EXISTENCE FOR WAVE EQUATIONS ON DOMAINS WITH ARBITRARY GROWING CRACKS

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Abstract. In this paper we formulate and study scalar wave equations on domains with arbitrary growing cracks. This includes a zero Neumann condition on the crack sets, and the only assumptions on these sets are that they have bounded surface measure and are growing in the sense of set inclusion. In particular, they may be dense, so the weak formulations must fall outside of the usual weak formulations using Sobolev spaces. We study both damped and undamped equations, showing existence and, for the damped equation, uniqueness and energy conservation.

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1. Introduction

The last fifteen years have seen significant advances in the mathematical analysis of quasi-static fracture [1, 14, 11, 12, 7, 13, 9, 8, 17, 10, 4]. While the mechanical and physical justifications for the underlying models have been open to some question, it seems that quasi-static models were a good place to begin the analysis of fracture evolution, as certain mathematical issues were clearly identified and treated, and good numerical methods were developed. However, there is little doubt that much better mechanical and physical support will be available for models of dynamic fracture, which will apply in a much broader range of circumstances, and which can be used in the end to clarify the appropriateness of different quasi-static models.

Quasi-static models are based on the assumption that whatever is driving the motion, e.g., loading, varies slowly in time compared to the elastic wave speed of the material. More precisely, for a given varying load $f(t)$ on a time interval $[0,T]$, one can consider the rescaled problem corresponding to $f_\varepsilon(t) := f(\varepsilon t)$ on $[0,T/\varepsilon]$. If the corresponding physical solution (presumably to the dynamic problem) is $u_\varepsilon(t)$, one needs to rescale again in order to take the limit, since the limit of $f_\varepsilon(t)$ is constant in time. Therefore, it is natural to define $u^\varepsilon(t) := u_\varepsilon(t/\varepsilon)$ for $t \in [0,T]$. Setting $u(t)$ to be the limit as $\varepsilon \searrow 0$, it is reasonable to suppose (assuming some damping in the dynamics) that $u(t)$ is in elastic equilibrium at every t, corresponding to the load $f(t)$. This idea underlies all quasi-static models, with the only debate being over whether the overall state, made up of both the displacement and crack set, should be a global minimizer of the total energy, a local minimizer, or something in between.

The main problem with the quasi-static fracture models concerns jumps in time of the crack set, for which the quasi-static assumption – that while the crack grows the material is always in elastic equilibrium – is dubious. The point is that if in the $\varepsilon \searrow 0$ limit the crack jumps, there is no reason to think that $u_\varepsilon(t)$ varies slowly, even though $f_\varepsilon(t)$ does. Hence, it

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is generally agreed that dynamic models need to be considered, and then quasi-static limits can be analyzed. This would help clarify whether cracks jump as soon as the material is not a global minimizer, as proposed in [14], or if jumps only occur to ensure the material is a local minimizer, or if jumps occur based on a condition somewhere in between global and local minimality, as in [17].

At this point, unfortunately, there are no generally accepted fundamental mathematical models for dynamic fracture (by which we mean models with no assumptions on regularity necessary to define things such as stress intensity factors and J-integrals). Still, we believe that there can be no real disagreement that any reasonable model must contain three principles: i) elastodynamics off the crack, ii) energy-dissipation balance (including the surface energy dissipated by the crack), and iii) a principle dictating when a crack must grow. Conditions i) and ii) follow, e.g., from [15], and a principle like iii) is necessary, since otherwise a stationary crack with elastodynamics off of it will always be a solution. In [16] this is discussed in some more detail, and a maximal dissipation condition is proposed for iii), but it is too early to claim any acceptance of this principle.

We note that, based on the success of numerical methods for quasi-static fracture using the Ambrosio-Tortorelli approximation (see, e.g., [3]), a numerical algorithm was proposed in [5] for dynamic fracture, which was shown in [18] to converge, as the time step tends to zero, to a solution obeying the appropriate elastodynamics, the total energy (stored elastic, kinetic, and the surface energy of the crack set) is conserved, and the field modeling the crack set satisfies a minimality analogous to that in the quasi-static setting. For these phase-field models, this minimality provides the principle iii), requiring the "crack" to run so as to maintain minimality. However, in the sharp-interface limit, there is no corresponding minimality, and so the formulation of this third principle is open.

In this paper, we consider a preliminary issue, namely, given initial conditions and *given* a growing-in-time crack set $\Gamma(t)$ with no assumptions other than finite surface measure (which corresponds to finite surface energy), does there exist a solution to the corresponding elastodynamics? We show that there does, both for undamped and damped dynamics. In particular, we look at weak versions of equations of the form

$$\ddot{u}(t) - \Delta u(t) - \gamma \Delta \dot{u}(t) = f(t)$$

on $\Omega \setminus \Gamma(t)$, with a zero Neumann condition on $\partial\Omega \cup \Gamma(t)$, where the dots denote derivatives with respect to time and the Laplace operator Δ acts on the space variables. We treat both the case $\gamma > 0$, corresponding to the damped wave equation, and $\gamma = 0$, corresponding to the undamped equation.

We also show an energy balance and uniqueness for the damped problem, but we were unable to show this for the undamped problem. Indeed, the energy balance we were able to show for the damped problem is a conservation of kinetic plus elastic plus dissipated energy due to the damping, but we note that this balance is inconsistent with models we have in mind for fracture, which balance energy only when the surface energy dissipated by the crack is also included.

This paper is organized as follows. First, in Section 2, we need to introduce new function spaces, V_t, containing both our solutions $u(t)$ as well as the test functions at time t. These spaces are somewhat technical and based on SBV functions with jump set contained in $\Gamma(t)$. However, to read this paper, one can think of V_t as the Sobolev space $H^1(\Omega \setminus \Gamma(t))$, and in fact, if $\Gamma(t)$ is closed, this is exactly V_t. The most serious mathematical issues arise because these spaces are increasing in time, so that test functions at some time t are not necessarily admissible test functions for times $s < t$.

In Section 3 we consider the damped wave equation, and we prove existence using discrete time approximations and passing to the limit as the time step tends to zero. Precisely, the function $u_n(t_{i+1})$ is the minimizer in $V_{t_{i+1}}$ of

$$u \mapsto \left\| \frac{u - u_n^i}{\tau_n} - \frac{u_n^i - u_n^{i-1}}{\tau_n} \right\|^2 + \|\nabla u\|^2 + \frac{\gamma}{\tau_n}\|\nabla u - \nabla u_n^i\|^2 - 2\langle f_n^i, u\rangle,$$

where $u_n^i = u_n(t_i)$, all norms and inner products are L^2, and τ_n is the time step. We are then able to pass to the limit as $\tau_n \to 0$, and show that the limit u is a weak solution. Furthermore, we are able to show uniqueness and energy balance.

Finally, in Section 4, we show existence for the undamped equation, following the same argument as in the damped case. However, we are unable to prove uniqueness or energy balance. We note that a lack of energy balance, where the energy includes only the kinetic and elastic energies, is in fact desirable, as only then can the total energy, including the surface energy of the crack, be balanced, as in the models formulated in [16]. We also note that a natural idea for proving existence for those models, which in addition to balance of the total energy have a maximality property of $\Gamma(t)$, would be to find $u_n(t_{i+1})$ and $\Gamma_n(t_{i+1})$ by minimizing

$$(u, \Gamma) \mapsto \left\| \frac{u - u_n^i}{\tau_n} - \frac{u_n^i - u_n^{i-1}}{\tau_n} \right\|^2 + \|\nabla u\|^2 + \mathcal{H}^{N-1}(\Gamma) - 2\langle f_n^i, u\rangle,$$

with the only restriction that $\Gamma \supset \Gamma_n(t_i)$ and u has jump set in Γ. This would generate u_n and Γ_n, but methods to prove appropriate properties of limits u and Γ are far from clear at this time.

2. Notation and preliminary results

For the definition of the space $GSBV(\Omega)$ we refer to [2, Definition 4.26]. For every $v \in GSBV(\Omega)$ the symbol ∇v denotes its weak approximate differential according to [2, Definition 4.31 and Theorem 4.34], defined for a.e. $x \in \Omega$, while S_v is the approximate discontinuity set of v according to [2, Definition 4.28 and Theorem 4.34]. $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure.

For any $\Gamma \subset \Omega$ with $\mathcal{H}^{N-1}(\Gamma) < \infty$, we define

$$GSBV_2^2(\Omega, \Gamma) := \{v \in GSBV(\Omega) \cap L^2(\Omega) : \nabla v \in L^2(\Omega; \mathbb{R}^N), S_v \subset \Gamma\}. \tag{2.1}$$

Here and below, by $A \subset B$ we mean $\mathcal{H}^{N-1}(A \setminus B) = 0$. We endow this space with the inner product

$$\langle u, v\rangle_{L^2} + \langle \nabla u, \nabla v\rangle_{L^2}. \tag{2.2}$$

The corresponding norm is denoted by $\|\cdot\|$. If Γ is closed in Ω, then $GSBV_2^2(\Omega, \Gamma)$ coincides with the Sobolev space $H^1(\Omega \setminus \Gamma)$.

Lemma 2.1. *The inner product space $GSBV_2^2(\Omega, \Gamma)$ is a Hilbert space.*

Proof. Let $\{v_n\}$ be a Cauchy sequence in $GSBV_2^2(\Omega, \Gamma)$. Then there exist $v \in L^2(\Omega)$, $w \in L^2(\Omega; \mathbb{R}^N)$ such that $v_n \to v$ and $\nabla v_n \to w$ strongly in L^2. By $GSBV$ compactness (see [2, Theorem 4.36]), we have $v \in GSBV(\Omega)$ and $w = \nabla v$. Moreover $S_v \subset \Gamma$. If Γ is closed, this inclusion is a consequence of the mentioned theorem, applied to the open set $\Omega \setminus \Gamma$; the general case can be obtained, e.g., from [9, Theorem 2.8]. Hence $v \in GSBV_2^2(\Omega, \Gamma)$ and $v_n \to v$ in the norm induced by (2.2). $\square$

The dual of this space, $GSBV_2^2(\Omega, \Gamma)^*$, will not be identified with the underlying Hilbert space, but instead will be endowed with a pairing consistent with the L^2 inner product, as is usually done for the duals of Sobolev spaces. Since

$$GSBV_2^2(\Omega, \Gamma) \subset L^2(\Omega)$$

is dense, we have

$$L^2(\Omega) = L^2(\Omega)^* \subset GSBV_2^2(\Omega, \Gamma)^*,$$

and $L^2(\Omega)$ is dense in $GSBV_2^2(\Omega, \Gamma)^*$.

We now fix $T > 0$ and $\Gamma \subset \Omega$ with

$$\mathcal{H}^{N-1}(\Gamma) < \infty \tag{2.3}$$

and $t \mapsto \Gamma(t)$ defined on $[0,T]$ such that $\Gamma(t)$ is an $\mathcal{H}^{N-1}$-measurable subset of Γ. We assume also that

$$\Gamma(s) \subset \Gamma(t) \quad \text{if} \quad s < t. \tag{2.4}$$

For simplicity of notation, from now on, we will denote $GSBV_2^2(\Omega,\Gamma)$ by V and $GSBV_2^2(\Omega,\Gamma(t))$ by V_t. The norm in V is $\|\cdot\|$ (see (2.2)), while the induced norm in V_t is $\|\cdot\|_t$. We will also denote $L^2(\Omega)$ by H and $L^2(\Omega;\mathbb{R}^N)$ by H_N. Note that for $s < t$ we have $V_s \subset V_t \subset V$. We also note that, since $V \subset H$ and V is dense in H, we have the embedding $H \subset V^*$ and the density of H in V^*. Similarly, H is a dense subspace of V_t^* for every $t \in [0,T]$. We denote the pairing between V^* and V by $\langle \cdot,\cdot \rangle$, producing the dual norm $\|\cdot\|^*$, and the pairing between V_t^* and V_t by $\langle \cdot,\cdot \rangle_t$, with dual norm $\|\cdot\|_t^*$. We note that these pairings are the unique continuous bilinear maps on $V^*\times V$ and $V_t^*\times V_t$ such that $\langle f,v\rangle = \langle f,v\rangle_H$ and $\langle f,v_t\rangle_t = \langle f,v_t\rangle_H$ whenever $f \in H$, $v \in V$, and $v_t \in V_t$.

If $s < t$, for every $f \in V_t^*$ we can consider the element $f|_{V_s}$ of V_s^* defined by $\langle f|_{V_s}, v\rangle_s = \langle f,v\rangle_t$ for every $v \in V_s$. The restriction map $f \mapsto f|_{V_s}$ is continuous and coincides with the adjoint of the embedding $V_s \hookrightarrow V_t$. Although $f \mapsto f|_{V_s}$ is not injective, in the rest of the paper we omit the notation $|_{V_s}$, since the restriction will be clear from the context.

Lemma 2.2. *Let $u \in W^{1,\infty}(0,T;H)$. Assume that there exists a constant c such that for every s, $t \in [0,T]$, with $s < t$, we have*

$$u \in W^{2,\infty}(t,T;V_s^*) \quad \textit{and} \quad \|\ddot{u}\|_{L^\infty(t,T;V_s^*)} \le c. \tag{2.5}$$

Then there exist a set $E \subset [0,T]$ of full measure and, for every $t \in E$, an element $w(t)$ of V_t^*, with

$$\|w(t)\|_t^* \le c, \tag{2.6}$$

such that for every $t \in E$ we have

$$\lim_{\substack{h\to 0\\ t+h\in E}} \frac{\dot{u}(t+h) - \dot{u}(t)}{h} = w(t) \tag{2.7}$$

weakly in V_t^* and strongly in V_s^* for every $s < t$. Moreover, for every $s \in [0,T]$ the functions $t \mapsto u(t)$ and $t \mapsto w(t)$, considered as a functions from $[s,T]$ to V_s^*, belong to $W^{2,\infty}(s,T;V_s^*)$ and $L^\infty(s,T;V_s^*)$, respectively, and satisfy $\ddot{u}(t) = w(t)$ in V_s^* for a.e. $t \in (s,T)$, so that (2.5) *holds with $s = t$.*

Before proving this, we give a short technical lemma about increasing sequences of subspaces of separable Hilbert spaces.

Lemma 2.3. *Let $\{X_t : t \in [0,T]\}$ be an increasing family of closed linear subspaces of a separable Hilbert space X. Then, there exists a countable set $S \subset [0,T]$ such that for all $t \in [0,T] \setminus S$, we have*

$$X_t = \overline{\bigcup_{s<t} X_s}.$$

Proof. For each $t \in [0,T]$, set X_{t-} to be the right hand side above. Then we have

$$X_s \subset X_{t-} \subset X_t \tag{2.8}$$

for every $s < t$. Set Y_t to be the orthogonal complement of X_{t-} in X_t. By (2.8), we have that $Y_s \perp Y_t$ for $s \neq t$. The separability of X implies that there is at most a countable set of t such that Y_t is nontrivial, which proves the lemma. □

Proof of Lemma 2.2. We first consider the set S from Lemma 2.3, and we fix a countable, dense set $D \subset [0,T]$. The differentiability properties of vector valued Sobolev functions (see, e.g., [6, Appendix]), together with (2.5), imply that there exists another set of measure zero $M \subset [0,T]$ such that

$$\frac{\dot u(t+h) - \dot u(t)}{h} \to \ddot u(t) \quad \text{strongly in } V_s^* \tag{2.9}$$

for all $t \notin M$ and for all $s < t$ with $s \in D$. The fact that (2.9) holds for all $s < t$ follows from the density of D and from the continuity properties of the restriction operator from V_σ^* to V_s^* for $s < \sigma$. Note that from (2.5) and (2.9) we have

$$\|\ddot u(t)\|_s^* \le c \tag{2.10}$$

for every $t \notin M$ and every $s < t$.

Now for $t \notin S \cup M$ choose $s_n \nearrow t$. For $\phi \in V_t$, Lemma 2.3 gives a sequence $\phi_n \in V_{s_n}$ such that $\phi_n \to \phi$ in V_t. We claim that $w(t) \in V_t^*$ given by

$$\langle w(t), \phi\rangle_t := \lim_{n\to\infty} \langle \ddot u(t), \phi_n\rangle_{s_n}$$

is well defined. The fact that this limit exists follows immediately from the uniform bound on $\|\ddot u(t)\|_s^*$ (see (2.10)) and the strong convergence of ϕ_n to ϕ. This also implies that the limit is independent of the choice of ϕ_n, as well as the linearity and boundedness of the limit. This gives the claim and proves (2.6).

Note that if $\phi \in V_s$ for some $s < t$, by taking $\phi_n = \phi$ above for n large enough, we actually have

$$\langle w(t), \phi\rangle_s = \langle w(t), \phi\rangle_t = \langle \ddot u(t), \phi\rangle_s, \tag{2.11}$$

which implies that $t \mapsto w(t)$, considered as a function from $[s,T]$ to V_s^*, belongs to $L^\infty(s,T;V_s^*)$ and satisfies the last assertion of the lemma.

Next, we claim that

$$\frac{\dot u(t+h) - \dot u(t)}{h} \rightharpoonup w(t) \text{ weakly in } V_t^*.$$

For $\phi \in V_t$, we choose again $s_n \nearrow t$ and $\phi_n \in V_{s_n}$ such that $\phi_n \to \phi$ strongly in V_t.

Then we have

$$\Big\langle w(t) - \frac{\dot u(t+h) - \dot u(t)}{h}, \phi\Big\rangle_t = \Big\langle w(t) - \frac{\dot u(t+h) - \dot u(t)}{h}, \phi - \phi_n\Big\rangle_t + \Big\langle w(t) - \frac{\dot u(t+h) - \dot u(t)}{h}, \phi_n\Big\rangle_t,$$

so that

$$\left|\Big\langle w(t) - \frac{\dot u(t+h) - \dot u(t)}{h}, \phi\Big\rangle_t\right| \le 2c\|\phi - \phi_n\|_t + \left|\Big\langle \ddot u(t) - \frac{\dot u(t+h) - \dot u(t)}{h}, \phi_n\Big\rangle_{s_n}\right|$$

by (2.10) and (2.11). Passing to the limit first in h (using (2.9)) and then in n we get (2.7). □

Definition 2.4. Under the assumptions of Lemma 2.2, the element $w(t)$ of V_t^* defined in (2.7) for a.e. $t \in [0,T]$ is denoted by $\ddot u(t)$.

The last sentence of Lemma 2.2 shows the relationships between this definition and the standard definition in the sense of distributions on (s,T). The point is that, under the assumptions of Lemma 2.2, the theory of distributions defines $\ddot u(t)$ as an element of V_s^* only for a.e. $t \in (s,T)$, while the study of the wave equation in cracking domains requires a precise definition of $\ddot u(t)$ as an element of V_t^*.

3. The Damped Wave Equation

Here we consider a certain weak formulation of the equation

$$\ddot{u} - \Delta u - \gamma \Delta \dot{u} = f \tag{3.1}$$

on a cracking domain, with $\gamma > 0$.

Definition 3.1. Assume (2.3), (2.4), and let $\gamma > 0$ and $f \in L^2(0,T;H)$. We say that u is a weak solution of the damped wave equation (3.1) on the time dependent cracking domain $t \mapsto \Omega \setminus \Gamma(t)$ with homogeneous Neumann boundary conditions if

$$u \in H^1(0,T;V) \cap W^{1,\infty}(0,T;H), \tag{3.2}$$

$$\text{for every } t \in [0,T] \text{ we have } u(t) \in V_t, \tag{3.3}$$

$$\text{for every } s \in [0,T) \text{ we have } u \in W^{2,\infty}(s,T;V_s^*), \tag{3.4}$$

$$\sup_{s\in[0,T)} \|\ddot{u}\|_{L^\infty(s,T;V_s^*)} < +\infty, \tag{3.5}$$

for every $s \in (0,T)$ the functions

$$t \mapsto \frac{1}{h}\|\dot{u}(t) - \dot{u}(t-h)\|_H^2\,, \quad h \in (0,s), \tag{3.6}$$

are equiintegrable on (s,T),

and for a.e. $t \in [0,T]$

$$\langle \ddot{u}(t), \phi\rangle_t + \langle \nabla u(t) + \gamma\nabla\dot{u}(t), \nabla\phi\rangle_{H_N} = \langle f(t), \phi\rangle_H \quad \text{for every } \phi \in V_t, \tag{3.7}$$

where $\ddot{u}(t)$ is given by Definition 2.4.

By (3.2) and (3.4) the functions $t \mapsto u(t)$ and $t \mapsto \dot{u}(t)$ are continuous from $[0,T]$ to V and V_0^* respectively, so that the initial values $u(0)$ and $\dot{u}(0)$ are well defined in V and V_0^* respectively. In the next theorem, given $u^{(1)} \in H$, we prescribe the initial condition for $\dot{u}(0)$ in the stronger form

$$\lim_{h\to 0+} \frac{1}{h}\int_0^h \|\dot{u}(t) - u^{(1)}\|_H^2 dt = 0. \tag{3.8}$$

Theorem 3.2. *Assume* (2.3) *and* (2.4). *Let* $u^{(0)} \in V_0$, $u^{(1)} \in H$, $\gamma > 0$, *and* $f \in L^2(0,T;H)$. *Then there exists a unique weak solution* u *of the damped wave equation considered in Definition 3.1 satisfying the initial conditions* $u(0) = u^{(0)}$ *and* (3.8). *Moreover* $t \mapsto \dot{u}(t)$ *is continuous from* $[0,T]$ *to* H *and* u *satisfies the energy balance*

$$\begin{aligned}&\frac{1}{2}\|\dot{u}(t)\|_H^2 + \frac{1}{2}\|\nabla u(t)\|_{H_N}^2 + \gamma\int_0^t \|\nabla\dot{u}(\tau)\|_{H_N}^2 d\tau \\ &= \frac{1}{2}\|u^{(1)}\|_H^2 + \frac{1}{2}\|\nabla u^{(0)}\|_{H_N}^2 + \int_0^t \langle f(\tau), \dot{u}(\tau)\rangle_H d\tau\end{aligned} \tag{3.9}$$

for every $t \in [0,T]$.

We shall see in Lemma 3.8 that the energy balance (3.9) implies the equinitegrability of (3.6). The proof of Theorem 3.2 will be obtained by combining several partial results proved in the following lemmas.

Lemma 3.3. *Under the assumptions of Theorem 3.2 there exists a function* u *satisfying* (3.2)-(3.5), (3.7), *the initial conditions* $u(0) = u^{(0)}$ *and* (3.8), *and the energy inequality*

$$\begin{aligned}&\frac{1}{2}\|\dot{u}(t)\|_H^2 + \frac{1}{2}\|\nabla u(t)\|_{H_N}^2 + \gamma\int_0^t \|\nabla\dot{u}(\tau)\|_{H_N}^2 d\tau \\ &\le \frac{1}{2}\|u^{(1)}\|_H^2 + \frac{1}{2}\|\nabla u^{(0)}\|_{H_N}^2 + \int_0^t \langle f(\tau), \dot{u}(\tau)\rangle_H d\tau\end{aligned} \tag{3.10}$$

for a.e. $t \in [0,T]$.

Proof. For $n \in \mathbb{N}$, we set $\tau_n := T/n$ and $t_n^i := i\tau_n$. For $i = 1, 2, \ldots, n$ we define $f_n^i \in H$ by

$$f_n^i := \frac{1}{\tau_n} \int_{t_n^{i-1}}^{t_n^i} f(t)dt \tag{3.11}$$

We define u_n^i for $i = -1, 0, ..., n$ inductively by the following: First,

$$u_n^0 := u^{(0)}, \quad u_n^{-1} := u^{(0)} - \tau_n u^{(1)}; \tag{3.12}$$

then, for $i = 0, 1, \ldots, n-1$, the function u_n^{i+1} is the minimizer in $V_{t_n^{i+1}}$ of

$$u \mapsto \left\| \frac{u - u_n^i}{\tau_n} - \frac{u_n^i - u_n^{i-1}}{\tau_n} \right\|_H^2 + \|\nabla u\|_{H_N}^2 + \frac{\gamma}{\tau_n} \|\nabla u - \nabla u_n^i\|_{H_N}^2 - 2\langle f_n^i, u\rangle_H$$

Note that the infimum of this functional is finite, since the sum of the first two terms is of the form $\frac{1}{\tau_n^2}\|u - a_n^i\|_H^2 + \|\nabla u\|_H^2$ for some $a_n^i \in H$, so that the functional can be bounded from below by $c_n\|u\|_{t_n^{i+1}}^2 - \frac{1}{\tau_n^2}\|a_n^i\|_H^2 - \|f_n^i\|_H \|u\|_{t_n^{i+1}}$, with $c_n = \min\{1, \frac{1}{2\tau_n^2}\}$. It follows that we have

$$\begin{aligned} \left\langle \frac{u_n^{i+1} - u_n^i}{\tau_n} - \frac{u_n^i - u_n^{i-1}}{\tau_n}, \frac{\phi}{\tau_n} \right\rangle_H &+ \langle \nabla u_n^{i+1}, \nabla \phi\rangle_{H_N} \\ +\frac{\gamma}{\tau_n}\langle \nabla u_n^{i+1} - \nabla u_n^i, \nabla\phi\rangle_{H_N} &= \langle f_n^i, \phi\rangle_H \end{aligned} \tag{3.13}$$

for every $\phi \in V_{t_n^{i+1}}$. Note that from (2.4) we can take $\phi = u_n^{i+1} - u_n^i$, and we get

$$\begin{aligned} \left\| \frac{u_n^{i+1} - u_n^i}{\tau_n} \right\|_H^2 - \left\langle \frac{u_n^{i+1} - u_n^i}{\tau_n}, \frac{u_n^i - u_n^{i-1}}{\tau_n} \right\rangle_H + \|\nabla u_n^{i+1}\|_{H_N}^2 \\ -\langle \nabla u_n^{i+1}, \nabla u_n^i\rangle_{H_N} + \frac{\gamma}{\tau_n}\|\nabla u_n^{i+1} - \nabla u_n^i\|_{H_N}^2 = \langle f_n^i, u_n^{i+1} - u_n^i\rangle_H. \end{aligned}$$

Using the fact that $\|a\|^2 - \langle a, b\rangle = \frac{1}{2}\|a\|^2 + \frac{1}{2}\|a - b\|^2 - \frac{1}{2}\|b\|^2$, multiplying by 2, and rearranging, we can write

$$\begin{aligned} \left\| \frac{u_n^{i+1} - u_n^i}{\tau_n} \right\|_H^2 + \left\| \frac{u_n^{i+1} - u_n^i}{\tau_n} - \frac{u_n^i - u_n^{i-1}}{\tau_n} \right\|_H^2 + \|\nabla u_n^{i+1}\|_{H_N}^2 + \|\nabla u_n^{i+1} - \nabla u_n^i\|_{H_N}^2 \\ +\frac{2\gamma}{\tau_n}\|\nabla u_n^{i+1} - \nabla u_n^i\|_{H_N}^2 = \left\| \frac{u_n^i - u_n^{i-1}}{\tau_n} \right\|_H^2 + \|\nabla u_n^i\|_{H_N}^2 + 2\langle f_n^i, u_n^{i+1} - u_n^i\rangle_H. \end{aligned} \tag{3.14}$$

Summing from $i = 0$ to j and using (3.12), we get

$$\begin{aligned} \left\| \frac{u_n^{j+1} - u_n^j}{\tau_n} \right\|_H^2 + \|\nabla u_n^{j+1}\|_{H_N}^2 + \sum_{i=0}^j \left\| \frac{u_n^{i+1} - u_n^i}{\tau_n} - \frac{u_n^i - u_n^{i-1}}{\tau_n} \right\|_H^2 + \sum_{i=0}^j \|\nabla u_n^{i+1} - \nabla u_n^i\|_{H_N}^2 \\ +\frac{2\gamma}{\tau_n}\sum_{i=0}^j \|\nabla u_n^{i+1} - \nabla u_n^i\|_{H_N}^2 = \|u^{(1)}\|_H^2 + \|\nabla u^{(0)}\|_{H_N}^2 + 2\sum_{i=0}^j \langle f_n^i, u_n^{i+1} - u_n^i\rangle_H. \end{aligned} \tag{3.15}$$

We now define $u_n, \tilde{u}_n, v_n : [0, T] \to V$ for $t \in (t_n^i, t_n^{i+1}]$ by

$$u_n(t) := u_n^i + (t - t_n^i)\frac{u_n^{i+1} - u_n^i}{\tau_n}, \tag{3.16}$$

$$\tilde{u}_n(t) := u_n^{i+1}, \qquad f_n(t) := f_n^i, \tag{3.17}$$

$$v_n(t) := \frac{u_n^i - u_n^{i-1}}{\tau_n} + \frac{t - t_n^i}{\tau_n}\left[\frac{u_n^{i+1} - u_n^i}{\tau_n} - \frac{u_n^i - u_n^{i-1}}{\tau_n}\right]. \tag{3.18}$$

Rewriting (3.15) using the above, for every $t \in (t_n^j, t_n^{j+1})$ we now have

$$\|\dot{u}_n(t)\|_H^2 + \|\nabla u_n(t_n^{j+1})\|_{H_N}^2 + \tau_n \int_0^{t_n^{j+1}} \|\dot{v}_n(t)\|_H^2\, dt + \tau_n \int_0^{t_n^{j+1}} \|\nabla \dot{u}_n(t)\|_{H_N}^2 dt$$

$$+2\gamma \int_0^{t_n^{j+1}} \|\nabla \dot{u}_n(t)\|_{H_N}^2\, dt = \|u^{(1)}\|_H^2 + \|\nabla u^{(0)}\|_{H_N}^2 + 2\int_0^{t_n^{j+1}} \langle f_n(t), \dot{u}_n(t)\rangle_H\, dt. \quad (3.19)$$

The right-hand side above is bounded as long as $M_n := \max_t \|\dot{u}_n(t)\|_H$ is bounded. From (3.19) we have that

$$M_n^2 \le \|u^{(1)}\|_H^2 + \|\nabla u^{(0)}\|_{H_N}^2 + 2\|f\|_{L^2(0,T;H)} T^{1/2} M_n\,.$$

This implies that M_n is bounded, and so is the right-hand side of (3.19).

We then have that

$$\nabla u_n(t) \text{ and } \nabla \tilde{u}_n(t) \text{ are bounded in } H_N \text{ uniformly in } t \text{ and } n, \quad (3.20)$$

$$\gamma \nabla \dot{u}_n \text{ is bounded in } L^2(0,T;H_N) \text{ uniformly in } n, \quad (3.21)$$

$$\dot{u}_n(t) \text{ and } v_n(t) \text{ are bounded in } H \text{ uniformly in } t \text{ and } n. \quad (3.22)$$

We note that (3.22) together with the fact that $u^{(0)} \in H$ implies that u_n is bounded in H uniformly in t and n. This together with (3.20) gives

$$u_n(t) \text{ is bounded in } V \text{ uniformly in } t \text{ and } n. \quad (3.23)$$

Furthermore, using (3.16), (3.17), and (3.18) in (3.13) gives that for all $t \in (t_n^i, t_n^{i+1})$,

$$\langle \dot{v}_n(t), \phi\rangle_H + \langle \nabla \tilde{u}_n(t) + \gamma \nabla \dot{u}_n(t), \nabla \phi\rangle_{H_N} = \langle f_n(t), \phi\rangle_H \quad (3.24)$$

for every $\phi \in V_{t_n^{i+1}}$. This together with (3.20) and (3.21) gives that for $t \in (t_n^i, t_n^{i+1})$,

$$\|\dot{v}_n(t)\|_{t_n^{i+1}}^* \le c \quad (3.25)$$

where c is independent of t, i, n. Using the fact that

$$\|\cdot\|_s^* \le \|\cdot\|_t^* \text{ for } s < t, \quad (3.26)$$

we get

$$\|\dot{v}_n(t)\|_s^* \le c \quad (3.27)$$

for all $s < t$, and for every n.

Using (3.21), (3.22), (3.23), and (3.27) we get

$$u_n \text{ is bounded in } H^1(0,T;V) \text{ and in } W^{1,\infty}(0,T;H), \quad (3.28)$$

$$v_n \text{ is bounded in } L^\infty(0,T;H), \quad (3.29)$$

$$v_n \text{ is bounded in } W^{1,\infty}(s,T;V_s^*) \text{ for every } s \in [0,T]. \quad (3.30)$$

Let us fix a countable dense subset D of $[0,T]$. By a diagonal argument we obtain a subsequence, not relabeled, such that

$$u_n \rightharpoonup u \text{ weakly in } H^1(0,T;V), \quad (3.31)$$

$$v_n \rightharpoonup v \text{ weakly in } L^2(0,T;H), \quad (3.32)$$

$$v_n \rightharpoonup v \text{ weakly in } H^1(s,T;V_s^*) \text{ for every } s \in D. \quad (3.33)$$

It is easy to see that in fact

$$u \in H^1(0,T;V) \cap W^{1,\infty}(0,T;H), \quad (3.34)$$

$$v \in L^\infty(0,T;H), \quad (3.35)$$

$$v \in W^{1,\infty}(t,T;V_s^*) \text{ for every } s,\, t \in [0,T] \text{ with } s < t. \quad (3.36)$$

Moreover (3.27) gives

$$\|\dot{v}\|_{L^\infty(t,T;V_s^*)} \le c \quad \text{for every } s,\, t \in [0,T] \text{ with } s < t. \quad (3.37)$$

We now show that

$$v(t) = \dot{u}(t) \text{ in } H \text{ for a.e. } t \in [0,T]. \tag{3.38}$$

First, for every $t \in (t_n^i, t_n^{i+1})$, we have $\dot{u}_n(t) = v_n(t_n^{i+1})$, so that using (3.25) we have

$$\|\dot{u}_n(t) - v_n(t)\|^*_{t_n^{i+1}} = \|v_n(t_n^{i+1}) - v_n(t)\|^*_{t_n^{i+1}} \le \int_{t_n^i}^{t_n^{i+1}} \|\dot{v}_n(\tau)\|^*_{t_n^{i+1}} d\tau \le c\tau_n.$$

From (3.26), we have $\|\dot{u}_n(t) - v_n(t)\|^*_s \le c\tau_n$ for all $s < t$. Together with (3.32), this shows that $\dot{u}_n \rightharpoonup v$ weakly in $L^2(s,T;V_s^*)$ for all $s \in [0,T]$. On the other hand, $\dot{u}_n \rightharpoonup \dot{u}$ weakly in $L^2(0,T;H)$, and so $v(t) = \dot{u}(t)$ in V_s^* for every $s \in [0,T]$ and for a.e. $t \in (s,T)$. Since $v(t)$ and $\dot{u}(t)$ belong to H, the previous equality means that $\langle v(t), \phi\rangle_H = \langle v(t), \phi\rangle_s = \langle \dot{u}(t), \phi\rangle_s = \langle \dot{u}(t), \phi\rangle_H$ for every $\phi \in V_s$. The density of V_s in H allows us to conclude that $v(t) = \dot{u}(t)$ as elements of H. This concludes the proof of (3.38). From (3.36) , (3.37), and (3.38) we deduce that

$$u \in W^{2,\infty}(t,T;V_s^*) \quad \text{and} \quad \|\ddot{u}\|_{L^\infty(t,T;V_s^*)} \le c \tag{3.39}$$

for every s, $t \in [0,T]$ with $s < t$. This gives (3.4) and (3.5) by Lemma 2.2.

From (3.28) we know that u_n is Lipschitz with values in H uniformly in n. Now, since $\tilde{u}_n(t) = u_n(t_n^{i+1})$ for $t \in (t_n^i, t_n^{i+1}]$ and , we have, as above, that

$$\tilde{u}_n \rightharpoonup u \text{ weakly in } L^2(0,T;H).$$

Since $\nabla\tilde{u}_n$ is bounded in $L^2(0,T;H_N)$ by (3.20), we obtain also that

$$\nabla\tilde{u}_n \rightharpoonup \nabla u \text{ weakly in } L^2(0,T;H_N). \tag{3.40}$$

This gives

$$\tilde{u}_n \rightharpoonup u \text{ in } L^2(0,T;V). \tag{3.41}$$

Furthermore, note that $\tilde{u}_n(t-\tau_n) \in V_t$ for every $t \in [0,T]$, and

$$\tilde{u}_n(\cdot - \tau_n) \rightharpoonup u \text{ weakly in } L^2(0,T;V).$$

Since the linear subspace $\{v \in L^2(0,T;V) : v(t) \in V_t \text{ for a.e. } t \in [0,T]\}$ is strongly closed, it is weakly closed in $L^2(0,T;V)$. Therefore $u(t) \in V_t$ for a.e. $t \in [0,T]$. For every $t \in (0,T]$ there exists $t_n \nearrow t$ such that $u(t_n) \in V_{t_n} \subset V_t$ for every n. Since $u(t_n) \to u(t)$ strongly in V by (3.34), we obtain $u(t) \in V_t$. Together with the inclusion $u(0) = u^{(0)} \in V_0$, this proves (3.3).

We now prove that (3.7) holds a.e. $t \in [0,T]$ for every $\phi \in V_t$. We first claim that for $s \in D$ and for all $\phi \in V_s$, we have

$$\langle \ddot{u}(t), \phi\rangle_s + \langle \nabla u(t) + \gamma\nabla\dot{u}(t), \nabla\phi\rangle_{H_N} = \langle f(t), \phi\rangle_H \text{ for a.e. } t > s. \tag{3.42}$$

We first fix $s \in D$ and $\phi \in V_s$. Using (3.24), we have that for a.e. $t > s$,

$$\langle \dot{v}_n(t), \phi\rangle_H + \langle \nabla\tilde{u}_n(t) + \gamma\nabla\dot{u}_n(t), \nabla\phi\rangle_{H_N} = \langle f_n(t), \phi\rangle_H.$$

Hence for every $s < t_1 < t_2 < T$ we have

$$\int_{t_1}^{t_2} \Big(\langle \dot{v}_n(t), \phi\rangle_H + \langle \nabla\tilde{u}_n(t) + \gamma\nabla\dot{u}_n(t), \nabla\phi\rangle_{H_N} - \langle f_n(t), \phi\rangle_H \Big) dt = 0. \tag{3.43}$$

Using (3.11) and (3.17) we obtain that $f_n \to f$ in $L^2(0,T;H)$, hence

$$\int_{t_1}^{t_2} \langle f_n(t), \phi\rangle_H \, dt \to \int_{t_1}^{t_2} \langle f(t), \phi\rangle_H \, dt$$

We know that $\dot{v}_n \rightharpoonup \dot{v}$ weakly in $L^2(s,T;V_s^*)$ from (3.33). Since $\dot{u} = v$ in $H^1(s,T;V_s^*)$, we also have that $\ddot{u} = \dot{v}$ in $L^2(s,T;V_s^*)$. Using also (3.31) and (3.40), we can pass to the limit in (3.43) and we get

$$\int_{t_1}^{t_2} \Big(\langle \ddot{u}(t), \phi\rangle_s + \langle \nabla u(t) + \gamma\nabla\dot{u}(t), \nabla\phi\rangle_{H_N} - \langle f(t), \phi\rangle_H \Big) dt = 0.$$

This implies (3.42).

Notice that since V_s is separable, the set N_s of $t > s$ for which (3.42) does not hold can be taken independent of ϕ. We set W to be the union over $s \in D$ of the sets N_s, so that W also has measure zero. It follows that for every $t \notin W$ and for every $s \in D$, $s < t$, we have

$$\langle \ddot{u}(t), \phi \rangle_s + \langle \nabla u(t) + \gamma \nabla \dot{u}(t), \nabla \phi \rangle_{H_N} = \langle f(t), \phi \rangle_H. \tag{3.44}$$

Using Lemma 2.3, it follows that for a.e. $t \notin S$, for every $\phi \in V_t$, and for every $s_n \nearrow t$, with $s_n \in D$, there exists $\phi_n \in V_{s_n}$ such that $\phi_n \to \phi$ strongly in V_t. Now note that

$$\begin{aligned} &\langle \ddot{u}(t), \phi_n \rangle_t + \langle \nabla u(t) + \gamma \nabla \dot{u}(t), \nabla \phi_n \rangle_{H_N} - \langle f(t), \phi_n \rangle_H \\ &= \langle \ddot{u}(t), \phi_n \rangle_{s_n} + \langle \nabla u(t) + \gamma \nabla \dot{u}(t), \nabla \phi_n \rangle_{H_N} - \langle f(t), \phi \rangle_H = 0. \end{aligned}$$

The convergence of the ϕ_n to ϕ gives (3.7).

For every $t \in (0,T)$ and $n \in \mathbb{N}$ there exists a unique j such that $t_n^j < t \le t_n^{j+1} =: t_n^*$ By (3.11) and (3.17) the sequence f_n converges to f strongly in $L^2(0,T;H)$. By (3.31) the sequence $\dot{u}_n$ converges to $\dot{u}$ strongly in $L^2(0,T;H)$. Therefore

$$\int_0^{t_n^*} \langle f_n(\tau), \dot{u}_n(\tau) \rangle_H d\tau \to \int_0^t \langle f(\tau), \dot{u}(\tau) \rangle_H d\tau$$

By (3.31) and (3.40), from (3.19) we obtain (3.10) by weak lower semicontinuity.

It remains to prove (3.8). It is enough to show that, if t_k are Lebesgue points for $t \mapsto \|\dot{u}(t)\|_H^2$ and $t_k \to 0$, then

$$\dot{u}(t_k) \to u^{(1)} \quad \text{strongly in } H. \tag{3.45}$$

By (3.4) $\dot{u}$ belongs to $W^{1,\infty}(0,T;V_0^*)$, so that, if $t_k \to 0$, then $\dot{u}(t_k) \to u^{(1)}$ in V_0^*. Since $\dot{u}(t_k)$ is bounded in H by (3.2) and $V_0 \subset H$ is dense, it follows that $\dot{u}(t_k) \rightharpoonup u^{(1)}$ weakly in H. Therefore (3.45) is equivalent to

$$\limsup_{k\to\infty} \|\dot{u}(t_k)\|_H \le \|u^{(1)}\|_H. \tag{3.46}$$

By (3.2) and (3.10) there exists a constant $C > 0$ such that

$$\|\dot{u}(t)\|_H^2 + \|\nabla u(t)\|_{H_N}^2 \le \|u^{(1)}\|_H^2 + \|\nabla u^{(0)}\|_{H_N}^2 + Ct^{1/2} \tag{3.47}$$

for a.e. $t \in [0,T]$. Since $t \mapsto \nabla u(t)$ is continuous from $[0,T]$ to H_N by (3.2), inequality (3.47) holds in for all Lebesgue points of $t \mapsto \|\dot{u}(t)\|_H^2$, in particular for $t = t_k$. By continuity we have $\|\nabla u(t_k)\|_{H_N} \to \|\nabla u^{(0)}\|_{H_N}$, so that (3.46) follows from (3.47). This proves (3.45) and concludes the proof of (3.8). □

The following lemma provides an equivalent formulation of (3.6) in terms of the behavior of the functions $\|\dot{u}(t)\|_H^2 - \|\dot{u}(t-h)\|_H^2$.

Lemma 3.4. *Assume* (3.2)-(3.5). *Then there exists a constant* $C > 0$ *such that*

$$\begin{aligned} -C\|\dot{u}(t-h)\|_{t-h} &\le \frac{1}{h}(\|\dot{u}(t)\|_H^2 - \|\dot{u}(t-h)\|_H^2) \\ &\le \frac{1}{h}\|\dot{u}(t) - \dot{u}(t-h)\|_H^2 + C\|\dot{u}(t-h)\|_{t-h} \end{aligned} \tag{3.48}$$

for every $h \in (0,T)$ *and a.e.* $t \in (h,T)$. *In particular, the equiintegrability of* (3.6) *holds if and only if for every* $s \in (0,T)$ *the functions*

$$t \mapsto \frac{1}{h}(\|\dot{u}(t)\|_H^2 - \|\dot{u}(t-h)\|_H^2), \quad h \in (0,s),$$

are equiintegrable on (s,T).

Proof. For every $h \in (0,T)$ and for a.e. $t \in (h,T)$ we have

$$\begin{aligned}\frac{1}{h}(\|\dot u(t)\|_H^2 - \|\dot u(t-h)\|_H^2) &= \Big\langle \frac{\dot u(t)-\dot u(t-h)}{h}, \dot u(t)+\dot u(t-h)\Big\rangle_t \\ &= \frac{1}{h}\|\dot u(t)-\dot u(t-h)\|_H^2 + 2\Big\langle \frac{\dot u(t)-\dot u(t-h)}{h}, \dot u(t-h)\Big\rangle_{t-h}. \end{aligned} \tag{3.49}$$

By (3.3) and (3.4) the function $\tau \mapsto \langle \dot u(\tau), \dot u(t-h)\rangle_{t-h}$ belongs to $W^{1,\infty}(t-h,T)$ and its time derivative is $\tau \mapsto \langle \ddot u(\tau), \dot u(t-h)\rangle_{t-h}$. By (3.5) there exists a constant c independent of t and h such that the absolute value of the last duality product in (3.49) is bounded by $c\|\dot u(t-h)\|_{t-h}$. This implies (3.48). Since the family of functions $t \mapsto \|\dot u(t-h)\|_{t-h}$, $0<h<s$, is equiintegrable in (s,T) by (3.2), the conclusion follows. □

Lemma 3.5. *Under the assumptions of Theorem 3.2, suppose that u satisfies* (3.2)-(3.5), *and let s, $t \in (0,T]$ be Lebesgue points for $\|\dot u(\cdot)\|_H^2$, with $s<t$. Then*

$$\|\dot u(t)\|_H^2 - \|\dot u(s)\|_H^2 \geq 2\int_s^t \langle \ddot u(\tau), \dot u(\tau)\rangle_\tau \, d\tau. \tag{3.50}$$

Proof. Fix $0<h<s$. Integrating both sides of the first equality of (3.49) from s to t, we get

$$\frac{1}{h}\int_{t-h}^t \|\dot u(\tau)\|_H^2 d\tau - \frac{1}{h}\int_{s-h}^s \|\dot u(\tau)\|_H^2 d\tau = \int_s^t \Big\langle \frac{\dot u(\tau)-\dot u(\tau-h)}{h}, \dot u(\tau)+\dot u(\tau-h)\Big\rangle_\tau d\tau.$$

Take a sequence $h \to 0$ such that

$$\dot u(\tau)+\dot u(\tau-h) \to 2\dot u(\tau) \text{ strongly in } V$$

for a.e. τ, which we can do since $\dot u \in L^2(0,T;V)$. By Lemma 2.2 we also have

$$\frac{\dot u(\tau)-\dot u(\tau-h)}{h} \rightharpoonup \ddot u(\tau) \text{ weakly in } V_\tau^*$$

for a.e. τ, so that

$$\Big\langle \frac{\dot u(\tau)-\dot u(\tau-h)}{h}, \dot u(\tau)+\dot u(\tau-h)\Big\rangle_\tau \to \langle \ddot u(\tau), 2\dot u(\tau)\rangle_\tau.$$

By (3.48) and (3.49) we can apply the Fatou Lemma (with an equiintegrable minorant) and we get

$$\liminf_{h\to 0+} \int_s^t \Big\langle \frac{\dot u(\tau)-\dot u(\tau-h)}{h}, \dot u(\tau)+\dot u(\tau-h)\Big\rangle_\tau d\tau \geq \int_s^t \langle \ddot u(\tau), 2\dot u(t)\rangle_\tau \, d\tau,$$

while

$$\frac{1}{h}\int_{t-h}^t \|\dot u(\tau)\|_H^2 dt - \frac{1}{h}\int_{s-h}^s \|\dot u(\tau)\|_H^2 d\tau \to \|\dot u(t)\|_H^2 - \|\dot u(s)\|_H^2,$$

since s and t are Lebesgue points for $\|\dot u(\cdot)\|_H^2$. □

Lemma 3.6. *Under the assumptions of Theorem 3.2, suppose that u satisfies* (3.2)-(3.5) *and the initial condition* (3.8) *for some $u^{(1)} \in H$. Then*

$$\|\dot u(t)\|_H^2 - \|u^{(1)}\|_H^2 \geq 2\int_0^t \langle \ddot u(\tau), \dot u(\tau)\rangle_s ds. \tag{3.51}$$

for every Lebesgue point $t \in (0,T]$ of $\|\dot u(\cdot)\|_H^2$.

Proof. By (3.8) there exists a sequence h_n of positive numbers converging to 0 such that $\dot u(h_n t) \to u^{(1)}$ strongly in H for a.e. $t \in [0,T]$. Then we choose t so that $t_n := h_n t$ is a Lebesgue point of $\|\dot u(\cdot)\|_H^2$ for every n and $\dot u(t_n) \to u^{(1)}$ strongly in H. By Lemma 3.5 we have

$$\|\dot u(t)\|_H^2 - \|\dot u(t_n)\|_H^2 \geq 2\int_{t_n}^t \langle \ddot u(\tau), \dot u(\tau)\rangle_\tau d\tau.$$

Passing to the limit as $n \to \infty$ we get (3.51). $\square$

We now prove that the function u obtained in Lemma 3.3 satisfies the energy balance for a.e. $t \in (0,T)$.

Lemma 3.7. *Under the assumptions of Theorem 3.2, suppose that u satisfies (3.2)-(3.5), (3.7), (3.10), and the initial conditions $u(0) = u^{(0)}$ and (3.8). Then u satisfies the energy balance (3.9) for every Lebesgue point $t \in (0,T]$ of $\|\dot u(\cdot)\|^2_H$.*

Proof. By (3.7) and since $\dot u(t) \in V_t$ we have that

$$\langle \ddot u(t), \dot u(t)\rangle_t + \langle \nabla u(t), \nabla \dot u(t)\rangle_{H_N} + \gamma \|\nabla \dot u(t)\|^2_{H_N} = \langle f(t), \dot u(t)\rangle_H$$

for a.e. $t \in [0,T]$. Integrating from 0 to t and using Lemma 3.6, for every Lebesgue point $t \in [0,T]$ of $\|\dot u(\cdot)\|^2_H$ we get

$$\frac12 \|\dot u(t)\|^2_H - \frac12 \|u^{(1)}\|^2_H + \frac12 \int_0^t \frac{d}{d\tau}\|\nabla u(\tau)\|^2_{H_N} d\tau + \gamma \int_0^t \|\nabla \dot u(\tau)\|^2_{H_N} d\tau \geq \int_0^t \langle f(\tau), \dot u(\tau)\rangle_H ds.$$

Together with (3.10) this inequality gives (3.9). $\square$

Lemma 3.8. *Under the assumptions of Theorem 3.2, suppose that u satisfies (3.2) and (3.9) for a.e. $t \in (0,T)$. Then the functions (3.6) are equiintegrable.*

Proof. Let us fix $s \in (0,T)$. By (3.9) for every $h \in (0,s)$ and for a.e. $t \in (s,T)$ we have

$$\begin{aligned} \|\dot u(t)\|^2_H - \|\dot u(t-h)\|^2_H + \|\nabla u(t)\|^2_{H_N} - \|\nabla u(t-h)\|^2_{H_N} + 2\gamma \int_{t-h}^t \|\nabla \dot u(\tau)\|^2_{H_N} d\tau \\ = 2 \int_{t-h}^t \langle f(\tau), \dot u(\tau)\rangle_H d\tau \end{aligned}$$

By (3.2) there exists a constant $C > 0$ such that $\|\nabla u(t)\|_{H_N} \leq C$ for a.e. $t \in (0,T)$, so that the previous equality gives

$$\begin{aligned} \frac1h \Big| \|\dot u(t)\|^2_H - \|\dot u(t-h)\|^2_H \Big| \leq \frac{2C}{h} \int_{t-h}^t \|\nabla \dot u(\tau)\|_{H_N} d\tau + \frac{2\gamma}{h} \int_{t-h}^t \|\nabla \dot u(\tau)\|^2_{H_N} d\tau \\ + \frac2h \int_{t-h}^t \langle f(\tau), \dot u(\tau)\rangle_H d\tau \end{aligned}$$

Since the functions $\tau \mapsto \|\nabla \dot u(\tau)\|_{H_N}$, $\tau \mapsto \|\nabla \dot u(\tau)\|^2_{H_N}$, and $\tau \mapsto \langle f(\tau), \dot u(\tau)\rangle_H$ belong to $L^1(0,T)$, the right-hand side of the previous inequality converges in $L^1(s,T)$ as $h \to 0+$. Therefore it is equiintegrable on (s,T). By Lemma 3.4 this implies the equiintegrability of (3.6). $\square$

Lemma 3.9. *Under the assumptions of Theorem 3.2, only one weak solution of the damped wave equation considered in Definition 3.1 satisfies the initial conditions $u(0) = u^{(0)}$ and (3.8).*

Proof. Since (3.6) is preserved by linear combinations, the difference v between two solutions is a solution with $f = 0$ satisfying the initial conditions $v(0) = 0$ and

$$\lim_{h \to 0+} \frac1h \int_0^h \|\dot v(t)\|^2_H dt = 0.$$

By Theorem 3.7 we have

$$\frac12 \|\dot v(t)\|^2_H + \frac12 \|\nabla v(t)\|^2_{H_N} + \gamma \int_0^t \|\nabla \dot v(s)\|^2_{H_N} ds = 0$$

for a.e. $t \in [0,T]$. This implies that $\dot v(t) = 0$ in H for a.e. $t \in [0,T]$. Since $v \in W^{1,\infty}(0,T;H)$ by (3.2), and $v(0) = 0$, we conclude that $v(t) = 0$ for every $t \in [0,T]$. $\square$

Lemma 3.10. *Under the assumptions of Theorem 3.2, let u be the weak solution of the damped wave equation considered in Definition 3.1, with initial conditions $u(0) = u^{(0)}$ and (3.8). Then $t \mapsto \dot u(t)$ is continuous from $[0,T]$ to H and (3.9) holds for every $t \in [0,T]$.*

Proof. By (3.4) the function $t \mapsto \dot u(t)$ is continuous from $[0,T]$ to V_0^* and by (3.2) $\dot u(t) \in H$ for a.e. $t \in [0,T]$, and there exists a constant M such that

$$\|\dot u(t)\|_H \leq M \quad \text{for a.e. } t \in [0,T]. \tag{3.52}$$

We claim that

$$\dot u(t) \in H \ \text{ and } \ \|\dot u(t)\|_H \leq M \ \text{ for every } \ t \in [0,T], \tag{3.53}$$

$$t \mapsto \dot u(t) \ \text{ is weakly continuous from } [0,T] \text{ to } H. \tag{3.54}$$

Given $t \in [0,T]$, by (3.52) there exists a sequence t_n in $[0,T]$ converging to t such that $\dot u(t_n)$ is bounded in H. Since $\dot u(t_n) \to \dot u(t)$ strongly in V_0^* and the embedding $H \hookrightarrow V_0^*$ is continuous, we conclude that $\dot u(t) \in H$, $\|\dot u(t)\|_H \leq M$, and $\dot u(t_n) \rightharpoonup \dot u(t)$ weakly in H. This proves (3.53). The same argument with an arbitrary t_n converging to t gives (3.54).

By Lemmas 3.3, 3.7, 3.8, and 3.9 the function u satisfies the energy balance (3.9) for a.e. $t \in (0,T)$. Using (3.54) and the weak lower semicontinuity of the norm, we can approximate an arbitrary $t \in [0,T]$ and we obtain (3.10) for every $t \in [0,T]$.

Define $\Gamma(t) := \Gamma(T)$ for every $t > T$ and fix $t_0 \in (0,T]$. Since $u(t_0) \in V_{t_0}$ by (3.3) and $\dot u(t_0) \in H$, by the lemmas mentioned above there exists a unique weak solution v of the damped wave equation on the interval $[t_0, T+1]$ (Definition 3.1) with initial conditions $v(t_0) = u(t_0)$ and

$$\lim_{h \to 0+} \frac{1}{h} \int_{t_0}^{t_0+h} \|\dot v(\tau) - \dot u(t_0)\|_H^2 d\tau = 0.$$

Define $w\colon [0, T+1] \to V$ by $w(t) = u(t)$ for $t \leq t_0$ and by $w(t) = v(t)$ for $t > t_0$. It is easy to check that w satisfies (3.2)-(3.5), (3.7), and the initial conditions $w(0) = u^{(0)}$ and

$$\lim_{h \to 0+} \frac{1}{h} \int_0^h \|\dot w(t) - u^{(1)}\|_H^2 dt = 0.$$

Moreover it satisfies the energy inequality (3.10) for a.e. $t \in (0, T+1)$. By Lemmas 3.7 and 3.8 the function w satisfies also the equiintegrability condition (3.6), so that it is a weak solution of the damped wave equation on $[0, T+1]$ (Definition 3.1). Let $t \in (t_0, T+1]$ be a Lebesgue point of $\|\dot v(\cdot)\|_H^2$. By Theorem 3.7 we have

$$\begin{aligned} &\frac{1}{2}\|\dot w(t)\|_H^2 + \frac{1}{2}\|\nabla w(t)\|_{H_N}^2 + \gamma \int_0^t \|\nabla \dot w(\tau)\|_{H_N}^2 d\tau \\ &= \frac{1}{2}\|u^{(1)}\|_H^2 + \frac{1}{2}\|\nabla u^{(0)}\|_{H_N}^2 + \int_0^t \langle f(\tau), \dot w(\tau)\rangle_H \, d\tau \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}\|\dot v(t)\|_H^2 - \frac{1}{2}\|\dot u(t_0)\|_H^2 + \frac{1}{2}\|\nabla v(t)\|_{H_N}^2 - \frac{1}{2}\|\nabla u(t_0)\|_{H_N}^2 + \gamma \int_{t_0}^t \|\nabla \dot v(\tau)\|_{H_N}^2 d\tau \\ = \int_{t_0}^t \langle f(\tau), \dot v(\tau)\rangle_H \, d\tau. \end{aligned}$$

Subtracting the second equation from the first one we get

$$\begin{aligned} &\frac{1}{2}\|\dot u(t_0)\|_H^2 + \frac{1}{2}\|\nabla u(t_0)\|_{H_N}^2 + \gamma \int_0^{t_0} \|\nabla \dot u(\tau)\|_{H_N}^2 d\tau \\ &= \frac{1}{2}\|u^{(1)}\|_H^2 + \frac{1}{2}\|\nabla u^{(0)}\|_{H_N}^2 + \int_0^{t_0} \langle f(\tau), \dot u(\tau)\rangle_H \, d\tau. \end{aligned}$$

This shows that (3.9) holds for $t = t_0$. The arbitrariness of t_0 implies that (3.9) holds for every $t \in [0,T]$. Since $t \mapsto \|\nabla u(t)\|_{H_N}^2$ is continuous by (3.2), we deduce from (3.9) holds for that $t \mapsto \|\dot{u}(t)\|_H^2$ is continuous. Together with (3.54), this proves that $t \mapsto \dot{u}(t)$ is continuous from $[0,T]$ to H. □

Proof of Theorem 3.2. By Lemma 3.3 there exists a function u satisfying (3.2)-(3.5), (3.7), the initial conditions $u(0) = u^{(0)}$ and (3.8), and the energy inequality (3.10) for a.e. $t \in [0,T]$. By Lemma 3.7 this function satisfies also the energy equality (3.9) for a.e. $t \in [0,T]$. By Lemma 3.8 the functions (3.6) are equiintegrable, so that u is a weak solution of the damped wave equation according to Definition 3.1. The uniqueness is proved in Lemma 3.9. The continuity of $t \mapsto \dot{u}(t)$ from $[0,T]$ to H and the energy equality (3.9) for every $t \in [0,T]$ follow from Lemma 3.10. □

4. The Undamped Wave Equation

In this section we study weak solutions of the undamped wave equation

$$\ddot{u} - \Delta u = f \tag{4.1}$$

on a cracking domain.

Definition 4.1. Assume (2.3) and (2.4), and let $f \in L^2(0,T;H)$. We say that u is a weak solution of the wave equation (4.1) on the time dependent cracking domain $t \mapsto \Omega \setminus \Gamma(t)$ if

$$u \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T;H), \tag{4.2}$$

$$\text{for every } t \in [0,T] \text{ we have } u(t) \in V_t, \tag{4.3}$$

$$\text{for every } s \in [0,T) \text{ we have } u \in W^{2,\infty}(s,T;V_s^*), \tag{4.4}$$

$$\sup_{s\in[0,T)} \|\ddot{u}\|_{L^\infty(s,T;V_s^*)} < +\infty, \tag{4.5}$$

and for a.e. $t \in [0,T]$

$$\langle \ddot{u}(t), \phi \rangle_t + \langle \nabla u(t), \nabla \phi \rangle_{H_N} = \langle f(t), \phi \rangle_H \quad \text{for every } \phi \in V_t, \tag{4.6}$$

where $\ddot{u}(t)$ is given by Definition 2.4.

In the next theorem we will prove that there exist solutions satisfying the initial condition for u in the strong form

$$\lim_{h\to 0+} \frac{1}{h} \int_0^h \Big(\|u(t) - u^{(0)}\|_H^2 + \|\nabla u(t) - \nabla u^{(0)}\|_{H_N}^2 \Big)\, dt = 0, \tag{4.7}$$

and that for $\dot{u}$ in the sense of (3.8).

Theorem 4.2. *Assume* (2.3) *and* (2.4)*, and let* $u^{(0)} \in V_0$, $u^{(1)} \in H$, *and* $f \in L^2(0,T;H)$ *be given. Then there exists a weak solution of the wave equation considered in Definition 4.1 satisfying the initial conditions* (4.7) *and* (3.8)*.*

Proof. We proceed exactly as in the proof of Lemma 3.3. In fact, the proof is identical, with $\gamma = 0$, through (3.19).

We then continue as in Lemma 3.3 with $\gamma = 0$, where (3.21) is useless, and (3.28) is replaced by

$$u_n \text{ is bounded in } L^\infty(0,T;V) \text{ and in } W^{1,\infty}(0,T;H). \tag{4.8}$$

As a consequence of this change, (3.31) is replaced by

$$u_n \rightharpoonup u \text{ weakly in } L^2(0,T;V) \text{ and in } H^1(0,T;H), \tag{4.9}$$

and (3.34) is replaced by $u \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T;H)$. The proof that u is a solution proceeds as in the damped case.

It remains to prove that u satisfies the initial conditions (4.7) and (3.8). It is enough to show that, if t_k are Lebesgue points for both $t \mapsto \|\dot{u}(t)\|_H^2$ and $t \mapsto \|\nabla u(t)\|_{H_N}^2$, and $t_k \to 0$, then

$$\nabla u(t_k) \to \nabla u^{(0)} \quad \text{strongly in } H_N \quad \text{and} \quad \dot{u}(t_k) \to u^{(1)} \quad \text{strongly in } H. \tag{4.10}$$

As in the proof of Theorem 3.2 we can show that $\dot{u}(t_k) \rightharpoonup u^{(1)}$ weakly in H. Moreover (4.2) implies that $u(t_k) \to u(0) = u^{(0)}$ strongly in H and that $\nabla u(t_k)$ is bounded in H_N. This imples that $\nabla u(t_k) \rightharpoonup \nabla u^{(0)}$ weakly in H_N.

Therefore (4.10) is equivalent to

$$\limsup_{k\to\infty} \left(\|\nabla u(t_k)\|^2_{H_N} + \|\dot{u}(t_k)\|^2_H\right) \leq \|\nabla u^{(0)}\|^2_{H_N} + \|u^{(1)}\|^2_H. \tag{4.11}$$

For every $t \in (0,T)$ and $n \in \mathbb{N}$ there exists a unique j such that $t_n^j < t \leq t_n^{j+1} =: t_n^*$. From (3.19), we have that

$$\|\dot{u}_n(t)\|^2_H + \|\nabla \tilde{u}_n(t)\|^2_{H_N} \leq \|u^{(1)}\|^2_H + \|\nabla u^{(0)}\|^2_{H_N} + 2M\left(\int_0^{t_n^*} \|\dot{u}_n(s)\|^2_H ds\right)^{1/2},$$

where $M := \|f\|_{L^2(0,T;H)}$. Since $\|\dot{u}_n(t)\|_H$ is uniformly bounded by (4.8), there exists $C > 0$ such that

$$\|\dot{u}_n(t)\|^2_H + \|\nabla \tilde{u}_n(t)\|^2_{H_N} \leq \|u^{(1)}\|^2_H + \|\nabla u^{(0)}\|^2_{H_N} + C(t_n^*)^{1/2}.$$

Therefore, from (3.32) and (3.40) and the fact that the chosen t_k are Lebesgue points, we get

$$\|\dot{u}(t_k)\|^2_H + \|\nabla u(t_k)\|^2_{H_N} \leq \|u^{(1)}\|^2_H + \|\nabla u^{(0)}\|^2_{H_N} + Ct_k^{1/2}$$

for every k. This proves (4.11) and concludes the proof of the theorem. □

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References

[1] Ambrosio, A.; Braides, A.: Energies in SBV and variational models in fracture mechanics, in *Homogenization and Applications to Material Sciences* (Nice, 1995), GAKUTO Internat. Ser. Math. Sci. Appl., Vol. 9 (Gakkotosho, 1995), 1-22.

[2] Ambrosio L.; Fusco N.; Pallara D.: *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford University Press, Oxford, 2000.

[3] Bourdin, B.: Numerical implementation of the variational formulation for quasi-static brittle fracture. *Interfaces Free Bound.* **9** (2007), 411-430.

[4] Bourdin, B.; Francfort, G. A.; Marigo, J.J.: The variational approach to fracture. *J. Elasticity* **91** (2008), 5-148.

[5] Bourdin, B; Larsen, C. J.; Richardson, C.: A time-discrete model for dynamic fracture based on crack regularization. *Int. J. Fracture* **168** (2011), 133-143.

[6] Brezis H.: *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. North-Holland, Amsterdam-London; American Elsevier, New York, 1973.

[7] Chambolle, A.: A density result in two-dimensional linearized elasticity, and applications. *Arch. Ration. Mech. Anal.* **167** (2003), 211-233.

[8] Chambolle, A.; Giacomini, A.; Ponsiglione, M.: Crack initiation in brittle materials. *Arch. Ration. Mech. Anal.* **188** (2008), 309-349.

[9] Dal Maso G.; Francfort G.A.; Toader R.: Quasistatic crack growth in nonlinear elasticity. *Arch. Ration. Mech. Anal.* **176** (2005), 165-225.

[10] Dal Maso, G.; Lazzaroni, G.: Quasistatic crack growth in finite elasticity with non-interpenetration. *Ann. Inst. H. Poincar Anal. Non Linaire* **27** (2010), 257-290.

[11] Dal Maso, G.; Toader, R.: A model for the quasi-static growth of brittle fractures: existence and approximation results. *Arch. Ration. Mech. Anal.* **162** (2002), 101-135.

[12] Dal Maso, G.; Toader, R.: A model for the quasi-static growth of brittle fractures based on local minimization. *Math. Models Methods Appl. Sci.* **12** (2002), 1773-1799.

[13] Francfort, G.; Larsen, C. J.: Existence and convergence for quasi-static evolution in brittle fracture, *Comm. Pure Appl. Math.* **56** (2003), 1465-1500.

[14] Francfort, G. A.; Marigo, J.-J.: Revisiting brittle fracture as an energy minimization problem. *J. Mech. Phys. Solids* **46** (1998), 1319-1342.

[15] Freund, L. B.: *Dynamic Fracture Mechanics.* Cambridge Univ. Press, New York, 1990.

[16] Larsen, C. J.: Models for dynamic fracture based on Griffith's criterion, in *IUTAM Symposium on Variational Concepts with Applications to the Mechanics of Materials* (Klaus Hackl, ed.), Springer, 2010, 131-140.

[17] Larsen, C. J.: Epsilon-stable quasi-static brittle fracture evolution. *Comm. Pure Appl. Math.* **63** (2010), 630-654.

[18] Larsen, C. J.; Ortner, C.; Süli, E.: Existence of solutions to a regularized model of dynamic fracture. *Math. Models Methods Appl. Sci.* 20 (2010), 1021-1048

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