SELF-PROPELLED MICRO-SWIMMERS IN A BRINKMAN FLUID

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ABSTRACT. We extend the existence, uniqueness, and regularity result obtained in [3] for the case of a self-propelled swimmer in a viscous fluid to the case of a self-propelled swimmer immersed in a particulate medium, modeled as a Brinkman fluid.

Keywords: Brinkman equation, self-propelled motion, swimming, particulate media

CONTENTS

1. INTRODUCTION

Modeling the motion of living beings has stimulated scientists for many decades. The first attempts to study motion inside fluids date back to the pioneering works by Taylor [10] and Lighthill [8]. These papers and the 1977 paper by Purcell [9] point out that the description of motion in viscous fluids at low Reynolds number can involve some counterintuitive facts. The recent literature has been populated by new and more refined results, both theoretical and experimental. In a recent paper by S. Jung, the motion of *Caenorhabditis elegans* is observed in different environments: this nematode usually swims in saturated soil, and its behavior was studied in different saturation conditions as well as in a viscous fluid without solid particles.

Following the approach proposed in [7, III.C], we model the particulate medium surrounding the swimmer as a Brinkman fluid. We show that the framework we proposed in [3] also applies to the case of a Brinkman problem in an exterior domain. We prove the existence, uniqueness, and regularity of the solution to the equations of motion for a body swimming in such an environment, thus generalizing the result previously obtained for the Stokes system.

2. BRINKMAN EQUATION – FUNCTIONAL SETTING

In this section we present some results about Brinkman equation. It was originally proposed in [2] to model a fluid flowing through a porous medium as a correction to Darcy's law by the addition of a diffusive term. A rigorous mathematical derivation from the Navier-Stokes equation via homogenization can be found in [1].

In a Lipschitz domain $\Omega \subset \mathbb{R}^3$, the Brinkman system reads

$$\begin{cases} \nu\Delta u - \alpha^2 u = \nabla p & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = U & \text{on } \partial\Omega, \\ u = 0 & \text{at infinity.} \end{cases} \tag{2.1}$$

The positive constant α takes into account the permeability properties of the porous matrix and the viscosity of the fluid, the constant ν is an effective viscosity of the fluid, while the third equation in the system is the *no-slip* boundary condition. The condition $u = 0$ at

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infinity is significant, and necessary, only when the domain Ω is unbounded. From now on, we will get rid of the effective viscosity, upon a redefinition of α, by setting $\nu = 1$. A brief discussion on the constant ν can be found in Brinkman's paper [2].

In order to cast equation (2.1) in the weak form, we introduce the function spaces in which we will look for the weak solution. Define

$$\mathcal{X}(\Omega) := \{u \in H^1(\Omega;\mathbb{R}^3) : \operatorname{div} u = 0 \text{ in } \Omega\}, \qquad \mathcal{X}_0(\Omega) := \{u \in H^1_0(\Omega;\mathbb{R}^3) : \operatorname{div} u = 0 \text{ in } \Omega\}.$$

Both $\mathcal{X}(\Omega)$ and $\mathcal{X}_0(\Omega)$ are equipped with the standard H^1 norm but we introduce this equivalent one

$$\|u\|^2_{\mathcal{X}(\Omega)} := \alpha^2 \|u\|^2_{L^2(\Omega;\mathbb{R}^3)} + 2\,\|\mathrm{E}u\|^2_{L^2(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}})},$$

the equivalence being a consequence of Korn's inequality.

The weak formulation of equation (2.1) is now given by

$$\begin{cases} \text{find } u \in \mathcal{X}(\Omega) \text{ such that} & u = U \text{ on } \partial\Omega, \\ 2\displaystyle\int_\Omega \mathrm{E}u : \mathrm{E}w \,\mathrm{d}x + \alpha^2 \int_\Omega u \cdot w \,\mathrm{d}x = 0, & \text{for every } w \in \mathcal{X}_0(\Omega), \end{cases} \tag{2.2}$$

where the boundary velocity is a given function $U \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$, the solution being the unique minimum in $\mathcal{X}(\Omega)$ of the strictly convex energy functional

$$\mathcal{E}(u) := 2\int_\Omega |\mathrm{E}u|^2 \,\mathrm{d}x + \alpha^2 \int_\Omega |u|^2 \,\mathrm{d}x = \|u\|^2_{\mathcal{X}(\Omega)}.$$

Here and henceforth the symbol $\mathrm{E}u$ denotes the *symmetric gradient* of u, namely $\mathrm{E}u := \frac{1}{2}(\nabla u + (\nabla u)^T)$.

We call Ω an *exterior domain* with Lipschitz boundary if Ω is an unbounded, connected open set whose boundary $\partial\Omega$ is bounded and Lipschitz, see [3, Section 2]. If we consider the term $\alpha^2 u$ as a forcing term f in system (2.1), we can invoke a classical existence and uniqueness result

Theorem 2.1. *Let $U \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$. Then the following results hold:*

(a) *Let Ω be a bounded connected open subset of $\mathbb{R}^3$ with Lipschitz boundary. If*

$$\int_{\partial\Omega} U \cdot n \,\mathrm{d}S = 0, \tag{2.3}$$

there exists a unique solution u to problem (2.2). *Moreover, there exists $p \in L^2(\Omega)$ such that $\Delta u - \nabla p = f$ in $\mathcal{D}'(\Omega;\mathbb{R}^3)$.*

(b) *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with Lipschitz boundary. Then problem* (2.2) *has a solution. Moreover, there exists $p \in L^2_{\mathrm{loc}}(\Omega)$, with $p \in L^2(\Omega \cap \Sigma_\rho)$ for every $\rho > 0$, such that $\Delta u - \nabla p = f$ in $\mathcal{D}'(\Omega;\mathbb{R}^3)$.*

Proof. Part (a) is classical and can be found in [11, Lemma 2.1 and Theorem 2.4]. Part (b) follows from the density result stated in Theorem 2.2, the application of Lax-Milgram Lemma in the space $\mathcal{X}_0(\Omega)$, and Theorem 2.3 in [11] for finding the pressure p. We notice that to solve the exterior Brinkman problem we do not need condition (2.3). □

Theorem 2.2 (Density [6])**.** *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with Lipschitz boundary. Then, the space $\{u \in C^\infty_{\mathrm{c}}(\Omega;\mathbb{R}^3) : \operatorname{div} u = 0 \text{ in } \Omega\}$ is dense in $\mathcal{X}(\Omega)$ for the H^1 norm.* □

We now define some physically relevant quantities. The *stress tensor* associated with the velocity field u and the pressure p is given by

$$\sigma := -p\,\mathrm{I} + 2\mathrm{E}u. \tag{2.4}$$

The *drag force* is the resultant of the viscous forces acting on the boundary $\partial\Omega$ and is given by

$$F := \int_{\partial\Omega} \sigma(x) n(x) \,\mathrm{d}S(x). \tag{2.5}$$

Analogously, the *torque* is defined by

$$M := \int_{\partial\Omega} x \times \sigma(x) n(x) \,\mathrm{d}S(x). \tag{2.6}$$

Definitions (2.5) and (2.6) are valid under the condition that σn has a trace in $L^1(\partial\Omega;\mathbb{R}^3)$. Since, in general, this assumption is not fulfilled, we have to define the viscous drag force and torque in a different way, by introducing σn as an element of $H^{-1/2}(\partial\Omega;\mathbb{R}^3)$. This will lead to a consistent definition of the *power* of the drag force and torque. In order to do this, we introduce $\mathbb{M}^{3\times3}_{\rm sym}$, the space of 3×3 symmetric matrices, and recall that every $\sigma \in \mathbb{M}^{3\times3}_{\rm sym}$ can be orthogonally decomposed as

$$\sigma = \frac{\operatorname{tr}\sigma}{3}\,\mathrm{I} + \sigma_D$$

where the deviatoric part σ_D is traceless. We are now ready to give the following

Definition 2.3. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with Lipschitz boundary and let $\sigma \in L^1_{\rm loc}(\Omega;\mathbb{R}^3)$ be such that $\sigma_D \in L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym})$ and $\operatorname{div}\sigma \in L^2(\Omega;\mathbb{R}^3)$. The trace of σn, still denoted by σn, is defined as the unique element of $H^{-1/2}(\partial\Omega;\mathbb{R}^3)$ satisfying the equality*

$$\langle \sigma n, V\rangle_\Omega := \int_\Omega (\operatorname{div}\sigma)\cdot v\,\mathrm{d}x + \int_\Omega \sigma : \mathrm{E}v\,\mathrm{d}x, \tag{2.7}$$

where $\langle\cdot,\cdot\rangle_\Omega$ denotes the duality pairing between $H^{-1/2}(\partial\Omega;\mathbb{R}^3)$ and $H^{1/2}(\partial\Omega;\mathbb{R}^3)$, and v is any function in $\mathcal{X}(\Omega)$ such that $v = V$ on $\partial\Omega$.

If there is no risk of misunderstanding, the subscript Ω will be dropped whenever the domain of integration is clear. Notice that if σ is sufficiently smooth then integrating (2.7) by parts leads to the equality

$$\langle \sigma n, V\rangle_\Omega = \int_{\partial\Omega} \sigma n\cdot V\,\mathrm{d}S, \qquad \text{for every } V \in H^{1/2}(\partial\Omega;\mathbb{R}^3).$$

In the general case, the right-hand side of (2.7) is easily proved to be well defined, given the assumptions on σ. In fact, $\operatorname{div}\sigma \in L^2(\Omega;\mathbb{R}^3)$ and $v \in L^2(\Omega;\mathbb{R}^3)$ make the first integral well defined, while the second one is also good since $\sigma : \mathrm{E}v = \sigma_D : \mathrm{E}v$, because of the symmetry of $\mathrm{E}v$, and both σ_D and $\mathrm{E}v$ belong to $L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym})$. Lastly, the definition is independent of the choice of $v \in \mathcal{X}(\Omega)$, since the right-hand side vanishes for every $v \in \mathcal{X}_0(\Omega)$: this follows from the very same computation for the more regular case, by the Density Theorem 2.2. It is easy to see that (2.7) defines a continuous linear functional on $H^{1/2}(\partial\Omega;\mathbb{R}^3)$ by choosing $v \in \mathcal{X}(\Omega)$ an extension of V.

We now proceed in showing other useful properties of the duality pairing introduced in Definition 2.3. Let $U \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$ and let u be the solution to the Brinkman problem (2.2) with boundary datum U and let σ be the corresponding stress tensor. Since all the assumptions of Definition 2.3 are fulfilled, for any given $V \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$ we have

$$\begin{aligned}\langle \sigma n, V\rangle &= \int_\Omega (\operatorname{div}\sigma)\cdot v\,\mathrm{d}x + \int_\Omega \sigma : \mathrm{E}v\,\mathrm{d}x = \alpha^2\int_\Omega u\cdot v\,\mathrm{d}x + \int_\Omega [-p\,\mathrm{I} : \mathrm{E}v + 2\mathrm{E}u : \mathrm{E}v]\,\mathrm{d}x\\ &= \alpha^2\int_\Omega u\cdot v\,\mathrm{d}x - \int_\Omega p\operatorname{div}v\,\mathrm{d}x + 2\int_\Omega \mathrm{E}u : \mathrm{E}v\,\mathrm{d}x\\ &= \alpha^2\int_\Omega u\cdot v\,\mathrm{d}x + 2\int_\Omega \mathrm{E}u : \mathrm{E}v\,\mathrm{d}x,\end{aligned}\tag{2.8}$$

where v is an arbitrary element in $\mathcal{X}(\Omega)$ such that $v = V$ on $\partial\Omega$. If we take, in particular, v to be the solution to problem (2.2) with boundary datum V, we recover the well known *reciprocity condition*

$$\langle \sigma n, V\rangle = \langle \tau n, U\rangle,$$

with τ being the stress tensor associated with v. Moreover, by taking $U = V$ in (2.8) we obtain

$$\langle \sigma n, U\rangle = \alpha^2\,\|u\|^2_{L^2(\Omega;\mathbb{R}^3)} + 2\,\|\mathrm{E}u\|^2_{L^2(\Omega;\mathbb{M}^{3\times3}_{\rm sym})} = \|u\|^2_{\mathcal{X}(\Omega)}\,.$$

This equality allows us to show that the quadratic form $\langle \sigma n, U\rangle$ is positive definite: if $\langle \sigma n, U\rangle = 0$, then it follows that $u = 0$, and therefore $U = 0$.

We are now in a position to define the viscous drag force and torque in a rigorous way, by means of the duality product introduced in Definition 2.3. This is

Definition 2.4. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with Lipschitz boundary, let $u \in \mathcal{X}(\Omega)$ be the solution to the Brinkman problem* (2.2) *with boundary datum $U \in H^{1/2}(\partial\Omega;\mathbb{R}^3)$, let σ be the corresponding stress tensor defined by* (2.4)*, and let $\sigma n \in H^{-1/2}(\partial\Omega;\mathbb{R}^3)$ be the trace on $\partial\Omega$ defined according to* (2.7)*. The* drag force *exerted by the fluid on the boundary $\partial\Omega$ is defined as the unique vector $F \in \mathbb{R}^3$ such that*

$$F \cdot V = \langle \sigma n, V \rangle \quad \text{for every } V \in \mathbb{R}^3. \tag{2.9}$$

The torque *exerted by the fluid on the boundary $\partial\Omega$ is defined as the unique vector $M \in \mathbb{R}^3$ such that*

$$M \cdot \omega = \langle \sigma n, W_\omega \rangle \quad \text{for every } \omega \in \mathbb{R}^3, \tag{2.10}$$

where $W_\omega(x) := \omega \times x$ is the velocity field generated by the angular velocity ω.

3. Kinematics and the equations of motion

In this section we describe the kinematics of the swimmer. The *motion* of a swimmer is described by a map $t \mapsto \varphi_t$, where, for every fixed t, the *state* φ_t is an orientation preserving bijective C^2 map from the *reference* configuration $A \subset \mathbb{R}^3$ into the *current* configuration $A_t \subset \mathbb{R}^3$. Given a distinguished point $x_0 \in A$, for every fixed t, we consider the following factorization

$$\varphi_t = r_t \circ s_t\,, \tag{3.1}$$

where the *position* function r_t is a rigid deformation and the *shape* function s_t is such that

$$s_t(x_0) = x_0 \quad \text{and} \quad \nabla s_t(x_0) \text{ is symmetric.} \tag{3.2}$$

We allow the map $t \mapsto s_t$ to be chosen in a suitable class of admissible shape changes and use it as a control to achieve propulsion as a consequence of the viscous reaction of the fluid. By contrast, $t \mapsto r_t$ is *a priori* unknown and it must be determined by imposing that the resulting $\varphi_t = r_t \circ s_t$ satisfies the equations of motion.

Since the kinematics of the swimmer does not depend on the fluid the swimmer is surrounded by, we can adopt the same setting as in [3]. For the reader's convenience, we recall the results proved there, and refer the reader to the above mentioned paper and the references therein for a more detailed exposition.

The reference configuration of the swimmer $A \subset \mathbb{R}^3$ is a bounded connected open set of class C^2. The time-dependent deformation of A from the point of view of an external observer is described by a function $\varphi_t : \overline{A} \to \mathbb{R}^3$ with the following properties:

$$\varphi_t \in C^2(\overline{A};\mathbb{R}^3), \qquad \varphi_t \quad \text{is injective,} \qquad \det \nabla\varphi_t(x) > 0 \quad \text{for all } x \in \overline{A}, \tag{3.3}$$

for every t; here and henceforth ∇ denotes the gradient with respect to the space variable. Under these hypotheses, $A_t := \varphi_t(A)$ is a bounded connected open set of class C^2 and

$$\text{the inverse} \quad \varphi_t^{-1} : \overline{A}_t \to \overline{A} \quad \text{belongs to } C^2(\overline{A}_t;\mathbb{R}^3).$$

We also assume that

$$\text{the sets } \mathbb{R}^3 \setminus \overline{A}_t \text{ are connected for all } t \in [0,T]. \tag{3.4}$$

Concerning the regularity in time,

$$\text{the map } t \mapsto \varphi_t \text{ belongs to } \mathrm{Lip}([0,T];C^1(\overline{A};\mathbb{R}^3)) \cap L^\infty([0,T];C^2(\overline{A};\mathbb{R}^3)). \tag{3.5}$$

Condition (3.5) implies that for almost every t there exists $\dot\varphi_t \in \mathrm{Lip}(\overline{A};\mathbb{R}^3)$ such that

$$\frac{\varphi_{t+h} - \varphi_t}{h} \to \dot\varphi_t\,, \quad \text{uniformly on } \overline{A} \text{ as } h \to 0.$$

From this, the Eulerian velocity on the boundary ∂A_t, defined by

$$U_t := \dot\varphi_t \circ \varphi_t^{-1}$$

belongs to $\mathrm{Lip}(\partial A_t;\mathbb{R}^3)$ with Lipschitz constant independent of t.

We now introduce the description of the kinematics from the point of view of the swimmer. Let $x_0 \in A$ be a distinguished point and let us look for a factorization of φ_t of the form (3.1). The function $s_t : A \to \mathbb{R}^3$ satisfies properties (3.2), in view of which can be interpreted as a pure shape change from the point of view of an observer inertial with x_0, and the rigid motion $r_t : \mathbb{R}^3 \to \mathbb{R}^3$ is written in the form

$$r_t(z) = y_t + R_t z, \tag{3.6}$$

with $y_t \in \mathbb{R}^3$ and $R_t \in \mathrm{SO}(3)$, the set of orthogonal matrices with positive determinant. This allows us to say that the deformation φ_t, from the point of view of an external observer, is decomposed into a shape change followed by a rigid motion.

From (3.1), (3.3), and (3.6), the following properties of s_t can be inferred: for every t,

$$s_t \in C^2(\overline{A};\mathbb{R}^3), \qquad s_t \quad \text{is injective}, \qquad \det \nabla s_t(x) > 0 \quad \text{for all } x \in \overline{A}, \tag{3.7a}$$

$$\text{the inverse} \quad s_t^{-1} : \overline{B}_t \to \overline{A} \quad \text{belongs to } C^2(\overline{B}_t;\mathbb{R}^3), \tag{3.7b}$$

where $B_t := s_t(A)$, see Fig. 1. Note that (3.7b) is a consequence of (3.7a). Note also that

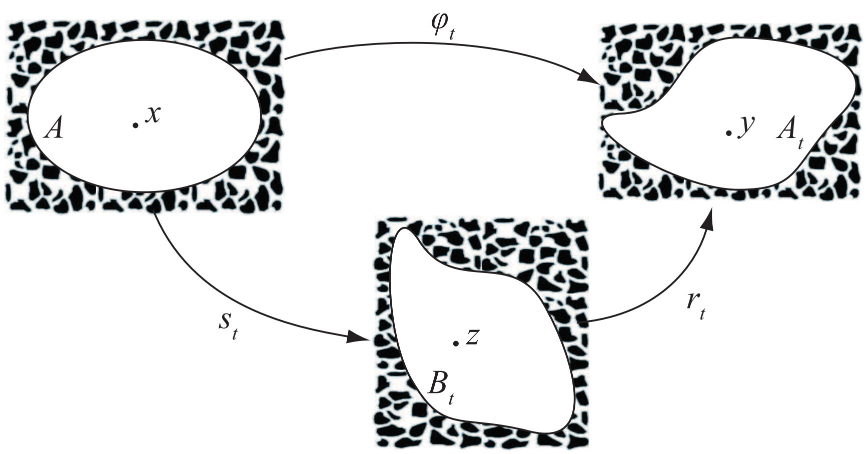

FIGURE 1. Notation for the kinematics.

B_t is a bounded connected open set of class C^2 and that $s_t(B_t) = A_t$ and $s_t(\partial B_t) = \partial A_t$. Moreover, since A is bounded and s_t is continuous, there exists a ball Σ_ρ centered at 0 with radius ρ such that

$$A \subset\subset \Sigma_{\rho-1} \qquad \text{and} \qquad B_t \subset\subset \Sigma_{\rho-1}\,.$$

Lastly, (3.4) implies that

$$\text{the sets } \Sigma_\rho \setminus \overline{B}_t \text{ are connected for all } t \in [0,T]. \tag{3.8}$$

By means of the Polar Decomposition Theorem and the factorization (3.1), it is possible to give explicit formulae for R_t and y_t that clearly show that the maps $t \mapsto R_t$ and $t \mapsto y_t$ are Lipschitz continuous. Since $s_t = r_t^{-1} \circ \varphi_t$,

$$\text{the map } t \mapsto s_t \text{ belongs to } \mathrm{Lip}([0,T];C^1(\overline{A};\mathbb{R}^3)) \cap L^\infty([0,T];C^2(\overline{A};\mathbb{R}^3)), \tag{3.9}$$

The third property in (3.7a) and (3.9) imply that $\|s_t^{-1}\|_{C^2(\overline{B}_t;\mathbb{R}^3)} \leqslant C < +\infty$, with C independent of t. Moreover, condition (3.9) yields the existence of $\dot{s}_t \in \mathrm{Lip}(\overline{A};\mathbb{R}^3)$ such that

$$\frac{s_{t+h} - s_t}{h} \to \dot{s}_t\,, \quad \text{uniformly on } \overline{A}, \text{ as } h \to 0.$$

Other properties of s_t that are worth mentioning, and whose full derivation can be found in [3, Section 3] are:

$$\text{the map } t \mapsto \dot{s}_t \text{ belongs to } L^\infty([0,T];H^{1/2}(\partial A;\mathbb{R}^3)),$$
$$\mathrm{Lip}(\dot{s}_t) \leqslant L, \text{ with } L \text{ independent of } t,$$
$$\text{for any fixed } x \in \overline{A}, \text{ the map } t \mapsto \dot{s}_t(x) \text{ is measurable.}$$

To conclude the description of the kinematics of the swimmer, we give the form of the boundary velocity on the intermediate configuration B_t. It turns out that, if we define $V_t(z) := R_t^T U_t(r_t(z))$ and $W_t(z) := \dot{s}_t(s_t^{-1}(z))$, for every $z \in \partial B_t$, an elementary computation shows that for almost every $t \in [0,T]$

$$V_t(z) = R_t^T \dot{y}_t + R_t^T \dot{R}_t z + W_t(z) \quad \text{for every } z \in \partial B_t\,.$$

We proceed now to the description of the motion of the swimmer. The motion $t \mapsto \varphi_t$ determines for almost every $t \in [0,T]$ the Eulerian velocity U_t through the formula

$$U_t(y) := \dot\varphi_t(\varphi_t^{-1}(y)) \quad \text{for almost every } y \in \partial A_t\,.$$

Notice that $U_t \in H^{1/2}(\partial A_t;\mathbb{R}^3)$ for almost every $t \in [0,T]$. By applying Theorem 2.1 (b) with $\Omega = A_t^{\rm ext} := \mathbb{R}^3 \setminus \overline{A}_t$ and, for almost every $t \in [0,T]$, we obtain a unique solution u_t to the problem

$$\begin{cases} \text{find } u_t \in \mathcal{X}(A_t^{\rm ext}) \text{ such that} & u_t = U_t \text{ on } \partial A_t\,, \\ 2\displaystyle\int_{A_t^{\rm ext}} \mathrm{E}u_t : \mathrm{E}w \,\mathrm{d}y + \alpha^2 \int_{A_t^{\rm ext}} u_t\cdot w\,\mathrm{d}y = 0 & \text{for every } w \in \mathcal{X}_0(A_t^{\rm ext}). \end{cases} \tag{3.10}$$

Let F_{A_t,U_t} and M_{A_t,U_t} be the drag force and torque determined by the velocity field U_t according to (2.9) and (2.10). By neglecting inertia and imposing the self-propulsion constraint, the equations of motion reduce to the vanishing of the viscous force and torque, i.e.,

$$F_{A_t,U_t} = 0 \quad \text{and} \quad M_{A_t,U_t} = 0 \quad \text{for almost every } t \in [0,T]. \tag{3.11}$$

By assuming that φ_t is factorized as $\varphi_t = r_t \circ s_t$, where r_t is a rigid motion as in (3.6) and $t \mapsto s_t$ is a prescribed shape function, our aim is to find $t \mapsto r_t$ so that the equations of motion (3.11) are satisfied. To this extent, we present Theorem 3.1 below, whose result is that (3.11) is equivalent to a system of ordinary differential equations where the unknown functions are the translation $t \mapsto y_t$ and the rotation $t \mapsto R_t$ of the map $t \mapsto r_t$.

The coefficients of these differential equations are defined starting from the intermediate configuration described by the sets $B_t = s_t(A)$ introduced before and the 3×3 matrices K_t , C_t , J_t , depending only on the geometry of B_t , whose entries are defined by

$$(K_t)_{ij} := \langle \sigma[e_j]n, e_i\rangle_{B_t^{\rm ext}}\,, \tag{3.12a}$$

$$(C_t)_{ij} := \langle \sigma[e_j]n, e_i\times z\rangle_{B_t^{\rm ext}}\,, \tag{3.12b}$$

$$(J_t)_{ij} := \langle \sigma[e_j\times z]n, e_i\times z\rangle_{B_t^{\rm ext}}\,, \tag{3.12c}$$

where $B_t^{\rm ext} := \mathbb{R}^3 \setminus \overline{B}_t$, the duality product is given in Definition 2.3 by formula (2.7), and $\sigma[W]$ denotes the stress tensor associated with the outer Brinkman problem in $B_t^{\rm ext}$ with boundary datum W. The notation $\sigma[W]$ is chosen to emphasize the linear dependence of σ on W. Formula (2.8) shows that K_t and J_t are symmetric. The matrix

$$\begin{bmatrix} K_t & C_t^T \\ C_t & J_t \end{bmatrix}$$

is often called in the literature *grand resistance matrix*, and is symmetric and invertible. It originally arises in the case of a Stokes system [5], but the adaptation to the Brinkman system is straightforward: it only share the structure with the original one, while the values of the entries are computed with a different formula, namely (2.8). Let

$$\begin{bmatrix} H_t & D_t^T \\ D_t & L_t \end{bmatrix} := \begin{bmatrix} K_t & C_t^T \\ C_t & J_t \end{bmatrix}^{-1} \tag{3.13}$$

be its inverse. For almost every $t \in [0,T]$, we defined $W_t = \dot s_t \circ s_t^{-1}$, and let $F_t^{\rm sh}$ and $M_t^{\rm sh}$ be the drag force and torque on ∂B_t determined by the boundary velocity field W_t. The components of $F_t^{\rm sh}$ and $M_t^{\rm sh}$ are given, according to (2.9) and (2.10), by

$$(F_t^{\rm sh})_i = \langle \sigma[W_t]n, e_i\rangle_{B_t^{\rm ext}}\,, \tag{3.14a}$$

$$(M_t^{\rm sh})_i = \langle \sigma[W_t]n, e_i\times z\rangle_{B_t^{\rm ext}}\,. \tag{3.14b}$$

Consider now the linear operator $\mathcal{A} : \mathbb{R}^3 \to \mathbb{M}^{3\times3}$ that associates to every $\omega \in \mathbb{R}^3$ the only skew-symmetric matrix $\mathcal{A}(\omega)$ such that $\mathcal{A}(\omega)z = \omega\times z$; therefore, ω is the axial vector of $\mathcal{A}(\omega)$. Finally, we define a vector b_t and a matrix Ω_t according to

$$b_t := H_tF_t^{\rm sh} + D_t^TM_t^{\rm sh}, \qquad \Omega_t := \mathcal{A}(D_tF_t^{\rm sh} + L_tM_t^{\rm sh}), \tag{3.15}$$

which depend on s_t and, most importantly on $\dot s_t$, via (3.14) and the definition of W_t.

Theorem 3.1. *Assume that the shape function $t \mapsto s_t$ satisfies* (3.7) *and* (3.9)*, and that the position function $t \mapsto r_t$ satisfies* (3.6) *and is Lipschitz continuous with respect to time. Then the following conditions are equivalent:*

(i) *the deformation function $t \mapsto \varphi_t := r_t \circ s_t$ satisfies the equations of motion* (3.11)*;*
(ii) *the functions $t \mapsto y_t$ and $t \mapsto R_t$ satisfy the system*

$$\dot{y}_t = R_t b_t\,, \qquad \dot{R}_t = R_t \Omega_t\,, \quad \textit{for almost every } t \in [0,T], \tag{3.16}$$

where b_t and Ω_t are defined in (3.15). □

The proof was given in [3] and need not be modified, so we skip it. It is developed by setting the problem in the intermediate configuration B_t, assuming the point of view of the coordinate system of the shape functions. Changing the variables according to $y = r_t(z)$, $z \in B_t^{\mathrm{ext}}$, the velocity field $v_t(z) := R_t^T u_t(r_t(z))$ is the solution to the problem

$$\begin{cases} \text{find } v_t \in \mathcal{X}(B_t^{\mathrm{ext}}) \text{ such that} & v_t = V_t \text{ on } \partial B_t\,, \\ 2\displaystyle\int_{B_t^{\mathrm{ext}}} \mathrm{E}v_t : \mathrm{E}w \,\mathrm{d}z + \alpha^2 \int_{B_t^{\mathrm{ext}}} v_t \cdot w \,\mathrm{d}z = 0, & \text{for every } w \in \mathcal{X}_0(B_t^{\mathrm{ext}}), \end{cases} \tag{3.17}$$

where $V_t(z) = R_t^T U_t(r_t(z))$, see Fig. 2.

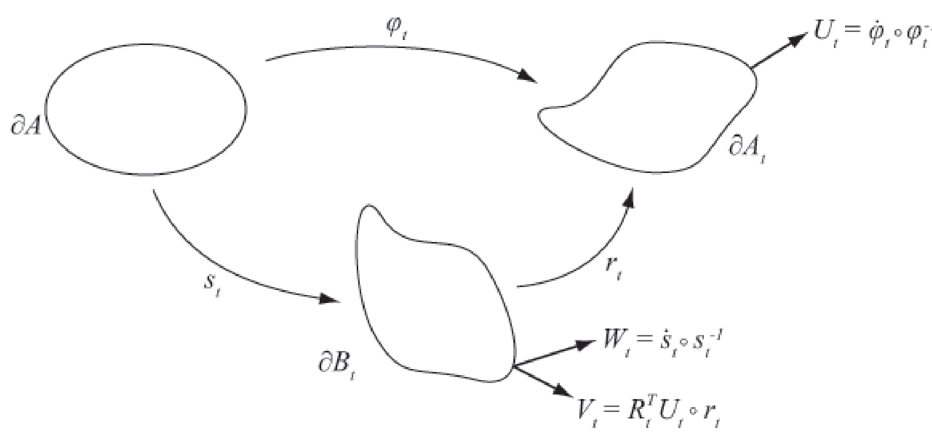

FIGURE 2. Notation for the boundary velocities (we neglect here the surrounding particulate medium).

Denote by F_{B_t,V_t} and M_{B_t,V_t} the drag force and torque on ∂B_t determined by the velocity field v_t according to (2.9) and (2.10), with $\Omega = B_t^{\mathrm{ext}}$. A straightforward computation yields $F_{B_t,V_t} = R_t^T F_{A_t,U_t}$ and $M_{B_t,V_t} = R_t^T M_{A_t,U_t}$, so that the equations of motion (3.11) reduce to

$$F_{B_t,V_t} = 0 \quad \text{and} \quad M_{B_t,V_t} = 0 \quad \text{for almost every } t \in [0,T].$$

Again by a simple manipulation we obtain the following form of the equations of motion

$$\begin{bmatrix} \dot{y}_t \\ \omega_t \end{bmatrix} = \begin{bmatrix} R_t & 0 \\ 0 & R_t \end{bmatrix} \begin{bmatrix} H_t & D_t^T \\ D_t & L_t \end{bmatrix} \begin{bmatrix} F_t^{\mathrm{sh}} \\ M_t^{\mathrm{sh}} \end{bmatrix} \quad \text{for almost every } t \in [0,T],$$

which read, by means of (3.15), as (3.16).

Now, the standard theory of ordinary differential equations with possibly discontinuous coefficients [4] ensures that the Cauchy problem for (3.16) has one and only one Lipschitz solution $t \mapsto R_t$, $t \mapsto y_t$, provided that the functions $t \mapsto \Omega_t$ and $t \mapsto b_t$ are measurable and bounded. By (3.13) and (3.15), this happens when the functions

$$t \mapsto K_t\,, \quad t \mapsto C_t\,, \quad t \mapsto J_t\,, \quad t \mapsto F_t^{\mathrm{sh}}, \quad t \mapsto M_t^{\mathrm{sh}} \tag{3.18}$$

are measurable and bounded. The continuity of the first three functions will be proved in the last part of this section. The proof of the measurability and boundedness of the last two functions in (3.18) requires some technical tools that will be developed in Section 4.

We need the notion of set convergence that we have already introduced in [3], and that we report here for convenience. Given a sequence of sets $(S_k)_k$, we say that S_k converge to S_∞, $S_k \to S_\infty$, if for every $\varepsilon > 0$ there exists m such that for every $k \geqslant m$

$$S_\infty^{-\varepsilon} \subset S_k \subset S_\infty^{+\varepsilon}, \tag{3.19}$$

where $S_\infty^{-\varepsilon} = \{y \in \mathbb{R}^3 : \mathrm{dist}(y, \mathbb{R}^3 \setminus S_\infty) \geqslant \varepsilon\}$ and $S_\infty^{+\varepsilon} = \{y \in \mathbb{R}^3 : \mathrm{dist}(y, S_\infty) \leqslant \varepsilon\}$. The next lemma states a continuity property of the set-valued function $t \mapsto B_t$.

Lemma 3.2 ([3]). *Let s_t satisfy* (3.9)*. Then if $t \to t_\infty$ the sets B_t converge to the set B_{t_∞} in the sense of* (3.19). □

Theorem 3.3. *Let w_t be the solution to the exterior Brinkman problem* (2.2) *on B_t^{ext} with boundary datum W on ∂B_t, where W can be either a constant vector $a \in \mathbb{R}^3$ or the rotation $W_\omega := \omega \times z$, with $\omega \in \mathbb{R}^3$. Define $\widetilde{w}_t$ to be the extension*

$$\widetilde{w}_t := \begin{cases} W & \text{on } B_t, \\ w_t & \text{on } B_t^{\mathrm{ext}}, \end{cases} \tag{3.20}$$

Assume that $t \mapsto s_t$ satisfies (3.9)*. Then the map $t \mapsto \widetilde{w}_t$ is continuous from $[0,T]$ into $\mathcal{X}(\mathbb{R}^3)$.*

Proof. Let $(t_k)_k \subset [0,T]$ be a sequence that converges to $t_\infty \in [0,T]$. Lemma 3.2 ensures the convergence of the sets B_{t_k} to B_{t_∞} in the sense of (3.19).

Since w_{t_k} are solutions to Brinkman problems, we have the bound $2\int_{B_{t_k}^{\mathrm{ext}}} |\mathrm{E}w_{t_k}|^2\,\mathrm{d}z + \alpha^2 \int_{B_{t_k}^{\mathrm{ext}}} |w_{t_k}|^2\,\mathrm{d}z \leqslant C$, which, in turn, implies that

$$2\int_{\mathbb{R}^3} |\mathrm{E}\widetilde{w}_{t_k}|^2\,\mathrm{d}z + \alpha^2 \int_{\mathbb{R}^3} |\widetilde{w}_{t_k}|^2\,\mathrm{d}z \leqslant C.$$

Therefore, $\widetilde{w}_t$ admits a subsequence that converges weakly to a function $w^* \in \mathcal{X}(\mathbb{R}^3)$. By the convergence of the B_{t_k}, it is easy to see that $w^* = W$ on B_{t_∞}. We now prove that $w^*|_{B_{t_\infty}^{\mathrm{ext}}}$ solves the exterior Brinkman problem on B_{t_∞}. Too see it, consider a test function $\varphi \in C_\mathrm{c}^\infty(B_{t_\infty}^{\mathrm{ext}}; \mathbb{R}^3)$. For k large enough, $\varphi \in C_\mathrm{c}^\infty(B_{t_k}^{\mathrm{ext}}; \mathbb{R}^3)$, so that

$$2\int_{\mathrm{spt}\,\varphi} \mathrm{E}w_{t_k} : \mathrm{E}\varphi\,\mathrm{d}z + \alpha^2 \int_{\mathrm{spt}\,\varphi} w_{t_k} \cdot \varphi\,\mathrm{d}z = 0.$$

This equality passes to the limit as $k \to \infty$, showing that $w^*|_{B_{t_\infty}^{\mathrm{ext}}}$ is a solution to the Brinkman problem at t_∞. Therefore, $w^* = \widetilde{w}_{t_\infty}$, and we have proved that $t \mapsto w_t$ is strongly continuous from $[0,T]$ into $\mathcal{X}(\mathbb{R}^3)$. □

We can now prove the following continuity result for the elements of the grand resistance matrix by means of Theorem 3.3.

Proposition 3.4. *Assume that s_t satisfies* (3.7) *and* (3.9)*. Then the functions*

$$t \mapsto K_t\,, \quad t \mapsto C_t\,, \quad t \mapsto J_t\,, \tag{3.21}$$

and consequently $t \mapsto H_t$, $t \mapsto D_t$, $t \mapsto L_t$, are continuous.

Proof. Formulae (3.12) and (2.8) provide us with an explicit form for the elements of the grand resistance matrix

$$(K_t)_{ij} = 2\int_{B_t^{\mathrm{ext}}} \mathrm{E}v_t^j : \mathrm{E}v_t^i\,\mathrm{d}z + \alpha^2 \int_{B_t^{\mathrm{ext}}} v_t^j \cdot v_t^i\,\mathrm{d}z, \tag{3.22a}$$

$$(C_t)_{ij} = 2\int_{B_t^{\mathrm{ext}}} \mathrm{E}v_t^j : \mathrm{E}\hat{v}_t^i\,\mathrm{d}z + \alpha^2 \int_{B_t^{\mathrm{ext}}} v_t^j \cdot \hat{v}_t^i\,\mathrm{d}z, \tag{3.22b}$$

$$(J_t)_{ij} = 2\int_{B_t^{\mathrm{ext}}} \mathrm{E}\hat{v}_t^j : \mathrm{E}\hat{v}_t^i\,\mathrm{d}z + \alpha^2 \int_{B_t^{\mathrm{ext}}} \hat{v}_t^j \cdot \hat{v}_t^i\,\mathrm{d}z, \tag{3.22c}$$

where v_t^i and $\hat{v}_t^i$ are the functions defined in (3.20) with $W = e_i$ and $W = e_i \times z$, respectively. We prove the result for K_t only, since the others are similar. We write

$$(K_t)_{ij} = 2\int_{\mathbb{R}^3} \mathrm{E}\widetilde{v}_t^j : \mathrm{E}\widetilde{v}_t^i\,\mathrm{d}z + \alpha^2 \int_{\mathbb{R}^3} \widetilde{v}_t^j \cdot \widetilde{v}_t^i\,\mathrm{d}z - \alpha^2 \int_{B_t} e_j \cdot e_i\,\mathrm{d}z,$$

where $\widetilde{v}_t^i$ and $\widetilde{v}_t^j$ are the extensions considered in (3.20). By Theorem 3.3, the first two integrals are continuous with respect to t. The continuity of the last integral is guaranteed by Lemma 3.2. □

The proof of the measurability and boundedness of $t \mapsto F_t^{\rm sh}$ and $t \mapsto M_t^{\rm sh}$ is a delicate issue. The difficulty arises from the fact that both the domains B_t and the boundary data $W_t = \dot{s}_t \circ s_t^{-1}$ depend on time. Moreover, since it is meaningful and interesting to consider boundary values W_t that might be discontinuous with respect to t, we cannot expect the functions $t \mapsto F_t^{\rm sh}$ and $t \mapsto M_t^{\rm sh}$ to be continuous.

To prove the measurability we start from an integral representation of $F_t^{\rm sh}$ and $M_t^{\rm sh}$, similar to (3.22). As $\int_{\partial B_t} W_t \cdot n \,\mathrm{d}S$ is not necessarily zero, we will not be able to compute integrals over the whole space $\mathbb{R}^3$, so we will have to work in the complement of an open ball $\Sigma_\varepsilon^0 \subset\subset B_t$. Since, in general, this inclusion holds only locally in time, we first fix $t_0 \in [0,T]$ and $z^0 \in B_{t_0}$ and select $\delta > 0$ and $\varepsilon > 0$ so that the open ball $\Sigma_\varepsilon^0 := \Sigma_\varepsilon(z^0)$ of radius ε centered at z^0 satisfies

$$\Sigma_\varepsilon^0 \subset\subset B_t\,, \quad \text{for all } t \in I_\delta(t_0) := [0,T] \cap (t_0 - \delta, t_0 + \delta). \tag{3.23}$$

This is possible thanks to the continuity properties of $t \mapsto s_t$ listed in the first part of this section.

Next we consider the solution w_t to the problem

$$\min \left\{ \|w\|^2_{\mathcal{X}(\Sigma_\varepsilon^{0,\rm ext})} : w \in \mathcal{X}(\Sigma_\varepsilon^{0,\rm ext}),\ w = W_t \text{ on } \partial B_t\,, \text{ and } w = \lambda_t (z - z^0)/\varepsilon^3 \text{ on } \partial\Sigma_\varepsilon^0 \right\}$$

In order for the flux condition (2.3) to be fulfilled on $\partial B_t \cup \partial\Sigma_\varepsilon^0$ by w_t , we choose

$$\lambda_t := -\frac{1}{4\pi} \int_{\partial B_t} W_t \cdot n \,\mathrm{d}S.$$

Finally, putting together (3.14) and (2.8), we obtain the following explicit integral representation of $F_t^{\rm sh}$ and $M_t^{\rm sh}$

$$(F_t^{\rm sh})_i = 2 \int_{\Sigma_\varepsilon^{0,\rm ext}} \mathrm{E}w_t : \mathrm{E}v_t^i \,\mathrm{d}z + \alpha^2 \int_{\Sigma_\varepsilon^{0,\rm ext}} w_t \cdot v_t^i \,\mathrm{d}z - \alpha^2 \int_{Q_{\varepsilon,t}} w_t \cdot v_t^i \,\mathrm{d}z$$

$$(M_t^{\rm sh})_i = 2 \int_{\Sigma_\varepsilon^{0,\rm ext}} \mathrm{E}w_t : \mathrm{E}\hat{v}_t^i \,\mathrm{d}z + \alpha^2 \int_{\Sigma_\varepsilon^{0,\rm ext}} w_t \cdot \hat{v}_t^i \,\mathrm{d}z - \alpha^2 \int_{Q_{\varepsilon,t}} w_t \cdot \hat{v}_t^i \,\mathrm{d}z$$

where v_t^i and $\hat{v}_t^i$ have been defined in the proof of Proposition 3.4 and $Q_{\varepsilon,t} := B_t \setminus \overline{\Sigma}_\varepsilon^0$. We deduce from Theorem 3.3 and Lemma 3.2 that the functions $t \mapsto v_t^i$ and $t \mapsto \hat{v}_t^i$ are continuous from $I_\delta(t_0)$ into $\mathcal{X}(\Sigma_\varepsilon^{0,\rm ext})$. Therefore, the measurability and boundedness of $t \mapsto F_t^{\rm sh}$ and $t \mapsto M_t^{\rm sh}$ will be proved once $t \mapsto w_t$ is proved to be measurable. We first show that $t \mapsto w_t$ is measurable and bounded from $I_\delta(t_0)$ into $\mathcal{X}(\Sigma_\varepsilon^{0,\rm ext})$ and eventually we will prove that the function $t \mapsto \int_{Q_{\varepsilon,t}} w_t \,\mathrm{d}z$ is continuous with respect to time. These two results are proved in the next Section.

4. Extensions of boundary data and main result

The following results have been proved in [3] and still hold in our setting.

Proposition 4.1 (Solenoidal extension operators)**.** *Assume that s_t satisfies* (3.7) *and* (3.9), *and let $t_0 \in [0,T]$ and $z^0 \in B_{t_0}$. Let $\delta > 0$ and $\varepsilon > 0$ be such that* (3.23) *holds true. Then there exists a uniformly bounded family $(\mathcal{T}_t)_{t \in I_\delta(t_0)}$ of continuous linear operators*

$$\mathcal{T}_t \colon H^{1/2}(\partial A; \mathbb{R}^3) \to \mathcal{X}(\Sigma_\rho \setminus \overline{\Sigma}_\varepsilon^0)$$

such that

(i) *for all $t \in I_\delta(t_0)$ and for all $\Phi \in H^{1/2}(\partial A; \mathbb{R}^3)$,*

$$\mathcal{T}_t(\Phi) = \Phi \circ s_t^{-1} \quad \text{on } \partial B_t\,,$$
$$\mathcal{T}_t(\Phi) = \lambda_t \frac{z}{|z|^3} \quad \text{on } \partial\Sigma_\rho\,,$$

(ii) *for every $\Phi \in H^{1/2}(\partial A; \mathbb{R}^3)$ the map $t \mapsto \mathcal{T}_t(\Phi)$ is continuous from $I_\delta(t_0)$ into $\mathcal{X}(\Sigma_\rho \setminus \overline{\Sigma}_\varepsilon^0)$.*

In particular, the following estimate holds

$$\|\mathcal{T}_t(\Phi)\|_{H^1(\Sigma_\rho\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)} \leqslant C\,\|\Phi\|_{H^{1/2}(\partial A;\mathbb{R}^3)}\,, \tag{4.1}$$

where the constant C is independent of t and Φ. □

Proposition 4.2. *Assume that s_t satisfies* (3.7), (3.8), *and* (3.9). *Let $t_0\in[0,T]$ and $z^0\in B_{t_0}$, and let Σ^0_ε and $I_\delta(t_0)$ be as in* (3.23). *Suppose, in addition, that for every $t\in I_\delta(t_0)$ there exists a C^2 diffeomorphism $\Psi^{t_0}_t:\Sigma_\rho\to\Sigma_\rho$ coinciding with the identity on $\Sigma_\rho\setminus\Sigma_{\rho-1}$, such that $\Psi^{t_0}_t=s_{t_0}\circ s_t^{-1}$ on B_t. Let the map $t\mapsto\Phi_t$ belong to $C^0(I_\delta(t_0);H^{1/2}(\partial A;\mathbb{R}^3))\cap L^\infty(I_\delta(t_0);\mathrm{Lip}(\partial A;\mathbb{R}^3))$. Let w_t be the solution to the problem*

$$\min\Big\{\|w\|^2_{\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)}:\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon),\ w=\Phi_t\circ s_t^{-1}\text{ on }\partial B_t\text{ and }w=\lambda_t(z-z^0)/\varepsilon^3\text{ on }\partial\Sigma^0_\varepsilon\Big\}, \tag{4.2}$$

where $\lambda_t:=-\frac{1}{4\pi}\int_{\partial B_t}(\Phi_t\circ s_t^{-1})\cdot n\,\mathrm{d}S$. Then $t\mapsto w_t$ belongs to $C^0(I_\delta(t_0);\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon))$.

Proof. The proof can be easily adapted from that of [3, Proposition 6.1]; the following important estimate provides a uniform bound for the norms of the w_t's in $\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$ that will also be useful in the proof of Proposition 4.3

$$\begin{aligned}2\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon}|\mathrm{E}w_{t_k}|^2\,\mathrm{d}z+\alpha^2\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon}|w_{t_k}|^2\,\mathrm{d}z&\leqslant 2\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon}|\mathrm{E}\psi_{t_k}|^2\,\mathrm{d}z+\alpha^2\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon}|\psi_{t_k}|^2\,\mathrm{d}z\\&\leqslant\|\psi_{t_k}\|^2_{H^1(\Sigma_\rho\setminus\overline{\Sigma}^0_\varepsilon;\mathbb{R}^3)}\leqslant C^2(\mathrm{Lip}(\Phi_{t_k})+\max|\Phi_{t_k}|)^2\leqslant(CM)^2,\end{aligned} \tag{4.3}$$

where $\psi_t\in\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$ is defined by

$$\psi_t:=\begin{cases}\mathcal{T}_t(\Phi_t) & \text{in }\Sigma_\rho\setminus\overline{\Sigma}^0_\varepsilon\\ \lambda_t\dfrac{z}{|z|^3} & \text{in }\Sigma^{\mathrm{ext}}_\rho\end{cases}$$

and is the function provided by Proposition 4.1 and extended on $\Sigma^{\mathrm{ext}}_\rho$, C is the constant in (4.1), and $M>0$ is a uniform upper bound of $\mathrm{Lip}(\Phi_{t_k})+\max|\Phi_{t_k}|$, whose existence is guaranteed by the fact that $t\mapsto\Phi_t$ belongs to $L^\infty(I_\delta(t_0);\mathrm{Lip}(\partial A;\mathbb{R}^3))$. □

Proposition 4.3. *Under the hypotheses of Proposition 4.2, recalling that $Q_{\varepsilon,t}=B_t\setminus\overline{\Sigma}^0_\varepsilon$, the maps*

$$t\mapsto\int_{Q_{\varepsilon,t}}w_t\,\mathrm{d}z,\qquad t\mapsto\int_{Q_{\varepsilon,t}}z\times w_t\,\mathrm{d}z \tag{4.4}$$

where $t\mapsto w_t\in\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)$ is the solution to the minimum problem (4.2) *as in Proposition 4.2, are continuous with respect to time in $I_\delta(t_0)$.*

Proof. We check the continuity with the definition

$$\begin{aligned}\left|\int_{Q_{t+h}}w_{t+h}\,\mathrm{d}z-\int_{Q_t}w_t\,\mathrm{d}z\right|&=\left|\int_{Q_{t+h}}(w_{t+h}-w_t)\,\mathrm{d}z+\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon}w_t(\chi_{Q_{t+h}}(z)-\chi_{Q_t}(z))\,\mathrm{d}z\right|\\&\leqslant\left(\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon}|w_{t+h}-w_t|^2\,\mathrm{d}z\right)^{\frac12}|Q_{t+h}|^{\frac12}+\left(\int_{\Sigma^{0,\mathrm{ext}}_\varepsilon}|w_t|^2\,\mathrm{d}z\right)^{\frac12}|Q_{t+h}\triangle Q_t|^{\frac12}\\&\leqslant\|w_{t+h}-w_t\|_{\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)}|Q_{t+h}|^{\frac12}+\|w_t\|_{\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)}|Q_{t+h}\triangle Q_t|^{\frac12}\\&\leqslant|\Sigma_\rho|^{\frac12}\|w_{t+h}-w_t\|_{\mathcal{X}(\Sigma^{0,\mathrm{ext}}_\varepsilon)}+CM\,|Q_{t+h}\triangle Q_t|^{\frac12}\quad\xrightarrow{h\to0}0.\end{aligned}$$

Here, χ_Q denotes the characteristic function of the set Q, $\triangle$ is the symmetric difference operator, and CM is the uniform (with respect to t) upper bound coming from (4.3). The continuity for the second map is achieved in the same way. □

Proposition 4.2 and Proposition 4.3 combined together give the continuity of $t\mapsto F^{\mathrm{sh}}_t$ and $t\mapsto M^{\mathrm{sh}}_t$ with respect to time, in the case of regular boundary data $\Phi_t\circ s_t^{-1}$ on ∂B_t, where the map $t\mapsto\Phi_t$ belongs to $C^0(I_\delta(t_0);H^{1/2}(\partial A;\mathbb{R}^3))\cap L^\infty(I_\delta(t_0);\mathrm{Lip}(\partial A;\mathbb{R}^3))$. The next results will prove that when the boundary data on ∂B_t are given by $\dot s_t\circ s_t^{-1}$, then the maps $t\mapsto F^{\mathrm{sh}}_t$ and $t\mapsto M^{\mathrm{sh}}_t$ are measurable and bounded.

Theorem 4.4. *Assume that s_t satisfies* (3.7)*,* (3.8)*, and* (3.9)*. Let $t_0 \in [0,T]$ and $z^0 \in B_{t_0}$, and let Σ_ε^0 and $I_\delta(t_0)$ be as in* (3.23)*. Suppose, in addition, that for every $t \in I_\delta(t_0)$ there exists a C^2 diffeomorphism $\Psi_t^{t_0} : \Sigma_\rho \to \Sigma_\rho$ coinciding with the identity on $\Sigma_\rho \setminus \Sigma_{\rho-1}$, such that $\Psi_t^{t_0} = s_{t_0} \circ s_t^{-1}$ on B_t. Let w_t be the solution to the problem*

$$\min\Big\{\|w\|^2_{\mathcal{X}(\Sigma_\varepsilon^{0,\mathrm{ext}})} : w \in \mathcal{X}(\Sigma_\varepsilon^{0,\mathrm{ext}}),\ w = \dot{s}_t \circ s_t^{-1} \text{ on } \partial B_t\,,\ \text{and } w = \lambda_t(z-z^0)/\varepsilon^3 \text{ on } \partial\Sigma_\varepsilon^0\Big\}.$$

Then the function $t \mapsto w_t$ is measurable and bounded from $I_\delta(t_0)$ into $\mathcal{X}(\Sigma_\varepsilon^{0,\mathrm{ext}})$. Moreover, also the functions (4.4) *considered in Proposition 4.3 are measurable and bounded in $I_\delta(t_0)$.*

Proof. It suffices to convolve the boundary datum with a suitable regularizing kernel and to apply Propositions 4.2 and 4.3. By passing to the limit, the continuity is lost but the functions turn out to be measurable and bounded. □

Proposition 3.4 and Theorem 4.4 give the regularity result for b_t and Ω_t in (3.15), as stated in the following result.

Theorem 4.5. *Assume that $t \mapsto s_t$ satisfies* (3.7)*,* (3.8)*, and* (3.9)*. Then the vector b_t and the matrix Ω_t in* (3.15) *are bounded and measurable with respect to t. If, in addition, the function $t \mapsto s_t$ belongs to $C^1([0,T];C^1(\overline{A};\mathbb{R}^3))$, then $t \mapsto (b_t\,,\Omega_t)$ belongs to $C^0([0,T];\mathbb{R}^3{\times}\mathbb{M}^{3\times3})$.* □

We are now in a position to state the existence, uniqueness, and regularity result for the equations of motion (3.16), whose proof can be found in [3].

Theorem 4.6. *Assume that $t \mapsto s_t$ satisfies* (3.7)*,* (3.8)*, and* (3.9)*. Let $y^* \in \mathbb{R}^3$ and $R^* \in \mathrm{SO}(3)$. Then* (3.16) *has a unique absolutely continuous solution $t \mapsto (y_t\,,R_t)$ defined in $[0,T]$ with values in $\mathbb{R}^3{\times}\mathrm{SO}(3)$ such that $y_0 = y^*$ and $R_0 = R^*$. In other words, there exists a unique rigid motion $t \mapsto r_t(z) = y_t + R_t z$ such that the deformation function $t \mapsto \varphi_t = r_t \circ s_t$ satisfies the equations of motion* (3.11)*.*

Moreover this solution is Lipschitz continuous with respect to t. If, in addition, the function $t \mapsto s_t$ belongs to $C^1([0,T];C^1(\overline{A};\mathbb{R}^3))$, then the solution $t \mapsto (y_t\,,R_t)$ belongs to $C^1([0,T];\mathbb{R}^3{\times}\mathrm{SO}(3))$. □

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