Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 259 (2010) 2165–2192
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Generic multiplicity for a scalar field equation on compact surfaces

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Received 12 January 2010; accepted 4 July 2010

Communicated by J. Coron

Abstract

We prove generic multiplicity of solutions for a scalar field equation on compact surfaces via Morse inequalities. In particular our result improves significantly the multiplicity estimate which can be deduced from the degree-counting formula in Chen and Lin (2003) [12]. Related results are derived for the prescribed Q-curvature equation.

Keywords: Scalar field equations; Geometric PDE's; Generic multiplicity; Morse inequalities

1. Introduction

Let (Σ, g) be a compact Riemannian surface (without boundary), $h \in C^2(\Sigma)$ be a positive function and ρ a real number. In this paper we consider the equation

$$-\triangle_g u + \frac{\rho}{\int_\Sigma dV_g} = \rho \frac{h(x)e^u}{\int_\Sigma h(x)e^u \, dV_g} \quad x \in \Sigma, \; u \in H_g^1(\Sigma), \tag{$*$}$$

where $\triangle_g$ is the Laplace–Beltrami operator on Σ. The above equation arises in statistical mechanics as a mean field equation for the Euler flow. More precisely, it has been proved in [3,26] that, according to Onsager's vortex theory, when the number of vortices is supposed to tend

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2010.07.003

to $+\infty$, the stream function satisfies $(*)$. In this interpretation the exponential is related to the Gibbs measure, which is finite provided $\rho > -8\pi$.

This PDE also concerns the description of self-dual condensates of some Chern–Simon–Higgs model; indeed via its solutions it is possible to describe the asymptotic behavior of a class of condensates (or multivortex) solutions which are relevant in theoretical physics and which were absent in the classical (Maxwell–Higgs) vortex theory (see [38,42,43] and references therein).

Another motivation for the study of $(*)$ is the problem of prescribing the Gauss curvature of a surface via a conformal transformation of the metric. Indeed, setting $\tilde{g} = e^{2w} g$ we have

$$\triangle_{\tilde{g}} = e^{-2w} \triangle_g; \qquad -\triangle_g w + K_g = K_{\tilde{g}} e^{2w},$$

where K_g and $K_{\tilde{g}}$ are the Gauss curvature of (Σ, g) and of $(\Sigma, \tilde{g})$. In this context, of particular interest is the classical Uniformization Theorem, which asserts that every compact surface carries a conformal metric with constant curvature. Viceversa, given a surface with constant curvature one may ask whether it is possible to obtain conformal metrics for which the Gauss curvature becomes a given function. The latter is known as the Kazdan–Warner problem, or as the Nirenberg problem when Σ is the standard sphere (see for example [5,7,25]).

Problem $(*)$ has a variational structure and solutions can be found as critical points of the functional

$$I_\rho(u) = \frac{1}{2} \int_\Sigma |\nabla_g u|^2 \, dV_g + \rho \fint_\Sigma u \, dV_g - \rho \log \int_\Sigma h(x) e^u \, dV_g, \quad u \in H_g^1(\Sigma). \tag{1.1}$$

Since Eq. $(*)$ is invariant when adding constants to u, we can restrict ourselves to the subspace of the functions with zero average $\bar{H}_g^1(\Sigma) := \{u \in H_g^1(\Sigma)\colon \fint_\Sigma u \, dV_g = 0\}$.

Because of the Moser–Trudinger inequality (see Lemma 2.5) one can easily prove the compactness and the coercivity of I_ρ when $\rho < 8\pi$ and thus one can find solutions of $(*)$ by minimization.

If $\rho = 8\pi$ the situation is more delicate since I_ρ still has a lower bound but it is not coercive anymore; in general when ρ is an integer multiple of 8π, the existence problem of $(*)$ is much harder (a far from complete list of references on the subject includes works by Chang and Yang [7], Chang, Gursky and Yang [5], Chen and Li [10], Nolasco and Tarantello [38], Ding, Jost, Li and Wang [14] and Lucia [30]).

For $\rho > 8\pi$, as the functional I_ρ is unbounded from below and from above, solutions have to be found as saddle points.

Li and Shafrir, exploiting an earlier work of Brezis and Merle [2], proved an important compactness property when ρ is not an integer multiple of 8π.

Theorem 1.1. *(See [28].) If $\rho \notin 8\pi\mathbb{N}$, then solutions of $(*)$ are bounded in $C^{2,\alpha}(\Sigma)$ for any $\alpha \in (0,1)$.*

When $\rho \neq 8k\pi$, this theorem permits to define the global Leray–Schauder degree of $(*)$. As a consequence of the homotopy invariance of the degree, it turns out that it is independent of h, of the parameter $\rho \in (8k\pi, 8(k+1)\pi)$ for $k \in \mathbb{N}$ and of the metric of Σ. In [27], Y.Y. Li first pointed out that the degree of $(*)$ only depends on $k \in \mathbb{N}$ (for $\rho \in (8k\pi, 8(k+1)\pi)$) and on the Euler characteristic of Σ, $\chi(\Sigma)$, so we will use $\mathrm{d}(k, \chi(\Sigma))$ to denote it. Chen and Lin in [12], analyzing the jump values of the degree after crossing the critical thresholds, obtained the following complete degree-counting formula, extending the results in [15,29].

Theorem 1.2. *(See [12].) Let* $\rho \in (8k\pi, 8(k+1)\pi)$, $k \in \mathbb{N}$, *then for any* (Σ, g) *and for any* $h \in C^2(\Sigma)^+$:

$$\mathrm{d}\big(k, \chi(\Sigma)\big) = \binom{k - \chi(\Sigma)}{k} \equiv \begin{cases} \frac{(k-\chi(\Sigma))\ldots(2-\chi(\Sigma))(1-\chi(\Sigma))}{k!} & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases} \tag{1.2}$$

In the latter statement we specified what we mean by the binomial coefficient because the upper term, $k - \chi(\Sigma)$, can be negative; clearly this definition extends the usual one.

Recently an alternative and direct proof of formula (1.2) has been obtained by Malchiodi [34], via a Morse-theoretical approach. He also provided a clear interpretation of the counting formula, related to the topology of high and low sublevels.

Remark 1.3. Notice that $\mathrm{d}(0,2) = 1$, $\mathrm{d}(1,2) = -1$, $\mathrm{d}(k,2) = 0$ for any $k \geqslant 2$ so if Σ has the homology of a sphere the degree does not suffice existence of a solution; while when Σ has the homology of a torus, since $\mathrm{d}(k,0) = 1$ for any $k \geqslant 0$, we can deduce existence but we have no information about multiplicity.

Finally, Djadli generalized these previous results establishing the existence also in the case of positive Euler characteristic.

Theorem 1.4. *(See [16].) If* $\rho \in (8k\pi, 8(k+1)\pi)$, *then for any* (Σ, g) *and for any* $h \in C^2(\Sigma)^+$ *there exists a solution of* $(*)$.

To do that, he deeply investigated the topology of low sublevels of I_ρ in order to perform a min–max scheme (already introduced in Djadli and Malchiodi [17]). A crucial observation, as noticed in [11], is that the constant in the Moser–Trudinger inequality (2.2) can be roughly divided by the number of regions where $\frac{e^u}{\int_\Sigma e^u}$ is *supported* (see Lemma 2.6 for details). As a consequence, if $\rho \in (8k\pi, 8(k+1)\pi)$ and if I_ρ attains large negative values, $\frac{e^u}{\int_\Sigma e^u}$ has to concentrate near at most k points of Σ, in the sense specified in Lemma 2.7. From these considerations one is led naturally to associate with $\frac{e^u}{\int_\Sigma e^u}$ a probability measure $\sum_{i=1}^k t_i \delta_{x_i}$ with $(x_i)_i \in \Sigma$ and $\sum_{i=1}^k t_i = 1$. The set of such objects, denoted here by Σ_k is known in literature as the formal set of formal barycenters of Σ of order k. It is in fact possible to prove that $\{I_\rho \leqslant -L\}$ has the same homology of Σ_k for L very large positive [34].

The purpose of this paper is to prove generic multiplicity of solutions also in the cases when $\chi(\Sigma) \leqslant 0$ (see Remark 1.3) and to improve for the other surfaces the estimate of the number of solutions which can be derived from Theorem 1.2. Our main result reads as follows.

Theorem 1.5. *Let* $\rho \in (8k\pi, 8(k+1)\pi)$, $k \in \mathbb{N}^*$. *Then, for a generic choice of the metric* g *and of the function* h (*namely for* (g,h) *in an open and dense subset of* $\mathcal{M}^2 \times C^2(\Sigma)^+$)

$$\#\{\textit{solutions of } (*)\} \geqslant \begin{cases} p_k & \text{if } \chi(\Sigma) = 2, \\ \sum_{r=0}^k \binom{k-r-\chi(\Sigma)+1}{k-r} p_r & \text{if } \chi(\Sigma) \leqslant 0, \end{cases} \tag{1.3}$$

where $p_0 = 1$, $p_{2m+1} = p_{2m} = \sum_{j=0}^m p_j$ *for any* $m \in \mathbb{N}^*$.

Moreover the latter estimate holds true also for (g,h) *in an open and dense subset of* $\mathcal{M}_1^2 \times C^2(\Sigma)^+$.

In the above statement $\mathcal{M}^2$ stands for the space of all C^2 Riemannian metrics on Σ equipped with the C^2 norm (see (4.1)), while $\mathcal{M}_1^2$ is the subset of $\mathcal{M}^2$ of the metrics g such that $\int_\Sigma dV_g = 1$.

Remark 1.6. In literature it is usually studied the case when $Vol_g(\Sigma) := \int_\Sigma dV_g = 1$, namely when $g \in \mathcal{M}_1^2$. It is for this reason that we specified that the set of (g,h) for which (1.3) holds true is dense not only in $\mathcal{M}^2 \times C^2(\Sigma)^+$ but also in $\mathcal{M}_1^2 \times C^2(\Sigma)^+$.

By direct calculation and an asymptotic formula for the sequence p_r, obtained in [32], from the latter theorem the following corollary can be derived.

Corollary 1.7. *Under the hypotheses of Theorem* 1.5, *for generic* $(g,h) \in \mathcal{M}^2 \times C^2(\Sigma)^+$:

1. *For any* Σ *and for any* $k \in \mathbb{N}^*$ (*except the case* $\chi(\Sigma) = 2$ *and* $k = 1$)

$$\#\big\{\textit{solutions of } (*)\big\} > \mathrm{d}\big(k, \chi(\Sigma)\big) \geqslant 0,$$

where by $\mathrm{d}(k, \chi(\Sigma))$ *we mean the Leray–Shauder degree of Eq.* $(*)$ (*see* (1.2)).
When $\chi(\Sigma) = 2$ *and* $k = 1$ *the right-hand side of formula* (1.3) *is simply equal to* $1 = |\mathrm{d}(1,2)|$.

2. *For any* Σ, *as* $k \geqslant k_0$, $k_0 \in \mathbb{N}^*$ (*independent of* Σ),

$$\#\big\{\textit{solutions of } (*)\big\} \geqslant C\left(\frac{[\frac{k}{2}]}{\log[\frac{k}{2}]}\right)^{\frac{1}{2l_2}\log(\frac{[\frac{k}{2}]}{\log[\frac{k}{2}]})+1+\frac{ll_2}{l_2}} \left[\frac{k}{2}\right]^{(\frac{1}{l_2}-\frac{1}{2})}, \tag{1.4}$$

where by $[\frac{k}{2}]$ *we mean the integer part of* $\frac{k}{2}$, $l_2 := \log 2$ *and* $ll_2 =: \log\log 2$; *so in particular for any* Σ

$$\#\big\{\textit{solutions of } (*)\big\} \to +\infty \quad \textit{as } k \to +\infty.$$

Moreover points 1 *and* 2 *hold true also for* (g,h) *in an open and dense subset of* $\mathcal{M}_1^2 \times C^2(\Sigma)^+$.

Remark 1.8. Actually it is not surprising that our estimate improves the one obtained with the degree. Indeed we tackle the problem using Morse inequalities and in general Morse theory gives more information about the structure of the critical points compared to degree theory, just because one includes the other as a particular case.

Besides it is worth pointing out that our estimate is not only better than the degree (point 1 of Corollary 1.7) but improves considerably the order of infinity, as $\rho \to +\infty$ of the number of solutions (point 2 of Corollary 1.7). Indeed for $\chi(\Sigma) \geqslant 0$ $|\mathrm{d}(k, \chi(\Sigma))| \leqslant 1$ and for $\chi(\Sigma) < 0$ the degree is just a polynomial in k, more precisely $\mathrm{d}(k, \chi(\Sigma)) = O_k(k^{-\chi(\Sigma)})$; while by means of the rough estimate $\frac{n}{\log(n)} \geqslant n^{\frac{1}{2}}$ (which holds for any $n \geqslant 2$) formula (1.4) implies that

$$\#\{\text{solutions of } (*)\} \geqslant C\left[\frac{k}{2}\right]^{\frac{1}{8l_2}\log[\frac{k}{2}]+\frac{2+ll_2}{2l_2}}.$$

To prove Theorem 1.5 we first show that we are in position to apply a transversality result (Theorem 4.3) which guarantees that for (g,h) in an open and dense subset of $\mathcal{M}^2 \times C^2(\Sigma)^+$ all the critical points of I_ρ are non-degenerate. Then we just need to derive the estimate (1.3) under the further assumption that all the critical points of I_ρ are non-degenerate, i.e. that I_ρ is a Morse functional. In these hypotheses we can exploit the weak Morse inequalities (Theorem 2.4), which, together with the exactness of the homology of a pair, permit to prove that

$$\#\{\text{solutions of } (*)\} \geqslant \sum_{q\geqslant 0} \dim H_q\big(\{I_\rho \leqslant b\}, \{I_\rho \leqslant -L\}; \mathbb{Z}_2\big). \tag{1.5}$$

Actually Morse inequalities require the Palais–Smale condition to hold, which is not known for I_ρ, but a deformation lemma from [34] (see also [31]) allows to overcome the problem. From formula (1.5) it is clear that the core of the analysis is the understanding of the homology groups of high and low sublevels. In [34] the author proved that for large values of b the sublevel $\{I_\rho \leqslant b\}$ has the homology of a point while for dealing with low sublevels we can take advantage of the aforementioned characterization in [16] (see Theorem 2.8).

From these considerations it can be deduced that the problem reduces to the computation of the following sum: $\sum_{q\geqslant 0}^{\infty} \dim \tilde{H}_q(\Sigma_k; \mathbb{Z}_2)$. To get it we use a theorem due to Kallel and Karoui [24] dealing with the homology of the set of formal barycenters on topological spaces (and so on manifolds, see Theorem 3.1), which in particular, combined with results in [36,37], permits to have a nice description of the homology of the family of formal barycenters on spheres of any dimension.

In four-dimensional geometry there exists a conformally covariant operator, the Paneitz operator (introduced in [39]), which enjoys analogous properties to the Laplace–Beltrami operator on surfaces, and to which is associated a natural concept of curvature: the Q-curvature (introduced in [1]). Let denote by $\mathrm{P_g}$ this operator and by Q_g the Q-curvature corresponding to a given 4-manifold (M,g). Their expressions in terms of the Ricci tensor Ric_g and of the scalar curvature R_g are as follows

$$\mathrm{P_g}(\varphi) = \Delta_g^2\varphi + div_g\left(\frac{2}{3}R_g g - 2\,Ric_g\right)d\varphi, \qquad Q_g = -\frac{1}{12}\big(\Delta_g R_g - R_g^2 + 3|Ric_g|^2\big),$$

and considering the conformal change of metric $\tilde{g} = e^{2u}g$, $Q_{\tilde{g}}$ is given by

$$\mathrm{P_g}u + 2Q_g = 2Q_{\tilde{g}}\frac{e^{4u}}{\int_M e^{4u}\,dV_g}. \tag{1.6}$$

Apart from the analogy with the prescribed Gauss curvature equation, there is an extension of the

Gauss–Bonnet formula involving the Weyl tensor W and the integral of Q_g, which is a conformal invariant:

$$4\pi^2\chi(M) = \int\limits_M \left(Q_g + \frac{1}{8}|W|^2\right) dV_g. \tag{1.7}$$

We refer to [6,9,22] for details.

As for the Uniformization theorem one can ask whether every four-manifold (M, g) carries a conformal metric $\tilde{g}$ for which the corresponding Q-curvature $Q_{\tilde{g}}$ is a constant. Writing $\tilde{g} = e^{2u}g$ the question amounts to solving (1.6) in u with $Q_{\tilde{g}}$ constant, namely the equation

$$\mathrm{P}_g u + 2Q_g = 2k_P \frac{e^{4u}}{\int_M e^{4u}\, dV_g}, \tag{\#}$$

where $k_P := \int_M Q_g\, dV_g$.

Concerning the Q-curvature equation, again applying Morse inequalities, we can prove the following multiplicity result.

Theorem 1.9. *Let (M, g) be a compact four-manifold such that the Paneitz operator P_g has $\bar{k}$ negative eigenvalues and only trivial kernel (the constant functions) and such that $k_P := \int_M Q_g\, dV_g \in (8k\pi^2, 8(k+1)\pi^2)$, for some $k \in \mathbb{N}^*$. If in addition all the solutions of (#) are non-degenerate, then*

$$\#\{\textit{solutions of } (\#)\} \geqslant \begin{cases} p_k & \text{if } \chi(M) = 2, \\ p_k + \sum_{r=0}^{k-1} \binom{k-r+\chi(\Sigma)-3}{k-r} p_r & \text{if } \chi(M) \geqslant 3, \end{cases} \tag{1.9}$$

where $p_0 = 1$, $p_{2m+1} = p_{2m} = \sum_{j=0}^{m} p_j$ for any $m \in \mathbb{N}^*$.

We point out that, for $k_P \in (8k\pi^2, 8(k+1)\pi^2)$, formula (1.7) implies $2k \leqslant \chi(M)$; in particular $\chi(M)$ is always greater or equal than 2 for any $k \geqslant 1$. Therefore the statement above takes into account all the possible situations which can occur with $k_P \in (8k\pi^2, 8(k+1)\pi^2)$, $k \in \mathbb{N}^*$.

Moreover when $k_P \notin 8\mathbb{N}\pi^2$ from a theorem in [33] (see also [19]) the equation is compact so it still makes sense to define its Leray–Schauder degree. In particular Malchiodi computed it in [34] and thanks to (1.7) it can be immediately seen that, contrary to what happens for Eq. $(*)$, the degree is always positive.

Although in the case of four-manifolds there is no any classification result in terms of the Euler characteristic, the latter result permits to improve the degree estimate, as specified in the following corollary.

Corollary 1.10. *For any (M, g) satisfying the hypotheses of Theorem 1.9 with $k_P := \int_M Q_g\, dV_g \in (8k\pi^2, 8(k+1)\pi^2)$, then, except for $\chi(M) = 2$ and $k = 1$,*

$$\#\{\textit{solutions of } (\#)\} > \left|\mathrm{d}_P\big(k, \bar{k}, \chi(M)\big)\right| > 0$$

where by $\mathrm{d}_P(k, \bar{k}, \chi(M))$ we mean the Leray–Shauder degree of Eq. (#) (see (4.7) in Section 4.1).

When $\chi(M)=2$ *and* $k=1$ *the r.h.s. of formula* (1.3) *is just equal to* $1=|\mathrm{d}_P(1,\bar{k},2)|$ *for any* $\bar{k}$.

Actually, exactly as in Corollary 1.7, it can also be proved that under these hypotheses the number of solutions of (#) for k large enough can be estimated from below by the r.h.s. of formula (1.4). But in fact this is not as relevant as for Eq. $(*)$ because now k and $\chi(M)$ are related by (1.7) and so it is not possible to fix M and let k tend to $+\infty$.

The structure of the paper is the following. In Section 2 we collect some notations and preliminary results concerning compactness properties for $(*)$ and the topological structure of I_ρ's sublevels. We also recall Morse inequalities and some basic notions in algebraic topology. Section 3 is devoted to get an explicit expression of the sum $\sum_{q\geqslant 0}^{\infty}\dim \tilde{H}_q(\Sigma_k;\mathbb{Z}_2)$. Then in Section 4 we prove Theorem 1.5 and Corollary 1.7 and we also deal with the case of Eq. (#), proving Theorem 1.9. Finally Appendix A contains some lemmas needed to prove the generic nondegeneracy.

2. Notations and preliminaries

In this section we collect some facts needed in order to obtain the multiplicity result. First of all we state a deformation lemma, proved in [31], and a compactness property of solutions of $(*)$ derived in [27]. These last results, for $\rho \neq 8k\pi$, allow us to overcome the possible failure of the Palais–Smale condition and to get a counterpart of the classical deformation lemma. Moreover we recall the Morse inequalities. Next, we consider some improvements of the Moser–Trudinger inequality which are useful to study the topological structure of the sublevels of I_ρ. Finally we collect some basic notions in algebraic topology.

Let now fix our notation. The symbol $B_r(p)$ denotes the metric ball of radius r and center p. As already specified we set $\bar{H}^1_g(\Sigma):=\{u\in H^1_g(\Sigma)\colon \fint_\Sigma u\,dV_g\}$ and the genus of Σ will be denoted as $g(\Sigma)$.

We want to stress that $(*)$ and I_ρ depend on ρ, g and h (as (#) depends on g) and sometimes to stress this dependence and to avoid any ambiguity we will write $I_{\rho,(g,h)}$ for I_ρ.

Large positive constants are always denoted by C, and the value of C is allowed to vary from formula to formula.

2.1. Deformation lemma, compactness and Morse inequalities

It is well known that, if $I\in C^1(H^1_g(\Sigma),\mathbb{R})$ satisfies the Palais–Smale condition, a classical deformation lemma ensures that we have the following alternative: either

1. $\{I\leqslant a\}$ is a deformation retract of $\{I\leqslant b\}$ $(a<b)$, or
2. there is a critical point $\bar{u}$ for the functional I, with $a\leqslant I(\bar{u})\leqslant b$.

This lemma, which is usually employed to derive existence of critical points, can be obtained by considering the pseudo-gradient vector field associated to I.

Unfortunately, for our functional I_ρ, the (PS)-condition is known to hold only for bounded sequences; Lucia in [31] bypassed this problem modifying the usual flow.

Lemma 2.1. *Given $a, b \in \mathbb{R}$, $a < b$, the following alternative holds: either*

1. $\exists(\rho_n, u_n) \subset \mathbb{R} \times \bar{H}^1_g(\Sigma)$ *satisfying*

$$I'_{\rho_n}(u_n) = 0 \quad \textit{for every } n,\ a \leqslant I_\rho(u_n) \leqslant b,\ \rho_n \to \rho,$$

2. *or the set* $\{I_\rho \leqslant a\}$ *is a deformation retract of* $\{I_\rho \leqslant b\}$.

By deformation retract onto $A \subset X$ we mean a continuous map $\eta : [0,1] \times X \to X$ such that $\eta(t, u_0) = u_0$ for every $(t, u_0) \in [0,1] \times A$ and such that $\eta(1, \cdot)_{|B}$ is contained in A.

This lemma is still too weak because it only guarantees that if sublevels are not homotopically equivalent, then there exists a sequence of solutions of perturbed problems. Nevertheless, if $\rho \neq 8k\pi$, as in our case, a compactness result due to Li, [27], comes to our rescue.

Theorem 2.2. *If $\rho \neq 8k\pi$, $k \in \mathbb{N}$, $\rho_n \to \rho$ and $(u_n)_n \subset H^1_g(\Sigma)$ is a sequence of solutions of $(*)$ relative to ρ_n such that $\int_\Sigma he^{u_n}\, dV_g = 1$, then $(u_n)_n$ admits a subsequence converging in C^2 to a solution of $(*)$ relative to ρ.*

So, employing together Lemma 2.1 and Theorem 2.2 (just considering the right normalization), it is immediate to establish a strong result concerning our functional I_ρ, through and through analogous to the classical aforementioned deformation lemma.

Corollary 2.3. *If $\rho \neq 8k\pi$ and if I_ρ has no critical levels inside some interval $[a, b]$, then $\{I_\rho \leqslant a\}$ is a deformation retract of $\{I_\rho \leqslant b\}$.*

Next we recall a classical result in Morse theory: Morse inequalities.

Theorem 2.4. *Let N be an Hilbert manifold, $f \in C^2(N; \mathbb{R})$ be a Morse function (i.e. all critical points are non-degenerate) satisfying the (PS)-condition. Let a, b $(a < b)$ be regular values for f and*

$$C_q := \#\big\{\textit{critical points of } f \textit{ in } \{a \leqslant f \leqslant b\} \textit{ with index } q\big\},$$
$$\beta_q(a,b) := \dim\big(H_q\big(\{f \leqslant b\}, \{f \leqslant a\}; \mathcal{F}\big)\big), \quad \textit{where } \mathcal{F} \textit{ is a field},$$

then

$$\sum_{q=0}^{n} (-1)^{n-q} C_q \geqslant \sum_{q=0}^{n} (-1)^{n-q} \beta_q(a,b), \quad n = 0, 1, 2, \ldots \ (\textit{strong inequalities}),$$
$$C_q \geqslant \beta_q(a,b), \quad q = 0, 1, 2, \ldots \ (\textit{weak inequalities}).$$

To prove the above inequalities the (*PS*)-condition is not necessarily needed, it only suffices that appropriate deformation lemmas for f hold true (see for example [4, Theorem 4.3, p. 3, Lemma 3.2, p. 21, and Theorem 3.2, p. 23]). Therefore this hypothesis can be replaced by the request that some proper deformation lemmas hold for f. We now want to point out that, despite the (PS)-condition is not known for I_ρ, is still possible to get Theorem 2.4 for $N = \bar{H}^1_g(\Sigma)$ and $f = I_\rho$, under the further assumption that all the critical points of I_ρ are non-degenerate.

In [34] (proof of Theorem 1.2) Malchiodi defined a new flow $\tilde{W}$, which is nothing but the steepest descent flow in a big ball of $\bar{H}^1_g(\Sigma)$, containing all the critical points of I_ρ (such a ball exists by Theorem 1.1), and which coincides with the flow W constructed by Lucia outside a bigger ball. More precisely:

$$\tilde{W}(u) := -\theta(u)\nabla I_\rho(u) + \big(1-\theta(u)\big)W(u) \tag{2.1}$$

where $\theta : \bar{H}^1_g(\Sigma) \to [0,1]$ is a radial cutoff function satisfying

$$\theta(u) = 1 \quad \text{for } u \in B_R; \qquad \theta(u) = 0 \quad \text{for } u \in \bar{H}^1_g(\Sigma) \setminus B_{2R}.$$

By means of $\tilde{W}$ it is still possible to get the alternative of Lemma 2.1, but this flow has been defined because, unlike W, it allows to adapt to I_ρ the classical deformation lemmas [4, Lemma 3.2, p. 21, and Theorem 3.2, p. 23] needed so that Theorem 2.4 can be applied.

To sum up, if I_ρ is a Morse functional and a and b are regular values for I_ρ, then the weak and the strong Morse inequalities are verified.

2.2. *Topology of sublevels*

First of all we recall the well-known Moser–Trudinger inequality on compact surfaces (see, e.g., [21]).

Lemma 2.5 *(Moser–Trudinger inequality). There exists a constant C, depending only on (Σ, g) such that for all $u \in H^1_g(\Sigma)$*

$$\int_\Sigma e^{\frac{4\pi(u-\bar{u})^2}{\int_\Sigma |\nabla_g u|^2\, dV_g}}\, dV_g \leqslant C, \tag{2.2}$$

where $\bar{u} := \fint_\Sigma u\, dV_g$. As a consequence one has that for any $p \geqslant 0$ and for all $u \in H^1_g(\Sigma)$

$$\log \int_\Sigma e^{p(u-\bar{u})}\, dV_g \leqslant \frac{p^2}{16\pi} \int_\Sigma |\nabla_g u|^2\, dV_g + C. \tag{2.3}$$

Chen and Li [11] showed from this result that if e^u has integral controlled from below (in terms of $\int_\Sigma e^u\, dV_g$) into $(l+1)$ distinct regions of Σ, the constant $\frac{1}{16\pi}$ can be basically divided by $(l+1)$, in the sense specified in the following result.

Lemma 2.6. *Let δ_0, γ be positive real numbers, and for a fixed integer l, let $\Omega_1, \ldots, \Omega_{l+1}$ be subsets of Σ satisfying $\mathrm{dist}(\Omega_i, \Omega_j) \geqslant \delta_0$, for $i \neq j$. Then for any $\tilde{\varepsilon} > 0$ there exists a constant $C = C(l, \tilde{\varepsilon}, \delta_0, \gamma_0)$ such that $\log \int_\Sigma e^{(u-\bar{u})}\, dV_g \leqslant C + \frac{1}{16(l+1)\pi - \tilde{\varepsilon}} \int_\Sigma |\nabla_g u|^2\, dV_g$ for all the functions $u \in H^1_g(\Sigma)$ satisfying $\int_{\Omega_i} e^u\, dV_g \geqslant \gamma_0 \int_\Sigma e^u\, dV_g$ for every $i \in \{1, \ldots, l+1\}$.*

Therefore if $\rho \in (8k\pi, 8(k+1)\pi)$ for some $k \in \mathbb{N}$ Lemma 2.6 implies that if $l \geqslant k$ then the functional I_ρ stays uniformly bounded from below. Qualitatively if I_ρ attains large negative values, $\frac{e^u}{\int_\Sigma e^u}$ has to concentrate near at most k points of Σ. Indeed, using the previous lemma and a covering argument, Ding, Jost, Li and Wang obtained (see [15] or [16]) the following result.

Lemma 2.7. *Assuming $\rho \in (8k\pi, 8(k+1)\pi)$ with $k \in \mathbb{N}$, the following property holds. For any $\varepsilon > 0$ and any $r > 0$ there exists a large positive constant $L = L(\varepsilon, r)$ such that for every $u \in H_g^1(\Sigma)$ with $I_\rho(u) \leqslant -L$, there exist k points $p_{1,u}, \dots, p_{k,u} \in \Sigma$ such that $\int_{\Sigma \setminus \bigcup_{i=1}^k B_r(p_{i,u})} e^u \, dV_g / \int_\Sigma e^u \, dV_g < \varepsilon$.*

By means of Lemma 2.7 one has that the probability measure $\frac{e^u}{\int_\Sigma e^u}$ is *close* to some formal barycenter $\sigma \in \Sigma_k$. We recall that

$$\Sigma_k = \left\{ \sum_{i=1}^k t_i \delta_{x_i} \;\middle|\; t_i \geqslant 0,\ \sum_{i=1}^k t_i = 1,\ x_i \in \Sigma \right\}, \tag{2.4}$$

where δ_{x_i} stands for the Dirac mass at x_i. It was indeed shown in [16] that is possible to map continuously low sublevels of the Euler functional into Σ_k and, viceversa, one can map Σ_k into arbitrarily low sublevels. The composition of the former map with the latter can be taken to be homotopic to the identity on Σ_k, and hence the following result holds true.

Proposition 2.8. *(See [34].) If $k \in \mathbb{N}$ and $\rho \in (8k\pi, 8(k+1)\pi)$, there exists $L > 0$ such that $\{I_\rho \leqslant -L\}$ has the same homology as Σ_k.*

On the other hand in [34] Corollary 2.3 is used to prove that, since I_ρ stays uniformly bounded on the solutions of $(*)$ (by Theorem 1.1), it is possible to retract the whole Hilbert space $\bar{H}_g^1(\Sigma)$ onto a high sublevel $\{I_\rho \leqslant b\}$, $b \gg 0$. More precisely:

Proposition 2.9. *(See [34].) If $\rho \in (8k\pi, 8(k+1)\pi)$ for some $k \in \mathbb{N}$ and if b is sufficiently large positive, the sublevel $\{I_\rho \leqslant b\}$ is a deformation retract of X, and hence it has the same homology of a point.*

Remark 2.10. Let notice that, since the set Σ_k is not contractible, Proposition 2.8 together with Proposition 2.9 and Corollary 2.3 permit to derive an alternative proof of the general existence result due to Djadli.

2.3. *Some notions in algebraic topology*

In this subsection we recall some well-known definitions in algebraic topology. Throughout, the sign $\simeq$ will refer to homotopy equivalences, while $\cong$ will refer to homeomorphisms between topological spaces or isomorphisms between groups. Given a pair of spaces (X, A) we will denote by $H_q(X, A)$ (resp. $H^q(X, A)$) the relative q-th homology (resp. cohomology) group and by $\tilde{H}_q(X) := H_q(X, x_0)$ (resp. $\tilde{H}^q(X) := H^q(X, x_0)$) the reduced homology (resp. cohomology), where $x_0 \in X$.

Join. The join of two spaces X and Y is the space of all segments "joining points" in X to points in Y. It is denoted by $X * Y$ and is the identification space

$$X * Y := X \times [0,1] \times Y/(x,0,y) \sim \big(x',0,y\big), (x,1,y) \sim \big(x,1,y'\big) \quad \forall x,x' \in X,\ \forall y,y' \in Y.$$

Wedge sum. Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, then the wedge sum $X \vee Y$ is the quotient of the disjoint union $X \sqcup Y$ obtained by identifying x_0 and y_0 to a single point. If $\{x_0\}$ (resp. $\{y_0\}$) is a closed subspace of X (resp. Y) that is a deformation retract of some neighbourhood in X (resp. Y), then $\tilde{H}_q(X \vee Y) \cong \tilde{H}_q(X) \oplus \tilde{H}_q(Y)$, provided that the wedge sum is formed at basepoints x_0 and y_0.

Smash product. Inside a product space $X \times Y$ there are copies of X and Y, namely $X \times \{y_0\}$ and $\{x_0\} \times Y$ for points $x_0 \in X$ and $y_0 \in Y$. These two copies of X and Y in $X \times Y$ intersect only at the point (x_0, y_0), so their union can be identified with the wedge sum $X \vee Y$. The smash product $X \wedge Y$ is then defined to be the quotient $X \times Y/X \vee Y$. For example $S^n \wedge S^m \cong S^{n+m}$.

Suspension. The k-fold (*unreduced*) suspension of X is defined to be $S^{k-1} * X$, while the k-fold *reduced* suspension is the smash product $S^k \wedge X$. A useful property of the reduced suspension is that, for any $q, n \geqslant 0$, $\tilde{H}_q(X) \cong \tilde{H}_{q+n}(S^n \wedge X)$. It is crucial to notice that reduced and unreduced constructions are homotopy equivalent constructions for the spaces we will deal with. In the following we will often use the latter property for replacing in some results of [24] the unreduced suspension by the reduced one.

Reduced symmetric product. We denote by $\overline{SP}^k(X)$ the k-th reduced symmetric product which is the symmetric smash product $X^{(k)}/\mathfrak{S}_k$, where $X^{(k)}$ is the k-fold smash product of X with itself and $\mathfrak{S}_k$ is the permutation group. We set $\overline{SP}^0(X) = S^0$. Let us recall also another characterization of the reduced symmetric product. Write $SP^k(X)$ for the k-th symmetric product of X obtained as the quotient of X^k by the permutation action of $\mathfrak{S}_k$. There is a topological embedding $SP^{k-1}(X) \hookrightarrow SP^k(X)$ which adjoins the basepoint to a configuration in $SP^{k-1}(X)$ and $\overline{SP}^k(X)$ is nothing but the cofiber of this embedding, $\overline{SP}^k(X) \cong SP^k(X)/SP^{k-1}(X)$. So a theorem by Dold [18, Theorem 7.2] on the homology of symmetric products of simplicial complexes implies that the homology of reduced symmetric products only depends on the homology of the underlying space. Moreover it has been proved that $\overline{SP}^k(X \vee Y) = \bigvee_{r+s=k} \overline{SP}^r(X) \wedge \overline{SP}^s(Y)$; finally in the case of the 2-sphere $\overline{SP}^k(S^2) \cong S^{2k}$ (see [24]).

Eilenberg–MacLane space. A space X having just one nontrivial homotopy group $\pi_n(X) \cong G$ (where G is a group and $n \in \mathbb{N}$) is called an Eilenberg–MacLane space $K(G,n)$. For any choice of G and n it is possible to build a $K(G,n)$ space and moreover the homotopy type of a $K(G,n)$ space is uniquely determined by G and n.

Steenrod squares. Steenrod defined some homomorphisms between cohomology groups: $Sq^i : H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z}_2)$ ($i \geqslant 0$), where X is any topological space. Properties of those homomorphisms can be found in [41] and references therein. To abbreviate notation we will denote the composition $Sq^{i_1} \circ Sq^{i_2} \circ \cdots \circ Sq^{i_m}$ by Sq^I, where $I = \{i_1, i_2, \ldots, i_m\}$.

Finally let us recall a basic result in homology (see [23, Theorem 2.13, p. 114, and Proposition 2.22, p. 124]).

Theorem 2.11. *If X is a space and A is a nonempty closed subspace that is a deformation retract of some neighbourhood in X, then there is an exact sequence*

$$\cdots \to \tilde{H}_q(A) \to \tilde{H}_q(X) \to H_q(X, A) \to \tilde{H}_{q-1}(A) \to \cdots \to \tilde{H}_0(X, A) \to 0.$$

3. Computation of $\sum_{q\geqslant 0}^{\infty} \dim \tilde{H}_q(\Sigma_k; \mathbb{Z}_2)$

We now focus on the homology with coefficient in $\mathbb{Z}_2$ of Σ_k, i.e. the set formal barycenters of a surface Σ of order k, defined in (2.4). We will present the main steps of the procedure, performed in [24], to achieve a description of $H_*((S^2)_k; \mathbb{Z}_2)$ and we will derive from that the description in the case of any surface.

Then we will compute the sum of the dimensions of the homology groups of Σ_k, the real interest of this section. More precisely we will show that $\sum_{q\geqslant 0}^{\infty} \dim \tilde{H}_q(\Sigma_k; \mathbb{Z}_2)$ equals the r.h.s. of formula (1.3).

First of all the main theorems in [24], dealing with the space of formal barycenters on topological spaces, imply in particular that

Theorem 3.1. *For any manifold M, let M_k denote the set of formal sums*

$$M_k := \left\{ \sum_{i=0}^{k} t_i \delta_{x_i} \colon \sum_{i=0}^{k} t_i = 1,\ t_i \geqslant 0,\ x_i \in M \right\} \tag{3.1}$$

endowed with the weak topology of distributions.

Then for any $q \geqslant 0$, $\tilde{H}_q(M_k; \mathbb{Z}_2) \cong H_{q+1}(\overline{SP}^k(S^1 \wedge M); \mathbb{Z}_2)$.

Remark 3.2. A key point is that, thanks to the isomorphism above, in the case of a surface Σ, the homology of Σ_k only depends on the homology of Σ, in particular on its genus.

Let us consider two particular situations. When $M \cong S^n$, applying Theorem 3.1 we can immediately describe the reduced homology of the space of formal barycenters by means of the homology of a reduced symmetric product of S^n. With some more work we can also deal with the case when M is a surface of genus g, reducing again the comprehension of the homology of the formal barycenters to the understanding of the homology of a reduced symmetric product of S^3.

- Let $M \cong S^2$, then for any $q \geqslant 0$

$$\tilde{H}_q\big((S^2)_k; \mathbb{Z}_2\big) \cong H_{q+1}\big(\overline{SP}^k(S^3); \mathbb{Z}_2\big). \tag{3.2}$$

- Let $M \cong \Sigma_g$, a surface of genus g. Notice that $S^1 \wedge \Sigma_g$ has the same homology of $S^3 \vee (\bigvee_{j=1}^{2g} S^2)$; hence, recalling that the reduced symmetric product of a space only depends on its homology and using, in order, the properties of the reduced symmetric product, those of the homology of the wedge sum, the fact that $\overline{SP}^n(S^2) \cong S^{2n}$ and the properties of the homology of the reduced suspension, we obtain for any $q \geqslant 0$:

$$\begin{aligned}
\tilde{H}_q\big((\Sigma_g)_k;\mathbb{Z}_2\big) &\cong H_{q+1}\big(\overline{SP}^k\big(S^1\wedge\Sigma_g\big);\mathbb{Z}_2\big)\\
&\cong H_{q+1}\left(\overline{SP}^k\left(S^3\vee\left(\bigvee_{j=1}^{2g}S^2\right)\right);\mathbb{Z}_2\right)\\
&\cong H_{q+1}\left(\bigvee_{r+s_1+\cdots+s_{2g}=k}\left(\overline{SP}^r S^3\wedge\left(\bigwedge_{j=1}^{2g}\overline{SP}^{s_j}S^2\right)\right);\mathbb{Z}_2\right)\\
&\cong \bigoplus_{r+s_1+\cdots+s_{2g}=k} H_{q+1}\left(\overline{SP}^r S^3\wedge\left(\bigwedge_{j=1}^{2g}\overline{SP}^{s_j}S^2\right);\mathbb{Z}_2\right)\\
&\cong \bigoplus_{r+s_1+\cdots+s_{2g}=k} H_{q+1}\left(\overline{SP}^r S^3\wedge\left(\bigwedge_{j=1}^{2g}S^{2s_j}\right);\mathbb{Z}_2\right)\\
&\cong \bigoplus_{r+s_1+\cdots+s_{2g}=k} \tilde{H}_{q-2k+2r+1}\big(\overline{SP}^r\big(S^3\big);\mathbb{Z}_2\big).
\end{aligned} \tag{3.3}$$

In the last line we mean $\tilde{H}_{q-2k+2r+1}(\overline{SP}^r(S^3))$ to be 0 if $q<\max\{0,2k-2r-1\}$.

The above examples show that it is really useful to have a description of $\tilde{H}_*(\overline{SP}^r(S^{n+1});\mathbb{Z}_2)$ for $r\geqslant 1$, being $\overline{SP}^0(S^{n+1})=S^0$. Actually what we need is to estimate the dimensions of the homology groups $\tilde{H}_q(\overline{SP}^r(S^{n+1});\mathbb{Z}_2)$, seen as vector spaces. To do that it will be more convenient, at least for notations, to switch by duality to cohomology; namely to study the dual vector space $\tilde{H}^*(\overline{SP}^r(S^{n+1});\mathbb{Z}_2)$. In fact at the moment we are just interested in the case $n=2$, but the general case will be exploited in Section 4.1.

General facts about symmetric products [23, p. 483] show that

$$\tilde{H}^*\big(\overline{SP}^r\big(S^{n+1}\big);\mathbb{Z}_2\big)\hookrightarrow\bigotimes_{i\geqslant 0}H^*\big(K\big(\tilde{H}_i\big(S^n\big),i+1\big);\mathbb{Z}_2\big)=H^*\big(K(\mathbb{Z},n+1);\mathbb{Z}_2\big).$$

Actually we will just summarize how Kallel and Karoui found it, deeply using works of Milgram [36], Nakaoka [37] and Serre [41]. Using the Steenrod splitting it is possible to write

$$\tilde{H}^*\big(K(\mathbb{Z},n);\mathbb{Z}_2\big)\cong\bigoplus_{j\geqslant 1}\tilde{H}^*\big(\overline{SP}^j S^n;\mathbb{Z}_2\big);$$

therefore, if we are able to filter $\tilde{H}^*(K(\mathbb{Z},n);\mathbb{Z}_2)$ over the positive integers so that $\tilde{H}^*(\overline{SP}^j S^n;\mathbb{Z}_2)$ corresponds to the class of filtration degree precisely j, we are done. This procedure rely on the following result:

Theorem 3.3. *(See [41].) $H^*((\mathbb{Z},n);\mathbb{Z}_2)$ is the polynomial algebra with coefficients in $\mathbb{Z}_2$ generated by the iterated Steenrod squares $Sq^I(u_n)$, where u_n is the only generator of $H^n((\mathbb{Z},n);\mathbb{Z}_2)$ and $I=\{i_1,\dots,i_r\}$ is admissible, i.e. if I satisfies the conditions below*:

1. $i_1 - i_2 - \cdots - i_m < n$,
2. $i_k \geqslant 2i_{k+1}$, $k = 1, 2, \ldots, m-1$,
3. $i_m > 1$.

Finally the following theorem leads to the characterization of $\tilde{H}^*(\overline{SP}^r S^{n+1}; \mathbb{Z}_2)$.

Theorem 3.4. *(See [36,37].) Set the filtration degree of $Sq^I(u_n)$, $I = \{i_1, i_2, \ldots, i_m\}$, to be 2^m. Then $\tilde{H}^*(\overline{SP}^r S^{n+1}; \mathbb{Z}_2)$ corresponds to elements of exact filtration r in $H^*((\mathbb{Z}, n); \mathbb{Z}_2)$.*

In particular when $n = 3$:

$$\tilde{H}^*\left(\overline{SP}^r S^3; \mathbb{Z}_2\right) \cong \mathbb{Z}_2[f_{(3,1)}, f_{(5,2)}, \ldots, f_{(2^{i+1}+1,2^i)}, \ldots]_r \tag{3.4}$$

where $f_{(3,1)} = u_3$ and, for $i \geqslant 1$, $f_{(2^{i+1}+1,2^i)} = Sq^I u_3$ with $I = \{2^i, \ldots, 4, 2\}$.

Since after considering the filtration $H^*(\overline{SP}^r S^3; \mathbb{Z}_2)$ is a bigraded algebra over $\mathbb{Z}_2$, writing $f_{(q,m)}$ we want to emphasize that $f_{(q,m)}$ is an element of cohomological degree q and filtration degree m.

Clearly Theorem 3.1 together with Theorem 3.4 (see also (3.2)) yield by duality to a complete description of $\tilde{H}_*((S^2)_k; \mathbb{Z}_2)$. Notice also that our computations in (3.3) allow to describe $\tilde{H}_*(\Sigma_k; \mathbb{Z}_2)$ for any other Σ.

We can now turn to the estimate of $\sum_{q\geqslant 0}^{\infty} \dim \tilde{H}_q(\Sigma_k; \mathbb{Z}_2)$. By (3.2), (3.3) and using that for any $k \geqslant 1$ $\overline{SP}^k(S^3)$ is connected while $\overline{SP}^0(S^3) = S^0$ we obtain

$$\sum_{q\geqslant 0} \dim \tilde{H}_q(\Sigma_k; \mathbb{Z}_2) = \begin{cases} \sum_{q\geqslant 0} \dim(\tilde{H}_q(\overline{SP}^k S^3; \mathbb{Z}_2)) & \text{if } g(\Sigma) = 0, \\ \sum_{r=0}^{k} \binom{k-r+2g-1}{k-r} \sum_{q\geqslant 0} \dim(\tilde{H}_q(\overline{SP}^r S^3; \mathbb{Z}_2)) & \text{if } g = g(\Sigma) > 0. \end{cases} \tag{3.5}$$

In the last line the binomial coefficient $\binom{k-r+2g-1}{k-r}$ counts the number of tuples $(s_1, \ldots, s_{2g})$ such that $\sum_{j=1}^{2g} s_j = k - r$; instead we denote as $g(\Sigma)$ the genus of the surface Σ.

Formula (4.6) rewritten in terms of the Euler characteristic of Σ, $\chi(\Sigma) = 2 - 2g(\Sigma)$, becomes:

$$\sum_{q\geqslant 0} \dim \tilde{H}_q(\Sigma_k; \mathbb{Z}_2) = \begin{cases} \sum_{q\geqslant 0} \dim(\tilde{H}_q(\overline{SP}^k S^3; \mathbb{Z}_2)) & \text{if } \chi(\Sigma) = 2, \\ \sum_{r=0}^{k} \binom{k-r-\chi(\Sigma)+1}{k-r} \sum_{q\geqslant 0} \dim(\tilde{H}_q(\overline{SP}^r S^3; \mathbb{Z}_2)) & \text{if } \chi(\Sigma) \leqslant 0. \end{cases} \tag{3.6}$$

In order to estimate, given $r \geqslant 1$, the quantity $\sum_{q\geqslant 0} \dim(\tilde{H}_q(\overline{SP}^r S^3; \mathbb{Z}_2))$, we can first pass to cohomology by duality, being $\dim(\tilde{H}_q(\overline{SP}^r S^3; \mathbb{Z}_2)) = \dim(\tilde{H}^q(\overline{SP}^r S^3; \mathbb{Z}_2))$, and then exploit the isomorphism in (3.4) and compute how many elements of filtration degree r there are in $\mathbb{Z}_2[f_{(3,1)}, f_{(5,2)}, \ldots, f_{(2^{i+1}+1,2^i)}, \ldots]$. These elements are of the form

$$F(r,n,a_1,\dots,a_{i_n}) = f_{(3,1)}^{r-2n} f_{(2^{1+1}+1,2^1)}^{a_1} \cdots f_{(2^{i_n+1}+1,2^{i_n})}^{a_{i_n}}, \tag{3.7}$$

where

$$n \in \mathbb{N} \quad \text{s.t. } r-2n \geqslant 0, \qquad i_0 := 1, \qquad i_n = \max\{i \mid 2^i \leqslant 2n\},$$
$$a_j \in \mathbb{N} \quad \text{s.t. } \sum_{j=1}^{i_n} a_j 2^j = 2n. \tag{3.8}$$

Since the last condition can be rewritten as $\sum_{j=0}^{i_n-1} a_{j+1}2^j = n$, for any $n \in \{0,\dots,[\frac{r}{2}]\}$, there are as many i_n-tuples $(a_1,\dots,a_{i_n})$ as the partition of n into powers of 2.

Finding such number p_n (as a function of n) is a classical problem in combinatorics going back to Euler. Indeed Euler in [20] showed that p_n is described by the following recurrence formula:

$$p_0 = 1, \qquad p_{2m+1} = p_{2m} = \sum_{j=0}^{m} p_j, \quad \forall m \in \mathbb{N}.$$

In particular, since in our case n is varying in $\{0,\dots,[\frac{r}{2}]\}$, adding up over n we obtain that there are exactly $\sum_{n=0}^{[\frac{r}{2}]} p_n = p_r$ elements of the form (3.7), which are the generators of $\mathbb{Z}_2[f_{(3,1)}, f_{(5,2)},\dots,f_{(2^{i_{r,n}+1}+1,2^{i_{r,n}})}]_r$. Finally, this computation together with (3.6) permits to get an explicit formula for the sum in terms of the elements of the sequence $\{p_n\}_n$:

$$\begin{cases} p_k & \text{if } \chi(\Sigma)=2, \\ \sum_{r=0}^{k} \binom{k-r-\chi(\Sigma)+1}{k-r} p_r & \text{if } \chi(\Sigma) \leqslant 0. \end{cases} \tag{3.9}$$

4. Proofs of the main theorems

Proof of Theorem 1.5. We will first show that for (g,h) in an open and dense subset of $\mathcal{M}^2 \times C^2(\Sigma)^+$ (resp. $\mathcal{M}_1^2 \times C^2(\Sigma)^+$) all the critical points of $I_{\rho,(g,h)}$ are non-degenerate. Then to conclude it will be enough to get the estimate (1.3) under the further assumption that (g,h) are such that $I_{\rho,(g,h)}$ is a Morse functional.

Step 1. As we just pointed out, our claim is:

for any $\rho \in (8k\pi, 8(k+1)\pi)$, then

$$\mathcal{D}(\rho) = \{(g,h) \in \mathcal{M}^2 \times C^2(\Sigma)^+ \colon \text{ all critical points of } I_{\rho,(g,h)} \text{ are non-degenerate}\}$$

is an open and dense subset of $\mathcal{M}^2 \times C^2(\Sigma)^+$ and

$$\mathcal{D}_1(\rho) = \{(g,h) \in \mathcal{M}_1^2 \times C^2(\Sigma)^+ \colon \text{ all critical points of } I_{\rho,(g,h)} \text{ are non-degenerate}\}$$

is an open and dense subset of $\mathcal{M}_1^2 \times C^2(\Sigma)^+$.

The main tool to prove it is an abstract transversality theorem due to Saut and Temam [40]. In particular we will apply the following scheme performed by Micheletti and Pistoia in [35].

First of all we introduce the space $\mathcal{S}^2$ of all C^2 symmetric matrices on Σ. $\mathcal{S}^2$ is a Banach space endowed with the C^2 norm, defined in the following way. We fix a finite covering $\{V_\alpha\}_{\alpha\in L}$

of Σ such that the closure of V_α is contained in U_α, where $\{U_\alpha, \psi_\alpha\}$ is the open coordinate neighbourhood. If $g \in \mathcal{S}^2$ we denote by g_{ij} the components of g with respect to the coordinates $(x_1, \ldots, x_N)$ on V_α. We define

$$\|g\|_2 := \sum_{\alpha \in L} \sum_{|\beta| \leqslant 2} \sum_{i,j=1}^{N} \sup_{\psi_\alpha(V_\alpha)} \frac{\partial^2 g_{ij}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}}. \tag{4.1}$$

The set $\mathcal{M}^2$ of all C^2 Riemannian metrics on Σ is an open subset of $\mathcal{S}^2$.

We fix now $(\bar{g}, \bar{h}) \in \mathcal{M}^2 \times C^2(\Sigma)^+$.

It is easy to verify that there exists $\delta > 0$ such that if $g \in \mathcal{G}_\delta := \{g \in \mathcal{S}^2\colon \|g\|_2 < \delta\}$, $\bar{g} + g$ is a Riemannian metric and the sets $H^1_{\bar{g}+g}(\Sigma)$, $L^2_{\bar{g}+g}(\Sigma)$, $L^1_{\bar{g}+g}(\Sigma)$ coincide respectively with $H^1_{\bar{g}}(\Sigma)$, $L^2_{\bar{g}}(\Sigma)$, $L^1_{\bar{g}}(\Sigma)$ and the two norms are equivalent. Moreover we will choose δ sufficiently small in order to have that $\bar{g} + g \in \mathcal{M}^2$ for any $h \in \mathcal{H}_\delta := \{h \in C^2(\Sigma)\colon \|h\|_\infty < \delta\}$.

Definition 4.1. For $g \in \mathcal{G}_\delta$ we set $A(g) := A_g : L^2_{\bar{g}}(\Sigma) \to H^1_{\bar{g}}(\Sigma)$ to be the only linear operator such that

$$(A_g u, v)_{H^1_{\bar{g}+g}(\Sigma)} = (u, v)_{L^2_{\bar{g}+g}(\Sigma)}, \quad \forall v \in H^1_{\bar{g}}(\Sigma),\ \forall u \in L^2_{\bar{g}}(\Sigma). \tag{4.2}$$

Clearly

$$(A_g u, v)_{H^1_{\bar{g}+g}(\Sigma)} = (u, A_g v)_{H^1_{\bar{g}+g}(\Sigma)}, \quad \forall u, v \in H^1_{\bar{g}}(\Sigma),$$

moreover A_g is nothing but the adjoint operator $i^*_{\bar{g}+g}$ of the compact embedding $i_{\bar{g}+g} : H^1_{\bar{g}+g}(\Sigma) \to L^2_{\bar{g}+g}(\Sigma)$. Integrating by parts it can be easily checked that the main term of the explicit expression of A_g is the inverse of the laplacian operator. Let us notice that in the definition of A_g we used the fact that $H^1_{\bar{g}+g}(\Sigma)$ and $H^1_{\bar{g}}(\Sigma)$ (respectively $L^2_{\bar{g}+g}(\Sigma)$ and $L^2_{\bar{g}}(\Sigma)$) are the same as sets and that the two norms are equivalent.

For what concerns the regularity in g of $A(g)$ we have the following result.

Lemma 4.2. *The map $A : \mathcal{G}_\delta \to \mathcal{L}(L^{p'}_{\bar{g}}(\Sigma); H^1_{\bar{g}}(\Sigma))$ is of class C^1, where $\mathcal{L}(L^{p'}_{\bar{g}}(\Sigma); H^1_{\bar{g}}(\Sigma))$ stands for the space of linear operators from $L^{p'}_{\bar{g}}(\Sigma)$ to $H^1_{\bar{g}}(\Sigma)$.*

For the proof, see Lemma 2.3 of [35].

Moreover we can assume that δ is sufficiently small such that there exists $\bar{R} > 0$ such that for any $(g_0, h_0) \in \mathcal{G}_\delta \times \mathcal{H}_\delta$ all the critical points of $I_{\rho,(\bar{g}+g,\bar{h}+h)}$ are contained in the ball $\mathcal{B} := B_{\bar{R}}(0)$ of $\bar{H}^1_{\bar{g}}(\Sigma)$.

We are finally in position to introduce the map $F : \mathcal{G}_\delta \times \mathcal{H}_\delta \times \mathcal{B} \to \bar{H}^1_{\bar{g}}(\Sigma)$:

$$F(g, h, u) := S_g^{-1}\big(\tilde{F}_g\big(h, S_g(u)\big)\big), \tag{4.3}$$

where

$$\tilde{F}_g : \mathcal{H}_\delta \times \bar{H}^1_{\bar{g}+g}(\Sigma) \to \bar{H}^1_{\bar{g}+g}(\Sigma),$$

$$(h, w) \mapsto w - A_g\left(\rho \frac{(\bar{h}+h)e^w}{\int_\Sigma (\bar{h}+h)e^w \, dV_{\bar{g}+g}} - \frac{\rho}{\int_\Sigma dV_{\bar{g}+g}} + w\right),$$

while $S_g : \bar{H}^1_{\bar{g}}(\Sigma) \to \bar{H}^1_{\bar{g}+g}(\Sigma)$ is defined as $S_g(u) := u - \fint_\Sigma u \, dV_{\bar{g}+g}$. Clearly S_g is linear, invertible and the inverse is given by $S_g^{-1} : \bar{H}^1_{\bar{g}+g}(\Sigma) \to \bar{H}^1_{\bar{g}}(\Sigma)$, $S_g^{-1}(w) := w - \fint_\Sigma w \, dV_{\bar{g}}$.

By the regularity of the map A (see Lemma 4.2) we get that the map F is of class C^1.

It is easy to see that $(g, h, u) \in \mathcal{G}_\delta \times \mathcal{H}_\delta \times \mathcal{B}$ are such that $F(g, h, u) = 0$ if and only if u is a critical point of $I_{\rho,(\bar{g}+g,\bar{h}+h)}$. Taking into account this remark, to establish the claim we need the following transversality theorem.

Theorem 4.3. *(See [40].) Let X, Y, Z be three real Banach spaces and let $U \subset X$, $V \subset Y$ be open subsets. Let $F : V \times U \to Z$ be a C^k-map with $k \geqslant 1$ such that*

(i) *for any $y \in V$, $F(y, \cdot) : x \mapsto F(y, x)$ is a Fredholm map of index l with $l \leqslant k$;*
(ii) *z_0 is a regular value of F, that is the operator $F'(y_0, x_0) : Y \times X \to Z$ is onto at any point (y_0, x_0) such that $F(y_0, x_0) = z_0$;*
(iii) *the set of $x \in U$ such that $F(y_0, x_0) = z_0$ with y in a compact set of V is relatively compact in U.*

Then the set $\{y \in V\colon\ z_0$ is a regular value of $F(y, \cdot)\}$ is a dense open subset of V.

If we take as F the map defined in (4.3) and we set $X = Z = \bar{H}^1_{\bar{g}}(\Sigma)$, $Y = \mathcal{S}^2 \times C^2(\Sigma)$, $V = \mathcal{G}_\delta \times \mathcal{H}_\delta$, $U = \mathcal{B}$ and $z_0 = 0$, all the assumptions of Theorem 4.3 are fulfilled (see Appendix A, respectively Lemmas A.1, A.5 and A.3). Applying Theorem 4.3 we get that the following set is an open and dense subset of $\mathcal{G}_\delta \times \mathcal{H}_\delta$

$$\begin{aligned}
&\big\{(g, h) \in \mathcal{G}_\delta \times \mathcal{H}_\delta\colon\ F'_u(g, h, u) : \bar{H}^1_{\bar{g}} \to \bar{H}^1_{\bar{g}} \text{ is invertible at any point} \\
&\quad (g, h, u) \text{ such that } F(g, h, u) = 0 \text{ with } u \in \mathcal{B}\big\} \\
&\quad = \Bigg\{(g, h) \in \mathcal{G}_\delta \times \mathcal{H}_\delta\colon \text{ any } u \in \mathcal{B} \text{ solution of the equation} \\
&\qquad -\triangle_{\bar{g}+g} u + \frac{\rho}{\int_\Sigma dV_{\bar{g}+g}} = \rho \frac{(\bar{h}+h)e^u}{\int_\Sigma (\bar{h}+h)e^u \, dV_{\bar{g}+g}} \text{ is non-degenerate}\Bigg\} \\
&\quad = \Bigg\{(g, h) \in \mathcal{G}_\delta \times \mathcal{H}_\delta\colon \text{ any solution of the equation} \\
&\qquad -\triangle_{\bar{g}+g} u + \frac{\rho}{\int_\Sigma dV_{\bar{g}+g}} = \rho \frac{(\bar{h}+h)e^u}{\int_\Sigma (\bar{h}+h)e^u \, dV_{\bar{g}+g}} \text{ is non-degenerate}\Bigg\},
\end{aligned}$$

where the last equality follows from our choice of $\bar{R}$. Finally, since we have this for any $(\bar{g}, \bar{h}) \in \mathcal{M}^2 \times C^2(\Sigma)^+$, the proof of the first part of the claim is complete.

For what concerns $\mathcal{D}_1(\rho)$ the openness in $\mathcal{M}_1^2 \times C^2(\Sigma)^+$ follows immediately from the openness of $\mathcal{D}(\rho)$ in $\mathcal{M}_2 \times C^2(\Sigma)^+$. Actually the previous proof also implies the density indeed focusing on the statement of Lemma A.4 it can be easily understand that we proved that for any $(g,h) \in \mathcal{M}^2 \times C^2(\Sigma)^+$ there exists $\tilde{h}$ arbitrarily close to h such that $(g,\tilde{h}) \in \mathcal{D}_\rho$. Applying this remark to an element $(g,h) \in \mathcal{M}_1^2 \times C^2(\Sigma)^+$ we get the second part of the claim and this concludes Step 1.

Step 2. In order to prove Theorem 1.5, thanks to what we proved in Step 1, we can assume without loss of generality that $g \in \mathcal{M}^2$ and $h \in C^2(\Sigma)^+$ are such that all the critical points of $I_{\rho,(g,h)}$ are non-degenerate. Henceforth we will work assuming this property of g and h to hold and we will write I_ρ for $I_{\rho,(g,h)}$.

Let us fix two real positive numbers $b > 0$ and $L > 0$ sufficiently large so that the hypotheses of Propositions 2.8 and 2.9 are verified and such that b and $-L$ are regular values of I_ρ. Thanks to the considerations at the end of Section 2.1 we can apply Theorem 2.4 and we have that

$$\begin{aligned}&\#\{\text{critical points of } I_\rho \text{ in } -L \leqslant I_\rho \leqslant b\}\\&\quad\geqslant \sum_{q \geqslant 0} \beta_q(-L,b;\mathcal{F}) \equiv \sum_{q\geqslant 0} \dim H_q\big(\{I_\rho \leqslant b\},\{I_\rho \leqslant -L\};\mathcal{F}\big)\end{aligned} \tag{4.4}$$

where $\mathcal{F}$ is any field. Hence, to estimate from below the number of critical points we have to focus on the right-hand side of the previous inequality.

Since $-L$ is a regular value, by Corollary 2.3 we have that $\{I_\rho \leqslant -L\}$ is a deformation retract of some neighbourhood in $H_g^1(\Sigma)$ and so we can apply Theorem 2.11 obtaining

$$\begin{aligned}&\cdots \to \tilde{H}_q\big(\{I_\rho \leqslant -L\};\mathcal{F}\big) \to \tilde{H}_q\big(\{I_\rho \leqslant b\};\mathcal{F}\big)\\&\quad\to H_q\big(\{I_\rho \leqslant b\},\{I_\rho \leqslant -L\};\mathcal{F}\big) \to \tilde{H}_{q-1}\big(\{I_\rho \leqslant -L\};\mathcal{F}\big) \to \cdots.\end{aligned}$$

Then by Propositions 2.8, 2.9 and from the exactness of the latter homology sequence we get

$$\begin{cases} H_{q+1}\big(\{I_\rho \leqslant b\},\{I_\rho \leqslant -L\};\mathcal{F}\big) \cong \tilde{H}_q(\Sigma_k;\mathcal{F}), & q \geqslant 0,\\ H_0\big(\{I_\rho \leqslant b\},\{I_\rho \leqslant -L\};\mathcal{F}\big) = 0. \end{cases} \tag{4.5}$$

Finally (4.4), (4.5) and (3.9) imply that

$$\begin{aligned}&\big\{\text{solutions of } (*)\big\} \geqslant \#\{\text{critical points of } I_\rho \text{ in } -L \leqslant I_\rho \leqslant b\}\\&\quad\overset{(4.4)}{\geqslant} \sum_{q\geqslant 0} \dim H_q\big(\{I_\rho \leqslant b\},\{I_\rho \leqslant -L\};\mathbb{Z}_2\big) \overset{(4.5)}{\geqslant} \sum_{q\geqslant 0} \dim \tilde{H}_q(\Sigma_k;\mathbb{Z}_2)\\&\quad\overset{(3.9)}{=} \begin{cases} p_k & \text{if } \chi(\Sigma) = 2,\\ \sum_{r=0}^{k} \binom{k-r-\chi(\Sigma)+1}{k-r} p_r & \text{if } \chi(\Sigma) \leqslant 0. \end{cases}\end{aligned} \tag{4.6}$$

This concludes the proof of Theorem 1.5. □

Proof of Corollary 1.7. 1. Let denote by $N_{k,\chi(\Sigma)}$ the r.h.s. of formula (1.3). It will be enough to prove that (except in the case $\chi(\Sigma)=2$ and $k=1$) $N_{k,\chi(\Sigma)} > \mathrm{d}(k,\chi(\Sigma)) \geqslant 0$. This is trivial for $\chi(\Sigma)=2$, while in the remaining cases, since $p_r \geqslant 1$ for any $r \in \mathbb{N}$, we have

$$N_{k,\chi(\Sigma)} = \sum_{r=0}^{k} \binom{k-r-\chi(\Sigma)+1}{k-r} p_r \geqslant \binom{k-\chi(\Sigma)+1}{k} p_0$$
$$= \frac{k-\chi(\Sigma)+1}{-\chi(\Sigma)+1} \mathrm{d}\big(k,\chi(\Sigma)\big) > \mathrm{d}\big(k,\chi(\Sigma)\big).$$

2. To prove this point we will use a formula on the asymptotic behavior of p_{2n} derived by Mahler (see [32] and also [13]). Let recall his result in an explicit way:

$$p_{2n} = O_n(1)\left(\frac{n}{\log n}\right)^{\frac{1}{2l_2}\log(\frac{n}{\log n})+1+\frac{ll_2}{l_2}} n^{(\frac{1}{l_2}-\frac{1}{2})},$$

where $l_2 := \log 2$ and $ll_2 =: \log\log 2$.

Now just combining Theorem 1.5 with the previous asymptotic formula we obtain estimate (1.4). □

4.1. Generic multiplicity of conformal metrics with constant Q-curvature

The existence of a solution of (#) was proved in [8] under the assumptions $\mathrm{P_g} \geqslant 0$ and $k_P < 8\pi^2$, which are naively the counterpart of $\rho < 8\pi$ for $(*)$. Also in this case there is a variant of the Moser–Trudinger inequality, the Adams inequality, which makes the problem coercive.

In [17] an extension of this result was obtained for a large class of manifolds, assuming $k_P \neq 8k\pi^2$, $k \in \mathbb{N}$, and that $\mathrm{P_g}$ has no kernel. The proof relies on a direct min–max method based on the study of the topology of the sublevels of the associated Euler functional, on some improvement of the Adams inequality and on some compactness results in [34,19], which are the counterpart of Theorem 1.1. As mentioned in the Introduction, thanks to the boundedness of solutions it is possible to define the Leray–Schauder degree of Eq. (#) and the following counting formula was obtained.

Theorem 4.4. *Let (M,g) be a compact four-manifold such that the Paneitz operator $\mathrm{P_g}$ has $\bar{k}$ negative eigenvalues and only trivial kernel (the constant functions) and such that $k_P := \int_M Q_g\, dV_g \in (8k\pi^2, 8(k+1)\pi^2)$, for some $k \in \mathbb{N}^*$. Then the degree of (#) is given by*

$$\mathrm{d}\big(k,\bar{k},\chi(M)\big) = \begin{cases} (-1)^{\bar{k}} & \text{if } k_P < 8\pi^2; \\ (-1)^{\bar{k}} \frac{(k-\chi(M))\dots(2-\chi(M))(1-\chi(M))}{k!} & \text{if } k_P \in (8k\pi^2, 8(k+1)\pi^2),\ k \in \mathbb{N}^*. \end{cases} \tag{4.7}$$

Notice that under these hypotheses, since $\chi(M) \geqslant 2k$ (by formula (1.7)), the degree never vanishes.

We are now in position to prove Theorem 1.9 and Corollary 1.10.

Proof of Theorem 1.9. We can reason as in the proof of Theorem 1.5: the main difference is that the presence of negative eigenvalues for $\mathrm{P_g}$ affects the topology of the sublevels of the Euler functional. In [17] was shown that the counterpart of Proposition 2.8 holds true replacing Σ_k with $A_{k,\bar{k}} = M_k \times B_1^{\bar{k}}$.

Here M_k is the set of k-barycenters of M (defined in (3.1)), $B_1^{\bar{k}}$ the closed unit ball in $\mathbb{R}^{\bar{k}}$ while the equivalence relation $\sim$ means that $M_k \times \partial B_1^{\bar{k}}$ is identified with $\partial B_1^{\bar{k}}$, namely $M_k \times \{y\}$ for every fixed $y \in \partial B_1^{\bar{k}}$ is collapsed to a single point.

Therefore following exactly the previous proof, we find that

$$\#\big\{\text{solutions of } (\#)\big\} \geqslant \sum_{q\geqslant 0} \dim \tilde{H}_q(A_{k,\bar{k}}; \mathbb{Z}_2). \tag{4.8}$$

To compute the latter sum we can use the Mayer–Vietoris sequence, see for example [23, p. 149]. We can cover $A_{k,\bar{k}}$ with the two sets

$$\mathcal{A} = M_k \times B^{\bar{k}}_{\frac{3}{4}}, \qquad \mathcal{B} = M_k \times \big(B_1^{\bar{k}} \setminus B^{\bar{k}}_{\frac{1}{4}}\big),$$

where $B_r^{\bar{k}}$ stands for the closed ball of radius r in $\mathbb{R}^{\bar{k}}$. Clearly $\mathcal{A}$ has the homology type of M_k, $\mathcal{B}$ that of $S^{\bar{k}-1}$ and $\mathcal{A} \cap \mathcal{B}$ that of $M_k \times S^{\bar{k}-1}$. Therefore, by the exactness of the Mayer–Vietoris sequence and the Kunneth theorem we find the relation

$$\begin{cases} \tilde{H}_{\bar{k}+p}(A_{k,\bar{k}}) \cong \tilde{H}_p(M_k) & \text{for } p \geqslant 1, \\ \tilde{H}_q(A_{k,\bar{k}}) \cong 0 & \text{for } 0 \leqslant q \leqslant \bar{k}, \end{cases}$$

which implies

$$\sum_{q\geqslant 0} \dim \tilde{H}_q(A_{k,\bar{k}}; \mathbb{Z}_2) = \sum_{q\geqslant 0} \dim \tilde{H}_q(M_k; \mathbb{Z}_2). \tag{4.9}$$

From formulas (4.8) and (4.9) we deduce that the problem reduces to the computation of $\sum_{q\geqslant 0} \dim \tilde{H}_q(M_k; \mathbb{Z}_2)$. By Theorem 3.1 we immediately get

$$\sum_{q\geqslant 0} \dim \tilde{H}_q(M_k; \mathbb{Z}_2) = \sum_{q\geqslant 0} \dim H_{q+1}\big(\overline{SP}^k\big(S^1 \wedge M\big); \mathbb{Z}_2\big). \tag{4.10}$$

Since $S^1 \wedge M$ is a CW complex with top integral homology group $H_5(M;\mathbb{Z}) = \mathbb{Z}$ and $\operatorname{rank}(H_3(M;\mathbb{Z})) \geqslant \chi(M) - 2$, it has the homology of $S^5 \vee (\bigvee_{j=1}^{\chi(M)-2} S^3) \vee Y$ for some topological space Y. Thus, as we did in the case of a surface of genus $g > 0$, we can apply the properties of the reduced symmetric product and of the homology of the wedge sum to obtain

$$\begin{aligned} &H_{q+1}\big(\overline{SP}^k\big(S^1 \wedge M\big); \mathbb{Z}_2\big) \\ &\quad\cong \bigoplus_{r+s_1+\cdots+s_{\chi(M)-2}+t=k} H_{q+1}\left(\overline{SP}^r S^5 \wedge \left(\bigwedge_{j=1}^{\chi(M)-2} \overline{SP}^{s_j} S^3\right) \wedge \overline{SP}^t Y; \mathbb{Z}_2\right). \end{aligned}$$

Considering now the sum of the dimensions we have

$$\sum_{q\geqslant 0} \dim H_{q+1}\big(\overline{SP}^k\big(S^1\wedge M\big);\mathbb{Z}_2\big)$$
$$\geqslant \sum_{q\geqslant 0} \dim \tilde{H}_q\big(\overline{SP}^k S^5;\mathbb{Z}_2\big)$$
$$+\sum_{r=0}^{k-1}\ \sum_{\sum_{j=1}^{\chi(M)-2} s_j=k-r}\ \sum_{q\geqslant 0} \dim H_{q+1}\left(\overline{SP}^r S^5\wedge\left(\bigwedge_{j=1}^{\chi(M)-2}\overline{SP}^{s_j} S^3\right);\mathbb{Z}_2\right). \tag{4.11}$$

Recalling that by definition the smash product $X\wedge Y$ is the quotient $X\times Y/X\vee Y$ and using the exact sequence for relative homology it is possible to see that for any $(r,s_1,\dots,s_{\chi(M)-2})$ such that $\sum_{j=1}^{\chi(M)-2} s_j=k-r>0$

$$H_{5r+3\sum_{j=1}^{\chi(M)-2}s_j}\left(\overline{SP}^r S^5\wedge\left(\bigwedge_{j=1}^{\chi(M)-2}\overline{SP}^{s_j} S^3\right);\mathbb{Z}_2\right)\neq 0. \tag{4.12}$$

Clearly for $\chi(M)=2$ we just have

$$H_{q+1}\big(\overline{SP}^k\big(S^1\wedge M\big);\mathbb{Z}_2\big)\cong\bigoplus_{r+t=k} H_{q+1}\big(\overline{SP}^r S^5\wedge\overline{SP}^t Y;\mathbb{Z}_2\big)$$

and

$$\sum_{q\geqslant 0} \dim H_{q+1}\big(\overline{SP}^k\big(S^1\wedge M\big);\mathbb{Z}_2\big)\geqslant\sum_{q\geqslant 0} \dim \tilde{H}_q\big(\overline{SP}^k S^5;\mathbb{Z}_2\big). \tag{4.13}$$

Next collecting formulas (4.8), (4.9), (4.10), (4.11), (4.12) and (4.13) we get

$$\#\big\{\text{solutions of }(\#)\big\}\geqslant\begin{cases}\sum_{q\geqslant 0}\dim\tilde{H}_q(\overline{SP}^k S^5;\mathbb{Z}_2) & \text{if } \chi(M)\geqslant 2,\\ \sum_{q\geqslant 0}\dim\tilde{H}_q(\overline{SP}^k S^5;\mathbb{Z}_2)+\sum_{r=0}^{k-1}\binom{k-r+\chi(M)-3}{k-r} & \text{if } \chi(M)\geqslant 3,\end{cases}$$

where the binomial coefficient $\binom{k-r+\chi(M)-3}{k-r}$ counts the number of tuples $(s_1,\dots,s_{\chi(M)-2})$ such that $\sum_{j=1}^{\chi(M)-2} s_j=k-r$.

Finally, since all the admissible tuples $\{i_1,\dots,i_r\}$ for $n=3$ are also admissible for $n=5$, the elements of exact filtration k in $H^*(\overline{SP}^k(S^5))$ are at least as many as the elements of exact filtration k in $H^*(\overline{SP}^k(S^3))$. Then by Theorem 3.4 and duality we have the desired estimate. □

Proof of Corollary 1.10. This estimate follows immediately from Theorem 1.9 indeed it is sufficient to prove that the r.h.s. of formula (1.9) is greater then $\mathrm{d}(k,\bar{k},\chi(M))$ (except for the case $\chi(M)=2$). But this is trivial because for $\chi(M)\geqslant 3$

$$\sum_{r=0}^{k}\binom{k-r+\chi(\Sigma)-3}{k-r} \geqslant \binom{k+\chi(\Sigma)-3}{k} > \frac{(\chi(M)-k)\ldots(\chi(M)-2)(\chi(M)-1)}{k!}$$
$$= \left|\mathrm{d}\big(k,\bar{k},\chi(M)\big)\right|.$$

On the other hand if $\chi(M)=2$, then k should be 1 and then $p_1 = 1 = |\mathrm{d}(1,\bar{k},2)|$. $\square$

Acknowledgments

The author is grateful to Andrea Malchiodi for helpful discussions and for having proposed her this topic. Moreover she really wants to thank Sadok Kallel for kindly answering her questions and Angela Pistoia for her useful suggestions.

Appendix A

In this section we collect some technical lemmas needed to verify that we are in condition to apply Theorem 4.3. The scheme of the proofs will follow the one performed in the last section of [35], the main differences are between Lemmas A.5 and 4.3 of [35].

We will keep the notations of Section 4.

Lemma A.1. *For any $(g,h)\in\mathcal{G}_\delta\times\mathcal{H}_\delta$ the map $u\mapsto F(g,h,u)$ with $u\in\mathcal{B}$ is Fredholm of index* 0.

Proof. For $(g_0,h_0)\in\mathcal{G}_\delta\times\mathcal{H}_\delta$ and $v\in\bar{H}^1_{\bar{g}}(\Sigma)$ we have

$$\begin{aligned}
&F'_u(g_0,h_0,u_0)[v]\\
&\quad= S_{g_0}^{-1}\big((\tilde{F}_{g_0})'_w\big(h_0,S_{g_0}(u_0)\big)\big[S_{g_0}(v)\big]\big)\\
&\quad= S_{g_0}^{-1}\bigg(S_{g_0}(v)-A_{g_0}\bigg(\rho\frac{\tilde{h}e^{S_{g_0}(u_0)}S_{g_0}(v)\int_\Sigma \tilde{h}e^{S_{g_0}(u_0)}\,dV_{\tilde{g}}-\tilde{h}e^{S_{g_0}(u_0)}\int_\Sigma \tilde{h}e^{S_{g_0}(u_0)}S_{g_0}(v)\,dV_{\tilde{g}}}{(\int_\Sigma \tilde{h}e^{S_{g_0}(u_0)}\,dV_{\tilde{g}})^2}+S_{g_0}(v)\bigg)\bigg)\\
&\quad= v-S_{g_0}^{-1}\bigg(A_{g_0}\bigg(\rho\frac{\tilde{h}e^{u_0}v\int_\Sigma \tilde{h}e^{u_0}\,dV_{\tilde{g}}-\tilde{h}e^{u_0}\int_\Sigma \tilde{h}e^{u_0}v\,dV_{\tilde{g}}}{(\int_\Sigma \tilde{h}e^{u_0}\,dV_{\tilde{g}})^2}+v\bigg)\bigg)\\
&\quad:= v-K(v),
\end{aligned}$$

where $\tilde{g}:=\bar{g}+g_0$, $\tilde{h}:=\bar{h}+h_0$ and clearly

$$K(v)=S_{g_0}^{-1}\bigg(A_{g_0}\bigg(\rho\frac{\tilde{h}e^{u_0}v\int_\Sigma \tilde{h}e^{u_0}\,dV_{\tilde{g}}-\tilde{h}e^{u_0}\int_\Sigma \tilde{h}e^{u_0}v\,dV_{\tilde{g}}}{(\int_\Sigma \tilde{h}e^{u_0}\,dV_{\tilde{g}})^2}+v\bigg)\bigg).$$

We will verify that $K:\bar{H}^1_{\bar{g}}(\Sigma)\to\bar{H}^1_{\bar{g}}(\Sigma)$ is compact and this will end the proof.

If v_n is a bounded sequence in $\bar{H}^1_{\bar{g}}(\Sigma)$, then v_n is also bounded in $\bar{H}^1_{\tilde{g}}(\Sigma)$ (because $g_0\in\mathcal{G}_\delta$). Then up to a subsequence, v_n converges to v in $L^q_{\tilde{g}}(\Sigma)$ for any $q\geqslant 1$. So, we have

$$\left(\int_\Sigma \left|\rho\frac{(\int_\Sigma \tilde h e^{u_0}\,dV_{\tilde g})\tilde h e^{u_0}(v_n-v)-(\int_\Sigma \tilde h e^{u_0}(v_n-v)\,dV_{\tilde g})\tilde h e^{u_0}}{(\int_\Sigma \tilde h e^{u_0}\,dV_{\tilde g})^2}+(v_n-v)\right|^2 dV_{\tilde g}\right)^{\frac12}$$

$$\leqslant \rho\left(\frac{\|\tilde h e^{u_0}\|_{L^4_{\tilde g}}\|v_n-v\|_{L^4_{\tilde g}}}{\|\tilde h e^{u_0}\|_{L^1_{\tilde g}}}+\frac{\|\tilde h e^{u_0}\|^2_{L^2_{\tilde g}}\|v_n-v\|_{L^2_{\tilde g}}}{\|\tilde h e^{u_0}\|^2_{L^1_{\tilde g}}}\right)+\|v_n-v\|_{L^2_{\tilde g}}\to 0.$$

Therefore, by continuity of A_{g_0} and of $S_{g_0}^{-1}$, $K(v_n-v)\to 0$ in $\bar H^1_{\tilde g}(\Sigma)$ and so it converges to 0 in $\bar H^1_{\tilde g}(\Sigma)$. $\square$

Remark A.2. Arguing exactly in the same way we can also prove that for any $(g_0,h_0,u_0)\in\mathcal G_\delta\times\mathcal H_\delta\times\mathcal B$ the map $w\mapsto(\tilde F_g)'_w(h,S_{g_0}(u_0))[w]$ for $w\in\bar H^1_{\bar g+g}(\Sigma)$ is a Fredholm map of index 0.

Lemma A.3. *The set*

$$\{u\in\mathcal B\colon\ F(g_0,h_0,u_0)=0,\ (g_0,h_0)\ \textit{belongs to a compact subset of}\ \mathcal G_\delta\times\mathcal H_\delta\}$$

is relatively compact in $\mathcal B\subset\bar H^1_{\tilde g}(\Sigma)$.

Proof. We show that if $u_n\in\mathcal B$ is such that $F(g_n,h_n,u_n)=0$ with $g_n\to g_0$ and $h_n\to h_0$, then u_n possesses a converging subsequence.

Let us first notice that, thanks to the invertibility of $S_{g_n}^{-1}$ for any n, $F(g_n,h_n,u_n)=0$ implies $\tilde F_{g_n}(h_n,S_{g_n}(u_n))=0$, which in turn is equivalent to

$$u_n=A_{g_n}\left(\rho\frac{\tilde h_n e^{u_n}}{\int_\Sigma \tilde h_n e^{u_n}\,dV_{\tilde g_n}}-\frac{\rho}{\int_\Sigma dV_{\tilde g_n}}+u_n\right).$$

Since the sequence u_n is bounded in $H^1_{\bar g}(\Sigma)$ and also in $H^1_{\tilde g}(\Sigma)$ (being $g_0\in\mathcal G_\delta$), u_n (up to a subsequence) converges to a function u in $L^q_{\tilde g}(\Sigma)$ and in $L^q_{\bar g}(\Sigma)$ for any $q\geqslant 1$. If we are able to prove that

$$\left\|\rho\left(\frac{\tilde h_n e^{u_n}}{\int_\Sigma \tilde h_n e^{u_n}\,dV_{\tilde g_n}}-\frac{\tilde h e^{u}}{\int_\Sigma \tilde h e^{u}\,dV_{\tilde g}}\right)-\rho\left(\frac{1}{\int_\Sigma dV_{\tilde g_n}}-\frac{1}{\int_\Sigma dV_{\tilde g}}\right)+(u_n-u)\right\|_{L^2_{\tilde g}}\to 0, \tag{A.1}$$

where $\tilde g_n:=\bar g+g_n$ and $\tilde h_n:=\bar h+h_n$, then we will get the same convergence in $L^2_{\bar g}$ and so

$$i^*_{\tilde g}(f_n) = A_{g_0}(f_n) \xrightarrow{H^1_{\tilde g}(\Sigma)} A_{g_0}\left(\rho\frac{\tilde h e^u}{\int_\Sigma \tilde h e^u\, dV_{\tilde g}} - \frac{\rho}{\int_\Sigma dV_{\tilde g}} + u\right), \tag{A.2}$$

where $f_n := \rho\frac{\tilde h_n e^{u_n}}{\int_\Sigma \tilde h_n e^{u_n}\, dV_{\tilde g_n}} - \frac{\rho}{\int_\Sigma dV_{\tilde g_n}} + u_n$.

On the other hand by Lemma 4.2 we have that for some $\theta \in (0,1)$:

$$\begin{aligned}\left\|A_{g_n}(f_n) - A_{g_0}(f_n)\right\|_{H^1_{\tilde g}} &= \left\|A'\big(g_0 + \theta(g_n - g_0)\big)[g_n - g_0](f_n)\right\|_{H^1_{\tilde g}} \\ &\leqslant \|f_n\|_{L^2_{\tilde g}}\left\|A'\big(g_0 + \theta(g_n - g_0)\big)[g_n - g_0]\right\|_{\mathcal{L}(L^2_{\tilde g}, H^1_{\tilde g})} \\ &\leqslant \|f_n\|_{L^2_{\tilde g}}\left\|A'\big(g_0 + \theta(g_n - g_0)\big)\right\|_{\mathcal{L}(\mathcal{G}_\delta, \mathcal{L}(L^2_{\tilde g}, H^1_{\tilde g}))}\|g_n - g_0\|_2.\end{aligned} \tag{A.3}$$

From (A.2) and (A.3) we can deduce that

$$A_{g_n}(f_n) \xrightarrow{H^1_{\tilde g}(\Sigma)} A_{g_0}\left(\rho\frac{\tilde h e^u}{\int_\Sigma \tilde h e^u\, dV_{\tilde g}} - \frac{\rho}{\int_\Sigma dV_{\tilde g}} + u\right).$$

Therefore, since $u_n = A_{g_n}(f_n)$, we get the claim.

Finally to conclude it remains to verify (A.1); as $g_n \to g_0$ in $\|\cdot\|_2$ and $u_n \to u_0$ in $L^2_{\tilde g}$, it will be enough to prove

$$\left\|\left(\int_\Sigma \tilde h e^u\, dV_{\tilde g}\right)\tilde h_n e^{u_n} - \left(\int_\Sigma \tilde h_n e^{u_n}\, dV_{\tilde g_n}\right)\tilde h e^u\right\|_{L^2_{\tilde g}} \to 0.$$

Simply manipulating the integrands and using Holder's inequality we have

$$\begin{aligned}&\int_\Sigma \left[\left(\int_\Sigma \tilde h e^u\, dV_{\tilde g}\right)\tilde h_n e^{u_n} - \left(\int_\Sigma \tilde h_n e^{u_n}\, dV_{\tilde g_n}\right)\tilde h e^u\right]^2 dV_{\tilde g} \\ &\quad= \left(\int_\Sigma \tilde h_n e^{u_n}\, dV_{\tilde g_n}\right)^2 \int_\Sigma \tilde h^2 e^{2u}\left[\frac{\int_\Sigma \tilde h e^u\, dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n}\, dV_{\tilde g_n}}\frac{\tilde h_n}{\tilde h}e^{(u_n - u)} - 1\right]^2 dV_{\tilde g} \\ &\quad\leqslant \left(\int_\Sigma \tilde h_n e^{u_n}\, dV_{\tilde g_n}\right)^2 \left(\int_\Sigma \tilde h^4 e^{4u}\, dV_{\tilde g}\right)^{\frac12}\left(\int_\Sigma \left[\frac{\int_\Sigma \tilde h e^u\, dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n}\, dV_{\tilde g_n}}\frac{\tilde h_n}{\tilde h}e^{(u_n - u)} - 1\right]^4 dV_{\tilde g}\right)^{\frac12}.\end{aligned}$$

The first two terms are bounded according to the Moser–Trudinger inequality (2.3); let us consider the square of the third one and use the simple estimate $|e^x - 1| \leqslant |x|e^{|x|}$, the triangular inequality and Holder's inequality.

$$\int_{\Sigma}\big[e^{\log(\frac{\int_\Sigma \tilde h e^u\,dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n}\,dV_{\tilde g_n}}\frac{\tilde h_n}{\tilde h})+(u_n-u)}-1\big]^4\,dV_{\tilde g}$$

$$\leqslant \int_{\Sigma}\bigg[\bigg|\log\bigg(\frac{\int_\Sigma \tilde h e^u\,dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n}\,dV_{\tilde g_n}}\frac{\tilde h_n}{\tilde h}\bigg)+(u_n-u)\bigg|e^{|\log(\frac{\int_\Sigma \tilde h e^u\,dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n}\,dV_{\tilde g_n}}\frac{\tilde h_n}{\tilde h})+(u_n-u)|}\bigg]^4\,dV_{\tilde g}$$

$$\leqslant \int_{\Sigma}\bigg[\bigg|\log\bigg(\frac{\int_\Sigma \tilde h e^u\,dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n}\,dV_{\tilde g_n}}\frac{\tilde h_n}{\tilde h}\bigg)+(u_n-u)\bigg|^4\max\bigg\{\frac{\int_\Sigma \tilde h e^u\,dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n}\,dV_{\tilde g_n}}\frac{\tilde h_n}{\tilde h},\frac{\int_\Sigma \tilde h_n e^{u_n}\,dV_{\tilde g_n}}{\int_\Sigma \tilde h e^u\,dV_{\tilde g}}\frac{\tilde h}{\tilde h_n}\bigg\}^4 e^{4|u_n-u|}\bigg]\,dV_{\tilde g}$$

$$\leqslant \bigg(\int_{\Sigma}\bigg|\log\bigg(\frac{\int_\Sigma \tilde h e^u\,dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n}\,dV_{\tilde g_n}}\frac{\tilde h_n}{\tilde h}\bigg)+(u_n-u)\bigg|^{12}\,dV_{\tilde g}\bigg)^{\frac13}$$

$$\times\bigg(\int_{\Sigma}\max\bigg\{\frac{\int_\Sigma \tilde h e^u\,dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n}\,dV_{\tilde g_n}}\frac{\tilde h_n}{\tilde h},\frac{\int_\Sigma \tilde h_n e^{u_n}\,dV_{\tilde g_n}}{\int_\Sigma \tilde h e^u\,dV_{\tilde g}}\frac{\tilde h}{\tilde h_n}\bigg\}^{12}\,dV_{\tilde g}\bigg)^{\frac13}\bigg(\int_{\Sigma}e^{12|u_n-u|}\,dV_{\tilde g}\bigg)^{\frac13}.$$

Again the last two terms can be bounded using (2.3), while the cube of the first one can be controlled by

$$C\bigg[\int_{\Sigma}\bigg(\log\bigg(\frac{\int_\Sigma \tilde h e^u\,dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n}\,dV_{\tilde g_n}}\frac{\tilde h_n}{\tilde h}\bigg)\bigg)^{12}\,dV_{\tilde g}+\|u_n-u\|^{12}_{L^{12}_{\tilde g}}\bigg]$$

and this sequence converges to 0, as $n\to+\infty$, because $u_n\to u$ in $L^{12}_{\tilde g}(\Sigma)$ and

$$\bigg\|\frac{\int_\Sigma \tilde h e^u\,dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n}\,dV_{\tilde g_n}}\frac{\tilde h}{\tilde h_n}\bigg\|_\infty\to 1.$$

Indeed $\|\frac{\tilde h}{\tilde h_n}\|_\infty\to 1$ and

$$\begin{aligned}
\frac{\int_\Sigma \tilde h(e^u-e^{u_n})\,dV_{\tilde g}}{\int_\Sigma \tilde h_n e^{u_n}\,dV_{\tilde g_n}} &\leqslant \frac{\int_\Sigma \tilde h e^u(1-e^{(u_n-u)})\,dV_{\tilde g}}{C\int_\Sigma \tilde h_n e^{u_n}\,dV_{\tilde g}}\\
&\leqslant C\frac{(\int_\Sigma \tilde h^2 e^{2u}\,dV_{\tilde g})^{\frac12}}{\int_\Sigma \tilde h_n e^{u_n}\,dV_{\tilde g}}\bigg(\int_{\Sigma}\big(1-e^{(u_n-u)}\big)^2\,dV_{\tilde g}\bigg)^{\frac12}\\
&\leqslant C\bigg(\int_{\Sigma}|u_n-u|^2e^{2|u_n-u|}\,dV_{\tilde g}\bigg)^{\frac12}\\
&\leqslant C\|u_n-u\|_{L^4_{\tilde g}}\big\|e^{(u_n-u)}\big\|_{L^4_{\tilde g}}\to 0,
\end{aligned}$$

where we used one more time the Holder's inequality, the estimate $|e^x-1|\leqslant |x|e^{|x|}$, (2.3) and the fact that $u_n\to u$ in $L^4_{\tilde g}(\Sigma)$. $\square$

Lemma A.4. *For any $(g_0, h_0, u_0) \in \mathcal{G}_\delta \times \mathcal{H}_\delta \times \mathcal{B}$ it holds that if $w \in \mathrm{Ker}(\tilde{F}_{g_0})'_w(h_0, S_{g_0}(u_0)) \subset \bar{H}^1_{\tilde{g}}(\Sigma)$ and*

$$\big(S_{g_0}\big(F'_{(g,h)}(g_0, h_0, u_0)[0, h]\big), w\big)_{H^1_{\tilde{g}}} = 0, \quad \forall h \in C^2(\Sigma)$$

then $w = 0$.

Proof. By hypothesis

$$\begin{aligned} 0 &= \big(S_{g_0}\big(F'_{(g,h)}(g_0, h_0, u_0)[0, h]\big), w\big)_{H^1_{\tilde{g}}} = \big((\tilde{F}_{g_0})'_h\big(h_0, S_{g_0}(u_0)\big)[h], w\big)_{H^1_{\tilde{g}}} \\ &= -\frac{\rho}{(\int_\Sigma \tilde{h}e^{u_0}\,dV_{\tilde{g}})^2}\bigg(\int_\Sigma he^{u_0}\bigg[w\int_\Sigma \tilde{h}e^{u_0}\,dV_{\tilde{g}} - \int_\Sigma \tilde{h}e^{u_0}w\,dV_{\tilde{g}}\bigg]dV_{\tilde{g}}\bigg) \end{aligned}$$

for any $h \in C^2(\Sigma)$. This implies that $w\int_\Sigma \tilde{h}e^{u_0}\,dV_{\tilde{g}} - \int_\Sigma \tilde{h}e^{u_0}w\,dV_{\tilde{g}} = 0$, that is $w \equiv \frac{\int_\Sigma \tilde{h}e^{u_0}w\,dV_{\tilde{g}}}{\int_\Sigma \tilde{h}e^{u_0}\,dV_{\tilde{g}}}$ is constant. Finally by the fact that $w \in \bar{H}^1_{\tilde{g}}(\Sigma)$ we deduce $w = 0$. $\square$

Lemma A.5. *For any $(g_0, h_0, u_0) \in \mathcal{G}_\delta \times \mathcal{H}_\delta \times \mathcal{B}$ such that $F(g_0, h_0, u_0) = 0$ and for any $b \in \bar{H}^1_{\tilde{g}}(\Sigma)$ there exists $(g_b, h_b, v_b) \in \mathcal{S}^2 \times C^2(\Sigma) \times \bar{H}^1_{\tilde{g}}(\Sigma)$ such that*

$$F'_{(g,h)}(g_0, h_0, u_0)[g_b, h_b] + F'_u(g_0, h_0, u_0)[v_b] = b.$$

Proof. Let us take $b \in \bar{H}^1_{\tilde{g}}(\Sigma)$. In the following we will use the notations $\tilde{g} := \bar{g} + g_0$ and $\tilde{h} := \bar{h} + h_0$.

Since by Remark A.2 the selfadjoint operator

$$\begin{aligned} w &\mapsto (\tilde{F}_{g_0})'_w\big(h_0, S_{g_0}(u_0)\big)[w] \\ &= w - A_{g_0}\bigg(\rho\frac{(\int_\Sigma \tilde{h}e^{u_0}\,dV_{\tilde{g}})\tilde{h}e^{u_0}w - (\int_\Sigma \tilde{h}e^{u_0}w\,dV_{\tilde{g}})\tilde{h}e^{u_0}}{(\int_\Sigma \tilde{h}e^{u_0})^2} + w\bigg) \end{aligned}$$

is Fredholm of index 0, the following decomposition holds

$$\mathrm{Im}(\tilde{F}_{g_0})'_w\big(h_0, S_{g_0}(u_0)\big) \oplus \mathrm{Ker}(\tilde{F}_{g_0})'_w\big(h_0, S_{g_0}(u_0)\big) = \bar{H}^1_{\tilde{g}}(\Sigma).$$

We will denote by P_{Im} and $\mathrm{P}_{\mathrm{Ker}}$ the orthogonal projections from $\bar{H}^1_{\tilde{g}}(\Sigma)$ onto $\mathrm{Im}(\tilde{F}_{g_0})'_w(h_0, S_{g_0}(u_0))$ and $\mathrm{Ker}(\tilde{F}_{g_0})'_w(h_0, S_{g_0}(u_0))$, respectively. According to these notations we can decompose b as follows

$$b = S^{-1}_{g_0}\big(\mathrm{P}_{\mathrm{Im}}\big(S_{g_0}(b)\big)\big) + S^{-1}_{g_0}\big(\mathrm{P}_{\mathrm{Ker}}\big(S_{g_0}(b)\big)\big).$$

Let us show first that there exists $h_b \in C^2(\Sigma)$ such that

$$\mathrm{P}_{\mathrm{Ker}}\big(S_{g_0}(b)\big) = \mathrm{P}_{\mathrm{Ker}}\big(S_{g_0}\big(F'_{(g,h)}(g_0, h_0, u_0)[0, h_b]\big)\big). \tag{A.4}$$

Let $\{w_1, \dots, w_\nu\}$ be a basis of $\mathrm{Ker}(\tilde{F}_{g_0})'_w(h_0, S_{g_0}(u_0))$ and let us consider the linear functionals $f_i : C^2(\Sigma) \to \mathbb{R}$ defined by

$$f_i(h) := \big(F'_{(g,h)}(g_0, h_0, u_0)[0, h], w_i\big)_{H^1_{\tilde{g}}}, \quad i = 1, \dots, \nu.$$

By Lemma A.4 it follows that the f_i's are independent; then there exist ν linearly independent functions $h_1, \dots, h_\nu$ in $C^2(\Sigma)$ such that $f_i(h_i) = 1$ for $i = 1, \dots, \nu$ and so we are able to find $h_b \in C^2(\Sigma)$ verifying (A.4).

At this point we have

$$\begin{aligned}
b &= S_{g_0}^{-1}\big(\mathrm{P}_{\mathrm{Ker}}\big(S_{g_0}(b)\big)\big) + S_{g_0}^{-1}\big(\mathrm{P}_{\mathrm{Im}}\big(S_{g_0}(b)\big)\big) \\
&= S_{g_0}^{-1}\big(\mathrm{P}_{\mathrm{Ker}}\big(S_{g_0}\big(F'_{(g,h)}(g_0, h_0, u_0)[0, h_b]\big)\big)\big) + S_{g_0}^{-1}\big(\mathrm{P}_{\mathrm{Im}}\big(S_{g_0}(b)\big)\big) \\
&= S_{g_0}^{-1}\big(S_{g_0}\big(F'_{(g,h)}(g_0, h_0, u_0)[0, h_b]\big)\big) \\
&\quad + S_{g_0}^{-1}\big(\mathrm{P}_{\mathrm{Im}}\big(-S_{g_0}\big(F'_{(g,h)}(g_0, h_0, u_0)[0, h_b]\big) + S_{g_0}(b)\big)\big) \\
&= F'_{(g,h)}(g_0, h_0, u_0)[0, h_b] + S_{g_0}^{-1}\big(\mathrm{P}_{\mathrm{Im}}\big(S_{g_0}\big(-F'_{(g,h)}(g_0, h_0, u_0)[0, h_b] + b\big)\big)\big).
\end{aligned}$$

Now, since by definition $\mathrm{P}_{\mathrm{Im}}(S_{g_0}(-F'_{(g,h)}(g_0, h_0, u_0)[0, h_b] + b)) \in \mathrm{Im}(\tilde{F}_{g_0})'_w(h_0, S_{g_0}(u_0))$ it is clearly possible to find $w_b \in \bar{H}^1_{\tilde{g}}(\Sigma)$ such that $\mathrm{P}_{\mathrm{Im}}(S_{g_0}(-F'_{(g,h)}(g_0, h_0, u_0)[0, h_b] + b)) = (\tilde{F}_{g_0})'_w(h_0, S_{g_0}(u_0))[w_b]$.

Finally if we set $v_b := S_{g_0}^{-1}(w_b)$ we have

$$\begin{aligned}
S_{g_0}^{-1}\big(\mathrm{P}_{\mathrm{Im}}\big(S_{g_0}\big(-F'_{(g,h)}(g_0, h_0, u_0)[0, h_b] + b\big)\big)\big) &= S_{g_0}^{-1}\big((\tilde{F}_{g_0})'_w\big(h_0, S_{g_0}(u_0)\big)[w_b]\big) \\
&= F'_u(g_0, h_0, u_0)[v_b].
\end{aligned}$$

Therefore, taking $g_b = 0$, we get $b = F'_{(g,h)}(g_0, h_0, u_0)[g_b, h_b] + F'_u(g_0, h_0, u_0)[v_b]$.

The proof is thereby complete. □

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