Double resonance with Landesman-Lazer conditions for planar systems of ordinary differential equations

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Dedicated to Alan Lazer

Abstract

We prove the existence of periodic solutions for first order planar systems at resonance. The nonlinearity is indeed allowed to interact with two positively homogeneous Hamiltonians, both at resonance, and some kind of Landesman-Lazer conditions are assumed at both sides. We are thus able to obtain, as particular cases, the existence results proposed in the pioneering papers [27] by Lazer and Leach, and [18] by Frederickson and Lazer. Our theorem also applies in the case of asymptotically piecewise linear systems, and in particular generalizes Fabry's results in [10], for scalar equations with double resonance with respect to the Dancer-Fučik spectrum.

1 Introduction

In 1969, Lazer and Leach studied in [27] the periodic problem

$$\begin{cases} \ddot{x} + \lambda_N x + h(x) = e(t) \\ x(0) = x(T),\ \dot{x}(0) = \dot{x}(T), \end{cases}$$

where h is continuous and *bounded*, and $\lambda_N = (\frac{2\pi N}{T})^2$ for some positive integer N. In this setting, they proved that a sufficient condition for the existence of a solution is

$$\frac{2}{\pi}\Big(\liminf_{x\to+\infty} h(x) - \limsup_{x\to-\infty} h(x)\Big) > \sqrt{a_N^2 + b_N^2}\,,$$

where

$$a_N = \frac{2}{T}\int_0^T e(s)\cos\left(\frac{2\pi N}{T}s\right)\,ds \quad \text{and} \quad b_N = \frac{2}{T}\int_0^T e(s)\sin\left(\frac{2\pi N}{T}s\right)\,ds \qquad (1)$$

are the Fourier coefficients of the forcing term $e(t)$. Since λ_N is an eigenvalue of the differential operator, this situation is sometimes referred to as *nonlinear resonance*.

For the more general problem

$$\begin{cases} \ddot{x} + g(t,x) = 0 \\ x(0) = x(T),\ \dot{x}(0) = \dot{x}(T), \end{cases} \qquad (2)$$

where $g(t,x) = \lambda_N x + h(t,x)$, with h continuous and *bounded*, such a condition can be generalized as follows:

$$\int_{\{v>0\}} \liminf_{x\to+\infty} h(t,x)v(t)\,dt + \int_{\{v<0\}} \limsup_{x\to-\infty} h(t,x)v(t)\,dt > 0, \tag{3}$$

for every $v \neq 0$ which solves the homogeneous equation $\ddot{v} + \lambda_N v = 0$. Since 1970, when Landesman and Lazer introduced in [25] a similar condition for a Dirichlet problem associated to an elliptic operator, (3) has always been referred to as *Landesman-Lazer condition.* There have been generalizations in several directions, see for instance [1, 7, 8, 9, 10, 11, 12, 19, 21, 23, 28, 29].

In particular, Brezis and Nirenberg proposed, in [1], an abstract version of Landesman-Lazer results in a Hilbert space H. For a given $\mathcal{N} : H \to H$, they introduced the *recession function* $\mathcal{J}_\mathcal{N} : H \to \mathbb{R}$, defined by

$$\mathcal{J}_\mathcal{N}(z) = \liminf_{\substack{\lambda\to+\infty \\ w\to z}} (\mathcal{N}(\lambda w)|w), \tag{4}$$

where $(\cdot|\cdot)$ denotes the scalar product in H. They proved an existence result (cf. [1, Theorem III.1]) assuming that $\mathcal{J}_\mathcal{N}(v) > 0$ for every $v \neq 0$ belonging to the kernel of the linear operator appearing in their abstract equation. In the particular case of problem (2), with $g(t,x)$ as above, taking $H = L^2(0,T)$ and denoting by $\mathcal{N}$ the Nemytzkii operator associated to $h(t,x)$, they showed that

$$\mathcal{J}_\mathcal{N}(v) \geq \int_{\{v>0\}} \liminf_{x\to+\infty} h(t,x)v(t)\,dt + \int_{\{v<0\}} \limsup_{x\to-\infty} h(t,x)v(t)\,dt,$$

for every $v \neq 0$ satisfying $\ddot{v} + \lambda_N v = 0$, and they were able to recover the existence result in [25].

The boundedness assumption on $h(t,x)$ is not really necessary. It was already noticed, in the above quoted papers, that the function $g(t,x)$ can be asymptotically controlled by two lines having consecutive eigenvalues as slopes. For instance, in [11, 12] a "double resonance" situation has been considered, taking

$$g(t,x) = \gamma(t,x)x + r(t,x),$$

with $\lambda_N \leq \gamma(t,x) \leq \lambda_{N+1}$ and $r(t,x)$ bounded, and assuming that a Landesman-Lazer type condition holds with respect to both eigenvalues λ_N and λ_{N+1}. This situation has been further extended by Fabry in [10], where a double resonance situation was considered with respect to the Dancer-Fučik spectrum. He assumed that

$$g(t,x) = \gamma_1(t,x)x^+ - \gamma_2(t,x)x^- + r(t,x),$$

with γ_1 and γ_2 such that

$$a_+ \leq \gamma_1(t,x) \leq b_+, \qquad a_- \leq \gamma_2(t,x) \leq b_-,$$

and $r(t,x)$ bounded, being the points (a_-, a_+) and (b_-, b_+) on two consecutive curves of the Dancer-Fučik spectrum. The existence result was obtained by adding again Landesman-Lazer conditions on both sides.

In this paper, we want to generalize Fabry's result to the periodic problem associated to a more general planar system, like

$$\begin{cases} \dot{u} = F(t,u) \\ u(0) = u(T), \end{cases} \tag{5}$$

where F is controlled by two positively homogeneous functions, for which resonance occurs, with some kind of Landesman-Lazer conditions to be imposed at both sides. In order to do this, we assume that F has the following decomposition:

$$F(t,u) = -(1-\gamma(t,u))J\nabla H_1(u) - \gamma(t,u)J\nabla H_2(u) + r(t,u), \tag{6}$$

being J the standard 2×2 symplectic matrix, namely

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and $0 \leq \gamma(t,u) \leq 1$. Moreover, we assume $r(t,u)$ to be bounded by a L^2-function, and H_1, H_2 to be C^1-functions which are positively homogeneous of order 2 and positive, i.e.

$$0 < H_i(\lambda u) = \lambda^2 H_i(u), \text{ for every } u \neq 0 \text{ and } \lambda > 0,$$

for $i = 1, 2$. Hence, the origin is an isochronous center for the systems $J\dot{u} = \nabla H_i(u)$, $i = 1, 2$. If φ satisfies $J\dot{\varphi} = \nabla H_1(\varphi)$, and ψ satisfies $J\dot{\psi} = \nabla H_2(\psi)$, and they are nonzero, letting τ_φ and τ_ψ be their minimal periods, we suppose that there exists a positive integer N such that

$$\frac{T}{N+1} \leq \tau_\psi < \tau_\varphi \leq \frac{T}{N}. \tag{7}$$

When equalities hold in (7), this condition gives rise to double resonance.

It seems difficult to apply the Brezis-Nirenberg approach to this type of situation; however, we can consider some kind of recession function in $\mathbb{R}^2$ instead of H. More precisely, denoting by $\varphi_\omega(t)$ and $\psi_\omega(t)$ the functions $\varphi(t+\omega)$ and $\psi(t+\omega)$ respectively, by $\mathcal{N}_1$ the Nemytzkii operator associated to $JF - \nabla H_1$, and by $\mathcal{N}_2$ the Nemytzkii operator associated to $\nabla H_2 - JF$, we define

$$\tilde{\mathcal{J}}_1(t;\theta) = \liminf_{\substack{\lambda \to +\infty \\ \omega \to \theta}} \langle \mathcal{N}_1(\lambda\varphi_\omega)(t) | \varphi_\omega(t) \rangle, \tag{8}$$

and

$$\tilde{\mathcal{J}}_2(t;\theta) = \liminf_{\substack{\lambda \to +\infty \\ \omega \to \theta}} \langle \mathcal{N}_2(\lambda\psi_\omega)(t) | \psi_\omega(t) \rangle, \tag{9}$$

where $\langle \cdot | \cdot \rangle$ is the euclidean scalar product in $\mathbb{R}^2$. In order to generalize the Landesman-Lazer conditions at both sides, we thus assume

$$\int_0^T \tilde{\mathcal{J}}_1(t;\theta)\,dt > 0, \qquad \text{and} \qquad \int_0^T \tilde{\mathcal{J}}_2(t;\theta)\,dt > 0, \tag{10}$$

for every $\theta \in [0, T]$. In this setting, we are able to prove that problem (5) has a solution. We will show in Section 3 that our result generalizes Fabry's one. Indeed, for problem (2), assuming the classical Landesman-Lazer conditions at both sides implies (10).

Coming back to the scalar case, it is worth underlining that, under the hypothesis that $h(t, x)$ is bounded and *strictly increasing* in x, Lazer and Leach proved in [27] that condition (3) is indeed *necessary and sufficient* for the existence of a periodic solution. Hence, in this case, the Landesman-Lazer condition, the Brezis-Nirenberg condition and ours are all equivalent one with the other.

It is interesting to notice that in 1969, the same year of publication of the Lazer-Leach result, Frederickson and Lazer introduced in [18] a rather similar condition for second order equations of Liénard or Rayleigh type, where the nonlinearity depends on the derivative of the solution x. For instance, considering the Rayleigh periodic problem

$$\begin{cases} \ddot{x} + Q(\dot{x}) + x = e(t), \\ x(0) = x(T),\ \dot{x}(0) = \dot{x}(T), \end{cases} \tag{11}$$

with the assumption that $Q(x)$ is strictly increasing, and that

$$\frac{2}{\pi}\Big(\lim_{x \to +\infty} Q(x) - \lim_{x \to -\infty} Q(x) \Big) > \sqrt{a_N^2 + b_N^2}\,,$$

being a_N and b_N as in (1), they proved that (11) has a solution.

There is a qualitative difference between this situation and the one when the nonlinearity Q depends on x rather than $\dot{x}$ (see [2, 20, 22, 32], and the references therein). However, it is still possible to see some analogy between the result proved by Frederickson and Lazer and the one by Lazer and Leach. In order to understand this analogy, we introduce in Section 6 a planar system like

$$\begin{cases} \dot{u} = -\hat{\gamma}(t, u) J \nabla H(u) + r(t, u) \\ u(0) = u(T), \end{cases}$$

where H is positively homogeneous of order 2 and positive, $\alpha \le \hat{\gamma}(t, u) \le \beta$ for suitable positive constants α, β, and $r(t, u)$ is bounded by a L^2-function. This means that we are considering a system like (5), assuming that F has a decomposition like (6), but this time with H_1, H_2 being multiples of a single function H. In this setting, we are able to provide a condition which includes both the Landesman-Lazer and the Frederickson-Lazer ones. Again, since Frederickson and Lazer proved that their condition is also necessary when Q is strictly increasing, this turns to be another case of necessity of our condition.

The proofs of our results use degree theory, and the degree of the associated operator is proved to be equal to 1. In order to obtain the required a priori estimates, we exploit in several occasions the planar framework of our problem, so that some kind of polar coordinates can be used. We will show that the Landesman-Lazer condition is needed to control the angular component of the solutions, while the Frederickson-Lazer condition gives information on their radial component. In Sections 6 and 7, we will combine the information obtained from either of the two conditions, in order to generalize the above mentioned existence results.

We recall that, when $H_1 = H_2$, different types of conditions generalizing the ones in [18, 27] have been proposed in [4, 5, 13, 14, 16, 17]. The main point in these papers, however, is that the associated degree can also be an arbitrary negative number, and can sometimes take large positive values, as well. The possibility of obtaining this kind of results in the case of double resonance with two different Hamiltonians is still to be investigated.

2 Double resonance in the case of two Hamiltonian functions

We consider the problem

$$\text{(P)} \qquad \begin{cases} \dot{u} = F(t,u) \\ u(0) = u(T), \end{cases}$$

where $F : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ is a L^2-Carathéodory function, that is to say, F satisfies the following three conditions:

1. for every $u \in \mathbb{R}^2$, the function $t \mapsto F(t,u)$ is measurable;
2. for almost every $t \in [0,T]$, the function $u \mapsto F(t,u)$ is continuous;
3. for every $R > 0$, there exists $\eta_R \in L^2(0,T)$ such that, for almost every $t \in [0,T]$ and every $u \in \mathbb{R}^2$, with $|u| \leq R$,

$$|F(t,u)| \leq \eta_R(t).$$

Let us first recall some basilar facts about positively homogeneous Hamiltonian systems, referring to [15] for further details. First of all, if $H : \mathbb{R}^2 \to \mathbb{R}$ is a C^1-function which satisfies

$$0 < H(\lambda u) = \lambda^2 H(u), \qquad \text{for every } u \neq 0 \text{ and } \lambda > 0, \tag{12}$$

so that H is positively homogeneous of order 2, Euler's formula holds:

$$\langle \nabla H(u) | u \rangle = 2H(u), \qquad \text{for every } u \in \mathbb{R}^2.$$

Notice that this implies that the only equilibrium point for the autonomous Hamiltonian system $J\dot{u} = \nabla H(u)$ is $u = 0$. As a consequence, according to Corollary 1 in [31], for every $u_0 \in \mathbb{R}^2$ there is uniqueness for the Cauchy problem

$$\begin{cases} J\dot{u} = \nabla H(u) \\ u(0) = u_0, \end{cases}$$

even without assuming any Lipschitz continuity of the right-hand side. It can then be proved that the origin is an isochronous center for the autonomous system

$$J\dot{u} = \nabla H(u),$$

that is to say, all the solutions of this system are periodic with the same minimal period τ. Moreover, it can be seen that, if $\varphi \neq 0$ is a solution, every other solution has the form $u(t) = C\varphi(t+\omega)$, for suitable $C \geq 0$, $\omega \in [0, \tau[$.

Our aim is to consider $F(t,u)$ in some sense "lying between" two positively homogeneous Hamiltonian functions, say H_1 and H_2, which satisfy (12) and

$$H_1(u) \leq H_2(u), \quad \text{for every } u \in \mathbb{R}^2. \tag{13}$$

We will consider a situation where double resonance can occur. Let φ and ψ satisfy, respectively,

$$J\dot{\varphi} = \nabla H_1(\varphi), \quad \text{and} \quad J\dot{\psi} = \nabla H_2(\psi),$$

and let τ_φ, τ_ψ be their minimal periods. We will suppose that there exists a positive integer N such that

$$\frac{T}{N+1} \leq \tau_\psi < \tau_\varphi \leq \frac{T}{N}, \tag{14}$$

with possible equalities at both sides. Recall that $H_1(\varphi(t))$ and $H_2(\psi(t))$ are constant in t. In this setting, our statement is the following.

Theorem 2.1. *Assume that (14) holds. Moreover, suppose that*

1) *there exist L^2-Carathéodory functions $\gamma : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, with $0 \leq \gamma \leq 1$, and $r : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$F(t,u) = -(1-\gamma(t,u))J\nabla H_1(u) - \gamma(t,u)J\nabla H_2(u) + r(t,u), \tag{15}$$

   with r satisfying

$$|r(t,u)| \leq \eta(t), \tag{16}$$

   for a suitable $\eta \in L^2(0,T)$, for almost every $t \in \mathbb{R}$ and every $u \in \mathbb{R}^2$;

2) *for every $\theta \in [0,T]$, the following relations are satisfied:*

$$\int_0^T \liminf_{\substack{\lambda\to+\infty\\ \omega\to\theta}} \left[\langle JF(t,\lambda\varphi(t+\omega)) | \varphi(t+\omega)\rangle - 2\lambda H_1(\varphi(t))\right] dt > 0, \tag{17}$$

$$\int_0^T \liminf_{\substack{\lambda\to+\infty\\ \omega\to\theta}} \left[2\lambda H_2(\psi(t)) - \langle JF(t,\lambda\psi(t+\omega)) | \psi(t+\omega)\rangle\right] dt > 0. \tag{18}$$

Then problem (P) has a solution.

From now on, we will fix φ and ψ in such a way that

$$H_1(\varphi(t)) = H_2(\psi(t)) = \tfrac{1}{2}, \qquad \text{for every } t \in [0, T]. \tag{19}$$

This choice is not restrictive in view of the preceding remarks. Notice that the strict inequality $\tau_\psi < \tau_\varphi$ in (14) implicitly assumes that $H_1(\cos\theta, \sin\theta) < H_2(\cos\theta, \sin\theta)$ for some $\theta \in [0, 2\pi]$.

Before proving the theorem, we give the lemmas below.

Lemma 2.2. *Let $G : [0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ be a L^2-Carathéodory function such that*

$$|G(t, u)| \leq c(t)(1 + |u|),$$

for almost every $t \in [0, T]$, and every $u \in \mathbb{R}^2$, being $c(t)$ a suitable function in $L^1(0, T)$. Then, for every $R_0 > 0$ there exists $R_1 \geq R_0$ such that, if u satisfies

$$\dot{u} = G(t, u), \tag{20}$$

and $|u(\bar{t})| \leq R_0$ for some $\bar{t} \in [0, T]$, then $|u(t)| \leq R_1$ for every $t \in [0, T]$.

Proof. Fix $R_0 > 0$; we choose $R_1 > (R_0 + \|c\|_1)e^{\|c\|_1}$, and prove that this choice makes the statement true. Indeed, otherwise, by continuity there would exist $t_0, t_1 \in [0, T]$ such that $|u(t_0)| = R_0$, $|u(t_1)| = R_1$, and

$$R_0 < |u(t)| < R_1, \qquad \text{for every } t \in]t_0, t_1[$$

(possibly with $t_1 < t_0$). It is then possible to pass to polar coordinates (ρ, θ) in (20), obtaining

$$|\dot{\rho}(t)| = \left| \left\langle \dot{u}(t) \;\middle|\; \frac{u(t)}{|u(t)|} \right\rangle \right| \leq |G(t, u(t))| \leq c(t)(1 + \rho(t)),$$

for every $t \in [t_0, t_1]$. By Gronwall's lemma, then, the following estimate holds:

$$\rho(t) \leq (R_0 + \|c\|_1) \exp \left| \int_{t_0}^{t} c(s)\, ds \right| ,$$

for every $t \in [t_0, t_1]$. By our choice of R_1, this implies $\rho(t_1) < R_1$, hence a contradiction. $\square$

The property stated in the above lemma is sometimes referred to as the "elastic property": a quite laborious proof of it, in a more general context, can be found in [24] (proof of Theorem 6.5). As a counterpart of it, in the assumptions of the lemma, for every $R_2 > 0$ there exists $R_3 \geq R_2$ such that if $|u(\bar{t})| \geq R_3$ for some $\bar{t} \in [0, T]$, then $|u(t)| \geq R_2$ for every $t \in [0, T]$.

Lemma 2.3. *Assume that (14) is fulfilled. Then, if* $v \in H^1(0,T)$ *satisfies*

$$\begin{cases} J\dot{v} = \alpha(t)\nabla H_1(v) + (1-\alpha(t))\nabla H_2(v) \\ v(0) = v(T), \end{cases} \tag{21}$$

being $\alpha \in L^2(0,T)$, *with* $0 \le \alpha(t) \le 1$ *for almost every* $t \in [0,T]$, *then* v *solves either*

$$J\dot{v} = \nabla H_1(v),$$

or

$$J\dot{v} = \nabla H_2(v).$$

Proof. First of all, we observe that a nontrivial solution of (21) never reaches the origin. Indeed, if $v(t)$ solves (21) then also $sv(t)$ does, for every $s > 0$, thanks to the homogeneity of the right-hand side; moreover, since the right-hand side grows at most linearly in v, Lemma 2.2 holds. It follows that, if $v(\bar{t}) \neq 0$ for some $\bar{t} \in [0,T]$, then $v(t) \neq 0$ for every $t \in [0,T]$.
Consequently, the usual system of polar coordinates (ρ, θ) is well defined for system (21). Writing $v(t) = \rho(t)(\cos\theta(t), \sin\theta(t))$, by a standard computation (recall Euler's formula) we get

$$-\dot{\theta}(t) = 2\alpha(t)H_1(\cos\theta(t), \sin\theta(t)) + 2(1-\alpha(t))H_2(\cos\theta(t), \sin\theta(t)).$$

Being $0 \le \alpha(t) \le 1$, it follows that

$$-\frac{\dot{\theta}(t)}{2H_2(\cos\theta(t), \sin\theta(t))} \le 1 \le -\frac{\dot{\theta}(t)}{2H_1(\cos\theta(t), \sin\theta(t))}, \tag{22}$$

for almost every $t \in [0,T]$. Since v is T-periodic, it will perform an integer number of turns around the origin, say m. Recalling that

$$\int_0^{2\pi} \frac{d\theta}{2H_1(\cos\theta, \sin\theta)} = \tau_\varphi\,, \quad \int_0^{2\pi} \frac{d\theta}{2H_2(\cos\theta, \sin\theta)} = \tau_\psi\,,$$

integrating in (22) from 0 to T yields

$$m\tau_\psi \le T \le m\tau_\varphi,$$

from which, using (14),

$$N \le \frac{T}{\tau_\varphi} \le m \le \frac{T}{\tau_\psi} \le N+1.$$

Since m is integer, this gives a contradiction unless $m = N$ and $\tau_\varphi = T/N$ or $m = N+1$ and $\tau_\psi = T/(N+1)$.
Assume the first case. We pass to generalized polar coordinates in (21) by writing $v(t) = r(t)\varphi(t+\omega(t))$, and get the equations for $\dot{r}$ and $\dot{\omega}$:

$$\dot{r}(t) = -\,r(t)(1-\alpha(t))\langle\nabla H_2(\varphi(t+\omega(t)))|\dot{\varphi}(t+\omega(t))\rangle, \tag{23}$$

and

$$\dot{\omega}(t) = (1-\alpha(t))(2H_2(\varphi(t+\omega(t))) - 1). \tag{24}$$

Since $\omega(0) = \omega(T)$, integrating in (24) from 0 to T gives

$$0 = \int_0^T (1-\alpha(t))(2H_2(\varphi(t+\omega(t))) - 1)\, dt,$$

and since our hypotheses imply $(1-\alpha(t))(2H_2(\varphi(t+\omega(t))) - 1) \geq 0$ for almost every $t \in [0, T]$, it will be

$$(1-\alpha(t))(2H_2(\varphi(t+\omega(t))) - 1) = 0 \tag{25}$$

almost everywhere, that is to say, $\dot{\omega}(t) = 0$ for almost every $t \in [0, T]$. Thus, since $\omega(t)$ is absolutely continuous, there exists $\omega_0 \in \mathbb{R}$ such that $\omega(t) = \omega_0$ for every $t \in [0, T]$. Concerning (23), it follows that

$$\dot{r}(t) = -\, r(t)(1-\alpha(t))\langle \nabla H_2(\varphi(t+\omega_0)) | \dot{\varphi}(t+\omega_0)\rangle.$$

We want to prove that $\dot{r}(t) = 0$ for almost every $t \in [0, T]$. Indeed, if $t \in [0, T]$, (25) implies that either $\alpha(t) = 1$, or $H_2(\varphi(t+\omega_0)) = \frac{1}{2}$. If $\alpha(\bar{t}) = 1$, then $\dot{r}(\bar{t}) = 0$; on the other hand, if $\alpha(\bar{t}) < 1$, then $\bar{t}$ is a zero of the function $t \mapsto H_2(\varphi(t+\omega_0)) - H_1(\varphi(t+\omega_0))$, which is of class C^1 and nonnegative. Necessarily $\bar{t}$ is then a minimum of this function, and so

$$\frac{d}{dt} H_2(\varphi(t+\omega_0))\Big|_{t=\bar{t}} = \frac{d}{dt} H_1(\varphi(t+\omega_0))\Big|_{t=\bar{t}} = 0,$$

as H_1 is preserved along φ. It follows that $\langle \nabla H_2(\varphi(\bar{t}+\omega_0)) | \dot{\varphi}(\bar{t}+\omega_0)\rangle = 0$, so that $\dot{r}(\bar{t}) = 0$. Summing up, $\dot{r}(t) = 0$ for almost every $t \in [0, T]$, and, since $r(t)$ is absolutely continuous, this implies that $r(t)$ is constant; being $v(t) = R_0 \varphi(t+\omega_0)$ for some nonnegative constant R_0, it follows that v is a solution of

$$J\dot{v} = \nabla H_1(v).$$

The other case can be proved similarly. □

Remark 2.4. We notice that, if (21) has a nontrivial solution, it is not possible to say that $\alpha(t) = 0$ or $\alpha(t) = 1$ almost everywhere: this is a priori true only if $H_1(\cos\theta, \sin\theta) < H_2(\cos\theta, \sin\theta)$ for every $\theta \in [0, 2\pi]$. For instance, if $H_1(x, y) = \frac{1}{2}((x^+)^2 + a_-(x^-)^2 + y^2)$ and $H_2(x, y) = \frac{1}{2}((x^+)^2 + b_-(x^-)^2 + y^2)$, with $0 < a_- < b_-$, then $\alpha(t)$ does not affect the orbit of the solutions in the half-plane $\{x > 0\}$.

We are now ready to give the proof of our main theorem.

Proof of Theorem 2.1. The proof will consist in carrying out a continuation argument by means of performing a suitable homotopy. Consider the family of problems, parametrized by $\sigma \in [0, 1]$,

$$\begin{cases} \dot{u} = \sigma F(t, u) - \frac{1-\sigma}{2}\left(J\nabla H_1(u) + J\nabla H_2(u)\right) \\ u(0) = u(T). \end{cases} \tag{26}$$

In view of [3, Theorem 2], it will be sufficient to prove that the solutions of (26) are a priori L^∞-bounded (the bound not depending on the homotopy parameter σ), since, by [24, Lemma II.6.5],

$$\deg_B(\tfrac{1}{2}(J\nabla H_1 + J\nabla H_2), \Omega) = \deg_B(\nabla H_1 + \nabla H_2, \Omega) = 1,$$

for every bounded open subset Ω of $\mathbb{R}^2$ containing 0. Thus, by contradiction we assume that, for every $n \in \mathbb{N}$, there exist $\sigma_n \in [0,1]$, $u_n \in H^1(0,T)$ such that

$$\begin{cases} \dot u_n = \sigma_n F(t,u_n) - \frac{1-\sigma_n}{2}\left(J\nabla H_1(u_n) + J\nabla H_2(u_n)\right) \\ u_n(0) = u_n(T), \end{cases} \tag{27}$$

and $\|u_n\|_\infty \to +\infty$. We can assume $\sigma_n \to \bar\sigma \in [0,1]$; thanks to hypothesis (15), by setting $v_n = \frac{u_n}{\|u_n\|_\infty}$, (27) is equivalent to

$$\begin{cases} J\dot v_n = \left(\frac{1+\sigma_n}{2} - \sigma_n\gamma(t,u_n)\right)\nabla H_1(v_n) + \left(\frac{1-\sigma_n}{2} + \sigma_n\gamma(t,u_n)\right)\nabla H_2(v_n) + \sigma_n J\frac{r(t,u_n)}{\|u_n\|_\infty} \\ v_n(0) = v_n(T). \end{cases} \tag{28}$$

Since $(v_n)_n$ is bounded in $L^2(0,T)$, (28) implies that $(v_n)_n$ is bounded in $H^1(0,T)$ and so there exists a T-periodic function $v \in H^1(0,T)$ such that (up to subsequences) $v_n \to v$ uniformly and $v_n \rightharpoonup v$ weakly in $H^1(0,T)$; being $\|v_n\|_\infty = 1$ for every n, it is $v \neq 0$. Moreover, the sequence $(\gamma(\cdot,u_n(\cdot)))_n$ is bounded in $L^2(0,T)$, so (extracting a new subsequence) it weakly converges to a function $\Gamma \in L^2(0,T)$; as $\{w \in L^2(0,T) \mid 0 \le w(t) \le 1 \text{ for almost every } t \in [0,T]\}$ is a convex and closed subset of $L^2(0,T)$, it is weakly closed and this implies $0 \le \Gamma(t) \le 1$ for almost every $t \in [0,T]$. Passing to the weak limit in (28), noticing that the last term vanishes thanks to the L^2-boundedness of $r(t,u)$, we then get

$$\begin{cases} J\dot v = \left(\frac{1+\bar\sigma}{2} - \bar\sigma\Gamma(t)\right)\nabla H_1(v) + \left(\frac{1-\bar\sigma}{2} + \bar\sigma\Gamma(t)\right)\nabla H_2(v) \\ v(0) = v(T). \end{cases} \tag{29}$$

Notice that this excludes the case $\bar\sigma = 0$, since in this case v (which is nonzero) would be a solution of the periodic problem

$$\begin{cases} J\dot v = \frac{1}{2}(\nabla H_1(v) + \nabla H_2(v)) \\ v(0) = v(T), \end{cases}$$

which has only the trivial solution. Being the right-hand side of the differential equation in (29) a convex combination of $\nabla H_1(v)$ and $\nabla H_2(v)$ (recall that $0 \le \Gamma(t) \le 1$ for almost every $t \in [0,T]$), we can now use Lemma 2.3 to infer that v solves either

$$J\dot v = \nabla H_1(v),$$

or

$$J\dot v = \nabla H_2(v).$$

Let us assume this last case (the other being similar): for suitable $R_0 > 0$, $\omega_0 \in [0, \tau_\psi[$, it will be $v(t) = R_0\psi(t+\omega_0)$. Writing, in generalized polar coordinates, $u_n(t) = r_n(t)\psi(t+\omega_n(t))$, with $w_n(0) \in [0, \tau_\psi[$ for every n, (27) gives

$$\begin{aligned} \dot{\omega}_n(t) = \sigma_n \frac{\langle JF(t, r_n(t)\psi(t+\omega_n(t))) | \psi(t+\omega_n(t))\rangle}{r_n(t)} + \\ +(1-\sigma_n)(H_1(\psi(t+\omega_n(t))) + H_2(\psi(t+\omega_n(t)))) - 1. \end{aligned} \tag{30}$$

Since v performs $N+1$ turns around the origin in the time T, and the sequence of T-periodic functions v_n tends to v uniformly, for n sufficiently large, every v_n performs $N+1$ turns around the origin, and so every u_n, since $u_n = \|u_n\|_\infty v_n$. As a consequence, for such n it is $\omega_n(0) = \omega_n(T)$, thus integrating in (30) gives 0. Using (13) and (19), it follows that

$$0 \geq \int_0^T \sigma_n \frac{r_n(t) - \langle JF(t, r_n(t)\psi(t+\omega_n(t))) | \psi(t+\omega_n(t))\rangle}{r_n(t)} \, dt,$$

from which we obtain, for n large (being $\bar{\sigma} \neq 0$),

$$0 \geq \int_0^T \frac{r_n(t) - \langle JF(t, r_n(t)\psi(t+\omega_n(t))) | \psi(t+\omega_n(t))\rangle}{r_n^V(t)} \, dt, \tag{31}$$

where $r_n^V(t) = r_n(t)/\|u_n\|_\infty$. Hypotheses (15) and (16) now allow us to apply Fatou's lemma, which gives

$$0 \geq \int_0^T \liminf_{n\to+\infty} \frac{r_n(t) - \langle JF(t, r_n(t)\psi(t+\omega_n(t))) | \psi(t+\omega_n(t))\rangle}{r_n^V(t)} \, dt;$$

using standard properties of the inferior limit, taking into account that, since $v_n \to v$ uniformly, also $r_n^V \to R_0$ uniformly, this yields

$$0 \geq \int_0^T \liminf_{n\to+\infty} [r_n(t) - \langle JF(t, r_n(t)\psi(t+\omega_n(t))) | \psi(t+\omega_n(t))\rangle] \, dt.$$

Moreover, using again the fact that $v_n \to v$ uniformly, we can assume without loss of generality that $\omega_n(t) \to \omega_0$ uniformly, passing, if necessary, to a further subsequence. Thus, recalling (19), for every fixed $t \in [0, T]$ we are computing the inferior limit which appears in (18) along the particular subsequence $(r_n(t), \omega_n(t))$, for which $\omega_n(t) \to \omega_0$ and $r_n(t) = \|u_n\|_\infty r_n^V(t) \to +\infty$. We deduce that

$$0 \geq \int_0^T \liminf_{\substack{\lambda\to+\infty \\ \omega\to\omega_0}} [\lambda - \langle JF(t, \lambda\psi(t+\omega)) | \psi(t+\omega)\rangle] \, dt,$$

which contradicts (18). $\square$

Remark 2.5. In the previous proof, we have been able to apply Fatou's lemma thanks to (15) and (16), which guarantee that

$$\langle JF(t,\lambda w)|w\rangle - 2\lambda H_1(w) \geq -\eta(t), \tag{32}$$

and

$$2\lambda H_2(w) - \langle JF(t,\lambda w)|w\rangle \geq -\eta(t), \tag{33}$$

for almost every $t \in [0,T]$, every $w \in \mathbb{R}^2$ with $|w| \leq 1$, and $\lambda \geq 1$, being $\eta \in L^2(0,T)$. If we replace the assumption that $r(t,u)$ is L^2-bounded with the following condition of sublinearity:

- for every $\epsilon > 0$, there exists $\eta_\epsilon \in L^2(0,T)$ such that, for almost every $t \in [0,T]$ and every $u \in \mathbb{R}^2$,

$$|r(t,u)| \leq \epsilon|u| + \eta_\epsilon(t),$$

then the statement still holds true, provided that (32) and (33) are assumed as hypotheses.

Remark 2.6. It will often be useful, in the sequel, to write assumption (18) as

$$\int_0^T \limsup_{\substack{\lambda\to+\infty \\ \omega\to\theta}} \left[\langle JF(t,\lambda\psi(t+\omega))|\psi(t+\omega)\rangle - 2\lambda H_2(\psi(t))\right] dt < 0. \tag{34}$$

The integrals appearing in (17) and (34) depend on $\theta \in [0,T]$, and in the sequel they will often be denoted by $\Gamma_1^-(\theta)$ and $\Gamma_1^+(\theta)$, respectively.

The following corollary is a straightforward consequence of Theorem 2.1. We will denote respectively by $\varphi_\omega(t)$ and $\psi_\omega(t)$ the functions $\varphi(t+\omega)$ and $\psi(t+\omega)$.

Corollary 2.7. *Assume that (14), (15) and (16) hold. Moreover, assume that*

$$\int_0^T \liminf_{\substack{\lambda\to+\infty \\ \omega\to\theta}} \langle Jr(t,\lambda\varphi_\omega(t))|\varphi_\omega(t)\rangle\, dt > 0, \tag{35}$$

and

$$\int_0^T \limsup_{\substack{\lambda\to+\infty \\ \omega\to\theta}} \langle Jr(t,\lambda\psi_\omega(t))|\psi_\omega(t)\rangle\, dt < 0, \tag{36}$$

for every $\theta \in [0,T]$. Then, problem (P) has a solution.

Proof. Being $H_1 \leq H_2$, it holds

$$\langle JF(t,\lambda w) - Jr(t,\lambda w)|w\rangle - 2\lambda H_1(w) \geq 0,$$

$$2\lambda H_2(w) - \langle JF(t,\lambda w) - Jr(t,\lambda w)|w\rangle \geq 0$$

(recall Euler's formula). Consequently, (35) and (36) imply (17) and (18). □

The corollary can be useful in the applications: from a practical point of view, indeed, we can first check if the part which has lower order satisfies the hypotheses of the theorem.

We conclude this section with two remarks, which link our theorem respectively with the results proved by Brezis and Nirenberg in [1] and with a typical tool in Calculus of Variations, the Γ-limit of a sequence of functions.

Remark 2.8. Introducing the two functions $\tilde{\mathcal{J}}_1, \tilde{\mathcal{J}}_2 : [0,T] \times [0,T] \to \mathbb{R}$ as in (8), (9), and recalling that $H_1(\varphi(t))$ and $H_2(\psi(t))$ are constant in t, we can write in an equivalent way conditions (17) and (18):

$$\int_0^T \tilde{\mathcal{J}}_1(t;\theta)\,dt > 0, \quad \text{and} \quad \int_0^T \tilde{\mathcal{J}}_2(t;\theta)\,dt > 0,$$

for every $\theta \in [0,T]$. Writing explicitly,

$$\tilde{\mathcal{J}}_1(t;\theta) = \liminf_{\substack{\lambda\to+\infty \\ \omega\to\theta}} [\langle JF(t,\lambda\varphi(t+\omega))|\varphi(t+\omega)\rangle - 2\lambda H_1(\varphi(t+\omega))],$$

and

$$\tilde{\mathcal{J}}_2(t;\theta) = \liminf_{\substack{\lambda\to+\infty \\ \omega\to\theta}} [2\lambda H_2(\psi(t+\omega)) - \langle JF(t,\lambda\psi(t+\omega))|\psi(t+\omega)\rangle].$$

It is possible to see some kind of relation between our conditions and the ones introduced by Brezis and Nirenberg in [1]. As we have already recalled in the introduction, they defined, in the abstract setting of Hilbert spaces, the concept of *recession function,* according to (4). In this particular case, the recession functions would be

$$\mathcal{J}_{\mathcal{N}_1}(\theta) = \liminf_{\substack{\lambda\to+\infty \\ w\to z}} \int_0^T [\langle JF(t,\lambda w(t))|w(t)\rangle - 2\lambda H_1(w(t))]\,dt,$$

and

$$\mathcal{J}_{\mathcal{N}_2}(\theta) = \liminf_{\substack{\lambda\to+\infty \\ w\to z}} \int_0^T [2\lambda H_2(w(t)) - \langle JF(t,\lambda w(t))|w(t)\rangle]\,dt,$$

where $w \to z$ in $L^2([0,T];\mathbb{R}^2)$. In some sense, the functions $\tilde{\mathcal{J}}_1(t;\theta)$ and $\tilde{\mathcal{J}}_2(t;\theta)$ can be thought as particular recession functions in $\mathbb{R}^2$ instead of L^2, and depending on t (and thus still to be integrated in order to write a Landesman-Lazer type condition). From our point of view, this approach gives the advantage of providing conditions which are easier to handle.

Remark 2.9. Conditions (17) and (18) can also be written in terms of the Γ-liminf of the generalized sequences of functions

$$L^t_\lambda(\omega) = \langle JF(t,\lambda\varphi(t+\omega))|\varphi(t+\omega)\rangle - 2\lambda H_1(\varphi(t)),$$

and

$$U^t_\lambda(\omega) = 2\lambda H_2(\psi(t)) - \langle JF(t,\lambda\psi(t+\omega))|\psi(t+\omega)\rangle$$

(at t fixed), which are defined, as usual, by

$$(\Gamma\text{-}\liminf_{\lambda\to+\infty} L^t_\lambda)(\theta) = \sup_{V\in\mathcal{I}_\theta} \liminf_{\lambda\to+\infty} \inf_{\omega\in V} L^t_\lambda(\omega),$$

and

$$(\Gamma\text{-}\liminf_{\lambda\to+\infty} U^t_\lambda)(\theta) = \sup_{V\in\mathcal{I}_\theta} \liminf_{\lambda\to+\infty} \inf_{\omega\in V} U^t_\lambda(\omega),$$

being $\mathcal{I}_\theta$ the filter of neighbourhoods of $\theta \in [0,T]$. Indeed, for every $t,\theta \in [0,T]$ it is known that the following equalities hold:

$$\liminf_{\substack{\lambda\to+\infty\\ \omega\to\theta}} [\langle JF(t,\lambda\varphi(t+\omega))|\varphi(t+\omega)\rangle - 2\lambda H_1(\varphi(t))] = (\Gamma\text{-}\liminf_{\lambda\to+\infty} L^t_\lambda)(\theta), \tag{37}$$

and

$$\liminf_{\substack{\lambda\to+\infty\\ \omega\to\theta}} [2\lambda H_2(\psi(t)) - \langle JF(t,\lambda\psi(t+\omega))|\psi(t+\omega)\rangle] = (\Gamma\text{-}\liminf_{\lambda\to+\infty} U^t_\lambda)(\theta). \tag{38}$$

For the reader's convenience, let us recall the proof of the first one, the computations for (38) being the same. It is

$$\begin{aligned}(\Gamma\text{-}\liminf_{\lambda\to+\infty} L^t_\lambda)(\theta) &= \sup_{\delta>0}\sup_{\gamma>0}\inf_{\lambda\geq\gamma}\inf_{|\omega-\theta|\leq\delta} L^t_\lambda(\omega)\\ &= \sup_{\delta,\gamma>0}\inf_{\substack{\lambda\in[\gamma,+\infty)\\ \omega\in[\theta-\delta,\theta+\delta]}} [\langle JF(t,\lambda\varphi(t+\omega))|\varphi(t+\omega)\rangle - 2\lambda H_1(\varphi(t))].\end{aligned}$$

Since a fundamental system of neighbourhoods for the ordered pair $(+\infty,\theta)$ is given by the family $\{[\gamma,+\infty)\times[\theta-\delta,\theta+\delta]\}_{\delta,\gamma>0}$, and taking the inferior limit over a fundamental system of neighbourhoods does not change its value, it follows that

$$(\Gamma\text{-}\liminf_{\lambda\to+\infty} L^t_\lambda)(\theta) = \liminf_{\substack{\lambda\to+\infty\\ \omega\to\theta}} L^t_\lambda(\omega),$$

and (37) is proved.
Consequently, (17) and (18) can be written in the following equivalent way:

$$\int_0^T \Gamma\text{-}\liminf_{\lambda\to+\infty} L^t_\lambda(\theta)\,dt > 0, \quad \text{and} \quad \int_0^T \Gamma\text{-}\liminf_{\lambda\to+\infty} U^t_\lambda(\theta)\,dt > 0.$$

3 Scalar second order equations without damping

We now want to consider the scalar case, namely we will focus on the problem

$$\begin{cases} \ddot{x} + g(t,x) = 0\\ x(0) = x(T),\ \dot{x}(0) = \dot{x}(T), \end{cases} \tag{39}$$

where $g : [0,T]\times\mathbb{R}\to\mathbb{R}$ is a L^2-Carathéodory function. To begin with, assume

$$g(t,x) = \mu x^+ - \nu x^- + f(t,x),$$

where μ and ν are positive constants such that the pair (μ,ν) belongs to the Dancer-Fučik spectrum (see [6, 19]) of the T-periodic problem. The equation can then be written as

$$\begin{cases} \dot{x} = y\\ \dot{y} = -\mu x^+ + \nu x^- - f(t,x). \end{cases}$$

Setting $u = (x, y)$, we define $\mathcal{F}(t, u) = \begin{pmatrix} 0 \\ -f(t,x) \end{pmatrix}$, and $H(x, y) = \frac{1}{2}(\mu(x^+)^2 + \nu(x^-)^2 + y^2)$. We will assume the following hypothesis on f:

(LL_1) For every $v \neq 0$ satisfying the homogeneous equation

$$\ddot{x} + \mu x^+ - \nu x^- = 0, \tag{40}$$

the following inequality holds:

$$\int_{\{v>0\}} \liminf_{x\to+\infty} f(t,x)v(t)\,dt + \int_{\{v<0\}} \limsup_{x\to-\infty} f(t,x)v(t)\,dt > 0.$$

Recall that, if $v(t)$ solves (40), then also $Cv(t + \theta)$ does, for every $C \geq 0$ and $\theta \in [0, T[$.

Proposition 3.1. *Assume hypothesis (LL_1). Then, for every $\theta \in [0, T]$, the following relation is satisfied:*

$$\int_0^T \liminf_{\substack{\lambda\to+\infty \\ \omega\to\theta}} \langle J\mathcal{F}(t, \lambda\varphi_\omega(t)) | \varphi_\omega(t)\rangle\,dt > 0,$$

being $\varphi \neq 0$ such that

$$J\dot{\varphi} = \nabla H(\varphi). \tag{41}$$

Proof. In view of the particular structure of (41), we can write $\varphi = (v, \dot{v})$, for a suitable v satisfying (40). Let $\theta \in [0, T]$ be fixed; setting $v_\theta(t) = v(t + \theta)$, we can write

$$[0, T] = \{v_\theta > 0\} \cup \{v_\theta < 0\} \cup N_0,$$

where, as it is well known, the Lebesgue measure of $N_0 = \{v_\theta = 0\}$ is equal to 0 (N_0 is made up by a finite number of points, as it can be easily seen by computing explicitly v_θ). Let us fix $t \in \{v_\theta > 0\}$ and consider

$$\liminf_{\substack{\lambda\to+\infty \\ \omega\to\theta}} \langle J\mathcal{F}(t, \lambda\varphi_\omega(t)) | \varphi_\omega(t)\rangle = \liminf_{\substack{\lambda\to+\infty \\ \omega\to\theta}} f(t, \lambda v(t+\omega))v(t+\omega).$$

Since $\lim_{\omega\to\theta} v(t+\omega) = v(t+\theta) > 0$, we have, by standard properties of the inferior limits,

$$\liminf_{\substack{\lambda\to+\infty \\ \omega\to\theta}} f(t, \lambda v(t+\omega))v(t+\omega) \geq \liminf_{x\to+\infty} f(t,x)\,v(t+\theta).$$

Fix now $t \in \{v_\theta < 0\}$; noticing that, for ω close to θ, the sign of v_ω will now be negative, a similar argument yields

$$\liminf_{\substack{\lambda\to+\infty \\ \omega\to\theta}} f(t, \lambda v(t+\omega))v(t+\omega) \geq \limsup_{x\to-\infty} f(t,x)\,v(t+\theta).$$

So,

$$\int_{\{v_\theta>0\}} \liminf_{\substack{\lambda\to+\infty\\ \omega\to\theta}} f(t,\lambda v(t+\omega))v(t+\omega)\,dt \geq \int_{\{v_\theta>0\}} \liminf_{x\to+\infty} f(t,x)\,v(t+\theta)\,dt,$$

and

$$\int_{\{v_\theta<0\}} \liminf_{\substack{\lambda\to+\infty\\ \omega\to\theta}} f(t,\lambda v(t+\omega))v(t+\omega)\,dt \geq \int_{\{v_\theta<0\}} \limsup_{x\to-\infty} f(t,x)\,v(t+\theta)\,dt.$$

By assumption (LL_1), we immediately get

$$\int_0^T \liminf_{\substack{\lambda\to+\infty\\ \omega\to\theta}} f(t,\lambda v(t+\omega))v(t+\omega)\,dt > 0,$$

and the assertion is proved. □

As a counterpart, consider the following assumption on f:

(LL_2) For every $v \neq 0$ satisfying the homogeneous equation (40), the following inequality holds:

$$\int_{\{v>0\}} \limsup_{x\to+\infty} f(t,x)v(t)\,dt + \int_{\{v<0\}} \liminf_{x\to-\infty} f(t,x)v(t)\,dt < 0.$$

The following proposition can be proved in the same way as Proposition 3.1.

Proposition 3.2. *Assume hypothesis (LL_2). Then, for every $\theta \in [0,T]$, the following relation is satisfied:*

$$\int_0^T \limsup_{\substack{\lambda\to+\infty\\ \omega\to\theta}} \langle J\mathcal{F}(t,\lambda\psi_\omega(t))|\psi_\omega(t)\rangle\,dt < 0,$$

being $\psi \neq 0$ such that

$$J\dot{\psi} = \nabla H(\psi).$$

We are now ready to show that Theorem 2.1 includes the main result proved by Fabry in [10].

Corollary 3.3 (Fabry 1995)**.** *Let $g : [0,T]\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function such that the following conditions hold:*

$$a_+x - \eta(t) \leq g(t,x) \leq b_+x + \eta(t) \quad \textit{for every } x \geq 0, \textit{ and a.e. } t\in[0,T], \tag{42}$$

$$b_-x - \eta(t) \leq g(t,x) \leq a_-x + \eta(t) \quad \textit{for every } x \leq 0, \textit{ and a.e. } t\in[0,T], \tag{43}$$

being a_-, a_+, b_-, b_+ positive numbers such that

$$\frac{1}{\sqrt{a_+}} + \frac{1}{\sqrt{a_-}} = \frac{T}{N\pi}\,, \tag{44}$$

$$\frac{1}{\sqrt{b_+}} + \frac{1}{\sqrt{b_-}} = \frac{T}{(N+1)\pi}\,, \tag{45}$$

for some positive integer N, and $\eta \in L^2(0,T)$. Moreover, assume that for every nontrivial solutions ϕ, ξ of

$$\ddot{\phi} + a_+\phi^+ - a_-\phi^- = 0, \quad \textit{and} \quad \ddot{\xi} + b_+\xi^+ - b_-\xi^- = 0,$$

respectively, the following conditions are satisfied:

$$\int_{\{\phi>0\}} [\liminf_{x\to+\infty}(g(t,x) - a_+x)]\,\phi(t)\,dt + \int_{\{\phi<0\}} [\limsup_{x\to-\infty}(g(t,x) - a_-x)]\,\phi(t)\,dt > 0,$$

and

$$\int_{\{\xi>0\}} [\limsup_{x\to+\infty}(g(t,x) - b_+x)]\,\xi(t)\,dt + \int_{\{\xi<0\}} [\liminf_{x\to-\infty}(g(t,x) - b_-x)]\,\xi(t)\,dt < 0.$$

Then problem (39) has a solution.

Proof. It can be shown (see e.g. [10, Lemma 1]) that, under conditions (42) and (43), one can write

$$g(t,x) = \gamma_1(t,x)x^+ - \gamma_2(t,x)x^- + h(t,x), \tag{46}$$

where $h(t,x)$ is L^2-bounded and

$$a_+ \le \gamma_1(t,x) \le b_+, \quad a_- \le \gamma_2(t,x) \le b_-, \tag{47}$$

for almost every $t \in [0,T]$, and every $x \in \mathbb{R}$. Defining

$$H_1(u) = \tfrac{1}{2}(a_+(x^+)^2 + a_-(x^-)^2 + y^2), \quad H_2(u) = \tfrac{1}{2}(b_+(x^+)^2 + b_-(x^-)^2 + y^2),$$

$$r(t,u) = \begin{pmatrix} 0 \\ -h(t,x) \end{pmatrix}, \quad F(t,u) = \begin{pmatrix} y \\ -g(t,x) \end{pmatrix},$$

we see that Theorem 2.1 applies. Indeed, (14) is straightly implied by (44) and (45), while (15) and (16) hold thanks to (46) and (47). Condition (17) follows from Proposition 3.1, with $\mu = a_+$ and $\nu = a_-$, applied to $f(t,x) = g(t,x) - a_+x^+ + a_-x^-$, and condition (18) follows from Proposition 3.2, with $\mu = b_+$ and $\nu = b_-$, applied to $f(t,x) = g(t,x) - b_+x^+ + b_-x^-$. □

4 A piecewise linear-controlled system

In this section we will show a further application of Theorem 2.1 to a class of planar systems which are, in some sense, asymptotically controlled by piecewise linear functions.

For $u = (x, y) \in \mathbb{R}^2$, let us write $u^+ = (x^+, y^+)$ and $u^- = (x^-, y^-)$. We consider the problem

$$\begin{cases} J\dot{u} = [(1-\gamma(t,u))\mathbb{A}_+ + \gamma(t,u)\mathbb{B}_+]u^+ - [(1-\gamma(t,u))\mathbb{A}_- + \gamma(t,u)\mathbb{B}_-]u^- + r(t,u) \\ u(0) = u(T), \end{cases} \tag{48}$$

where $\gamma(t, u)$ and $r(t, u)$ are L^2-Carathéodory functions such that $0 \le \gamma(t, u) \le 1$ for almost every $t \in [0, T]$ and every $u \in \mathbb{R}^2$, and $r(t, u)$ is L^2-bounded. Moreover, we assume that

$$r(t,u) = \begin{pmatrix} r_{1,1}(t,x) + r_{1,2}(t,y) \\ r_{2,1}(t,x) + r_{2,2}(t,y) \end{pmatrix}, \tag{49}$$

and

$$\mathbb{A}_+ = \begin{pmatrix} a_+ & c \\ c & A_+ \end{pmatrix}, \mathbb{B}_+ = \begin{pmatrix} b_+ & c \\ c & B_+ \end{pmatrix}, \mathbb{A}_- = \begin{pmatrix} a_- & c \\ c & A_- \end{pmatrix}, \mathbb{B}_- = \begin{pmatrix} b_- & c \\ c & B_- \end{pmatrix},$$

for positive numbers $a_\pm, A_\pm, b_\pm, B_\pm$ satisfying $a_\pm \le b_\pm$, $A_\pm \le B_\pm$, with at least one of these inequalities strict, and $c \in \mathbb{R}$ such that

$$c^2 < \min\{a_+A_+, a_+A_-, a_-A_+, a_-A_-\},$$

in order to ensure that the two Hamiltonians

$$H_1(u) = \tfrac{1}{2}\left(a_+(x^+)^2 + a_-(x^-)^2 + A_+(y^+)^2 + A_-(y^-)^2 + cxy\right),$$
$$H_2(u) = \tfrac{1}{2}\left(b_+(x^+)^2 + b_-(x^-)^2 + B_+(y^+)^2 + B_-(y^-)^2 + cxy\right)$$

are positive. The particular form of the system is due to the fact that we want the right-hand side of (48) to be (up to r) a convex combination of the gradients of the two comparison Hamiltonians. For the sake of simplicity, we will only search for conditions which allow us to apply Corollary 2.7.

It is immediately seen that condition (15) holds. Concerning the Landesman-Lazer conditions, fix a solution $\varphi = (\varphi^{(1)}, \varphi^{(2)})$ of the Hamiltonian system associated to H_1, and a solution $\psi = (\psi^{(1)}, \psi^{(2)})$ of the Hamiltonian system associated to H_2. We will ask a condition which is slightly stronger than (35) and (36), but has the advantage of being more understandable. Define, for $i, j = 1, 2$,

$$L_{i,j}(\theta) = \int_{\{\varphi_\theta^{(j)}>0\}} \liminf_{s\to+\infty} r_{i,j}(t,s)\varphi_\theta^{(i)}(t)\,dt + \int_{\{\varphi_\theta^{(j)}<0\}} \limsup_{s\to-\infty} r_{i,j}(t,s)\varphi_\theta^{(i)}(t)\,dt,$$

and

$$U_{i,j}(\theta) = \int_{\{\psi_\theta^{(j)}>0\}} \limsup_{s\to+\infty} r_{i,j}(t,s)\psi_\theta^{(i)}(t)\,dt + \int_{\{\psi_\theta^{(j)}<0\}} \liminf_{s\to-\infty} r_{i,j}(t,s)\psi_\theta^{(i)}(t)\,dt.$$

Setting

$$\tilde{\Gamma}_1^-(\theta) = L_{1,1}(\theta) + L_{1,2}(\theta) + L_{2,1}(\theta) + L_{2,2}(\theta), \tag{50}$$

and

$$\tilde{\Gamma}_1^+(\theta) = U_{1,1}(\theta) + U_{1,2}(\theta) + U_{2,1}(\theta) + U_{2,2}(\theta), \tag{51}$$

to fulfill conditions (35) and (36) we will then ask

$$\tilde{\Gamma}_1^-(\theta) > 0 > \tilde{\Gamma}_1^+(\theta), \tag{52}$$

for every $\theta \in [0, T]$. For the computation of the periods of the solutions of the comparison systems

$$\begin{cases} -\dot{y} = a_+ x^+ - a_- x^- + cy \\ \dot{x} = cx + A_+ y^+ - A_- y^-, \end{cases} \qquad \text{and} \qquad \begin{cases} -\dot{y} = b_+ x^+ - b_- x^- + cy \\ \dot{x} = cx + B_+ y^+ - B_- y^-, \end{cases}$$

we refer to [17, Section 4]. In the particular case $c = 0$, they have the following simple expressions:

$$\tau_\varphi = \frac{\pi}{2}\left[\frac{1}{\sqrt{a_+ A_+}} + \frac{1}{\sqrt{a_- A_+}} + \frac{1}{\sqrt{a_- A_-}} + \frac{1}{\sqrt{a_+ A_-}}\right],$$

and

$$\tau_\psi = \frac{\pi}{2}\left[\frac{1}{\sqrt{b_+ B_+}} + \frac{1}{\sqrt{b_- B_+}} + \frac{1}{\sqrt{b_- B_-}} + \frac{1}{\sqrt{b_+ B_-}}\right],$$

respectively. Summing up, we infer:

Corollary 4.1. *Assume that conditions (14) and (52) hold. Then problem (48) has a solution.*

We have thus proved a double resonance existence result, which, in the scalar case without damping, corresponding to $r_{1,2} \equiv r_{2,1} \equiv r_{2,2} \equiv 0$, $A_\pm = B_\pm = 1$, and $c = 0$, is strongly related to Fabry's one in [10]. As particular cases of system (48), one can also consider scalar second order equations of Liénard or Rayleigh type (see [17] for details).

5 Simple resonance and nonresonance

The technique used to prove Theorem 2.1 can be adapted to more specific cases, in particular when some of the inequalities in (14) are strict. First of all, we show that it is possible to deduce, as an immediate corollary, an existence result for the case of simple resonance, that is, when the nonlinearity interacts only with one resonant Hamiltonian.

Corollary 5.1 (Simple resonance)**.** *Assume that condition 1) of Theorem 2.1 holds, and*

$$\frac{T}{N+1} < \tau_\psi \le \tau_\varphi \le \frac{T}{N} \tag{53}$$

(with the same notations as in Section 2). If, moreover, (17) holds, then problem (P) has a solution.

Proof. The result can be obtained following the lines of the proof of Theorem 2.1, performing a homotopy of the type

$$\dot{u} = -\sigma((1-\gamma(t,u))J\nabla H_1(u) + \gamma(t,u)J\nabla H_2(u)) - (1-\sigma)J\nabla H_2(u),$$

for $\sigma \in [0,1]$. In this case, the normalized sequence v_n will necessarily converge to a solution of $J\dot{v} = \nabla H_1(v)$. We omit the details for briefness. □

Clearly, we have a similar statement if we replace (53) by

$$\frac{T}{N+1} \le \tau_\psi \le \tau_\varphi < \frac{T}{N},$$

and in this case we will assume (18) instead of (17).

On the other hand, if we want to investigate the case when all the inequalities in (14) are strict, it is even possible to drop some of the hypotheses of Theorem 2.1, still performing a similar proof, as we are going to show.

Theorem 5.2 (Nonresonant case)**.** *Assume that $F : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ grows at most linearly in the second variable, i.e.*

$$|F(t,u)| \le c(t)(1+|u|),$$

with $c \in L^2(0,T)$, and that (32), (33) hold. If there exists a positive integer N such that

$$\frac{T}{N+1} < \tau_\psi \le \tau_\varphi < \frac{T}{N} \tag{54}$$

(with the same notations as in Section 2), then problem (P) has a solution.

Proof. As in the proof of Theorem 2.1, we argue by contradiction, assuming that (27) holds for an unbounded (in L^∞-norm) sequence $(u_n)_n$, i.e.

$$\begin{cases} \dot{u}_n = \sigma_n F(t,u_n) - \frac{1-\sigma_n}{2}\left(J\nabla H_1(u_n) + J\nabla H_2(u_n)\right) \\ u_n(0) = u_n(T). \end{cases}$$

By the elastic property, $\min|u_n(t)| \to \infty$ for $n \to \infty$. Consequently, it is possible to introduce polar coordinates, writing $u_n(t) = \rho_n(t)(\cos\theta_n(t), \sin\theta_n(t))$, and we know that u_n will perform an integer number m_n of rotations around the origin in the time T. A direct computation of $\dot{\theta}_n$, together with the use of (32), (33) gives

$$\frac{\dot{\theta}_n(t)}{2H_2(\cos\theta_n(t),\sin\theta_n(t))} \ge \frac{-\eta(t)}{\rho_n^2(t)2H_2(\cos\theta_n(t),\sin\theta_n(t))} - 1, \tag{55}$$

and

$$\frac{\dot{\theta}_n(t)}{2H_1(\cos\theta_n(t),\sin\theta_n(t))} \le \frac{\eta(t)}{\rho_n^2(t)2H_1(\cos\theta_n(t),\sin\theta_n(t))} - 1. \tag{56}$$

Since, as we have already recalled,

$$\int_0^{2\pi} \frac{d\theta}{2H_1(\cos\theta, \sin\theta)} = \tau_\varphi \,, \qquad \int_0^{2\pi} \frac{d\theta}{2H_2(\cos\theta, \sin\theta)} = \tau_\psi \,,$$

integrating in (55) and (56) from 0 to T yields

$$T \geq m_n\, \tau_\psi + \int_0^T \frac{-\eta(t)}{\rho_n^2(t) 2H_2(\cos\theta_n(t), \sin\theta_n(t))}\, dt,$$

and

$$T \leq m_n\, \tau_\varphi + \int_0^T \frac{\eta(t)}{\rho_n^2(t) 2H_1(\cos\theta_n(t), \sin\theta_n(t))}\, dt.$$

However, since $\rho_n \to \infty$ uniformly, the contribution of the two terms

$$\int_0^T \frac{-\eta(t)}{\rho_n^2(t) 2H_2(\cos\theta_n(t), \sin\theta_n(t))}\, dt \quad \text{and} \quad \int_0^T \frac{\eta(t)}{\rho_n^2(t) 2H_1(\cos\theta_n(t), \sin\theta_n(t))}\, dt$$

vanishes for $n \to \infty$. As a consequence, in view of (54) we will have, for a suitable $\epsilon > 0$, to be chosen sufficiently small,

$$N < \frac{T}{\tau_\varphi} - \epsilon \leq m_n \leq \frac{T}{\tau_\psi} + \epsilon < N + 1,$$

for n sufficiently large. Since m_n is integer, this is a contradiction. □

As already observed in Remark 2.5, if condition 1) of Theorem 2.1 is satisfied, then F has at most linear growth in the second variable, and (32), (33) hold. Notice that the Landesman-Lazer conditions, namely (17) and (18), are not needed, in view of the nonresonance hypothesis (54).

6 A possible relaxing of the double resonance conditions

We now focus on the special case when double resonance occurs with two multiples of the same Hamiltonian function. Let $H : \mathbb{R}^2 \to \mathbb{R}$ be a C^1-function satisfying (12), and let α, β be two positive constants such that $\alpha < \beta$. We will take $H_1(u) = \alpha H(u)$ and $H_2(u) = \beta H(u)$. Let ζ be a solution of

$$J\dot\zeta = \nabla H(\zeta),$$

satisfying

$$H(\zeta(t)) = \tfrac{1}{2}\,, \qquad \text{for every } t \in [0, T],$$

and let τ be its minimal period. Hence, $\varphi(t) = \zeta(\alpha t)$ and $\psi(t) = \zeta(\beta t)$ solve $J\dot\varphi = \alpha\nabla H(\varphi)$ and $J\dot\psi = \beta\nabla H(\psi)$, respectively. Denoting by $\tau_\varphi = \frac{\tau}{\alpha}$, and $\tau_\psi = \frac{\tau}{\beta}$ their minimal periods, respectively, we assume that

$$\frac{T}{N+1} \leq \tau_\psi < \tau_\varphi \leq \frac{T}{N}\,, \tag{57}$$

for some positive integer N. Consider the problem

$$\begin{cases} \dot{u} = -\hat{\gamma}(t,u) J\nabla H(u) + r(t,u) \\ u(0) = u(T), \end{cases} \tag{58}$$

being $\alpha \leq \hat{\gamma}(t,u) \leq \beta$ for almost every $t \in [0,T]$, and every $u \in \mathbb{R}^2$, and $r(t,u)$ a L^2-bounded function. In this setting, assuming hypotheses (17) and (18), Theorem 2.1 straightly applies; indeed, conditions (15) and (16) are plainly satisfied, since $\hat{\gamma}(t,u)\nabla H(u)$ can be written as a convex combination of the gradients of the Hamiltonians $H_1(u) = \alpha H(u)$ and $H_2(u) = \beta H(u)$.

However, it is possible to prove a better result which includes this one, as we are going to show. Recalling that $H(\zeta(t)) = \frac{1}{2}$ for every $t \in [0,T]$, as we have already announced in Remark 2.6 we define the functions $\Gamma_1^{\pm}$ by

$$\Gamma_1^-(\theta) = \int_0^T \liminf_{\substack{\lambda\to+\infty \\ \omega\to\theta}} \left[\lambda(\hat{\gamma}(t,\lambda\varphi(t+\omega)) - \alpha) + \langle Jr(t,\lambda\varphi(t+\omega)) | \varphi(t+\omega)\rangle\right] dt,$$

$$\Gamma_1^+(\theta) = \int_0^T \limsup_{\substack{\lambda\to+\infty \\ \omega\to\theta}} \left[\lambda(\hat{\gamma}(t,\lambda\psi(t+\omega)) - \beta) + \langle Jr(t,\lambda\psi(t+\omega)) | \psi(t+\omega)\rangle\right] dt.$$

Notice that conditions (17) and (18) can be written, in this particular setting, as $\Gamma_1^-(\theta) > 0$ and $\Gamma_1^+(\theta) < 0$ for every $\theta \in [0,T]$, respectively. We now introduce the new functions $\Gamma_2^{\pm}$ and $\Gamma_3^{\pm}$, defined by

$$\Gamma_2^-(\theta) = \int_0^T \liminf_{\substack{\lambda\to+\infty \\ \omega\to\theta}} \langle Jr(t,\lambda\varphi(t+\omega)) | \dot{\varphi}(t+\omega)\rangle\, dt,$$

$$\Gamma_2^+(\theta) = \int_0^T \limsup_{\substack{\lambda\to+\infty \\ \omega\to\theta}} \langle Jr(t,\lambda\psi(t+\omega)) | \dot{\psi}(t+\omega)\rangle\, dt,$$

and

$$\Gamma_3^-(\theta) = \int_0^T \limsup_{\substack{\lambda\to+\infty \\ \omega\to\theta}} \langle Jr(t,\lambda\varphi(t+\omega)) | \dot{\varphi}(t+\omega)\rangle\, dt,$$

$$\Gamma_3^+(\theta) = \int_0^T \liminf_{\substack{\lambda\to+\infty \\ \omega\to\theta}} \langle Jr(t,\lambda\psi(t+\omega)) | \dot{\psi}(t+\omega)\rangle\, dt.$$

Theorem 6.1. *Suppose that (57) holds. Moreover, assume that, for every $\theta \in [0,T]$,*

$$\Gamma_1^-(\theta) > 0 \quad \text{or} \quad \Gamma_2^-(\theta) > 0 \quad \text{or} \quad \Gamma_3^-(\theta) < 0, \tag{59}$$

and

$$\Gamma_1^+(\theta) < 0 \quad \text{or} \quad \Gamma_2^+(\theta) < 0 \quad \text{or} \quad \Gamma_3^+(\theta) > 0. \tag{60}$$

Then problem (58) has a solution.

Proof. Following the lines of the proof of Theorem 2.1, we proceed by perfoming a suitable homotopy. Assume by contradiction that an unbounded (in L^∞-norm) sequence $(u_n)_n$ satisfies

$$\begin{cases} \dot{u}_n = \sigma_n(-\hat{\gamma}(t,u_n)J\nabla H(u_n) + r(t,u_n)) - (1-\sigma_n)\delta(J\nabla H(u_n)) \\ u_n(0) = u_n(T), \end{cases} \tag{61}$$

where $\sigma_n \in [0,1]$, and $\delta \in \mathbb{R}$ is a fixed number such that $\alpha < \delta < \beta$ (for example, $\delta = \frac{1}{2}(\alpha+\beta)$); without loss of generality, we can suppose $\sigma_n \to \bar{\sigma} \in [0,1]$. We can show that $\bar{\sigma} \neq 0$ exactly as in the proof of Theorem 2.1. Moreover, setting $v_n = \frac{u_n}{\|u_n\|_\infty}$, for a subsequence, v_n converges uniformly to a function v which has the form $v(t) = R_0\varphi(t+\omega_0)$ or $v(t) = R_0\psi(t+\omega_0)$, for suitable constants $R_0 > 0$, $\omega_0 \in [0,T[$. For example, suppose that this second situation occurs; we pass to generalized polar coordinates in (61), writing $u_n(t) = r_n(t)\psi(t+\omega_n(t))$, with $\omega_n(0) \in [0,\tau_\psi[$ for every n. For a subsequence, we have that $\omega_n(t) \to \omega_0$ uniformly. We have already seen, in the proof of Theorem 2.1, that the result holds if $\Gamma_1^+(\omega_0) < 0$. Assume now $\Gamma_1^+(\omega_0) \geq 0$. We have

$$-\dot{r}_n(t) = \frac{1}{\beta}\sigma_n\langle Jr(t, r_n(t)\psi(t+\omega_n(t)))|\dot{\psi}(t+\omega_n(t))\rangle.$$

In view of the T-periodicity of u_n, we have

$$0 = \int_0^T \sigma_n\langle Jr(t, r_n(t)\psi(t+\omega_n(t)))|\dot{\psi}(t+\omega_n(t))\rangle\, dt.$$

By a straight use of Fatou's lemma, since r is L^2-bounded, (notice that $\bar{\sigma} \neq 0$), it follows that

$$0 \geq \int_0^T \liminf_{n\to+\infty} \langle Jr(t, r_n(t)\psi(t+\omega_n(t)))|\dot{\psi}(t+\omega_n(t))\rangle\, dt,$$

and

$$0 \leq \int_0^T \limsup_{n\to+\infty} \langle Jr(t, r_n(t)\psi(t+\omega_n(t)))|\dot{\psi}(t+\omega_n(t))\rangle\, dt,$$

whence $\Gamma_3^+(\omega_0) \leq 0 \leq \Gamma_2^+(\omega_0)$, in contradiction with the hypothesis. □

Remark 6.2. Let us give a geometrical interpretation of conditions (59) and (60). Defining the two curves $\Gamma^\pm : [0,T] \to \mathbb{R}^3$ as

$$\Gamma^-(\theta) = (\Gamma_1^-(\theta), \Gamma_2^-(\theta), \Gamma_3^-(\theta)), \quad \Gamma^+(\theta) = (\Gamma_1^+(\theta), \Gamma_2^+(\theta), \Gamma_3^+(\theta)),$$

condition (59) requires that $\Gamma^-(\theta)$ never enters the sector $\{(x,y,z) \in \mathbb{R}^3 \mid x \leq 0,\ y \leq 0,\ z \geq 0\}$, while condition (60) imposes that $\Gamma^+(\theta)$ never enters the sector $\{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0,\ y \geq 0,\ z \leq 0\}$. Recall that, in Theorem 2.1, we assumed, in a more restrictive way, that $\Gamma^-(\theta)$ always had to remain in the half-space $\{x > 0\}$ and $\Gamma^+(\theta)$ in $\{x < 0\}$.

Remark 6.3. It could happen that $\Gamma_2^-(\theta) = \Gamma_3^-(\theta)$ for every $\theta \in [0,T]$ (or $\Gamma_2^+(\theta) = \Gamma_3^+(\theta)$ for every $\theta \in [0,T]$). In this case, there is no need to define the curve Γ^- in $\mathbb{R}^3$, and one could define, instead, $\Gamma^- : [0,T] \to \mathbb{R}^2$ as $\Gamma^-(\theta) = (\Gamma_1^-(\theta), \Gamma_2^-(\theta))$. Then, condition (59) requires that $\Gamma^-(\theta)$ never touches the half-line $\{(x,y) \in \mathbb{R}^2 \mid x \leq 0, y = 0\}$. Clearly, in such a situation, the winding number of the curve Γ^-, with respect to the origin, is equal to 0. In the case of simple resonance, it was shown in [16] that this winding number $\mathrm{rot}(\Gamma^-, 0)$ is related to the topological degree associated to the considered periodic problem. Different examples were given (see [4, 5, 13, 14, 16, 17]) where $\mathrm{rot}(\Gamma^-, 0) \neq 0$. It can indeed be proved that the degree associated to the problem is equal to $1 - \mathrm{rot}(\Gamma^-, 0)$, see [16]. This agrees with the fact that, in our situation, the degree is equal to 1.

As in Section 4, we now show some possible applications to a particular class of planar systems. Consider the T-periodic problem

$$\begin{cases} J\dot{u} = \hat{\gamma}(t,u)[\mathbb{A}_+ u^+ - \mathbb{A}_- u^-] + r(t,u) \\ u(0) = u(T), \end{cases} \tag{62}$$

being $\alpha \leq \hat{\gamma}(t,u) \leq \beta$ for some positive constants $\alpha < \beta$, for almost every $t \in [0,T]$ and every $u \in \mathbb{R}^2$, and $r(t,u)$ a L^2-bounded function of the form (49). Moreover, we assume that

$$\mathbb{A}_+ = \begin{pmatrix} a_+ & c \\ c & A_+ \end{pmatrix}, \qquad \text{and} \qquad \mathbb{A}_- = \begin{pmatrix} a_- & c \\ c & A_- \end{pmatrix},$$

for positive constants $a_\pm$, $A_\pm$, and $c \in \mathbb{R}$ such that

$$c^2 < \alpha^2 \min\{a_+A_+, a_+A_-, a_-A_+, a_-A_-\}.$$

Notice that, without loss of generality, we can assume $\alpha = 1$. Hence, we are dealing with a particular case of the systems treated in Section 4, with $\mathbb{B}_+ = \beta\mathbb{A}_+$, and $\mathbb{B}_- = \beta\mathbb{A}_-$; as a consequence, the functions $\tilde{\Gamma}_1^-$ and $\tilde{\Gamma}_1^+$ can be explicitly written as in (50) and (51). However, in view of Theorem 6.1, it is possible to improve Corollary 4.1. Being $\varphi = (\varphi^{(1)}, \varphi^{(2)})$, $\psi = (\psi^{(1)}, \psi^{(2)})$, $\tilde{\Gamma}_1^-(\theta)$ and $\tilde{\Gamma}_1^+(\theta)$ as in Section 4, we define, for $i,j = 1,2$,

$$\mathcal{L}_{i,j}(\theta) = \int_{\{\dot{\varphi}_\theta^{(j)} > 0\}} \liminf_{s\to+\infty} r_{i,j}(t,s)\dot{\varphi}_\theta^{(i)}(t)\,dt + \int_{\{\dot{\varphi}_\theta^{(j)} < 0\}} \limsup_{s\to-\infty} r_{i,j}(t,s)\dot{\varphi}_\theta^{(i)}(t)\,dt,$$

$$\mathcal{U}_{i,j}(\theta) = \int_{\{\dot{\psi}_\theta^{(j)} > 0\}} \limsup_{s\to+\infty} r_{i,j}(t,s)\dot{\psi}_\theta^{(i)}(t)\,dt + \int_{\{\dot{\psi}_\theta^{(j)} < 0\}} \liminf_{s\to-\infty} r_{i,j}(t,s)\dot{\psi}_\theta^{(i)}(t)\,dt,$$

and

$$\mathcal{M}_{i,j}(\theta) = \int_{\{\dot{\varphi}_\theta^{(j)} > 0\}} \limsup_{s\to+\infty} r_{i,j}(t,s)\dot{\varphi}_\theta^{(i)}(t)\,dt + \int_{\{\dot{\varphi}_\theta^{(j)} < 0\}} \liminf_{s\to-\infty} r_{i,j}(t,s)\dot{\varphi}_\theta^{(i)}(t)\,dt,$$

$$\mathcal{V}_{i,j}(\theta) = \int_{\{\dot{\psi}_\theta^{(j)} > 0\}} \liminf_{s\to+\infty} r_{i,j}(t,s)\dot{\psi}_\theta^{(i)}(t)\,dt + \int_{\{\dot{\psi}_\theta^{(j)} < 0\}} \limsup_{s\to-\infty} r_{i,j}(t,s)\dot{\psi}_\theta^{(i)}(t)\,dt.$$

Moreover, we set

$$\tilde{\Gamma}_2^-(\theta) = \mathcal{L}_{1,1}(\theta) + \mathcal{L}_{1,2}(\theta) + \mathcal{L}_{2,1}(\theta) + \mathcal{L}_{2,2}(\theta),$$

$$\tilde{\Gamma}_2^+(\theta) = \mathcal{U}_{1,1}(\theta) + \mathcal{U}_{1,2}(\theta) + \mathcal{U}_{2,1}(\theta) + \mathcal{U}_{2,2}(\theta),$$

and

$$\tilde{\Gamma}_3^-(\theta) = \mathcal{M}_{1,1}(\theta) + \mathcal{M}_{1,2}(\theta) + \mathcal{M}_{2,1}(\theta) + \mathcal{M}_{2,2}(\theta),$$

$$\tilde{\Gamma}_3^+(\theta) = \mathcal{V}_{1,1}(\theta) + \mathcal{V}_{1,2}(\theta) + \mathcal{V}_{2,1}(\theta) + \mathcal{V}_{2,2}(\theta).$$

To satisfy (59) and (60), we will then ask, for every $\theta \in [0, T]$,

$$\tilde{\Gamma}_1^-(\theta) > 0 \quad \text{or} \quad \tilde{\Gamma}_2^-(\theta) > 0 \quad \text{or} \quad \tilde{\Gamma}_3^-(\theta) < 0, \tag{63}$$

and

$$\tilde{\Gamma}_1^+(\theta) < 0 \quad \text{or} \quad \tilde{\Gamma}_2^+(\theta) < 0 \quad \text{or} \quad \tilde{\Gamma}_3^+(\theta) > 0. \tag{64}$$

For the computation of the periods of the comparison Hamiltonian systems, we refer again to [17]. With a direct application of Theorem 6.1, we now obtain, in this particular framework, the following improvement of Corollary 4.1.

Corollary 6.4. *Assume that conditions (57), (63) and (64) hold. Then problem (62) has a solution.*

7 Scalar equations with damping in a case of simple resonance

In this section, we consider a special case of simple resonance and show some applications to the periodic problem associated to some scalar second order equations. Let us state the following immediate consequence of Theorem 6.1. We will use the notations introduced in Section 6.

Corollary 7.1. *Consider the problem*

$$\begin{cases} \dot{u} = -J\nabla H(u) + r(t, u) \\ u(0) = u(T), \end{cases} \tag{65}$$

being $r(t, u)$ a L^2-bounded function. Suppose that there exists a positive integer N such that

$$\tau = \frac{T}{N},$$

being τ the minimal period of the solutions of $J\dot{u} = \nabla H(u)$. Moreover, assume that, for every $\theta \in [0, T]$,

$$\Gamma_1^-(\theta) > 0, \quad \text{or} \quad \Gamma_2^-(\theta) > 0, \quad \text{or} \quad \Gamma_3^-(\theta) < 0. \tag{66}$$

Then problem (65) has a solution.

Clearly, assumption (66) can be replaced by the following one:

$$\Gamma_1^+(\theta) < 0, \quad \text{or} \quad \Gamma_2^+(\theta) < 0, \quad \text{or} \quad \Gamma_3^+(\theta) > 0. \tag{67}$$

Notice that, in this situation, since φ and ψ coincide and they both solve $J\dot{u} = \nabla H(u)$, we have that

$$\Gamma_1^-(\theta) \le \Gamma_1^+(\theta), \quad \text{and} \quad \Gamma_2^-(\theta) = \Gamma_3^+(\theta) \le \Gamma_3^-(\theta) = \Gamma_2^+(\theta),$$

for every $\theta \in [0, T]$.

We now examine an application of this corollary to a scalar equation with damping, which fits in the framework of system (62). For simplicity, we will consider only the symmetric case, namely $\mathbb{A}_+ = \mathbb{A}_-$, assuming $\alpha = \beta = 1$, and $T = 2\pi$. The same arguments would apply to the asymmetric case, as well. We will take in consideration the following two problems:

$$\begin{cases} \ddot{x} + q(x)\dot{x} + N^2 x = e(t) \\ x(0) = x(2\pi), \ \dot{x}(0) = \dot{x}(2\pi), \end{cases} \tag{68}$$

and

$$\begin{cases} \ddot{x} + Q(\dot{x}) + N^2 x = e(t), \\ x(0) = x(2\pi), \ \dot{x}(0) = \dot{x}(2\pi), \end{cases} \tag{69}$$

being N a positive integer, and $e(t)$ continuous and 2π-periodic. The differential equation appearing in (68) is a *Liénard* equation, while the one appearing in (69) is a *Rayleigh* equation. Clearly, similar considerations would hold for the T-periodic problem, with N^2 replaced by the corresponding λ_N.
The differential equations in (68) and (69) are equivalent to the systems

$$\begin{cases} \dot{x} = y - Q(x) \\ \dot{y} = -N^2 x + e(t), \end{cases} \tag{70}$$

and

$$\begin{cases} \dot{x} = y \\ \dot{y} = -N^2 x - Q(y) + e(t), \end{cases} \tag{71}$$

respectively, where in (70) we have set $Q(x) = \int_0^x q(s)\, ds$. They are thus included in our framework, with $H(x,y) = \frac{1}{2}(N^2x^2 + y^2)$. As a structural hypothesis, we assume that Q is a *bounded* function. For simplicity, in the following we will deal only with (68), and hence with (70). The 2π-periodic problem associated to the Rayleigh equation can be treated in the same way, yielding similar results.

Let us observe that, as a matter of fact, Theorem 2.1 is not suitable to deal with this kind of systems. Considering (70), if we assume that $Q(x)$ is strictly increasing, there always exists $\theta \in [0, 2\pi]$ such that neither condition (17) nor (18) is satisfied. To see this, for instance for what concerns (17), set

$$\phi(t) = \frac{1}{N}\cos(Nt),$$

and $\varphi(t) = (\phi(t), \dot{\phi}(t))$; after some computations we see that, if (17) holds, the quantity

$$\int_0^{2\pi} \Big(\liminf_{\substack{\lambda \to +\infty \\ \omega \to \theta}} [-Q(\lambda\phi_\omega(t))\dot{\phi}_\omega(t)] - e(t)\phi_\theta(t) \Big)\, dt$$

has to be strictly positive for every $\theta \in [0, 2\pi]$. Noticing that, since Q has finite limits at $\pm\infty$, the inferior limit which appears under the integral sign is indeed a finite limit, this is true if and only if

$$L(\theta) = \int_0^{2\pi} e(t)\phi_\theta(t)\, dt < 0, \qquad \text{for every } \theta \in [0, T].$$

Such a condition, however, is never satisfied, due to the form of ϕ_θ: explicitly, it should be

$$\cos(N\theta) \int_0^{2\pi} e(t)\cos(Nt)\, dt - \sin(N\theta) \int_0^{2\pi} e(t)\sin(Nt)\, dt < 0,$$

for every $\theta \in [0, T]$, which is clearly impossible. In the same way, we see that also (18) fails.

We now show how it is possible to overcome this problem using Corollary 7.1. Consider system (70): setting

$$Q_-(+\infty) = \liminf_{x \to +\infty} Q(x), \quad Q_+(+\infty) = \limsup_{x \to +\infty} Q(x),$$

$$Q_-(-\infty) = \liminf_{x \to -\infty} Q(x), \quad Q_+(-\infty) = \limsup_{x \to -\infty} Q(x),$$

and

$$\Delta Q(+\infty) = Q_+(+\infty) - Q_-(+\infty), \quad \Delta Q(-\infty) = Q_+(-\infty) - Q_-(-\infty),$$

the following result holds true:

Corollary 7.2. *Assume that, for every $\theta \in [0, 2\pi]$,*

$$\begin{aligned} &\int_0^{2\pi} e(t)\phi_\theta(t)\, dt < -\tfrac{1}{N}(\Delta Q(+\infty) + \Delta Q(-\infty)), \quad \text{or} \\ &\int_0^{2\pi} e(t)\dot{\phi}_\theta(t)\, dt < 2(Q_-(+\infty) - Q_+(-\infty)), \;\; \text{or} \\ &\int_0^{2\pi} e(t)\dot{\phi}_\theta(t)\, dt > 2(Q_+(+\infty) - Q_-(-\infty)). \end{aligned} \tag{72}$$

Then problem (68) has a solution.

Notice that the statement follows from Corollary 7.1, since (72) implies (66). A symmetrical result can be stated assuming, for every $\theta \in [0, 2\pi]$,

$$\begin{aligned} &\int_0^{2\pi} e(t)\phi_\theta(t)\,dt > \tfrac{1}{N}(\Delta Q(+\infty) + \Delta Q(-\infty)), \quad \text{or} \\ &\int_0^{2\pi} e(t)\dot{\phi}_\theta(t)\,dt < 2(Q_-(+\infty) - Q_+(-\infty)), \qquad \text{or} \\ &\int_0^{2\pi} e(t)\dot{\phi}_\theta(t)\,dt > 2(Q_+(+\infty) - Q_-(-\infty)), \end{aligned} \tag{73}$$

since (73) implies (67).

The last part of this section will be dedicated to compare Corollary 7.2, and its symmetric version with (73) instead of (72), with the following result proved by Frederickson and Lazer in [18], in the particular case $N = 1$.

Theorem 7.3 (Frederickson-Lazer 1969)**.** *Assume that $N = 1$ and that $Q(x)$ is strictly increasing. Then, setting*

$$Q(+\infty) = \lim_{x\to+\infty} Q(x), \quad \text{and} \quad Q(-\infty) = \lim_{x\to-\infty} Q(x),$$

the condition

$$\left|\int_0^{2\pi} e(t)e^{-it}\,dt\right| < 2(Q(+\infty) - Q(-\infty)) \tag{74}$$

is both necessary and sufficient for the existence of a solution of (68).

We thus consider, in our framework, the case when Q is increasing. Since Q has finite limits at $\pm\infty$ (recall that we are assuming Q to be bounded), the inferior limits appearing under the integral sign in our hypotheses are finite limits. So, by Corollary 7.2, if for every $\theta \in [0, 2\pi]$,

$$\int_0^{2\pi} e(t)\phi_\theta(t)\,dt < 0 \quad \text{or} \quad \int_0^{2\pi} e(t)\dot{\phi}_\theta(t)\,dt \neq 2(Q(+\infty) - Q(-\infty)), \tag{75}$$

being $\phi(t) = \cos t$, then problem (68) has a solution. It is straightly seen that this hypothesis follows from the Frederickson-Lazer condition, which implies indeed

$$\int_0^{2\pi} e(t)\dot{\phi}_\theta(t)\,dt < 2(Q(+\infty) - Q(-\infty)),$$

for every $\theta \in [0, 2\pi]$. Apparently, however, (75) seems to be more general, which looks strange, as the Frederickson-Lazer condition was also proved to be necessary. We now prove that the two statements are indeed equivalent. Suppose, for simplicity, $e(t) = \cos t$, and assume that (75) holds, namely

$$\cos\theta < 0, \quad \text{or} \quad -\pi\sin\theta \neq 2(Q(+\infty) - Q(-\infty)),$$

for every $\theta \in [0,T]$. We claim that, necessarily, it will be

$$2(Q(+\infty) - Q(-\infty)) > \pi; \tag{76}$$

otherwise, we could always find $\theta_0 \in [0, 2\pi]$ such that

$$2(Q(+\infty) - Q(-\infty)) = -\pi \sin \theta_0, \quad \text{and} \quad \cos \theta_0 \geq 0$$

hold at the same time, making (75) fail. Being

$$\left| \int_0^{2\pi} (\cos t) e^{-it}\, dt \right| = \pi,$$

we have that (76) implies the Frederickson-Lazer condition, so we are done in the particular case $e(t) = \cos t$. The reasoning works, in the same way, for $e(t) = \cos kt$ and $e(t) = \sin kt$, for every $k \in \mathbb{N}$. Using the fact that $\{\cos kt, \sin kt\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(0, 2\pi)$, the previous considerations can be extended to every continuous forcing term $e(t)$. Summing up, if Q is bounded and increasing, Corollary 7.2 generalizes Frederickson and Lazer's result.

Remark 7.4. By the above discussion, we can conclude that Corollary 7.1 generalizes, for the periodic problem, both the Lazer and Leach existence result and the Frederickson and Lazer one, in the case when Q is bounded (see also [4, 5, 17, 26]). Notice, however, that, in [18], Q was not assumed to be bounded, and the almost periodic problem was also considered, obtaining a similar existence result.

Remark 7.5. The above arguments can be adapted to the case when Q is not bounded, but has sublinear growth, provided that the functions $\Gamma_1^{\pm}$, $\Gamma_2^{\pm}$ and $\Gamma_3^{\pm}$ are well defined. Even in this case, if Q is increasing, we have that the Frederickson-Lazer condition and ours turn out to be equivalent.

8 Final remarks

In this last section, we are interested in the case where the inferior and superior limits which appear in the Landesman-Lazer conditions are equal to 0, and so conditions (17) and (18) do not hold. This problem has already been studied in the scalar setting, see e.g. [10, Theorem 2], and [30]. We propose here a possible generalization of this result, based on the main theorem of Section 2, and consisting in refining conditions (17) and (18). We will use again the notations introduced there. Moreover, we will also assume as hypotheses the corresponding refinements of conditions (32) and (33) (the idea is that $|r(t,u)|$ has to be controlled by some negative power of $|u|$).

Theorem 8.1. *Let us assume that there exist two C^1-functions $H_1, H_2 : \mathbb{R}^2 \to \mathbb{R}$, satisfying (12), such that (14), (15) and (16) hold. Moreover, assume that there exists $k \geq 0$ such that the following conditions are satisfied:*

- *there exists a positive function* $\eta \in L^2(0,T)$ *with*

$$\lambda^k(\langle JF(t,\lambda w)|w\rangle - 2\lambda H_1(w)) \geq -\eta(t), \tag{77}$$

$$\lambda^k(2\lambda H_2(w) - \langle JF(t,\lambda w)|w\rangle) \geq -\eta(t), \tag{78}$$

  for almost every $t \in [0,T]$, *every* $w \in \mathbb{R}^2$ *with* $|w| \leq 1$ *and every* $\lambda \geq 1$;

- *for every* $\theta \in [0,T]$,

$$\int_0^T \liminf_{\substack{\lambda\to+\infty \\ \omega\to\theta}} \lambda^k[\langle JF(t,\lambda\varphi(t+\omega))|\varphi(t+\omega)\rangle - 2\lambda H_1(\varphi(t))]\,dt > 0, \tag{79}$$

$$\int_0^T \liminf_{\substack{\lambda\to+\infty \\ \omega\to\theta}} \lambda^k[2\lambda H_2(\psi(t)) - \langle JF(t,\lambda\psi(t+\omega))|\psi(t+\omega)\rangle]\,dt > 0. \tag{80}$$

Then problem (P) has a solution.

Proof. It is sufficient to follow the lines of the proof of Theorem 2.1, noticing that, in view of (19), we have that (31) yields

$$0 \geq \int_0^T r_n(t)^k \, \frac{r_n(t) - \langle JF(t, r_n(t)\psi(t+\omega_n(t)))|\psi(t+\omega_n(t))\rangle}{(r_n^V(t))^{k+1}}\,dt;$$

using Fatou's lemma, thanks to (77), (78), this implies that

$$0 \geq \int_0^T \liminf_{n\to+\infty} r_n(t)^k\,[r_n(t) - \langle JF(t, r_n(t)\psi(t+\omega_n(t)))|\psi(t+\omega_n(t))\rangle]\,dt,$$

and this contradicts (80). □

Notice that Theorem 2.1 is a particular case of this result (for $k = 0$). In a similar way, moreover, it is possible to obtain, also in this framework, results analogous to the ones proved in Sections 4, 5, 6 and 7.

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