (52) we get an equation for $\widetilde{K}$:

$$K\left(\frac{\widetilde{K}}{C_4}\,(C_3\,K)^2+\frac{3}{10}\left(\frac{\widetilde{K}}{C_4}\right)^{10/3}(C_3\,K)^{13/3}+\frac{1}{2}\left(\frac{\widetilde{K}}{C_4}\right)^2(C_3\,K)^3\right)\leq 2\,.$$

So, if we take $\widetilde{K}$ small enough, we obtain $T\leq T^*$, as claimed.

r:life **Remark 5.1.** Let us notice that, in the classical case (constant density), the lifespan of a solution was controlled by below by

$$T_{cl}\,:=\,C\left(\|\Omega_0\|_{L^q\cap L^\infty}\,\log\left(e+\frac{\|\Omega_0\|_{\mathcal{C}^\varepsilon_{X_0}}}{\|\Omega_0\|_{L^q\cap L^\infty}}\right)\right)^{-1}$$

(see also [12]). We have just proved that in our case the lifespan is given by (52), instead. As

$$V'(0)=\|u_0\|_{L^p}+\|\Omega_0\|_{L^q\cap L^\infty}+\|\partial_{X_0}\Omega_0\|_{\mathcal{C}^{\varepsilon-1}}\;\geq\;c\,\|\Omega_0\|_{L^q\cap L^\infty}\,\log\left(e+\frac{\|\Omega_0\|_{\mathcal{C}^\varepsilon_{X_0}}}{\|\Omega_0\|_{L^q\cap L^\infty}}\right)\,,$$

it's quite evident that $T\leq T_{cl}$ (up to multiplication by a constant). Note also that the logarithmic dependence disappeared thanks to estimate (48), which really simplifies our complicated computations but is maybe quite rough.

6 Generalizing vortex patches

First of all, let us prove conservation of conormal regularity.

Given a compact hypersurface $\Sigma\subset\mathbb{R}^N$ of class $\mathcal{C}^{1+\varepsilon}$, we can always find, in a canonical way, a family X of $m=N(N+1)/2$ vector-fields such that the inclusion $\mathcal{C}^\eta_\Sigma\subset\mathcal{C}^\eta_X$ holds for all $\eta\in[\varepsilon,1+\varepsilon]$. For completeness, let us recall the result (see proposition 5.1 of [12]), which turns out to be important in the sequel.

p:con->stri **Proposition 6.1.** *Let Σ be a compact hypersurface of class $\mathcal{C}^{1+\varepsilon}$.*

Then there exists a non-degenerate family of $m=N(N+1)/2$ vector-fields $X\subset\mathcal{T}^\varepsilon_\Sigma$ such that $\mathcal{C}^\eta_\Sigma\subset\mathcal{C}^\eta_X$ for all $\eta\in[\varepsilon,1+\varepsilon]$.

Hence, thanks to theorem 2.3 we propagate striated regularity with respect to this family. Finally, in a classical way, from this fact one can recover conormal properties of the solution, and so get the thesis of theorem 2.5 (see e.g. [17] and [12] for the details).

Actually, in the case of space dimension $N=2\,,\,3$ (finally, the only relevant ones from the physical point of view) one can improve the statement of theorem 2.5. To avoid traps coming from differential geometry, let us clarify our work setting.

In considering a submanifold $\Sigma\subset\mathbb{R}^N$ of dimension k and of class $\mathcal{C}^{1+\varepsilon}$ (for some $\varepsilon>0$), we mean that Σ is a manifold of dimension k endowed with the differential structure inherited from its inclusion in $\mathbb{R}^N$, and the transition maps are of class $\mathcal{C}^{1+\varepsilon}$.
In particular, for all $x\in\Sigma$ there is an open ball $B\subset\mathbb{R}^N$ containing x, and a $\mathcal{C}^{1+\varepsilon}$ local parametrization $\varphi:\mathbb{R}^k\to B\cap\Sigma$ with inverse of class $\mathcal{C}^{1+\varepsilon}$. This is equivalent to require local equations $H:B\to\mathbb{R}^k$ of class $\mathcal{C}^{1+\varepsilon}$ such that $H_{|B\cap\Sigma}\equiv 0$.

Given a local parametrization φ on $U:=\Sigma\cap B$, its differential $\varphi_*:T\mathbb{R}^k\to TU\cong T\Sigma$ induces, in each point $x\in\mathbb{R}^k$, a linear isomorphism between the tangent spaces, $\varphi_{*,x}:T_x\mathbb{R}^k\to T_{\varphi(x)}\Sigma$. Moreover, the dependence of this map on the point $x\in\mathbb{R}^k$ is of class $\mathcal{C}^\varepsilon$: in coordinates, φ_* is given by the Jacobian matrix $\nabla\varphi$.

Finally, we say that a function f defined on Σ is (locally) of class $\mathcal{C}^\alpha$ (for $\alpha > 0$) if the composition $f \circ \varphi : \mathbb{R}^k \to \mathbb{R}$ is α-Hölder continuous for any local parametrization φ.

Before stating our claim, some preliminary results are in order. Let us start with a very simple lemma.

l:D->Hold **Lemma 6.2.** *Let $f \in L^\infty(\mathbb{R}^N)$ such that its gradient is α-Hölder continuous for some $\alpha > 0$.*
Then $f \in \mathcal{C}^{1+\alpha}(\mathbb{R}^N)$.

Proof. It's obvious using dyadic characterization of Hölder spaces and Bernstein's inequalities. □

Now, by analogy, one may ask if this property still holds true for a function defined on a submanifold, with Hölder continuous tangential derivatives. In fact, with some additional hypothesis on the submanifold, one can prove that also in this case there is a gain of regularity.

l:Hold-man **Lemma 6.3.** *Let $\Sigma \subset \mathbb{R}^N$ be a submanifold of dimension k and of class $\mathcal{C}^{1+\varepsilon}$, for some $\varepsilon > 0$. Moreover, let us suppose Σ to be compact.*
Let us consider a function $f : \Sigma \to \mathbb{R}$, bounded on Σ, such that $\partial_X f \in \mathcal{C}^\varepsilon(\Sigma)$ for all vector-fields X of class $\mathcal{C}^\varepsilon$ tangent to Σ.
Then $f \in \mathcal{C}^{1+\varepsilon}(\Sigma)$.

Proof. Let us fix a coordinate set $U := B \cap \Sigma$ (for some open ball $B \subset \mathbb{R}^N$) with its $\mathcal{C}^{1+\varepsilon}$ local parametrization $\varphi : \mathbb{R}^k \to U$, and let us define $g := f \circ \varphi : \mathbb{R}^k \to \mathbb{R}$.
Obviously, $g \in L^\infty(\mathbb{R}^k)$.
Moreover, for all $1 \leq i \leq k$ let us set $\varphi_*(\partial_i) = X_i$: then, X_i is obviously of class $\mathcal{C}^\varepsilon$. Hence we have $\partial_i g(x) = X_i(f)(\varphi(x))$, i.e. $\partial_i g$ in a point x is the derivation X_i applied to the function f, and evaluated in the point $\varphi(x)$. In our notations, we get $\partial_i g = (\partial_{X_i} f) \circ \varphi$.
Therefore, from our hypothesis it follows that $\nabla g \in \mathcal{C}^\varepsilon$, and so, by lemma 6.2, $g \in \mathcal{C}^{1+\varepsilon}(\mathbb{R}^k)$.
In conclusion, we have proved that f composed with any local parametrization φ is of class $\mathcal{C}^{1+\varepsilon}$ on $\mathbb{R}^k$. Moreover, as Σ is compact, we can bound its Hölder norm gobally on Σ, that is to say in a way independent of the fixed open set U, and we get that $f \in \mathcal{C}^{1+\varepsilon}(\Sigma)$. □

r:Hold-man **Remark 6.4.** Let us note that the operator ∂_X depends linearly on the vector-field X. Hence, in the hypothesis of previous lemma it's enough to assume that one can find, locally on Σ, a family $\{X_1, \ldots, X_k\}$ of linearly independent vector-fields of class $\mathcal{C}^\varepsilon$ such that $\partial_{X_i} f \in \mathcal{C}^\varepsilon(\Sigma)$ for all $1 \leq i \leq k$.

c:Hold-man **Corollary 6.5.** *Let $\Sigma \subset \mathbb{R}^N$ be a compact hypersurface of class $\mathcal{C}^{1+\varepsilon}$, and let $f \in \mathcal{C}^1_*(\mathbb{R}^N)$.*
If $f \in \mathcal{C}^{1+\varepsilon}_\Sigma$, then $f_{|\Sigma} \in \mathcal{C}^{1+\varepsilon}(\Sigma)$.

Proof. By proposition 6.1 and non-degeneracy condition, we can find, locally on Σ, $N-1$ linearly independent vector-fileds $X_1 \ldots X_{N-1}$, defined on the whole $\mathbb{R}^N$ and of class $\mathcal{C}^\varepsilon$, which are tangent to Σ and such that $\mathrm{div}\,(f\,X_i) \in \mathcal{C}^\varepsilon(\mathbb{R}^N)$ for all $1 \leq i \leq N-1$.
Moreover, also the divergence of these vector-fields is ε-Hölder continuous; therefore, using also Bony's paraproduct decomposition, we gather that

$$\partial_{X_i} f \;=\; \mathrm{div}\,(f\,X_i) \;-\; f\;\mathrm{div}\,X_i \;\in\; \mathcal{C}^\varepsilon(\mathbb{R}^N) \qquad \forall\, 1 \leq i \leq N-1\,,$$

and hence this regularity is preserved if we restrict $\partial_{X_i} f$ only to Σ.
So, lemma 6.3 and remark 6.4 both imply that $f_{|\Sigma} \in \mathcal{C}^{1+\varepsilon}(\Sigma)$. □

Now, let us come back to the situation of theorem 2.5. Moreover, let us suppose that the hypersurface Σ_0 is also connected: then it separates the whole space $\mathbb{R}^N$ into two connected components, the first one bounded and the other one unbounded. In dimension 2, this is nothing but the Jordan curve theorem, while in the case $N = 3$ it's a consequence of Alexander duality theorem (see e.g. [19]).

So, let us set D_0 to be the bounded domain of $\mathbb{R}^N$ whose boundary is $\partial D_0 = \Sigma_0$ and let us define $D(t) = \psi_t(D_0)$. As the flow ψ_t is a diffeomorphism for every fixed time t, we have that $\partial D(t) = \Sigma(t)$ and also the complementary region is transported by ψ: $D(t)^c = \psi_t(D_0^c)$.

Let us denote by χ_A the characteristic function of a set A.

p:conorm **Proposition 6.6.** *Under hypothesis of theorem 2.5, suppose also that the initial data can be decomposed in the following way:*

$$\rho_0(x) \,=\, \rho_0^i(x)\,\chi_{D_0}(x) \,+\, \rho_0^e(x)\,\chi_{D_0^c}(x) \qquad \textit{and} \qquad \Omega_0(x) \,=\, \Omega_0^i(x)\,\chi_{D_0}(x) \,+\, \Omega_0^e(x)\,\chi_{D_0^c}(x)\,,$$

with $\rho_0^i \in \mathcal{C}^{1+\varepsilon}(D_0)$ and $\Omega_0^i \in \mathcal{C}^{\varepsilon}(D_0)$.

Then, the previous decomposition still holds for the solution at every time $t \in [0,T]$:

eq:dec_rho
$$\rho(t,x) \;=\; \rho^i(t,x)\,\chi_{D(t)}(x) \,+\, \rho^e(t,x)\,\chi_{D(t)^c}(x) \tag{53}$$

eq:dec_vort
$$\Omega(t,x) \;=\; \Omega^i(t,x)\,\chi_{D(t)}(x) \,+\, \Omega^e(t,x)\,\chi_{D(t)^c}(x)\,. \tag{54}$$

Moreover, Hölder continuity in the interior of the domain $D(t)$ is preserved, uniformly on $[0,T]$: at every time t, we have

- $\rho^i(t) \,\in\, \mathcal{C}^{1+\varepsilon}(D(t))$ *and*
- $\Omega^i(t) \,\in\, \mathcal{C}^{\varepsilon}(D(t))$,

and regularity on $D(t)$ propagates also for the velocity field and the pressure term: $u(t)$ and $\nabla\Pi(t)$ both belong to $\mathcal{C}^{1+\varepsilon}(D(t))$.

Proof. First of all, let us recall that, by theorem 2.5, on $[0,T]$ we have

est:L^1_t-Du
$$\int_0^T \|\nabla u(t)\|_{L^\infty}\,dt \,\leq\, C\,. \tag{55}$$

Thanks to first equation of (1), relation (53) obviously holds, with

$$\rho^{i,e}(t,x) \,=\, \rho_0^{i,e}\left(\psi_t^{-1}(x)\right)\,.$$

So, we immediately get that $\rho^i(t)$ belongs to the space $\mathcal{C}^{1+\varepsilon}(D(t))$. Let us observe also that a decomposition analogous to (53) holds also for $a = 1/\rho$, and its components $a^{i,e}$ have the same properties of the corresponding ones of ρ.

Now let us handle the vorticity term. We can always decompose the solution in a component localised on $D(t)$ and the other one supported on the complementary set, defining

$$\Omega^i(t,x) \,:=\, \Omega(t,x)\,\chi_{D(t)}(x)\,, \qquad \Omega^e(t,x) \,:=\, \Omega(t,x)\,\chi_{D(t)^c}(x)\,,$$

and therefore obtain relation (54). By virtue of this fact, equation (6) restricted on the domain $D(t)$ reads as follows:

$$\partial_t\Omega^i \,+\, u\cdot\nabla\Omega^i \,=\, -\left(\Omega^i\cdot\nabla u \,+\, {}^t\nabla u\cdot\Omega^i \,+\, \nabla a^i\wedge\nabla\Pi\right),$$

which gives us the estimate (keep in mind also (55))

$$\left\|\Omega^i(t)\right\|_{\mathcal{C}^\varepsilon} \,\leq\, C\left(\left\|\Omega_0^i\right\|_{\mathcal{C}^\varepsilon} \,+\, \int_0^t \left(\left\|\Omega^i\cdot\nabla u + {}^t\nabla u\cdot\Omega^i\right\|_{\mathcal{C}^\varepsilon} + \left\|\nabla a^i\wedge\nabla\Pi\right\|_{\mathcal{C}^\varepsilon}\right)d\tau\right).$$

We claim that the first term under the integral can be controlled in $\mathcal{C}^\varepsilon$. As a matter of facts, by (3) we know that the velocity field satisfies the elliptic equation

$$-\,\Delta u^k \,=\, \sum_{j=1}^N \partial_j\Omega_{kj}^i$$

in $D(t)$, with the boundary condition (by theorem 2.5 and corollary 6.5) $u_{|\partial D(t)} \in \mathcal{C}^{1+\varepsilon}(\partial D(t))$. So (see theorem 8.33 of [18]) we have that $u \in \mathcal{C}^{1+\varepsilon}(D(t))$ and the following inequality holds:

$$\|u\|_{\mathcal{C}^{1+\varepsilon}(D(t))} \,\leq\, C\left(\left\|u_{|\partial D(t)}\right\|_{\mathcal{C}^{1+\varepsilon}(\partial D(t))} \,+\, \left\|\Omega^i\right\|_{\mathcal{C}^{\varepsilon}(D(t))}\right).$$

Let us note that, as pointed out in [18], a priori the constant C depends on $\partial D(t)$ through the $\mathcal{C}^{1+\varepsilon}$ norms of its local parametrizations, so finally on $\exp\left(\int_0^t \|\nabla u\|_{L^\infty} d\tau\right)$. However relation (55) allows us to control it uniformy on $[0,T]$. Therefore, in $D(t)$ one gets the following inequality:

$$\left\|\Omega^i \cdot \nabla u + {}^t\nabla u \cdot \Omega^i\right\|_{\mathcal{C}^{\varepsilon}(D(t))} \,\leq\, C\,\left\|\Omega^i\right\|_{\mathcal{C}^{\varepsilon}(D(t))}\,\|u\|_{\mathcal{C}^{1+\varepsilon}(D(t))}\,,$$

which proves our claim.
Finally, let us handle the pressure term: we will argue as just done for the velocity field. From what we have proved, we have that ∇a^i is in $\mathcal{C}^\varepsilon$; moreover, $\nabla\Pi$ satisfies the equation

$$-\,\Delta\Pi \,=\, \nabla(\log a^i)\cdot\nabla\Pi \,+\, \rho^i\,\nabla u : \nabla u$$

in the bounded domain $D(t)$, provided with the boundary conditions (again thanks to theorem 2.5 and corollary 6.5) $\nabla\Pi_{|\partial D(t)} \in \mathcal{C}^{1+\varepsilon}(\partial D(t))$. So we get (see again [18]) $\nabla\Pi \in \mathcal{C}^{1+\varepsilon}(D(t))$ and its norm in this space can be bounded (recall (55) again) by

$$\left\|\nabla\Pi_{|\partial D(t)}\right\|_{\mathcal{C}^{1+\varepsilon}(\partial D(t))} + \left\|\nabla a^i\right\|_{\mathcal{C}^{\varepsilon}(D(t))}\|\nabla\Pi\|_{\mathcal{C}^{\varepsilon}(D(t))} + \left\|\rho^i\right\|_{\mathcal{C}^{1+\varepsilon}(D(t))}\|\nabla u\|^2_{\mathcal{C}^{\varepsilon}(D(t))}\;;$$

applying interpolation inequality for Hölder spaces for the inclusions $\mathcal{C}^{1+\varepsilon} \hookrightarrow \mathcal{C}^\varepsilon \hookrightarrow L^\infty$ leads us to the control of $\|\nabla\Pi\|_{\mathcal{C}^{1+\varepsilon}(D(t))}$.

Putting all these inequalities together and applying Gronwall's lemma, we finally get a control for the $\mathcal{C}^\varepsilon$ norm of Ω^i in the interior of $D(t)$, and this completes the proof of the corollary. $\square$

References

A [1] S. Alinhac: Interaction d'ondes simples pour des équations complètement non linéaires, *Annales Scientifiques de l'École Normale Supérieure*, **21**, 1988, p. 91-133.

B-C-D [2] H. Bahouri, J.-Y. Chemin, R. Danchin: *Fourier Analysis and Nonlinear Partial Differential Equations,* Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), **343**, Springer, Heidelberg, 2011.

Bo1979 [3] J.-M. Bony: Propagation des singularités pour des équations aux dérivées partielles non linéaires, *Séminaires Équations aux Dérivées Partielles de l'École Polytechnique*, 1979-1980.

Bo1981 [4] J.-M. Bony: Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, *Annales de l'École Normale Supérieure*, **14**, 1981, p. 209-246.

Ch1988 [5] J.-Y. Chemin: Calcul paradifférentiel précisé et application à des équations aux dérivées partielles non linéaires, *Duke Mathematical Journal*, **51**(1), 1988, p. 431-469.

Ch1991 [6] J.-Y. Chemin: Sur le mouvement des particules d'un fluide parfait, incompressible, bidimensionel, *Inventiones Mathematicae*, **103**, 1991, p. 599-629.

Ch1993 [7] J.-Y. Chemin: Persistance des structures géométriques liées aux poches de tourbillon, *Annales de l'École Normale Supérieure*, **26**(4), 1993, p. 1-26.

C1995 [8] J.-Y. Chemin: *Fluides parfaits incompressibles*, Astérisque, **230**, 1995.

C1996 [9] J.-Y. Chemin: A remark on the inviscid limit for two-dimensional incompressible fluid, *Comm. Partial Diff. Equations*, **21**, 1996, p. 1771-1779.

D1997 [10] R. Danchin: Poches de tourbillon visqueuses, *J. Math. Pures Appl.*, **76**, 1997, p. 609-647.

D1997-sing [11] R. Danchin: Évolution temporelle d'une poche de tourbillon singulière, *Comm. Partial Diff. Equations*, **22**, 1997, p. 685-721.

D1999 [12] R. Danchin: Persistance de structures géométriques et limite non visqueuse pour les fluides incompressibles en dimension quelconque, *Bull. Soc. Math. France*, **127**, 1999, no. 2.

D2010 [13] R. Danchin: On the well-posedness of the incompressible density-dependent Euler equations in the L^p framework, *Journal of Differential Equations*, **248**(8), 2010, p. 2130-2170.

D-F [14] R. Danchin, F. Fanelli: The well-posedness issue for the density-dependent Euler equations in endpoint Besov spaces, to appear in *Journal de Mathématiques Pures et Appliquées*.

Dep [15] N. Depauw: Poche de tourbillon pour Euler 2D dans un ouvert à bord, *Journal de Mathématiques Pures et Appliquées*, **78**, 1999, p. 313-351.

Dut [16] A. Dutrifoy: On 3-D vortex patches in bounded domains, *Comm. Partial Diff. Equations*, **28**(7-8), 2003, p. 1237-1263.

G-SR [17] P. Gamblin, X. Saint-Raymond: On three-dimensional vortex patches, *Bull. Soc. Math. France*, **123**, 1995, p. 375-424.

G-T [18] D. Gilbarg, N. S. Trudinger: *Elliptic Partial Differential Equations of Second Order*, 3rd edition, Springer-Verlag, New York, 1985.

Hat [19] A. Hatcher: *Algebraic Topology*, Cambridge University Press, 2002.

Hm [20] T. Hmidi: Régularité höldérienne de poches de tourbillon visqueuses, *Journal de Mathématiques Pures et Appliquées*, **84**(11), 2005, p. 1455-1495.

Hu [21] C. Huang: Singular integral system approach to regularity of 3D vortex patches, *Indiana University Mathematical Journal*, **50**(1), 2001, p. 509-552.

M [22] A. Majda: Vorticity and the mathematical theory of an incompressible fluid flow, *Communications in Pure and Applied Mathematics*, **38**, 1986, p. 187-220.

S [23] P. Serfati: Régularité stratifiée et équation d'Euler 3D à temps grand, *C.R. Acad. Sci. Paris* Sér I Math., **318**, 1994, p. 925-928.

Y [24] V. Yudovich: Non stationary flow of an ideal incompressible fluid, *Zh. Vychisl. Mat. i Mat. Fiz.*, **3**, 1963, p. 1032-1066.