Multiplicity of solutions for a mean field equation on compact surfaces

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ABSTRACT

We consider a scalar field equation on compact surfaces which has variational structure. When the surface is a torus and a physical parameter ρ belongs to $(8\pi, 4\pi^2)$ we show under some extra assumptions that, as conjectured in [9], the functional admits at least three saddle points other than a local minimum.

Key Words: Scalar field equations, Geometric PDE's, Multiplicity result.

1 Introduction

Let (Σ, g) be a compact Riemann surface (without boundary), $h \in C^2(\Sigma)$ be a positive function and ρ a positive real parameter. We consider the equation

$$-\triangle_g u + \rho = \rho \frac{h(x)e^u}{\int_\Sigma h(x)e^u dV_g} \qquad x \in \Sigma,\ u \in H_g^1(\Sigma), \tag{$*$}$$

where $\triangle_g$ is the Laplace-Beltrami operator on Σ.

When (Σ, g) is a flat torus equation $(*)$ is related to the study of some Chern–Simons–Higgs models; indeed via its solutions it is possible to describe the asymptotic behavior of a class of condensates (or multivortex) solutions which are relevant in theoretical physics and which were absent in the classical (Maxwell-Higgs) vortex theory (see [24], [27], [28] and references therein). This PDE arises also in conformal geometry; when (Σ, g) is the standard sphere and $\rho = 8\pi$, the geometric meaning of this problem is that from a solution u we can obtain a new conformal metric $e^u g$ which has curvature $\frac{\rho}{2}h$; the latter is known as the Kazdan-Warner problem, or as the Nirenberg problem and has been studied for example in [3], [4] and [17]. Moreover this problem arises in statistical mechanics. Indeed, when formulated on bounded domains of $\mathbb{R}^2$ with Dirichlet boundary conditions, equation $(*)$ was considered in [1] and [16] as the mean field limit as point vortices for the two–dimensional Euler equation.

Problem $(*)$ has a variational structure and solutions can be found as critical points of the functional

$$I_\rho(u) = \frac{1}{2}\int_\Sigma |\nabla_g u|^2 dV_g + \rho\int_\Sigma u\, dV_g - \rho \log \int_\Sigma h(x)e^u dV_g \qquad u \in H_g^1(\Sigma). \tag{1.1}$$

Since equation $(*)$ is invariant when adding constants to u, we can restrict ourselves to the subspace of the functions with zero average $\bar{H}_g^1(\Sigma) := \{u \in H_g^1(\Sigma) : \int_\Sigma u\, dv_g = 0\}$.

By virtue of the Moser-Trudinger inequality (see Lemma 2.2) one can easily prove the compactness and the coercivity of I_ρ when $\rho < 8\pi$ and thus one can find solutions of $(*)$ by minimization.

If $\rho = 8\pi$ the situation is more delicate since I_ρ still has a lower bound but it is not coercive anymore; in general when ρ is an integer multiple of 8π, the existence problem of $(*)$ is much harder (a far from complete list of references on the subject includes works by Chang and Yang [4], Chang, Gursky and Yang [3], Chen and Li [5], Nolasco and Tarantello [24], Ding, Jost, Li and Wang [12] and Lucia [21]).

For $\rho > 8\pi$, as the functional I_ρ is unbounded from below and from above, solutions have to be found as saddle points.
In [11] Ding, Jost, Li and Wang proved that, assuming $\rho \in (8\pi, 16\pi)$ and assuming that the genus of the surface is greater or equal than 1, there exists a solution to $(*)$. In [19] Yan Yan Li initiated a program to find solutions for $\rho > 8\pi$ by using the topological degree theory. He proved an uniform bound for solutions to equation $(*)$ whenever ρ is contained in a compact set of $(8k\pi, 8(k+1)\pi)$, where $k \geq 0$ is an integer. Therefore, the Leray–Schauder degree for $(*)$ remains the same when ρ is in the interval $(8k\pi, 8(k+1)\pi)$. Few years ago this program was completed by Chen and Lin in [7] using a finite-dimensional reduction to compute the jump values. The authors obtained a complete degree-counting formula, extending the results in [20], where the case $\Sigma = S^2$ and $k = 1$ was studied. Finally, when $\rho \notin 8\mathbb{N}\pi$, Djadli [13] generalized these previous results establishing the existence of a solution for any (Σ, g); to do that he deeply investigated the topology of low sublevels of I_ρ in order to perform a min-max scheme (already introduced in Djadli and Malchiodi [14]).

Not much is known about multiplicity. Recently the author in [10], via Morse inequalities, improved significantly the multiplicity estimate which can be deduced from the degree-counting formula in [7].

Besides, the case of the flat torus, which is a relevant situation from the physical point of view, has been treated by Struwe and Tarantello under the assumptions that $h \equiv 1$ and $\rho \in (8\pi, 4\pi^2)$. In these hypotheses, $u = 0$ is clearly a critical point for I_ρ. Moreover, $u = 0$ is a strict local minimum, since the second variation in the direction $v \in \bar{H}^1_g(T)$ can be estimated as follows

$$D^2 I_\rho(0)[v, v] = \|v\|^2 - \rho \int_\Sigma v^2 dx \geq \left(1 - \frac{\rho}{4\pi^2}\right) \|v\|^2 . \tag{1.2}$$

Under these conditions, the functional possesses a mountain pass geometry and by thanks to this structure the existence of a saddle point of I_ρ has been detected by Struwe and Tarantello.

Theorem 1.1. *([26]) Let Σ be the flat torus and $h \equiv 1$. Then, for any $\rho \in (8\pi, 4\pi^2)$, there exists a non-trivial solution u_ρ of $(*)$ satisfying $I_\rho(u_\rho) \geq (1-\rho/4\pi^2)c_0$ for some constant $c_0 > 0$ independent of ρ.*

As g is the flat metric and h is constant, if u is a solution of $(*)$, the functions $u_{x_0}(x) := u(x - x_0)$ still solve $(*)$, for any $x_0 \in T$; so from Theorem 1.1 we can deduce the existence of an infinite number of solutions of $(*)$.

Perturbing g and h there is still a local minimum, $\bar{u}$, close to $u = 0$ and the same procedure of [26] ensures the presence of a saddle point, but on the other hand, if u is a non-trivial solution, the criticality of the translated functions u_{x_0} is not anymore guaranteed. In [9] the author improved this result stating that apart from $\bar{u}$ there are at least two critical points, see Theorem 3.1 in Section 3.

The strategy of the proof consists in defining a deformed functional $\tilde{I}_\rho$, having the same saddle points of I_ρ but a greater topological complexity of its low sublevels, and in estimating from below the number of saddle points of $\tilde{I}_\rho$ using the notion of Lusternik-Schnirelmann relative category (roughly speaking a natural number measuring how a set is far from being contractible, when a subset is fixed).

Always in [9] the author conjectured that apart from the minimum and the two saddle points another critical point should exist. In fact this turns out to be true.

Theorem 1.2. *If $\rho \in (8\pi, 4\pi^2)$ and $\Sigma = T$ is the torus, if the metric g is sufficiently close in $C^2(T; S^{2\times 2})$ to dx^2 and h is uniformly close to the constant 1, I_ρ admits a point of strict local minimum and at least three different saddle points.*

In the above statement $S^{2\times 2}$ stands for the symmetric matrices on T. To prove Theorem 3.1 we exploit the following inequality derived in [9]:

$$\#\{\text{solutions of } (*)\} \geq \mathrm{Cat}_{X,\partial X} X,$$

where X is the topological cone over T. Next, applying a classical result we are able to estimate from below the previous relative category by one plus the cup–length of the pair $(T\times[0,1], T\times(\{0\}\cup\{1\}))$. The cup–length of a topological pair (Y,Z), denoted by $\mathrm{CL}(Y,Z)$, is the maximum number of elements of the cohomology ring $H^*(Y)$ having positive dimensions and whose cup product do not "annihilate" the ring $H^*(Y,Z)$; we refer to the next section for a rigorous definition. Finally, to obtain the thesis, we show that $\mathrm{CL}(T\times[0,1], T\times(\{0\}\cup\{1\})) \geq \mathrm{CL}(T) = 2$.

Remark 1.3. *Since all the arguments only use the presence of a strict local minimum and the fact that X is the topological cone over T, whenever on some (Σ, g) the functional I_ρ possesses a strict local minimum, the theorem holds true, more precisely I_ρ has at least $CL(\Sigma)+1$ critical points other than the minimum.*

In section 2 we collect some useful material concerning the topological structure of I_ρ and we recall some definitions and some classical results in algebraic topology; besides, we focus on the notion of Lusternik-Schnirelmann relative category and its relation with the cuplength. In section 3 we present briefly the result in [9] and prove our multiplicity result.

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2 Notation and preliminaries

In this section we collect some facts needed in order to obtain the multiplicity result.
First of all we consider some improvements of the Moser-Trudinger inequality which are useful to study the topological structure of the sublevels of I_ρ. Next, we collect some basic notions in algebraic topology and we recall the definition of Lusternik-Schnirelmann relative category stating also some results relating the category to both the cup-length and the existence of critical points.

Let now fix our notation. The symbol $B_r(p)$ denotes the metric ball of radius r and center p.
As already specified we set $\bar{H}^1_g(\Sigma) := \left\{u \in H^1_g(\Sigma) \,:\, \int_\Sigma u\, dv_g = 0\right\}$.
Large positive constants are always denoted by C, and the value of C is allowed to vary from formula to formula. Moreover, given a smooth functional $I : H^1_g(\Sigma) \to \mathbb{R}$ and a real number c, we set $I^c := \{u \in H^1_g(\Sigma) \,|\, I(u) \leq c\}$.
Finally, given a pair of topological spaces (X,A) we will denote by $\mathrm{H}^q(X,A)$ the relative q-th cohomology group with coefficients in $\mathbb{R}$ and by $\mathrm{H}^*(X,A)$ the direct sum of the cohomology groups, $\bigoplus_{q=0}^{\infty} \mathrm{H}^q(X,A)$.

2.1 Variational Structure

Even though the Palais-Smale is not known to hold for our functional, employing together a deformation lemma proved by Lucia in [22] and a compactness result due to Li and Shafrir [18] it is possible to establish for I_ρ a strong result through and through analogous to the classical deformation lemma.

Proposition 2.1. *If $\rho \neq 8k\pi$ and if I_ρ has no critical levels inside some interval $[a,b]$, then $\{I_\rho \leq a\}$ is a deformation retract of $\{I_\rho \leq b\}$.*

To understand the topology of sublevels of I_ρ it is useful to recall the well-known Moser-Trudinger inequality on compact surfaces.

Lemma 2.2 (*Moser-Trudinger inequality*)**.** *There exists a constant C, depending only on (Σ, g) such that for all $u \in H^1_g(\Sigma)$*

$$\int_\Sigma e^{\frac{4\pi(u-\bar{u})^2}{\int_\Sigma |\nabla_g u|^2 dV_g}} \leq C. \tag{2.1}$$

where $\bar{u} := \fint_\Sigma u dV_g$. As a consequence one has for all $u \in H^1_g(\Sigma)$

$$\log \int_\Sigma e^{(u-\bar{u})} dV_g \leq \frac{1}{16\pi} \int_\Sigma |\nabla_g u|^2 dV_g + C. \tag{2.2}$$

Chen and Li [6] from this result showed that if e^u has integral controlled from below (in terms of $\int_\Sigma e^u dV_g$) into $(l+1)$ distinct regions of Σ, the constant $\frac{1}{16\pi}$ can be basically divided by $(l+1)$. Since we are interested in the behavior of the functional when $\rho \in (8\pi, 16\pi)$, it is sufficient to consider the case $l = 1$.

Lemma 2.3. *[6] Let Ω_1, Ω_2 be subsets of Σ satisfying $\operatorname{dist}(\Omega_1, \Omega_2) \geq \delta_0$, where δ_0 is a positive real number, and let $\gamma_0 \in (0, \frac{1}{2})$. Then, for any $\tilde{\varepsilon} > 0$ there exists a constant $C = C(\tilde{\varepsilon}, \delta_0, \gamma_0)$ such that $\log \int_\Sigma e^{(u-\bar{u})} dV_g \leq C + \frac{1}{32\pi - \tilde{\varepsilon}} \int_\Sigma |\nabla_g u|^2 dV_g$ for all the functions satisfying $\frac{\int_{\Omega_i} e^u dV_g}{\int_\Sigma e^u dV_g} \geq \gamma_0$, for $i = 1, 2$.*

Therefore if $\rho \in (8\pi, 16\pi)$ Lemma 2.3 implies that if "e^u" is spread in at least two regions then the functional I_ρ stays uniformly bounded from below. Qualitatively if I_ρ attains large negative values, $\frac{e^u}{\int_\Sigma e^u}$ has to concentrate at a point of Σ. Indeed, using the previous Lemma and a covering argument, Ding, Jost, Li and Wang obtained (see [11] or [13]) the following result.

Lemma 2.4. *Assuming $\rho \in (8\pi, 16\pi)$, the following property holds. For any $\varepsilon > 0$ and any $r > 0$ there exists a large positive constant $L = L(\varepsilon, r)$ such that for every $u \in H^1_g(\Sigma)$ with $I_\rho(u) \leq -L$, there exist a point $p_u \in \Sigma$ such that $\int_{\Sigma \setminus B_r(p_u)} e^u dV_g / \int_\Sigma e^u dV_g < \varepsilon$.*

By means of Lemma 2.4 it is possible to map continuously low sublevels of the Euler functional into Σ, roughly speaking associating to u the point p_u (see [13] for details); in the following we will denote this map $\Psi : I_\rho^{-L} \to \Sigma$. Viceversa, one can map Σ into arbitrarily low sublevels, associating to $x \in \Sigma$ the function $\varphi_{\lambda,x} := \tilde{\varphi}_{\lambda,x} - \overline{\tilde{\varphi}_{\lambda,x}}$, where $\tilde{\varphi}_{\lambda,x}(y) := \log \left(\frac{\lambda}{1+\lambda^2 \operatorname{dist}^2(x,y)} \right)^2$ and λ is a sufficiently large positive real parameter. The composition of the former map with the latter can be taken to be homotopic to the identity on Σ, and hence the following result holds true.

Proposition 2.5. *[23] If $\rho \in (8\pi, 16\pi)$, there exists $L > 0$ such that $\{I_\rho \leq -L\}$ has the same homology as Σ.*

On the other hand in [23] Proposition 2.1 is used to prove that, since I_ρ stays uniformly bounded on the solutions of $(*)$ (again by the compactness result due to Li), it is possible to retract the whole Hilbert space $\bar{H}^1_g(\Sigma)$ onto a high sublevel $\{I_\rho \leq b\}$, $b \gg 0$. More precisely:

Proposition 2.6. *[23] If $\rho \in (8\pi, 16\pi)$ for some $k \in \mathbb{N}$ and if b is sufficiently large positive, the sublevel $\{I_\rho \leq b\}$ is a deformation retract of X, and hence it has the same homology of a point.*

Remark 2.7. *Let notice that, since Σ is not contractible, Proposition 2.5 together with Proposition 2.6 and Proposition 2.1 permits to derive an alternative proof of the general existence result due to Djadli.*

2.2 Notions in algebraic topology

Let now recall some well known definitions and results in algebraic topology.
First, we recall the Kunneth Theorem for cohomology in a particular case.

Theorem 2.8. *([2], page 8) If* $(X \times Y', Y \times X')$ *is an excisive couple in* $X \times X'$ *and* $H^*(X,Y)$ *is of finite type, i.e.* $H^q(X,Y)$ *is finitely generated for each* q*, then the map*

$$\mu : H^*(X,Y) \otimes H^*(X',Y') \longrightarrow H^*((X,Y) \times (X',Y')), \tag{2.3}$$

defined as $\mu(u \otimes v) := u \times v \in H^{p+q}((X,Y) \times (X',Y'))$*, for any* $u \in H^p(X,Y)$ *and* $v \in H^q(X',Y')$*, is an isomorphism.*

Cup product. We recall that it is possible to endow the direct sum of the cohomology groups, $H^*(X) = \bigoplus_q H^q(X)$, with an associative and graded multiplication, namely the cup product $\bigcup : H^p(X) \times H^q(X) \to H^{p+q}(X)$. This multiplication turns $H^*(X)$ into a ring; in fact it is naturally a $\mathbb{Z}$-graded ring with the integer q serving as degree and the cup product respects this grading. This definition can be extended to topological pairs; in particular, if (Y_1, Y_2) is an excisive couple in X, it is possible to define the cup product

$$\cup : H^p(X, Y_1) \times H^q(X, Y_2) \longrightarrow H^{p+q}(X, Y_1 \cup Y_2)$$

In de Rham cohomology the cup product of differential forms is also known as the wedge product.

Proposition 2.9. *([25], page 253) Let* $(X \times Y', Y \times X')$ *be an excisive couple in* $X \times X'$*, and let* $p_1 : (X,Y) \times X' \to (X,Y)$ *and* $p_2 : X \times (X',Y') \to (X',Y')$ *be the projections. Given* $u \in H^p(X,Y)$ *and* $v \in H^q(X',Y')$*, then in* $H^{p+q}((X,Y) \times (X',Y'))$ *we have*

$$u \times v = p_1^*(u) \cup p_2^*(v).$$

Cup–length. A numerical invariant derived from the cohomology ring is the cup-length, which for a topological space X is defined as follows:

$$\mathrm{CL}(X) = \max \{\, l \in \mathbb{N} \mid \exists\, c_1, \ldots, c_l \in H^*(X), \text{ with } \dim(c_i) > 0,\ i = 1, 2, \ldots, l, \\ \text{such that } c_1 \cup \ldots \cup c_l \neq 0 \}.$$

For example the cup–length of the 2-torus is equal to 2; too see it one can think to the volume form in de Rham cohomology.
More generally, we define the cup length for a topological pair (X, Y).

$$\mathrm{CL}(X,Y) = \max \{\, l \in \mathbb{N} \mid \exists\, c_0 \in H^*(X,Y),\ \exists\, c_1, \ldots, c_l \in H^*(X), \text{ with } \dim(c_i) > 0 \\ \text{for } i = 1, 2, \ldots, l, \text{ such that } c_0 \cup c_1 \cup \ldots \cup c_l \neq 0 \}.$$

In the case where $Y = \emptyset$, we just take $c_0 \in H^0(X)$; thus the two definitions are the same.

2.3 Lusternik-Schnirelmann relative category

We recall the definition of Lusternik-Schnirelmann category (category, for short); then, following [15], we introduce a more powerful notion. In fact, to be precise, it is not a notion but rather a family of (Lusternik-Schnirelmann) relative categories. In this family we choose only two for their special properties, which are given in Proposition 2.12. We will see that the category is a useful tool in critical point theory to obtain multiplicity results.

Definition 2.10. Let X be a topological space and A a subset of X. The *category of* A *with respect to* X, denoted by $\mathrm{Cat}_X A$, is the least integer k such that $A \subset A_1 \cup \ldots \cup A_k$, with A_i $(i = 1, \ldots, k)$ closed and contractible in X. We set $\mathrm{Cat}_X \emptyset = 0$ and $\mathrm{Cat}_X A = +\infty$ if there are no integers satisfying the demand.

Definition 2.11. Let X be a topological space and Y a closed subset of X. A closed subset A of X *is of the k-th (strong) category relative to* Y (we write $\operatorname{Cat}_{X,Y} A = k$) if k is the least positive integer such that there exist $A_i \subset A$ closed and $h_i : A_i \times [0,1] \to X$, $i = 0, \dots, k$, satisfying the following properties:

(i) $A = \cup_{i=0}^{k} A_i$,

(ii) $h_i(x, 0) = x \quad \forall\, x \in A_i \;\; 0 \le i \le k$,

(iii) $h_0(x, 1) \in Y \quad \forall x \in A_0$ and $h_0(y, t) = y \quad \forall y \in Y \quad \forall t \in [0, 1]$,

(iv) $\forall\, i \ge 1 \;\; \exists\, x_i \in X$ such that $h_i(x, 1) = x_i$,

(v) $\forall\, i \ge 1 \;\; h_i(A_i \times [0, 1]) \cap Y = \emptyset$.

We say that A *is of the k-th weak category relative to* Y, written $\operatorname{cat}_{X,Y} A = k$, if k is minimal verifying conditions $(i) - (iv)$.
If one such k does not exist, we set $\operatorname{Cat}_{X,Y} A = +\infty$ (respectively $\operatorname{cat}_{X,Y} A = +\infty$).

Starting from the above definition, it is easy to check that the following properties hold true.

Proposition 2.12. *[15] Let A, B and Y be closed subsets of X:*

1. *if $Y = \emptyset$, then $\operatorname{cat}_{X,\emptyset} A = \operatorname{Cat}_{X,\emptyset} A = \operatorname{Cat}_X A$;*
2. $\operatorname{Cat}_{X,Y} A \ge \operatorname{cat}_{X,Y} A$;
3. *if $A \subset B$, then $\operatorname{Cat}_{X,Y} A \le \operatorname{Cat}_{X,Y} B$;*
4. *if there exists an homeomorphism $\phi : X \to X'$ such that $Y' = \phi(Y)$ and $A' = \phi(A)$, then $\operatorname{Cat}_{X',Y'} A' = \operatorname{Cat}_{X,Y} A$;*
5. *if $X' \supset X \supset A$ and $r : X' \to X$ is a retraction such that $r^{-1}(Y) = Y$ and $r^{-1}(A) \supset A$, then $\operatorname{Cat}_{X',Y} A \ge \operatorname{Cat}_{X,Y} A$.*

Usually, the notion of category is employed to find critical points of a functional I on a manifold X, in connection with the topological structure of X. Moreover a classical theorem by Lusternik-Schnirelmann shows that either there are at least $\operatorname{Cat}_X X$ critical points of I on X, or at some critical level of I there is a continuum of critical points.
This result cannot directly help us because, since we look for critical points on $\bar{H}^1_g(T)$, we would take $X = \bar{H}^1_g(T)$ which, clearly, has category equal to 1 (being contractible).
So we will need a generalization of such a theorem which involves relative category of sublevels. In particular a Theorem in [15] can be adapted to our functional.

Theorem 2.13. *If $-\infty < a < b < +\infty$ and a, b are regular value for I_ρ, then*

$$\#\{\text{critical points of } I_\rho \text{ in } a \le I_\rho \le b\} \ge \operatorname{Cat}_{\{I_\rho \le b\},\{I_\rho \le a\}} \{I_\rho \le b\}.$$

In its original formulation the previous statement dealt with C^1 functionals verifying the Palais-Smale condition, but, as pointed out in [9], the (PS)-condition is used in the proof only twice to apply the classical deformation lemma (see for example [8]). Thus, it is not hard to understand that Proposition 2.1 allows to extend the result to I_ρ.
Besides, in a particular case the relative category can be estimated by means of the cup-length of a pair in the following way:

Theorem 2.14. *[2] For any topological space X, if Y is a closed subset of X, then:*

$$\operatorname{cat}_{X,Y} X \ge \operatorname{CL}(X, Y) + 1.$$

3 Proof of Theorem 1.2

Before proving Theorem 1.2 we recall the previous result in [9] and we summarize its proof.

Theorem 3.1. *[9] If $\rho \in (8\pi, 4\pi^2)$ and $\Sigma = T$ is the torus, if the metric g is sufficiently close in $C^2(T; S^{2\times 2})$ to dx^2 and h is uniformly close to the constant 1, I_ρ admits a point of strict local minimum and at least two different saddle points.*

Let consider a new functional $\tilde{I}_\rho$ which coincides with I_ρ out of a small neighborhood of $\bar{u}$ and assumes large negative values near $\bar{u}$ (here we are exploiting the existence of a strict local minimum), then fix b and L conveniently, in particular such that $I_\rho^b = \tilde{I}_\rho^b$ and $\tilde{I}_\rho^{-L} = I_\rho^{-L} \amalg \{\text{neighb. of } \bar{u}\}$, I_ρ and $\tilde{I}_\rho$ have the same critical points of saddle type in $\tilde{I}_\rho^b \setminus \tilde{I}_\rho^{-L}$.

Let X denote the contractible cone over T and let ∂X be its boundary; they can be represented as $X = \frac{T\times[0,1]}{T\times\{0\}}$, $\partial X = \frac{T\times(\{0\}\cup\{1\})}{T\times\{0\}}$. To get the thesis it is sufficient to establish the following chain of inequalities:

$$\begin{aligned}
\#\{\text{critical points of } \tilde{I}_\rho \text{ in } -L \le \tilde{I}_\rho \le b\} &\overset{1}{\ge} \mathrm{Cat}_{\tilde{I}_\rho^b, \tilde{I}_\rho^{-L}} \tilde{I}_\rho^b \overset{2}{\ge} \mathrm{Cat}_{\tilde{I}_\rho^b, \phi(\partial X)} \tilde{I}_\rho^b \\
&\overset{3}{\ge} \mathrm{Cat}_{\tilde{I}_\rho^b, \phi(\partial X)} \phi(X) \overset{4}{\ge} \mathrm{Cat}_{\phi(X), \phi(\partial X)} \phi(X) \\
&\overset{5}{\ge} \mathrm{Cat}_{X, \partial X} X \overset{6}{\ge} 2,
\end{aligned} \tag{3.1}$$

where ϕ is the homeomorphism on the image defined as follows:

$$\begin{aligned}
\phi : \ X &\longrightarrow \bar{H}_g^1(T) \\
(x,t) &\longmapsto t\,\varphi_{\lambda,x},
\end{aligned}$$

with $\varphi_{\lambda,x}$ defined in Section 2.1 and L, λ, b suitable constants, clearly depending on ρ.

The first inequality follows immediately from Theorem 2.13, which as showed in [9] holds true also for $\tilde{I}_\rho$, while the third and the fifth can be easily derived from the properties of the relative category.

In order to prove 2 one has to construct a deformation retraction (in $\tilde{I}_\rho^b$) of $\tilde{I}_\rho^{-L}$ onto $\phi(\partial X)$. In particular, since I_ρ^{-L} has two connected components, one can deal separately with these two different regions. For what concerns the neighborhood of the minimum point $\bar{u}$, it is enough to combine the steepest descent flow with a deformation sending $\bar{u}$ into 0; while, in I_ρ^{-L}, the map $\Psi : I_\rho^{-L} \to \Sigma$ has to be composed with the map which realizes the deformation of $\bar{H}_g^1(T)$ on $\tilde{I}_\rho^b$.

Moreover, just perturbing Ψ, it is possible to obtain a new continuous map $\tilde{\Psi} : \tilde{I}_\rho^{-L} \to \phi(\partial X)$ verifying $\tilde{\Psi}_{|\phi(\partial X)} = \mathrm{Id}_{|\phi(\partial X)}$. The key point is that applying again (2.1), one is able to extend $\tilde{\Psi}$ to $\tilde{I}_\rho^b \setminus B_R$, $R = R(\rho, b)$. Then by means of $\tilde{\Psi}$, one can construct a new map $r : \tilde{I}_\rho^{-L} \to \phi(X)$ such that $r_{|\phi(X)} = \mathrm{Id}_{|\phi(X)}$ and $r^{-1}(\phi(\partial X)) = \phi(\partial X)$. Finally, category's properties allow to derive the fourth inequality from the existence of the latter map.

At last the sixth inequality has been tackled using a direct topological argument.

Proof of Theorem 1.2 Our aim will be to improve the last inequality of (3.1), proving that $\mathrm{Cat}_{X,\partial X} X \ge 3$.

To do that we are going to establish a new chain of inequalities, involving the notion of cup length.

$$\begin{aligned}
\mathrm{Cat}_{X,\partial X} X &\overset{a}{\ge} \mathrm{Cat}_{T\times[0,1], T\times(\{0\}\cup\{1\})}(T\times[0,1]) \\
&\overset{b}{\ge} \mathrm{cat}_{T\times[0,1], T\times(\{0\}\cup\{1\})}(T\times[0,1]) \\
&\overset{c}{\ge} \mathrm{CL}(T\times[0,1], T\times(\{0\}\cup\{1\})) + 1 \\
&\overset{d}{\ge} \mathrm{CL}(T) + 1 \overset{e}{=} 3.
\end{aligned} \tag{3.2}$$

Let us first prove point a. Let consider the A_i and h_i verifying the conditions for $\mathrm{Cat}_{X,\partial X} X$.

First of all, in order to show that A_0 is disconnected, let us denote by $X_0 := T \times \{0\}/T \times \{0\}$ and $X_1 := T \times \{1\}/T \times \{0\}$ the two disconnected components of ∂X. By definition we know that $X_0 \cup X_1 = \partial X \subset A_0$ and that there exists $h_0 : A_0 \times [0,1] \to X$ continuous with the properties: $h_0(A_0, 1) \subset \partial X$ and $h_{0|\partial X \times [0,1]} \equiv \mathrm{Id}_{\partial X}$. Now, if A_0 was connected we would get a contradiction because $h_0(A_0, 1)$ would be connected (by continuity of h_0) and disconnected being the union of X_0 and X_1.

Thus we can consider the connected component A_{00} of A_0 containing X_0 and its complementary in A_0, $A_{01} := A_0 \setminus A_{00}$. Then, we define

$$\tilde{A}_{0j} := \{(x,t) \,|\, x \in T,\, t \in [0,1],\, [(x,t)] \in A_{0j})\} \qquad j = 0, 1,$$

where $[(x,t)]$ stands for the equivalence class of (x,t) in X.
Let us set $\tilde{A}_0 := \tilde{A}_{00} \cup \tilde{A}_{01}$.
Next, we construct a continuous map $\tilde{h}_0 : \tilde{A}_0 \times [0,1] \to T \times [0,1]$ in the following way:

$$\tilde{h}_0((x,t), s) := \begin{cases} (x, (1-s)t) & (x,t) \in \tilde{A}_{00} \\ (x, (1-s)t + s) & (x,t) \in \tilde{A}_{01}. \end{cases}$$

Just to be rigorous we also define the sets

$$\tilde{A}_i := \{(x,t) \,|\, x \in T,\, t \in [0,1],\, [(x,t)] \in A_i)\} \qquad i \geq 1,$$

which are nothing but the A_i's seen as subsets of $T \times [0,1]$, without the equivalence relation.
Analogously we define the maps

$$\tilde{h}_i((x,t), s) := h_i([(x,t)], s)$$

which turn out to be well defined, being $A_i \cap \partial X = \emptyset$, for any $i \geq 1$ (see point (v) of Definition 2.11).

Now, it is easy to see that the sets $\tilde{A}_i$'s, together with the continuous maps $\tilde{h}_i$'s, satisfy the conditions of Definition 2.11 for $\mathrm{Cat}_{T\times[0,1], T\times(\{0\}\cup\{1\})}(T \times [0,1])$ and this concludes the proof of this first inequality.

Point b follows directly from property 2 of Proposition 2.12, while applying Theorem 2.14 we obtain inequality c.

To get step d, let us denote by k the cup–length of T. By definition there exist $\alpha_1, \ldots, \alpha_k \in H^*(T)$, with $\dim(\alpha_i) > 0$ for any $i \in \{1, \ldots, k\}$, such that

$$\alpha_1 \cup \ldots \cup \alpha_k \neq 0.$$

Since $H^1([0,1], \{0\} \cup \{1\}) = \mathbb{R}$, we can also choose $0 \neq \beta \in H^1([0,1], \{0\} \cup \{1\})$.
We are now in position to apply Theorem 2.8 with $X = [0,1]$, $Y = \{0\} \cup \{1\}$, $X' = T$ and $Y' = \emptyset$. By definition of μ, see (2.3), and its injectivity, we obtain

$$\beta \times (\alpha_1 \cup \alpha_k) = \mu(\beta \otimes (\alpha_1 \cup \alpha_k)) \neq 0. \tag{3.3}$$

Consider now the projections $p_1 : T \times ([0,1], \{0\} \cup \{1\}) \to ([0,1], \{0\} \cup \{1\})$ and $p_2 : T \times [0,1] \to T$. Applying Proposition 2.9, we find:

$$\beta \times (\alpha_1 \cup \alpha_k) = p_1^*(\beta) \cup p_2^*(\alpha_1 \cup \alpha_k) = p_1^*(\beta) \cup p_2^*(\alpha_1) \cup \ldots \cup p_2^*(\alpha_k). \tag{3.4}$$

Notice that $p_1^*(\beta) \in H^*(T \times [0,1], T \times (\{0\} \cup \{1\}))$ and, for any $i \in \{1, \ldots, k\}$, $p_2^*(\alpha_i) \in H^*(T \times [0,1])$, with $\dim(p_2^*(\alpha_i)) > 0$.

In conclusion, by virtue of (3.3) and (3.4), we proved exactly that $\mathrm{CL}(T) \leq \mathrm{CL}(T \times [0,1], T \times (\{0\} \cup \{1\}))$.

Finally, the equality named e is just due to the well known fact that $\mathrm{CL}(T) = 2$. The proof is thereby complete. ■

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