SINGULAR PERTURBATION MODELS IN PHASE TRANSITIONS FOR SECOND ORDER MATERIALS

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ABSTRACT. A variational model proposed in the physics literature to describe the onset of pattern formation in two-component bilayer membranes and amphiphilic monolayers leads to the analysis of a Ginzburg-Landau type energy, precisely,

$$u \mapsto \int_\Omega \left[W(u) - q\,|\nabla u|^2 + \left|\nabla^2 u\right|^2 \right] dx.$$

When the stiffness coefficient $-q$ is negative, one expects curvature instabilities of the membrane and, in turn, these instabilities generate a pattern of domains that differ both in composition and in local curvature. Scaling arguments motivate the study of the family of singular perturbed energies

$$u \mapsto F_\varepsilon(u,\Omega) := \int_\Omega \left[\frac{1}{\varepsilon} W(u) - q\varepsilon |\nabla u|^2 + \varepsilon^3 |\nabla^2 u|^2 \right] dx.$$

Here, the asymptotic behavior of $\{F_\varepsilon\}$ is studied using Γ-convergence techniques. In particular, compactness results and an integral representation of the limit energy are obtained.

1. Introduction

In [10], [21], (see also [11, 18]) Andelman, Kawasaki, Kawakatsu, and Taniguchi introduced a nonlocal variational model for the shape deformation of unilamellar membranes undergoing an inplane phase separation. A simplified local version of this model (see [18]) leads to the study of a Ginzburg-Landau energy

$$\int_\Omega \left[W(u) - q\,|\nabla u|^2 + \left|\nabla^2 u\right|^2 \right] dx, \tag{1.1}$$

where the order parameter u is a real-valued function on a domain $\Omega \subset \mathbb{R}^2$, W is a nonnegative double-well potential and $q > 0$. Here, and in what follows, ∇u and $\nabla^2 u$ denote the gradient vector and the Hessian matrix of u, respectively. Since the stiffness coefficient $-q$ is negative, one expects instability of the membrane and pattern formation.

The model (1.1) in the one-dimensional case was independently proposed by Coleman, Marcus, and Mizel in [4] in connection with the study of periodic or quasiperiodic layered structures. There is a vast literature of the qualitative properties of local minimizers of (1.1) in this setting (see [2], [4], [9], [12], [14] [17], [19]).

In this paper we use Γ-convergence techniques to characterize the singular perturbation limit of the family of rescaled energies

$$F_\varepsilon(u,\Omega) := \int_\Omega \left[\frac{1}{\varepsilon} W(u) - q\varepsilon |\nabla u|^2 + \varepsilon^3 |\nabla^2 u|^2 \right] dx, \tag{1.2}$$

in dimension $N \geq 1$. The limiting functional provides the effective energy on the phase transition surface. Our analysis will be carried out without any assumption on the sign of q. When $q = 0$, the problem was already studied in [7] under weaker hypotheses on W. The case $q < 0$ is easier, and

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may be obtained using simpler techniques (see [9], where $|\nabla^2 u|^2$ is replaced by $|\Delta u|^2$). However, it is also covered by our arguments, which are developed for the more difficult case $q > 0$.

Throughout the paper we assume that Ω is a bounded open set of $\mathbb{R}^N$ with C^1 boundary and $W\colon \mathbb{R} \to [0, +\infty)$ is a continuous function satisfying the following conditions for a suitable constant $c_0 \geq 1$:

$(H1)$ $W(s) = 0$ if and only if $s \in \{-1, 1\}$;
$(H2)$ $W(s) \geq (|s| - 1)^2$ for all $s \in \mathbb{R}$;
$(H3)$ $W(s) \leq c_0 W(t) + c_0$ for every $t \in \mathbb{R}$ and every $s \in \mathbb{R}$ with $|s| \leq |t|$.

A prototype for W is

$$W(s) := (s^2 - 1)^2.$$

We begin by stating a compactness result for sequences with finite energy.

Theorem 1.1. *Let Ω be a bounded open set of $\mathbb{R}^N$ with C^1 boundary and assume $(H1)$ and $(H2)$. Then there exists a constant $q^* > 0$, independent of Ω, such that, if $-\infty < q < q^*/N$, if $\{\varepsilon_n\}$ converges to zero, and $\{u_n\} \subset H^2(\Omega)$ satisfies*

$$\liminf_{n\to+\infty} F_{\varepsilon_n}(u_n, \Omega) < +\infty,$$

then there exist a subsequence $\{u_{n_k}\} \subset \{u_n\}$ and $u \in BV(\Omega; \{-1, 1\})$ such that $u_{n_k} \to u$ in $L^2(\Omega)$.

Here and in what follows $BV(\Omega; \{-1, 1\})$ denotes the set of all functions u of bounded variation in Ω such that $u(x) \in \{-1, 1\}$ for $\mathcal{L}^N$-a.e. $x \in \Omega$.

A major difficulty in the proof of the previous theorem is the fact that the energy may have a negative term. To overcome this problem and to obtain apriori bounds from below, we prove the following interpolation result that determines q^* in Theorem 1.1.

Theorem 1.2. *Let Ω be a bounded open set of $\mathbb{R}^N$ with C^1 boundary and assume $(H1)$ and $(H2)$. Then there exists a constant $q^* > 0$, independent of Ω, such that for every $-\infty < q < q^*/N$ there exists $\varepsilon_0 = \varepsilon_0(\Omega, q) > 0$ such that*

$$q\varepsilon^2 \int_\Omega |\nabla u|^2\, dx \leq \int_\Omega W(u)\, dx + \varepsilon^4 \int_\Omega |\nabla^2 u|^2\, dx \tag{1.3}$$

for every $\varepsilon \in (0, \varepsilon_0)$ and every $u \in H^2(\Omega)$.

Note that if $q \leq 0$, then $\varepsilon_0 = \infty$. To state the Γ-convergence result, given $-\infty < q < q^*/N$, we define

$$\mathbf{m}_N := \inf\left\{ \int_Q \left[\frac{1}{\varepsilon} W(u) - q\varepsilon|\nabla u|^2 + \varepsilon^3 |\nabla^2 u|^2\right] dx \,:\, 0 < \varepsilon \leq 1,\, u \in \mathcal{A} \right\}, \tag{1.4}$$

where $Q := (-1/2, 1/2)^N$ and

$$\begin{aligned}\mathcal{A} := \Big\{ & u \in H^2_{\mathrm{loc}}(\mathbb{R}^N) : \ u(x) = -1 \text{ near } x \cdot e_N = -\frac{1}{2}, \ \ u(x) = 1 \text{ near } x \cdot e_N = \frac{1}{2}, \\ & u(x) = u(x + e_i) \text{ for all } x \in \mathbb{R}^N,\, i = 1, \dots, N-1 \Big\}.\end{aligned} \tag{1.5}$$

Here and in what follows $\{e_1, \dots, e_N\}$ is the canonical basis of $\mathbb{R}^N$, and by "near" we mean "in a neighborhood of".

We will show that $\mathbf{m}_N > 0$ for $-\infty < q < q^*/N$ (see Proposition 3.9 below).

Theorem 1.3. *Let Ω be a bounded open set of $\mathbb{R}^N$ with C^1 boundary, let $-\infty < q < q^*/N$, and assume (H1)–(H3). Then the sequence of functionals $\mathcal{F}_\varepsilon\colon L^2(\Omega) \to \mathbb{R}\cup\{+\infty\}$, defined by*

$$\mathcal{F}_\varepsilon(u) := \begin{cases} F_\varepsilon(u,\Omega) & \text{if } u \in H^2(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega)\setminus H^2(\Omega), \end{cases} \tag{1.6}$$

Γ-converges as $\varepsilon \to 0^+$ to the functional $\mathcal{F}\colon L^2(\Omega) \to \mathbb{R}\cup\{+\infty\}$ defined by

$$\mathcal{F}(u) := \begin{cases} \mathbf{m}_N \operatorname{Per}_\Omega(\{u=1\}) & \text{if } u \in BV(\Omega;\{-1,1\}), \\ +\infty & \text{if } u \in L^2(\Omega)\setminus BV(\Omega;\{-1,1\}). \end{cases} \tag{1.7}$$

In view of Proposition 8.1 in [5], for every sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0^+$, and every $u \in L^2(\Omega)$, we have that

$$\Gamma - \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u) = \inf\left\{ \liminf_{n\to+\infty} \mathcal{F}_{\varepsilon_n}(u_n) : \{u_n\} \subset L^2(\Omega),\, u_n \to u \text{ in } L^2(\Omega) \right\},$$

$$\Gamma - \limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u) = \inf\left\{ \limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) : \{u_n\} \subset L^2(\Omega),\, u_n \to u \text{ in } L^2(\Omega) \right\}.$$

Thus, also by Theorem 1.1, to prove Theorem 1.3, it suffices to show:

(i) Lower bound: For every $u \in L^2(\Omega)$, for every sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0^+$, and every sequence $\{u_n\} \subset H^2(\Omega)$ such that $u_n \to u$ in $L^2(\Omega)$,
$$\mathcal{F}(u) \le \liminf_{n\to+\infty} \mathcal{F}_{\varepsilon_n}(u_n). \tag{1.8}$$

(ii) Upper bound: For every $\eta > 0$, every $u \in BV(\Omega;\{-1,1\})$, and every sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0^+$, there exists $\{u_n\} \subset H^2(\Omega)$ such that $u_n \to u$ in $L^2(\Omega)$ and
$$\limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) \le (1+\eta)\,\mathcal{F}(u) + \eta. \tag{1.9}$$

Remark 1.4. By standard properties of Γ-convergence (see e.g. Corollary 7.20 in [5]), Theorems 1.1 and 1.3 imply that minimizers of $\mathcal{F}_{\varepsilon_n}$, up to a subsequence, converge in $L^2(\Omega)$ to a minimizer of $\mathcal{F}$.

Remark 1.5. As usual (see [15], [20]), one can also impose the constraint $\int_\Omega u(x)\, dx = m$, for some $m \in \left[-\mathcal{L}^N(\Omega), \mathcal{L}^N(\Omega)\right]$. We omit the details.

This paper can be considered as a first step towards the analysis of the original nonlocal model mentioned at the beginning of the introduction, which will be studied in forthcoming papers.

This paper is organized as follows. In Section 2 we introduce some of the notation used in the paper and we give some preliminary results. In Section 3 we prove Theorem 1.2, while in Section 4 we prove Theorems 1.1 and 1.3. Section 5 is devoted to the study of the properties of the one-dimensional constant $\mathbf{m}_1$ defined in (1.4) with $N = 1$.

Near the completion of this work, we became aware that a detailed analysis of the one-dimensional case had been obtained independently and at the same time by Cicalese, Spadaro and Zeppieri in [3].

2. Preliminaries

Throughout the paper, Ω is a bounded open set of $\mathbb{R}^N$ with C^1 boundary. The symbols $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ denote the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$, respectively. We shall label the first $N-1$ coordinates of a point $x \in \mathbb{R}^N$ by x', and the N-th one by x_N, so that $x = (x', x_N)$. Given $\nu \in \mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$, the symbol Q_ν denotes an open unit cube centered at the origin with two of its faces normal to ν, i.e.,

$$Q_\nu := \left\{ x \in \mathbb{R}^N : |x\cdot\nu| < \frac{1}{2},\, |x\cdot\nu_i| < \frac{1}{2},\, i = 1,\dots,N-1 \right\}, \tag{2.1}$$

where $\{\nu_1, \ldots, \nu_{N-1}, \nu\}$ is an orthonormal basis of $\mathbb{R}^N$. If $x_0 \in \mathbb{R}^N$ and $r > 0$, then $Q_\nu(x_0, r) := x_0 + rQ_\nu$. If $\{\nu_1, \ldots, \nu_{N-1}, \nu\}$ is the canonical basis, we drop the dependence on e_N, i.e., $Q(x_0, r) := x_0 + r(-1/2, 1/2)^N = x_0 + rQ$, where Q is the open unit cube centered at the origin with faces normal to the coordinates axes. A rectangle in $\mathbb{R}^N$ is a set of the form $R = I_1 \times \cdots \times I_N$, where I_i is an interval of $\mathbb{R}$, $i = 1, \ldots, N$.

Given $\Psi \in C_c^\infty\left(\mathbb{R}^N\right)$ with $\operatorname{supp}\Psi \subset B(0,1)$ and $\int_{\mathbb{R}^N} \Psi(x)\, dx = 1$, for every $\varepsilon > 0$ we define the mollifier

$$\Psi_\varepsilon(x) := \frac{1}{\varepsilon^N}\Psi\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^N. \tag{2.2}$$

Note that $\operatorname{supp}\Psi_\varepsilon \subset B(0,\varepsilon)$. As usual, given $u \in L^1_{\mathrm{loc}}\left(\mathbb{R}^N\right)$, we define

$$u_\varepsilon(x) := (u * \Psi_\varepsilon)(x) = \int_{\mathbb{R}^N} \Psi_\varepsilon(x-y)\, u(y)\, dy \tag{2.3}$$

for $x \in \mathbb{R}^N$. Then $u_\varepsilon \in C^\infty\left(\mathbb{R}^N\right)$ and, if u is bounded, then $u_\varepsilon \to u$ in $L^p_{\mathrm{loc}}\left(\mathbb{R}^N\right)$ for every $1 \le p < +\infty$, with

$$\|u_\varepsilon\|_\infty \le \|u\|_\infty, \quad \|\nabla u_\varepsilon\|_\infty \le C\|u\|_\infty/\varepsilon, \quad \left\|\nabla^2 u_\varepsilon\right\|_\infty \le C\|u\|_\infty/\varepsilon^2. \tag{2.4}$$

The space of all bounded Radon measures on Ω is denoted by $\mathcal{M}_b(\Omega)$. It is identified with the dual of $C_0(\Omega)$, the space of continuous functions vanishing on $\partial\Omega$.

In the sequel, the letter C will denote a generic constant that may change from line to line.

3. Interpolation Inequalities Involving W

3.1. **The One-Dimensional Case.** Let I be an open interval of $\mathbb{R}$, $u\colon I \to \mathbb{R}$, and let $\varepsilon > 0$. We set

$$F_\varepsilon(u, I) := \int_I \left[\frac{1}{\varepsilon}W(u) - q\varepsilon(u')^2 + \varepsilon^3(u'')^2\right] dx, \tag{3.1}$$

and

$$F(u, I) := \int_I \left[W(u) - q(u')^2 + (u'')^2\right] dx. \tag{3.2}$$

By the change of variable $v_\varepsilon(x) := u(\varepsilon x)$, we have

$$F_\varepsilon(u, I) = F(v_\varepsilon, \varepsilon^{-1} I). \tag{3.3}$$

The following two results are due to Gagliardo. For the proof we refer to [8, Lemma 1.I, Lemma 1.II].

Lemma 3.1. *There exists a constant $c_1 > 0$ such that*

$$\sup_{x\in I}\left|u'(x)\right| \le c_1 \left[\int_I u^2\, dx\right]^{1/8} \left[\int_I (u'')^2\, dx\right]^{3/8} \tag{3.4}$$

for every bounded open interval I of $\mathbb{R}$ and every $u \in H^2(I)$ with u' vanishing at some point of I.

Lemma 3.2. *There exist two constants $c_2 > 0$ and $c_3 > 0$ such that*

$$\sup_{x\in I}\left|u'(x)\right| \le c_2 \left[\int_I u^2\, dx\right]^{1/8} \left[\int_I (u'')^2\, dx\right]^{3/8} + c_3|I|^{-3/2}\left[\int_I u^2\, dx\right]^{1/2} \tag{3.5}$$

for every bounded open interval I of $\mathbb{R}$ and every $u \in H^2(I)$.

The following lemma is a modified version of Nirenberg's inequality (2.7) in [16], which was given without proof. For the reader's convenience, we give here a complete proof.

Lemma 3.3. *There exist two constants $c_4, c_5 > 0$ such that*

$$\int_I (u')^2\, dx \le c_4 \lambda^{-2} \int_I u^2\, dx + c_5 \lambda^2 \int_I (u'')^2\, dx \tag{3.6}$$

for every open interval I of $\mathbb{R}$ and for every $0 < \lambda \le |I|$ and $u \in H^2(I)$.

Proof. Step 1: Assume first that $u \in C^\infty(I)$ and that $I = (0, \lambda)$ for some $\lambda > 0$. Fix $s \in \left(0, \frac{1}{3}\lambda\right)$ and $t \in \left(\frac{2}{3}\lambda, \lambda\right)$. By the Mean Value Theorem, there exists $\xi \in (s, t)$ such that

$$u'(\xi) = \frac{u(t) - u(s)}{t - s}.$$

Hence, by the Fundamental Theorem of Calculus, for all $x \in (0, \lambda)$,

$$u'(x) = u'(\xi) + \int_\xi^x u''(y)\, dy = \frac{u(t) - u(s)}{t - s} + \int_\xi^x u''(y)\, dy.$$

Since $t - s \ge \frac{\lambda}{3}$, it follows that

$$|u'(x)| \le \frac{3}{\lambda}\left(|u(t)| + |u(s)|\right) + \int_0^\lambda |u''(y)|\, dy,$$

and, by Hölder's inequality, we obtain

$$|u'(x)| \le \frac{3}{\lambda}\left(|u(t)| + |u(s)|\right) + \lambda^{1/2}\left(\int_0^\lambda (u''(y))^2\, dy\right)^{1/2}.$$

Using the convexity of the function $\tau \mapsto \tau^2$, we have that

$$(u'(x))^2 \le \frac{3^3}{\lambda^2}\left((u(t))^2 + (u(s))^2\right) + 3\lambda \int_0^\lambda (u''(y))^2\, dy.$$

By averaging first in s over $(0, \lambda/3)$ and then in t over $(2\lambda/3, \lambda)$, we get

$$\begin{aligned} (u'(x))^2 &\le \frac{3^4}{\lambda^3}\left(\int_{\frac{2}{3}\lambda}^\lambda (u(t))^2\, dt + \int_0^{\frac{1}{3}\lambda} (u(s))^2\, ds\right) + 3\lambda \int_0^\lambda (u''(y))^2\, dy \\ &\le \frac{3^4}{\lambda^3} \int_0^\lambda (u(y))^2\, dy + 3\lambda \int_0^\lambda (u''(y))^2\, dy. \end{aligned}$$

Finally, we integrate in x over $(0, \lambda)$, to obtain

$$\int_I (u'(x))^2\, dx \le \frac{3^4}{\lambda^2} \int_I (u(y))^2\, dy + 3\lambda^2 \int_I (u''(y))^2\, dy. \tag{3.7}$$

Step 2: If I has infinite length and $u \in C^\infty(I)$, let $\lambda > 0$ and subdivide I in subintervals of length λ. Then (3.7) holds in each subinterval, and adding these inequalities yields

$$\int_I (u'(x))^2\, dx \le \frac{3^4}{\lambda^2} \int_I (u(y))^2\, dy + 3\lambda^2 \int_I (u''(y))^2\, dy.$$

On the other hand, if I has finite length, then for $0 < \lambda \le |I|$ let $m \in \mathbb{N}$ be the integer part of $\frac{|I|}{\lambda}$ and divide I into m subintervals of length $\lambda_1 := \frac{1}{m}|I|$. Then by Step 1, (3.7) holds in each of the m subintervals of I with λ_1 in place of λ. To conclude, it suffices to add these inequalities and to observe that $\lambda \le \lambda_1 \le 2\lambda$.

Step 3: To remove the additional hypothesis that $u \in C^\infty(I)$, one can use standard mollifiers. We omit the details. ☐

We now present the main result of this section: An interpolation inequality involving the double well potential W.

Theorem 3.4. *Assume (H1) and (H2). There exists a constant $q^* > 0$ such that for every $-\infty < q < q^*$ the following inequalities hold:*

(i) *For every open interval I, with $|I| \geq 1$,*

$$q \int_I (u')^2\, dx \leq \int_I W(u)\, dx + \int_I (u'')^2\, dx \tag{3.8}$$

for all $u \in H^2_{\mathrm{loc}}(I)$.

(ii) *For every open interval I,*

$$q \int_I (u')^2\, dx \leq \int_I W(u)\, dx + \int_I (u'')^2\, dx \tag{3.9}$$

for every $u \in H^2_{\mathrm{loc}}(I)$ such that u' vanishes at some point of I.

Proof. It is enough to prove (3.4) and (3.9) for bounded intervals I. Indeed, in the case of an unbounded interval I, it suffices to subdivide I into subintervals of length one, apply (i) on each subinterval, and then add all these inequalities. Thus, without loss of generality, we assume that $I = (0, \ell)$, and by density that $u \in C^2(I)$ and has a finite number M of zeros.

(i) In this case $\ell \geq 1$. We divide the proof into three cases depending on M.

Case 1. Assume that $M = 0$ and, without loss of generality, that $u > 0$ on I. Applying (3.6) to $u - 1$, with $\lambda = 1 \leq \ell$, we obtain

$$\begin{aligned} \int_I (u')^2\, dx &\leq c_4 \int_I (u-1)^2\, dx + c_5 \int_I (u'')^2\, dx \\ &\leq c_4 \int_I W(u)\, dx + c_5 \int_I (u'')^2\, dx, \end{aligned} \tag{3.10}$$

where the last inequality follows by $(H2)$, in view of the fact that $u > 0$.

Case 2. Assume now that $M = 1$ and let ℓ_1 be the unique zero of u. If $\ell_1 \geq 1/2$ and $\ell - \ell_1 \geq 1/2$, we can repeat the proof of Case 1 on each subinterval $(0, \ell_1)$ and (ℓ_1, ℓ), taking $\lambda = 1/2$ in (3.6). Then

$$\int_I (u')^2\, dx \leq 4c_4 \int_I W(u)\, dx + \frac{1}{4} c_5 \int_I (u'')^2\, dx. \tag{3.11}$$

Consider now the case $\ell_1 < 1/2$ and $\ell - \ell_1 \geq 1/2$ (the case $\ell_1 \geq 1/2$ and $\ell - \ell_1 < 1/2$ is analogous). Without loss of generality, we may assume that $u < 0$ in $(0, \ell_1)$ and $u > 0$ in (ℓ_1, ℓ). Since $\ell - \ell_1 \geq 1/2$, as in (3.11), we obtain

$$\int_{\ell_1}^{\ell} (u')^2\, dx \leq 4c_4 \int_{\ell_1}^{\ell} W(u)\, dx + \frac{1}{4} c_5 \int_{\ell_1}^{\ell} (u'')^2\, dx. \tag{3.12}$$

On the other hand, by the Fundamental Theorem of Calculus and Young's inequality, we have

$$\int_0^{\ell_1} (u')^2\, dx \leq (u'(\ell_1))^2 + \int_0^{\ell_1} (u'')^2\, dx. \tag{3.13}$$

We apply (3.5) to $u - 1$ in (ℓ_1, ℓ) to obtain

$$\begin{aligned} \sup_{[\ell_1, L]} |u'| &\leq c_2 \left[\int_{\ell_1}^{\ell} (u-1)^2\, dx \right]^{1/8} \left[\int_{\ell_1}^{\ell} (u'')^2\, dx \right]^{3/8} + c_3 (\ell - \ell_1)^{-3/2} \left[\int_{\ell_1}^{\ell} (u-1)^2\, dx \right]^{1/2} \\ &\leq c_2 \left[\int_{\ell_1}^{\ell} (u-1)^2\, dx \right]^{1/8} \left[\int_{\ell_1}^{\ell} (u'')^2\, dx \right]^{3/8} + 2^{3/2} c_3 \left[\int_{\ell_1}^{\ell} (u-1)^2\, dx \right]^{1/2}, \end{aligned} \tag{3.14}$$

where we have used the fact that $\ell - \ell_1 \geq 1/2$. Thus, by Young's inequality, $(H2)$, and the fact that $u > 0$ in (ℓ_1, ℓ),

$$\begin{aligned} \big(u'(\ell_1)\big)^2 &\leq 2c_2^2 \left[\int_{\ell_1}^{\ell} (u-1)^2\,dx\right]^{1/4} \left[\int_{\ell_1}^{\ell} (u'')^2\,dx\right]^{3/4} + 2^4 c_3^2 \int_{\ell_1}^{\ell} (u-1)^2\,dx \\ &\leq \frac{1}{2}c_2^2 \int_{\ell_1}^{\ell} W(u)\,dx + \frac{3}{2}c_2^2 \int_{\ell_1}^{\ell} (u'')^2\,dx + 2^4 c_3^2 \int_{\ell_1}^{\ell} W(u)\,dx. \end{aligned} \tag{3.15}$$

Inserting (3.15) into (3.13) and adding the resulting inequality to (3.12) yields the conclusion.

Case 3. Assume now that $M \geq 2$ and let $\ell_1 < \cdots < \ell_M$ be the zeros of u in I. We fix $i = 1, \dots, M-1$, and we obtain the estimate in the interval $J_i := (\ell_i, \ell_{i+1})$. Without loss of generality, we assume that $u > 0$ on J_i. Since $u(\ell_i) = u(\ell_{i+1})$, the derivative u' vanishes at some point of J_i, and so we can apply (3.4) to $u - 1$ to obtain

$$\begin{aligned} \sup_{J_i}(u')^2 &\leq c_1^2 \left[\int_{J_i} (u-1)^2\,dx\right]^{1/4} \left[\int_{J_i} (u'')^2\,dx\right]^{3/4} \\ &\leq \frac{1}{4}c_1^2 \int_{J_i} W(u)\,dx + \frac{3}{4}c_1^2 \int_{J_i} (u'')^2\,dx, \end{aligned} \tag{3.16}$$

where we have used Young's inequality, $(H2)$, and the fact that $u > 0$ in J_i.

If $|J_i| \leq 1/2$, then by (3.16),

$$\int_{J_i} (u')^2\,dx \leq \frac{1}{2}\sup_{J_i}(u')^2 \leq \frac{1}{8}c_1^2 \int_{J_i} W(u)\,dx + \frac{3}{8}c_1^2 \int_{J_i} (u'')^2\,dx. \tag{3.17}$$

If $|J_i| > 1/2$, then as in (3.11), we get

$$\int_{J_i} (u')^2\,dx \leq 4c_4 \int_{J_i} W(u)\,dx + \frac{1}{4}c_5 \int_{J_i} (u'')^2\,dx. \tag{3.18}$$

It remains to prove the estimate in the intervals $J_0 := (0, \ell_1)$ and $J_M := (\ell_M, \ell)$. We only treat the interval J_0. If $|J_0| \leq 1/2$, by (3.13) and (3.16) applied to J_1 we obtain

$$\begin{aligned} \int_{J_0} (u')^2\,dx &\leq (u'(\ell_1))^2 + \int_{J_0} (u'')^2\,dx \\ &\leq \frac{1}{4}c_1^2 \int_{J_1} W(u)\,dx + \frac{3}{4}c_1^2 \int_{J_1} (u'')^2\,dx + \int_{J_0} (u'')^2\,dx. \end{aligned} \tag{3.19}$$

If $|J_0| > 1/2$, inequality (3.18) holds.

The inequality in the interval I is obtaining by summing the appropriate inequalities among (3.17), (3.18) and (3.19). We observe that in Case 3 we never used the hypothesis $\ell \geq 1$.

(ii) In view of (i), it suffices to consider the case $\ell < 1$. As before, we divide the proof into three cases depending on the number M of zeros.

Case 1. Assume that $M = 0$ and, without loss of generality, that $u > 0$ on I. We can apply (3.4) to I and, arguing as in (3.16) and (3.17), we get

$$\int_I (u')^2\,dx \leq \sup_I (u')^2 \leq \frac{1}{4}c_1^2 \int_I W(u)\,dx + \frac{3}{4}c_1^2 \int_I (u'')^2\,dx.$$

Case 2. Assume that $M = 1$ and let ℓ_1 be the unique zero of u in I. Without loss of generality, we can assume that $u > 0$ on $(0, \ell_1)$ and that u' vanishes at a point of $[\ell_1, \ell)$. Since $\ell < 1$, by the Fundamental Theorem of Calculus,

$$\int_0^{\ell} (u')^2\,dx \leq 2(u'(\ell_1))^2 + 2\int_0^{\ell} (u'')^2\,dx.$$

If $u'(\ell_1) = 0$, then the proof is concluded. Thus assume that u' vanishes at a point of (ℓ_1, ℓ). Then, by (3.4) applied to $u - 1$ in (ℓ_1, ℓ), arguing as in (3.16), we obtain

$$\big(u'(\ell_1)\big)^2 \leq \frac{1}{4} c_1^2 \int_{\ell_1}^{\ell} W(u)\, dx + \frac{3}{4} c_1^2 \int_{\ell_1}^{\ell} (u'')^2\, dx.$$

Combining the last two inequalities, we conclude Case 2.

Case 3. If $M \geq 2$, we can proceed as in Case 3 of Part (i). □

Corollary 3.5. *Assume* $(H1)$ *and* $(H2)$. *For every open interval* I, *every* $0 < \varepsilon \leq |I|$, *and every* $-\infty < q < q^*$,

$$q\varepsilon^2 \int_I (u')^2\, dx \leq \int_I \left[W(u) + \varepsilon^4 (u'')^2\right] dx,$$

for all $u \in H^2_{\mathrm{loc}}(I)$.

Proof. By Theorem 3.4 (i) and (3.3), for every $\varepsilon \leq |I|$, $q < q^*$, and $u \in H^2_{\mathrm{loc}}(I)$, we have

$$0 \leq F(v_\varepsilon, \varepsilon^{-1} I) = \int_{\varepsilon^{-1} I} [W(v_\varepsilon) - q(v_\varepsilon')^2 + (v_\varepsilon'')^2]\, dy = \frac{1}{\varepsilon} \int_I \left[W(u) - q\varepsilon^2 (u')^2 + \varepsilon^4 (u'')^2\right] dx. \quad (3.20)$$

This concludes the proof. □

The next corollary extends the previous result to the case of an open set $\Omega \subset \mathbb{R}$ with a finite number of connected components.

Corollary 3.6. *Assume* $(H1)$ *and* $(H2)$. *Let* Ω *be an open subset of* $\mathbb{R}$ *with a finite number of connected components, and let* $\varepsilon_0 = \varepsilon_0(\Omega)$ *be the length of the shortest connected component of* Ω. *Then*

$$q\varepsilon^2 \int_\Omega (u')^2\, dx \leq \int_\Omega \left[W(u) + \varepsilon^4 (u'')^2\right] dx \quad (3.21)$$

for every $0 < \varepsilon \leq \varepsilon_0$, *every* $-\infty < q < q^*$, *and every* $u \in H^2_{\mathrm{loc}}(\Omega)$. *In particular, Theorem 1.2 holds for* $N = 1$.

Proof. By Corollary 3.5 the inequality holds for each connected component. To conclude it suffices to add all these inequalities.

Since a bounded open set Ω of $\mathbb{R}$ with C^1 boundary has finitely many connected components, Theorem 1.2 holds for $N = 1$. □

If Ω has an infinite number of connected components, then an inequality like (3.21) does not hold. More precisely, we have the following result.

Proposition 3.7. *Assume* $(H1)$. *Assume also that* Ω *is a bounded open subset of* $\mathbb{R}$ *with infinitely many connected components. Then for every* $q > 0$ *and every* $\varepsilon > 0$ *there exists* $u \in H^2(\Omega)$ *such that*

$$q\varepsilon^2 \int_\Omega (u')^2\, dx > \int_\Omega \left[W(u) + \varepsilon^4 (u'')^2\right] dx. \quad (3.22)$$

Proof. Fix $q > 0$ and $\varepsilon > 0$. Since W is continuous and $W(1) = 0$, there exists $\delta > 0$ such that

$$W(s) < q\varepsilon^2 \quad \text{for } 1 \leq s \leq 1 + \delta. \quad (3.23)$$

Let $\{I_n\}$ be the family of connected components of Ω and, for every n, let $a_n < b_n$ be the endpoints of I_n. Since $|I_n| \to 0$ as $n \to \infty$, there exists m such that $|I_n| \leq \delta$ for every $n \geq m$. We define $u \colon \Omega \to \mathbb{R}$ by $u(x) := 1$ if $x \in I_n$ for some $n < m$, and by $u(x) := 1 + x - a_n$ if $x \in I_n$ for some $n \geq m$. Then $u \in H^2(\Omega)$. By construction, for every $n \geq m$ we have $1 < u < 1 + |I_n| \leq 1 + \delta$ on

I_n, hence by (3.23) $W(u) < q\varepsilon^2$ on I_n. On the other hand, $W(u) = W(1) = 0$ on the intervals I_n with $n < m$. Therefore

$$q\varepsilon^2 \int_\Omega (u')^2\, dx = q\varepsilon^2 \sum_{n=m}^{+\infty} |I_n| > \sum_{n=m}^{+\infty} \int_{I_n} W(u)\, dx = \int_\Omega \left[W(u) + \varepsilon^4 (u'')^2\right]\, dx,$$

which concludes the proof of (3.23). □

Remark 3.8. Note that the function $u \in H^2(\Omega)$ constructed in the previous proof belongs to $C^\infty(\Omega)$ but, in general, not to $C^\infty(\overline{\Omega})$. However, if $b_n < a_{n+1}$ for every n (or if $a_n > b_{n+1}$ for every n), then u belongs to $C^\infty(\overline{\Omega})$. Indeed, since $u'' = 0$ in Ω, it is enough to show that, if $\{x_j\} \subset \Omega$ is a sequence converging to a point of the boundary, then the limits of $\{u(x_j)\}$ and $\{u'(x_j)\}$ exist and are finite. If $\{x_j\}$ is contained in a finite number of intervals I_n, then there is nothing to prove. In the opposite case, we have $u(x_j) \to 1$ and $u'(x_j) \to 1$.

3.2. **The N-Dimensional Case.** In order to prove Theorem 1.2 for $N \geq 2$, we will use slicing techniques. Given $\xi \in \mathbb{S}^{N-1}$, let Π^ξ be the hyperplane through the origin orthogonal to ξ, i.e.,

$$\Pi^\xi := \{x \in \mathbb{R}^N : x \cdot \xi = 0\}.$$

For every open set Ω of $\mathbb{R}^N$ and every $y \in \Pi^\xi$ we define the slice Ω^ξ_y by

$$\Omega^\xi_y := \{t \in \mathbb{R} : y + t\xi \in \Omega\}.$$

The orthogonal projection Ω^ξ of Ω onto Π^ξ is given by

$$\Omega^\xi := \{y \in \Pi^\xi : \Omega^\xi_y \neq \varnothing\}.$$

For every function $u\colon \Omega \to \mathbb{R}^m$ and for every $y \in \Omega^\xi$, we define $u^\xi_y\colon \Omega^\xi_y \to \mathbb{R}^m$ by

$$u^\xi_y(t) := u(y + t\xi).$$

It is well-known (see, e.g., [13]) that, if $u \in H^2(\Omega)$, then for every $\xi \in \mathbb{S}^{N-1}$ and for $\mathcal{H}^{N-1}$-a.e. $y \in \Pi^\xi$ we have $u^\xi_y \in H^2(\Omega^\xi_y)$ and

$$\begin{aligned} \left(u^\xi_y\right)'(t) &= (\nabla u)^\xi_y(t) \cdot \xi && \text{for } \mathcal{L}^1\text{-a.e. } t \in \Omega^\xi_y, \\ \left(u^\xi_y\right)''(t) &= (\nabla^2 u)^\xi_y(t)\xi \cdot \xi && \text{for } \mathcal{L}^1\text{-a.e. } t \in \Omega^\xi_y. \end{aligned} \tag{3.24}$$

We turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $q < q^*/N$ and fix $\widetilde{q} \in (q, q^*/N)$. Given $\ell > 0$, $\xi \in \mathbb{S}^{N-1}$, and $y \in \Omega^\xi$, we define $\Omega^{\xi,\ell}_y$ as the union of all the connected components of Ω^ξ_y with length greater than ℓ, i.e., $\Omega^{\xi,\ell}_y$ is given by all points $x \in \Omega$ belonging to a segment, with length greater than ℓ, parallel to ξ and contained in Ω. Set

$$\Omega(\xi, \ell) := \{y + t\xi : y \in \Omega^\xi,\ t \in \Omega^{\xi,\ell}_y\}.$$

Note that the slices satisfy

$$\Omega(\xi, \ell)^\xi_y = \Omega^{\xi,\ell}_y.$$

Hence, $\Omega(\xi, \ell)^\xi_y$ is the union of a finite family of open intervals with length greater than ℓ. Therefore, for $\mathcal{H}^{N-1}$-a.e. $y \in \Omega^\xi$, we are in a position to apply Corollary 3.5 to obtain that for every $\varepsilon \leq \ell$,

$$N\widetilde{q}\varepsilon^2 \int_{\Omega(\xi,\ell)^\xi_y} \left(\left(u^\xi_y\right)'(t)\right)^2 dt \leq \int_{\Omega(\xi,\ell)^\xi_y} W\left(u^\xi_y(t)\right) dt + \varepsilon^4 \int_{\Omega(\xi,\ell)^\xi_y} \left(\left(u^\xi_y\right)''(t)\right)^2 dt. \tag{3.25}$$

Integrating both sides of the previous inequality with respect to y over Ω^ξ, using (3.24) and Fubini's Theorem, we get

$$N\widetilde{q}\varepsilon^2 \int_{\Omega(\xi,\ell)} \left(\nabla u(x) \cdot \xi\right)^2 dx \leq \int_\Omega W(u(x))\, dx + \varepsilon^4 \int_\Omega \left|\nabla^2 u(x)\right|^2 dx,$$

which implies

$$N\widetilde{q}\varepsilon^2\int_\Omega \big(\nabla u(x)\cdot\xi\big)^2\,dx - N\widetilde{q}\varepsilon^2\int_{A(\xi,\ell)}\big|\nabla u(x)\big|^2\,dx \le \int_\Omega W(u(x))\,dx + \varepsilon^4\int_\Omega\big|\nabla^2 u(x)\big|^2\,dx,$$

where $A(\xi,\ell) := \Omega\setminus\Omega(\xi,\ell)$. Note that for every $\alpha\in\mathbb{R}^N$,

$$\int_{\mathbb{S}^{N-1}}(\alpha\cdot\xi)^2\,d\mathcal{H}^{N-1}(\xi) = \frac{1}{N}|\alpha|^2\sigma_{N-1},$$

where $\sigma_{N-1} := \mathcal{H}^{N-1}(\mathbb{S}^{N-1})$. Averaging both sides of the previous inequality in the variable ξ over $\mathbb{S}^{N-1}$ and using Fubini's Theorem, we get, for every $\varepsilon\le\ell$,

$$\begin{aligned}\widetilde{q}\varepsilon^2\int_\Omega\big|\nabla u(x)\big|^2\,dx - \frac{N\widetilde{q}}{\sigma_{N-1}}\varepsilon^2\int_{\mathbb{S}^{N-1}}\int_{A(\xi,\ell)}\big|\nabla u(x)\big|^2\,dx\,d\mathcal{H}^{N-1}(\xi)&\\ \le\int_\Omega W(u(x))\,dx+\varepsilon^4\int_\Omega\big|\nabla^2u(x)\big|^2\,dx.&\end{aligned}\tag{3.26}$$

By Fubini's Theorem

$$\int_{\mathbb{S}^{N-1}}\int_{A(\xi,\ell)}\big|\nabla u(x)\big|^2\,dx\,d\mathcal{H}^{N-1}(\xi) = \int_\Omega\big|\nabla u(x)\big|^2\mathcal{H}^{N-1}\big(D(x,\ell)\big)\,dx,\tag{3.27}$$

where $D(x,\ell) := \{\xi\in\mathbb{S}^{N-1} : x\in A(\xi,\ell)\}$. To conclude the proof, we have to choose $\ell>0$ such that $\mathcal{H}^{N-1}(D(x,\ell))$ is small, uniformly in x.

For every $x\in\partial\Omega$ and $r>0$, let $C_r(x)$ be the right circular cylinder centered at x with height $2r$, radius r, and axis parallel to the normal $\nu(x)$ of $\partial\Omega$ at x. Fix $\eta>0$ such that

$$q<\widetilde{q}\Big(1-\frac{N\eta}{\sigma_{N-1}}\Big).\tag{3.28}$$

Since the boundary of Ω is of class C^1, there exists $r>0$ such that for every $x\in\partial\Omega$ the intersection $C_r(x)\cap\partial\Omega$ is the graph of a C^1 function defined on the basis of the cylinder. By taking r smaller, if necessary, we can also assume that for every $x\in\partial\Omega$,

$$\mathcal{H}^{N-1}(\{\xi\in\mathbb{S}^{N-1} : \xi\in T_{\partial\Omega}(y),\ y\in C_r(x)\cap\partial\Omega\})<\eta,\tag{3.29}$$

where $T_{\partial\Omega}(y)$ is the tangent space to $\partial\Omega$ at x. We observe that r depends on Ω and η, which, in turn, depends on q.

We now fix $\ell<r/2$. In particular, ℓ depends on Ω and q. We want to show that $\mathcal{H}^{N-1}\big(D(x,\ell)\big)<\eta$ for every $x\in\Omega$. Fix $x\in\Omega$. If $D(x,\ell)=\varnothing$, then there is nothing to prove. Otherwise, let $\xi_0\in D(x,\ell)$. By the definition $A(\xi,\ell)$, and by the characterization of its complement $\Omega(\xi,\ell)$ in Ω, the point x belongs to a segment parallel to ξ_0 with length less than or equal to ℓ, and with endpoints on $\partial\Omega$. Let x_0 be one of these endpoints. Note that $|x-x_0|<\ell$. If ξ belongs to $D(x,\ell)$, the point x also belongs to a segment parallel to ξ with length less than or equal to ℓ, and with endpoints x_1 and x_2 on $\partial\Omega$. Since $\ell<r/2$, this segment is contained in $C_r(x_0)$. Consider the 2-dimensional plane containing x and parallel to the vectors ξ and $\nu(x_0)$. Then this plane intersects $C_r(x_0)\cap\partial\Omega$ on a C^1 curve containing x_1 and x_2. Then, by the Mean Value Theorem, there exists a point y on the curve such that ξ is tangent to the curve at y and, therefore, $\xi\in T_{\partial\Omega}(y)$. We conclude that

$$D(x,\ell)\subset\big\{\xi\in\mathbb{S}^{N-1} : \xi\in T_{\partial\Omega}(y),\ y\in C_r(x)\cap\partial\Omega\big\},$$

hence $\mathcal{H}^{N-1}\big(D(x,\ell)\big)<\eta$. This, together with (3.26), (3.27), and (3.28), gives (1.3) for every $\varepsilon\le\ell$. $\square$

In the proof of Theorem 1.3, we will also need to consider rectangles.

Proposition 3.9. *Assume* $(H1)$ *and* $(H2)$. *Let* Ω *be an open set that can be written as the union of finitely many pairwise disjoint open rectangles and of a set of Lebesgue measure zero, and let* $\varepsilon_0=\varepsilon_0(\Omega)$ *be the smallest side-length of these rectangles. Then* (1.3) *holds for every* $-\infty<q<q^*/N$, *every* $0<\varepsilon\le\varepsilon_0$, *and every* $u\in H^2(\Omega)$.

Proof. Assume first that Ω is given by a single rectangle $R = (a_1, b_1) \times \cdots \times (a_N, b_N)$. Let $u \in H^2(R)$ and let $q < q^*/N$. Fix $i = 1, \ldots, N$. Since $b_i - a_i \geq \varepsilon_0$, by Corollary 3.5,

$$Nq\varepsilon^2 \int_{a_i}^{b_i} \left(\frac{\partial u}{\partial x_i}\right)^2 dx_i \leq \int_{a_i}^{b_i} \left[W(u) + \varepsilon^4 \left(\frac{\partial^2 u}{\partial x_i^2}\right)^2\right] dx_i \leq \int_{a_i}^{b_i} \left[W(u) + \varepsilon^4 \left|\nabla^2 u\right|^2\right] dx_i.$$

Set $R_i := (a_1, b_1) \times \cdots \times (a_{i-1}, b_{i-1}) \times (a_{i+1}, b_{i+1}) \times \cdots \times (a_N, b_N)$, and integrate the previous inequality over R_i to obtain

$$Nq\varepsilon^2 \int_R \left(\frac{\partial u}{\partial x_i}\right)^2 dx \leq \int_R \left[W(u) + \varepsilon^4 \left|\nabla^2 u\right|^2\right] dx.$$

Summing over $i = 1, \ldots, N$ and then dividing by N, we get

$$q\varepsilon^2 \int_R \left|\nabla u\right|^2 dx \leq \int_R \left[W(u) + \varepsilon^4 \left|\nabla^2 u\right|^2\right] dx. \tag{3.30}$$

Next, if Ω is the union of finitely many pairwise disjoint open rectangles and of a set with Lebesgue measure zero, we obtain (3.30) in each rectangle and then add all the inequalities. $\square$

4. Proof of Theorems 1.1 and 1.3

In this section we prove Theorems 1.1 and 1.3. As we will see below, Theorem 1.1 is a consequence of Theorem 1.2 and standard compactness results for the Modica–Mortola functional.

Proof of Theorem 1.1. Let $-\infty < q < q^*/N$, where q^* is the constant given in Theorem 1.2, and let $\sigma \in (0, 1)$ be so small that $(q + \sigma)/(1 - \sigma) < q^*/N$. Let $u \in H^2(\Omega)$ and write

$$\begin{aligned} &W(u) - q\varepsilon^2|\nabla u|^2 + \varepsilon^4|\nabla^2 u|^2 \\ &= (1 - \sigma)\Big(W(u) - \frac{q + \sigma}{1 - \sigma}\varepsilon^2|\nabla u|^2 + \varepsilon^4|\nabla^2 u|^2\Big) + \sigma\Big(W(u) + \varepsilon^2|\nabla u|^2 + \varepsilon^4|\nabla^2 u|^2\Big). \end{aligned} \tag{4.1}$$

By Theorem 1.2, there exists $\varepsilon_0 = \varepsilon_0(\Omega, q) > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ we have

$$\int_\Omega \left[\frac{1}{\varepsilon}W(u) - \frac{q + \sigma}{1 - \sigma}\varepsilon|\nabla u|^2 + \varepsilon^3|\nabla^2 u|^2\right] dx \geq 0,$$

which implies that

$$\int_\Omega \left[\frac{1}{\varepsilon}W(u) - q\varepsilon|\nabla u|^2 + \varepsilon^3|\nabla^2 u|^2\right] dx \geq \sigma \int_\Omega \left[\frac{1}{\varepsilon}W(u) + \varepsilon|\nabla u|^2 + \varepsilon^3|\nabla^2 u|^2\right] dx. \tag{4.2}$$

Hence,

$$\int_\Omega \left[\frac{1}{\varepsilon}W(u) - q\varepsilon|\nabla u|^2 + \varepsilon^3|\nabla^2 u|^2\right] dx \geq \sigma \int_\Omega \left[\frac{1}{\varepsilon}W(u) + \varepsilon|\nabla u|^2\right] dx. \tag{4.3}$$

Now consider $\{u_n\} \subset H^2(\Omega)$ such that

$$\liminf_{n\to+\infty} F_{\varepsilon_n}(u_n, \Omega) < +\infty.$$

By (4.3), and using standard compactness results for Modica–Mortola type functionals (see, e.g., [15], [20]), there exist a subsequence $\{u_{n_k}\} \subset \{u_n\}$ and $u \in BV(\Omega; \{-1, 1\})$ such that $u_{n_k} \to u$ in $L^1(\Omega)$. In turn, by $(H2)$, for every measurable set $E \subset Q$ and for all k sufficiently large,

$$\int_E |u_{n_k}|^2 dx \leq 2\mathcal{L}^N(E) + 2\int_E W(u_{n_k}) dx \leq C\big(\mathcal{L}^N(E) + \varepsilon_{n_k}\big).$$

This implies, in particular, that the sequence $\{|u_{n_k}|^2\}$ is equi-integrable. Since $\{|u_{n_k}|\}$ converges to u in measure, by Vitali's Convergence Theorem it follows that

$$\lim_{k\to+\infty} \int_\Omega |u_{n_k} - u|^2 dx = 0. \tag{4.4}$$

$\square$

Proposition 4.1. *Assume (H1) and (H2). Then* $\mathbf{m}_N > 0$ *for every* $-\infty < q < q^*/N$.

Proof. Let $-\infty < q < q^*/N$ and $\sigma \in (0,1)$ be so small that $(q+\sigma)/(1-\sigma) < q^*/N$. Arguing as in the proof of Theorem 1.1, using Proposition 3.9 instead of Theorem 1.2, we obtain that

$$\int_Q \left[\frac{1}{\varepsilon}W(u) - q\varepsilon|\nabla u|^2 + \varepsilon^3|\nabla^2 u|^2\right] dx \geq \sigma \int_Q \left[\frac{1}{\varepsilon}W(u) + \varepsilon|\nabla u|^2\right] dx \geq 2\sigma \int_Q \sqrt{W(u)}|\nabla u|\, dx$$
$$\geq 2\sigma \int_{Q'} \int_{-1/2}^{1/2} \sqrt{W(u(x',x_N))}\left|\frac{\partial u}{\partial x_N}(x',x_N)\right| dx_N\, dx'$$

for every $u \in H^2(Q)$ and every $0 < \varepsilon \leq 1$, where $Q' := (-1/2,1/2)^{N-1}$. Since $u(x', \pm 1/2) = \pm 1$ for every $u \in \mathcal{A}$ and for every $x' \in Q'$ (see (1.5)), a change of variables shows that

$$\int_{-1/2}^{1/2} \sqrt{W(u(x',x_N))}\left|\frac{\partial u}{\partial x_N}(x',x_N)\right| dx_N \geq \int_{-1}^{1} \sqrt{W(s)}\, ds\,.$$

Therefore, the previous inequalities give

$$\int_Q \left[\frac{1}{\varepsilon}W(u) - q\varepsilon|\nabla u|^2 + \varepsilon^3|\nabla^2 u|^2\right] dx \geq 2\sigma \int_{-1}^{1} \sqrt{W(s)}\, ds > 0$$

for every $u \in \mathcal{A}$ and every $0 < \varepsilon \leq 1$. By (1.4) this implies $\mathbf{m}_N > 0$. □

Next we prove Theorem 1.3. We will use a blow-up argument that will reduce the problem to the case in which the target function is of the type

$$u_0(x) := \begin{cases} 1 & \text{if } x\cdot\nu > 0, \\ -1 & \text{if } x\cdot\nu < 0, \end{cases} \tag{4.5}$$

where $\nu \in \mathbb{S}^{N-1}$. The lemma below allows us to replace a sequence $\{v_n\}$ converging to u_0 by a sequence $\{w_n\}$ of functions still converging to u_0, satisfying $w_n = u_0$ on the faces of cube Q_ν orthogonal to ν, and without increasing the limiting energy.

Let Ψ_ε be defined as in (2.2). Then

$$(u_0 * \Psi_\varepsilon)(x) = 1 \quad \text{if } x\cdot\nu > \varepsilon, \qquad (u_0 * \Psi_\varepsilon)(x) = -1 \quad \text{if } x\cdot\nu < -\varepsilon, \tag{4.6}$$
$$\nabla(u_0 * \Psi_\varepsilon)(x) = 0 \quad \text{and} \quad \nabla^2(u_0 * \Psi_\varepsilon)(x) = 0 \quad \text{if } |x\cdot\nu| > \varepsilon. \tag{4.7}$$

Lemma 4.2. *Let* $-\infty < q < q^*/N$. *Assume (H1)–(H3). For every sequence* $\{\varepsilon_n\}$, $\varepsilon_n \to 0^+$, *and every sequence* $\{v_n\} \subset H^2(Q_\nu)$ *converging in* $L^2(Q_\nu)$ *to* u_0, *there exists a sequence* $\{w_n\} \subset H^2(Q_\nu)$ *such that* $w_n \to u_0$ *in* $L^2(Q_\nu)$, $w_n = u_0 * \Psi_{\varepsilon_n}$ *near* ∂Q_ν, *and*

$$\liminf_{n\to+\infty} \int_{Q_\nu} \left[\frac{1}{\varepsilon_n}W(w_n) - q\varepsilon_n|\nabla w_n|^2 + \varepsilon_n^3|\nabla^2 w_n|^2\right] dx$$
$$\leq \liminf_{n\to+\infty} \int_{Q_\nu} \left[\frac{1}{\varepsilon_n}W(v_n) - q\varepsilon_n|\nabla v_n|^2 + \varepsilon_n^3|\nabla^2 v_n|^2\right] dx.$$

Proof. For simplicity, we assume that $\nu = e_N$, the general case being completely analogous. If the right-hand side of the previous inequality is infinite, then it suffices to take $w_n := u_0 * \Psi_{\varepsilon_n}$. Thus, by extracting a subsequence (not relabeled), without loss of generality, we may assume that the following limit exists, that

$$\lim_{n\to+\infty} \int_Q \left[\frac{1}{\varepsilon_n}W(v_n) - q\varepsilon_n|\nabla v_n|^2 + \varepsilon_n^3|\nabla^2 v_n|^2\right] dx = C < +\infty, \tag{4.8}$$

and that $v_n(x) \to u_0(x)$ for $\mathcal{L}^N$-a.e. $x \in Q$. Let $\sigma \in (0,1)$ be so small that $(q+\sigma)/(1-\sigma) < q^*/N$. As in (4.2) and (4.8) (using Proposition 3.9 in place of Theorem 1.2), we deduce from (4.8) that

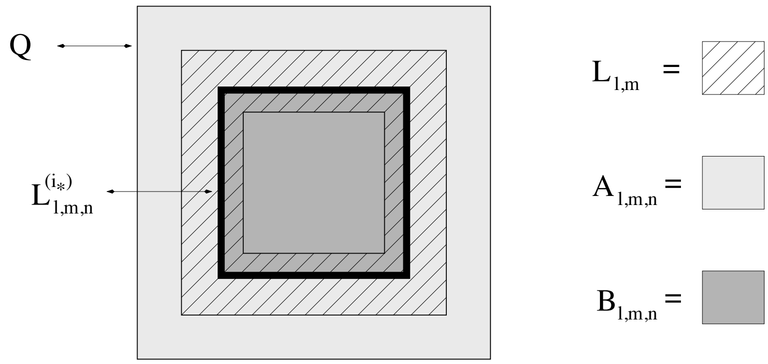

FIGURE 4.1. Geometry of the sets $A_{l,m,n}$, $B_{l,m,n}$ and $L^{(i_*)}_{l,m,n}$.

for n large enough we have

$$\begin{aligned}&\int_Q \left[\frac{1}{\varepsilon_n}W(v_n)+\varepsilon_n|\nabla v_n|^2+\varepsilon_n^3|\nabla^2 v_n|^2\right]dx\\ &\qquad\le \frac{1}{\sigma}\int_Q\left[\frac{1}{\varepsilon_n}W(v_n)-q\varepsilon_n|\nabla v_n|^2+\varepsilon_n^3|\nabla^2 v_n|^2\right]dx\le\frac{1}{\sigma}(C+1).\end{aligned}\tag{4.9}$$

For every $l,m\in\mathbb{N}$, with $l,m\ge 4$, define

$$L_{l,m}:=\left\{x\in Q:\ \frac{1}{l}<\operatorname{dist}(x,\partial Q)\le\frac{1}{l}+\frac{1}{m}\right\}.$$

Let $\lceil\varepsilon_n^{-1}\rceil$ be the smallest integer greater than ε_n^{-1}. Then

$$\varepsilon_n/2\le\lceil\varepsilon_n^{-1}\rceil^{-1}\le\varepsilon_n,\tag{4.10}$$

for $\varepsilon_n<1$. Divide $L_{l,m}$ into $\lceil\varepsilon_n^{-1}\rceil$ pairwise disjoint layers of width $\dfrac{1}{m\lceil\varepsilon_n^{-1}\rceil}$,

$$L^{(i)}_{l,m,n}:=\left\{x\in Q:\ \frac{1}{l}+\frac{i-1}{m\lceil\varepsilon_n^{-1}\rceil}<\operatorname{dist}\left(x,\partial Q\right)\le\frac{1}{l}+\frac{i}{m\lceil\varepsilon_n^{-1}\rceil}\right\},\qquad i=1,\dots,\lceil\varepsilon_n^{-1}\rceil.$$

Set

$$\widetilde{u}_n:=u_0*\Psi_{\varepsilon_n}$$

and note that if $v_n=\widetilde{u}_n$ for infinitely many n, then there is nothing to prove. Otherwise, without loss of generality, we may assume that for every $n\in\mathbb{N}$, $\|v_n-\widetilde{u}_n\|_{L^2(Q)}>0$ and we have

$$\begin{aligned}&\sum_{i=1}^{\lceil\varepsilon_n^{-1}\rceil}\int_{L^{(i)}_{l,m,n}}\left[\frac{1}{\varepsilon_n}W(v_n)+\varepsilon_n|\nabla v_n|^2+\varepsilon_n^3|\nabla^2 v_n|^2+\frac{|v_n-\widetilde{u}_n|^2}{\|v_n-\widetilde{u}_n\|_{L^2(Q)}}\right]dx\\ &\qquad=\int_{L_{l,m}}\left[\frac{1}{\varepsilon_n}W(v_n)+\varepsilon_n|\nabla v_n|^2+\varepsilon_n^3|\nabla^2 v_n|^2+\frac{|v_n-\widetilde{u}_n|^2}{\|v_n-\widetilde{u}_n\|_{L^2(Q)}}\right]dx\le C,\end{aligned}$$

where in the last inequality we have used (4.9), and the fact that $v_n,\widetilde{u}_n\to u_0$ in $L^2(Q)$. Thus, also by (4.10), there exists $i_*=i_*(m,n)$ such that

$$\int_{L^{(i_*)}_{l,m,n}}\left[\frac{1}{\varepsilon_n}W(v_n)+\varepsilon_n|\nabla v_n|^2+\varepsilon_n^3|\nabla^2 v_n|^2\right]dx+\int_{L^{(i_*)}_{l,m,n}}\frac{|v_n-\widetilde{u}_n|^2}{\|v_n-\widetilde{u}_n\|_{L^2(Q)}}dx\le C\varepsilon_n.\tag{4.11}$$

Construct cut-off functions $\varphi_{l,m,n} \in C_c^\infty(Q;[0,1])$ such that

$$\begin{aligned} \varphi_{l,m,n} &= 0 \qquad \text{on } \left\{ x \in Q : \operatorname{dist}(x, \partial Q) \leq \frac{1}{l} + \frac{i_* - 1}{m\lceil \varepsilon_n^{-1} \rceil} \right\} =: A_{l,m,n}, \\ \varphi_{l,m,n} &= 1 \qquad \text{in } \left\{ x \in Q : \operatorname{dist}(x, \partial Q) > \frac{1}{l} + \frac{i_*}{m\lceil \varepsilon_n^{-1} \rceil} \right\} =: B_{l,m,n}, \end{aligned} \tag{4.12}$$

and

$$\|\nabla \varphi_{l,m,n}\|_\infty = O(m/\varepsilon_n), \qquad \|\nabla^2 \varphi_{l,m,n}\|_\infty = O(m^2/\varepsilon_n^2). \tag{4.13}$$

Define

$$w_{l,m,n} := \varphi_{l,m,n} v_n + (1 - \varphi_{l,m,n}) \widetilde{u}_n.$$

Note that Q is the disjoint union of $A_{l,m,n}$, $L_{l,m,n}^{(i_*)}$, and $B_{l,m,n}$, and that $w_{l,m,n} \in H^2(Q_\nu)$. Since $v_n, \widetilde{u}_n \to u_0$ in $L^2(Q)$, we have that

$$\lim_{n \to +\infty} \|w_{l,m,n} - u_0\|_{L^2(Q)} = 0 \tag{4.14}$$

for all $l, m \geq 4$. Moreover

$$\begin{aligned} &\int_Q \left[\frac{1}{\varepsilon_n} W(w_{l,m,n}) - q\varepsilon_n |\nabla w_{l,m,n}|^2 + \varepsilon_n^3 |\nabla^2 w_{l,m,n}|^2 \right] dx \\ &\quad \leq \int_{A_{l,m,n}} \left[\frac{1}{\varepsilon_n} W(\widetilde{u}_n) + |q|\, \varepsilon_n |\nabla \widetilde{u}_n|^2 + \varepsilon_n^3 |\nabla^2 \widetilde{u}_n|^2 \right] dx \\ &\quad\quad + \int_{L_{l,m,n}^{(i_*)}} \left[\frac{1}{\varepsilon_n} W(w_{l,m,n}) + |q|\, \varepsilon_n |\nabla w_{l,m,n}|^2 + \varepsilon_n^3 |\nabla^2 w_{l,m,n}|^2 \right] dx \\ &\quad\quad + \int_{B_{l,m,n}} \left[\frac{1}{\varepsilon_n} W(v_n) - q\varepsilon_n |\nabla v_n|^2 + \varepsilon_n^3 |\nabla^2 v_n|^2 \right] dx =: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \end{aligned} \tag{4.15}$$

By $(H1)$, (4.6), (4.7), the continuity of W, and (2.4), in this order, we get

$$\begin{aligned} \mathcal{I}_1 &= \int_{A_{l,m,n} \cap \{|x_N| < \varepsilon_n\}} \left[\frac{1}{\varepsilon_n} W(\widetilde{u}_n) + |q|\, \varepsilon_n |\nabla \widetilde{u}_n|^2 + \varepsilon_n^3 |\nabla^2 \widetilde{u}_n|^2 \right] dx \\ &\leq \frac{C}{\varepsilon_n} \mathcal{L}^N \left(\left\{ x \in Q : \operatorname{dist}(x, \partial Q) \leq \frac{1}{l} + \frac{1}{m},\, |x_N| < \varepsilon_n \right\} \right) \leq C \left(\frac{1}{m} + \frac{1}{l} \right) \end{aligned} \tag{4.16}$$

for all n sufficiently large.

By the continuity of W and $(H3)$, it follows that

$$W(s) \leq CW(t) + C \tag{4.17}$$

for every $t \in \mathbb{R}$ and every $s \in \mathbb{R}$ of the form $s = \theta t + (1 - \theta)\, t_0$, where $|t_0| \leq 1$ and $\theta \in [0,1]$. Indeed, if $|t| \geq 1$, then $|s| \leq \theta\, |t| + (1 - \theta)\, |t_0| \leq |t|$, and so (4.17) follows from $(H3)$, while if $|t| \leq 1$, then $|s| \leq 1$, and so $W(s) \leq \max_{|\tau| \leq 1} W(\tau)$.

Since $|\widetilde{u}_n| \leq 1$ and

$$\begin{aligned} \nabla w_{l,m,n} &= \varphi_{l,m,n} \nabla v_n + (1 - \varphi_{l,m,n}) \nabla \widetilde{u}_n + (v_n - \widetilde{u}_n) \nabla \varphi_{l,m,n}, \\ \nabla^2 w_{l,m,n} &= \varphi_{l,m,n} \nabla^2 v_n + (1 - \varphi_{l,m,n}) \nabla^2 \widetilde{u}_n + 2 \nabla \varphi_{l,m,n} \odot (\nabla v_n - \nabla \widetilde{u}_n) + (v_n - \widetilde{u}_n) \nabla^2 \varphi_{l,m,n}, \end{aligned}$$

where $\odot$ is the symmetrized tensor product, by (4.7), (4.13), (4.17), (2.4), and (4.11) we have

$$
\begin{aligned}
\mathcal{I}_2 &\le C\int_{L^{(i_*)}_{l,m,n}\cap\{|x_N|<\varepsilon_n\}}\left[(m^2+|q|)\,\varepsilon_n|\nabla\widetilde{u}_n|^2+\varepsilon_n^3|\nabla^2\widetilde{u}_n|^2\right]dx\\
&\quad+C\int_{L^{(i_*)}_{l,m,n}}\left[\frac{1}{\varepsilon_n}(W(v_n)+1)+(m^2+|q|)\,\varepsilon_n|\nabla v_n|^2\right]dx\\
&\quad+C\int_{L^{(i_*)}_{l,m,n}}\left[\varepsilon_n^3|\nabla^2 v_n|^2+\frac{m^4+|q|m^2}{\varepsilon_n}|v_n-\widetilde{u}_n|^2\right]dx\\
&\le\frac{C}{\varepsilon_n}(1+m^2)\mathcal{L}^N\left(L^{(i_*)}_{l,m,n}\cap\left\{x\in Q:\,|x_N|<\varepsilon_n\right\}\right)+\frac{C}{m\lceil\varepsilon_n^{-1}\rceil\varepsilon_n}\\
&\quad+C\int_{L^{(i_*)}_{l,m,n}}\left[\frac{1}{\varepsilon_n}W(v_n)+m^2\varepsilon_n|\nabla v_n|^2+\varepsilon_n^3|\nabla^2 v_n|^2\right]dx+\frac{Cm^4}{\varepsilon_n}\int_{L^{(i_*)}_{l,m,n}}|v_n-\widetilde{u}_n|^2\,dx\\
&\le C(1+m^2)\varepsilon_n+\frac{C}{m}+Cm^4\|v_n-\widetilde{u}_n\|_{L^2(Q)},
\end{aligned}
\tag{4.18}
$$

where we used also the inequalities $m^2+|q|\le Cm^2$ and $m^4+|q|m^2\le Cm^4$. Moreover, since $Q\setminus B_{l,m,n}$ can be written as a union of finitely many pairwise disjoint rectangles with the smallest side-length greater than $1/l$, by Proposition 3.9 and the fact that $-\infty<q<q^*/N$, for all n sufficiently large, we have

$$
\begin{aligned}
&\int_{Q\setminus B_{l,m,n}}\left[\frac{1}{\varepsilon_n}W(v_n)-q\varepsilon_n|\nabla v_n|^2+\varepsilon_n^3|\nabla^2 v_n|^2\right]dx\\
&\qquad=\int_{A_{l,m,n}\cup L^{i_*}_{l,m,n}}\left[\frac{1}{\varepsilon_n}W(v_n)-q\varepsilon_n|\nabla v_n|^2+\varepsilon_n^3|\nabla^2 v_n|^2\right]dx\ge 0.
\end{aligned}
\tag{4.19}
$$

Thus,

$$
\mathcal{I}_3\le\int_Q\left[\frac{1}{\varepsilon_n}W(v_n)-q\varepsilon_n|\nabla v_n|^2+\varepsilon_n^3|\nabla^2 v_n|^2\right]dx
$$

and, recalling (4.15), (4.16), (4.18), and (4.19), we obtain

$$
\begin{aligned}
&\lim_{l\to+\infty}\lim_{m\to+\infty}\lim_{n\to+\infty}\int_Q\left[\frac{1}{\varepsilon_n}W(w_{l,m,n})-q\varepsilon_n|\nabla w_{l,m,n}|^2+\varepsilon_n^3|\nabla^2 w_{l,m,n}|^2\right]dx\\
&\qquad\le\lim_{n\to+\infty}\int_Q\left[\frac{1}{\varepsilon_n}W(v_n)-q\varepsilon_n|\nabla v_n|^2+\varepsilon_n^3|\nabla^2 v_n|^2\right]dx.
\end{aligned}
\tag{4.20}
$$

In view of (4.14) and (4.20), the result now follows by a standard diagonalization argument. $\square$

Remark 4.3. Using a change of variables with a rotation, it can be shown that for every $\nu\in\mathbb{S}^{N-1}$,

$$
\mathbf{m}_N=\inf\left\{\int_{Q_\nu}\left[\frac{1}{\varepsilon}W(u)-q\varepsilon|\nabla u|^2+\varepsilon^3|\nabla^2 u|^2\right]dx\,:\,0<\varepsilon\le 1,\,u\in\mathcal{A}_\nu\right\},
\tag{4.21}
$$

where

$$
\begin{aligned}
\mathcal{A}_\nu:=\Big\{&u\in H^2_{\mathrm{loc}}(\mathbb{R}^N):\,u(x)=-1\text{ near }x\cdot\nu=-\frac{1}{2},\,u(x)=1\text{ near }x\cdot\nu=\frac{1}{2},\\
&u(x)=u(x+k\nu_i)\text{ for all }x\in\mathbb{R}^N,\,i=1,\dots,N-1,\text{ and }k\in\mathbb{Z}\Big\}
\end{aligned}
\tag{4.22}
$$

and $\{\nu_1,\dots,\nu_{N-1},\nu\}$ is an orthonormal basis of $\mathbb{R}^N$. Hence, the functions w_n constructed in the previous lemma belong to $\mathcal{A}_\nu$.

We now turn to the proof of Theorem 1.3.

Proof of Theorem 1.3. Step 1. We prove inequality (1.8). Let $u \in L^2(\Omega)$, let $\{\varepsilon_n\}$ be a sequence of positive numbers converging to 0, and let $\{u_n\} \subset H^2(\Omega)$ be such that $u_n \to u$ in $L^2(\Omega)$. If the right-hand side of (1.8) is infinite, then there is nothing to prove. Otherwise, we can extract a subsequence, not relabeled, such that $u_n \to u$ $\mathcal{L}^N$-a.e. in Ω and

$$\lim_{n\to+\infty} \int_\Omega \left[\frac{1}{\varepsilon_n} W(u_n) - q\varepsilon_n |\nabla u_n|^2 + \varepsilon_n^3 |\nabla^2 u_n|^2\right] dx$$

exists, is finite, and coincides with the lim inf of the original sequence. Using Theorem 1.1, we have that $u \in BV(\Omega, \{-1, 1\})$, and so that we may write

$$u = \chi_{E_0} - \chi_{\Omega\setminus E_0},$$

where $\mathrm{Per}_\Omega(E_0) < +\infty$.

Set $f_n := \frac{1}{\varepsilon_n} W(u_n) - q\varepsilon_n|\nabla u_n|^2 + \varepsilon_n^3|\nabla^2 u_n|^2$. By the previous inequality and Theorem 1.2, the sequence $\{f_n\}$ is bounded in $L^1(\Omega)$. Therefore, there exist a subsequence (not relabeled) and two bounded Radon measures μ, λ such that

$$f_n\, \mathcal{L}^N \lfloor \Omega \overset{*}{\rightharpoonup} \lambda, \quad |f_n|\, \mathcal{L}^N \lfloor \Omega \overset{*}{\rightharpoonup} \mu \qquad \text{in } \mathcal{M}_b(\Omega). \tag{4.23}$$

We claim that $\lambda \geq 0$. By the Besicovitch Derivation Theorem (see, e.g., [6, Theorem 1.155]), for $|\lambda|$-a.e. $x_0 \in \Omega$,

$$\frac{d\lambda}{d|\lambda|}(x_0) = \lim_{r\to 0^+} \frac{\lambda\big(Q(x_0, r)\big)}{|\lambda|\big(Q(x_0, r)\big)} \in \mathbb{R},$$

where $|\lambda|$ denotes the total variation of λ. Choose a sequence $\{r_k\}$ satisfying $\mu\big(\partial Q(x_0, r_k)\big) = 0$. Then (see, e.g., [6, Corollary 1.204]),

$$\begin{aligned}\frac{d\lambda}{d|\lambda|}(x_0) &= \lim_{k\to+\infty} \frac{\lambda\big(Q(x_0, r_k)\big)}{|\lambda|\big(Q(x_0, r_k)\big)} \\ &= \lim_{k\to+\infty} \lim_{n\to+\infty} \frac{1}{|\lambda|\big(Q(x_0, r_k)\big)} \int_{Q(x_0,r_k)} f_n(x)\, dx \geq 0,\end{aligned}$$

where in the last inequality we have used the fact that, by Proposition 3.9, for every fixed $k \in \mathbb{N}$, $\int_{Q(x_0,r_k)} f_n(x)\, dx \geq 0$ for all n sufficiently large (depending on k). Since λ is absolutely continuous with respect $|\lambda|$, this implies that $\lambda \geq 0$.

Consider the nonnegative measure

$$\zeta\,(E) := \mathcal{H}^{N-1}(E \cap \partial^* E_0)$$

defined over all Borel subsets $E \subset \Omega$, where $\partial^* E_0$ is the essential boundary of E_0 (see Definition 3.60 in [1]). Since $\mathrm{Per}_\Omega(E_0) < +\infty$, by Theorem 3.61 and (3.62) in [1], we have that

$$\zeta\,(\Omega) = \mathcal{H}^{N-1}(\Omega \cap \partial^* E_0) = \mathrm{Per}_\Omega(E_0) < +\infty,$$

so that ζ is a bounded Radon measure. Hence, we may use the Radon-Nikodym and Lebesgue Decomposition Theorem (see, e.g., [6, Theorem 1.180]) to decompose λ as $\lambda = g\zeta + \lambda_s$, where g is a nonnegative integrable function and $\lambda_s \geq 0$ is a bounded Radon measure, with λ_s and ζ mutually singular. We claim that

$$g(x_0) \geq \mathbf{m}_N \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x_0 \in \Omega \cap \partial^* E_0. \tag{4.24}$$

Assuming that (4.24) holds, the inequality $\lambda_s \geq 0$ gives

$$\begin{aligned}&\lim_{n\to+\infty} \int_\Omega \left[\frac{1}{\varepsilon_n} W(u_n) - q\varepsilon_n |\nabla u_n|^2 + \varepsilon_n^3 |\nabla^2 u_n|^2\right] dx \\ &\quad= \lim_{n\to+\infty} \int_\Omega f_n(x)\, dx \geq \lambda(\Omega) \geq \int_{\Omega\cap\partial^* E_0} g(x)\, d\mathcal{H}^{N-1}(x) \\ &\quad\geq \mathbf{m}_N \mathcal{H}^{N-1}(\Omega \cap \partial^* E_0) = \mathbf{m}_N\, \mathrm{Per}_\Omega(\{u = 1\}),\end{aligned}$$

which proves (1.8).

In the remaining of this step we show that (4.24) holds. By Theorem 3.59 and 3.61 in [1], for $\mathcal{H}^{N-1}$-a.e. $x_0 \in \Omega \cap \partial^* E_0$ we have

$$\lim_{r\to 0^+} \frac{\mathcal{L}^N\big(\{x \in Q_\nu(x_0,r)\setminus E_0 : (x-x_0)\cdot\nu < 0\}\big)}{r^N} = 0, \tag{4.25}$$

$$\lim_{r\to 0^+} \frac{\mathcal{L}^N\big(\{x \in Q_\nu(x_0,r)\cap E_0 : (x-x_0)\cdot\nu > 0\}\big)}{r^N} = 0, \tag{4.26}$$

$$\lim_{r\to 0^+} \frac{\mathcal{H}^{N-1}\left(Q_\nu(x_0,r)\cap \partial^* E_0\right)}{r^{N-1}} = 1, \tag{4.27}$$

where $\nu := \nu(x_0)$ is the outward normal to E_0 at x_0. Fix any such $x_0 \in \Omega$. In view of the Besicovitch Derivation Theorem (see, e.g., [6, Theorem 1.155]), we can also assume that

$$g(x_0) = \lim_{r\to 0^+} \frac{\lambda\left(Q_\nu(x_0,r)\right)}{\mathcal{H}^{N-1}\left(Q_\nu(x_0,r)\cap\partial^* E_0\right)} < +\infty. \tag{4.28}$$

Then, by (4.27) and choosing $r_k \to 0^+$ such that $\mu\big(\partial Q_\nu(x_0,r_k)\big) = 0$, we obtain (see, e.g., [6, Corollary 1.204]),

$$\begin{aligned} g(x_0) &= \lim_{k\to+\infty} \frac{\lambda\big(Q_\nu(x_0,r_k)\big)}{r_k^{N-1}} \\ &= \lim_{k\to+\infty}\lim_{n\to+\infty} \frac{1}{r_k^{N-1}} \int_{Q_\nu(x_0,r_k)} \left[\frac{1}{\varepsilon_n} W(u_n(x)) - q\varepsilon_n |\nabla u_n(x)|^2 + \varepsilon_n^3 |\nabla^2 u_n(x)|^2\right] dx \\ &= \lim_{k\to+\infty}\lim_{n\to+\infty} \int_{Q_\nu} \left[\frac{r_k}{\varepsilon_n} W(v_{n,k}(y)) - q\frac{\varepsilon_n}{r_k}|\nabla v_{n,k}(y)|^2 + \left(\frac{\varepsilon_n}{r_k}\right)^3 |\nabla^2 v_{n,k}(y)|^2\right] dy, \end{aligned} \tag{4.29}$$

where $v_{n,k} \in H^2(Q_\nu)$ is defined by

$$v_{n,k}(y) := u_n(x_0 + r_k y).$$

Since $u_n \to u$ in $L^2(\Omega)$, by (4.25) and (4.26), we have that

$$\lim_{k\to+\infty}\lim_{n\to+\infty} \|v_{n,k} - u_0\|_{L^2(Q_\nu)} = 0, \tag{4.30}$$

where u_0 is defined in (4.5). By (4.29), (4.30), and a diagonalization argument we may find a subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ such that

$$t_k := \frac{\varepsilon_{n_k}}{r_k} \to 0, \qquad v_k := v_{n_k,k} \to u_0 \text{ in } L^2(Q_\nu) \qquad \text{as } k\to+\infty,$$

and

$$g(x_0) = \lim_{k\to+\infty} \int_{Q_\nu} \left[\frac{1}{t_k} W(v_k) - q t_k |\nabla v_k|^2 + t_k^3 |\nabla^2 v_k|^2\right] dy. \tag{4.31}$$

Applying Lemma 4.2 to the sequences $\{v_k\}$ and $\{t_k\}$, we conclude that there exists a sequence $\{w_k\} \subset H^2(Q_\nu)$ such that $w_k \to u_0$ in $L^2(Q_\nu)$, $\{w_k\} \subset \mathcal{A}_\nu$ (see (4.22)) and

$$\begin{aligned} g(x_0) &= \lim_{k\to+\infty} \int_{Q_\nu} \left[\frac{1}{t_k} W(v_k) - q t_k |\nabla v_k|^2 + t_k^3 |\nabla^2 v_k|^2\right] dy \\ &\geq \liminf_{k\to+\infty} \int_{Q_\nu} \left[\frac{1}{t_k} W(w_k) - q t_k |\nabla w_k|^2 + t_k^3 |\nabla^2 w_k|^2\right] dy. \end{aligned} \tag{4.32}$$

Since each w_k belongs to $\mathcal{A}_\nu$ (see Remark 4.3), (4.24) follows from (4.21) and (4.32).

Step 2. We prove inequality (1.9). Given $u \in BV(\Omega; \{-1,1\})$, write

$$u = \chi_{E_0} - \chi_{\Omega\setminus E_0}, \tag{4.33}$$

where $\mathrm{Per}_\Omega(E_0) < +\infty$. If $\min\left\{\mathcal{L}^N(E_0), \mathcal{L}^N(\Omega \setminus E_0)\right\} = 0$, then $\mathrm{Per}_\Omega(E_0) = 0$, u is constantly equal either to 1 or -1, and it suffices to take $u_n = u$. Hence, in what follows we assume that u is not constant. By (1.5), for every fixed $\rho > 0$ there exist $\epsilon_0 > 0$ and $w \in \mathcal{A}$ such that

$$\int_Q \left[\frac{1}{\epsilon_0} W(w) - q\epsilon_0 |\nabla w|^2 + \epsilon_0^3 |\nabla^2 w|^2\right] dx < \mathbf{m}_N + \rho. \tag{4.34}$$

It suffices to show that for every sequence $\varepsilon_n \to 0^+$, there exists a sequence $\{u_n\} \in H^2(\Omega)$ such that $u_n \to u$ in $L^2(\Omega)$ as $n \to +\infty$ and

$$\begin{aligned}\limsup_{n\to+\infty} \int_\Omega \left[\frac{1}{\varepsilon_n} W(u_n) - q\varepsilon_n |\nabla u_n|^2 + \varepsilon_n^3 |\nabla^2 u_n|^2\right] dx \\ \le (\mathbf{m}_N + 2\rho)\, \mathrm{Per}_\Omega(\{u = 1\}).\end{aligned} \tag{4.35}$$

We divide the proof of (4.35) into three substeps.

Substep 2A. Consider first the case in which u has a flat interface orthogonal to a given direction $\nu \in \mathbb{S}^{N-1}$ and Ω has Lipschitz boundary that meets this interface orthogonally, i.e.,

$$u(x) := \begin{cases} 1 & \text{if } (x - x_0)\cdot\nu > 0, \\ -1 & \text{if } (x - x_0)\cdot\nu < 0, \end{cases} \tag{4.36}$$

for every $x \in \mathbb{R}^N$ and some $x_0 \in \Omega$, and

$$\text{the normal } \nu(x) \text{ to } \partial\Omega \text{ is orthogonal to } \nu \tag{4.37}$$

at all points $x \in \partial\Omega$ with $|(x - x_0)\cdot\nu|$ small enough.

Consider a rotation R such that $Re_N = \nu$. For every n, define $w_n \in H^2_{\mathrm{loc}}(\mathbb{R}^N)$ by

$$w_n(x) := \begin{cases} 1 & \text{if } (x - x_0)\cdot\nu \ge \dfrac{\varepsilon_n}{2\epsilon_0}, \\ w\left(\dfrac{\epsilon_0 R^T(x - x_0)}{\varepsilon_n}\right) & \text{if } |(x - x_0)\cdot\nu| \le \dfrac{\varepsilon_n}{2\epsilon_0}, \\ -1 & \text{if } (x - x_0)\cdot\nu \le -\dfrac{\varepsilon_n}{2\epsilon_0}. \end{cases} \tag{4.38}$$

Without loss of generality, we may assume that $x_0 = 0$, $\nu = e_N$, and that R is the identity. Using a change of variables and the periodicity of w in the first $N - 1$ coordinates, we can prove that

$$\|w_n\|_{L^2(\{x\in\Omega:\, |x_N| \le \varepsilon_n/(2\epsilon_0)\})} \le C\|w\|_{L^2(Q)} \frac{\varepsilon_n}{\epsilon_0}. \tag{4.39}$$

This inequality, together with (4.36) and (4.38), gives

$$\|w_n - u\|_{L^2(\Omega)} \le C\|w\|_{L^2(Q)} \frac{\varepsilon_n}{\epsilon_0} + \mathcal{L}^N\left(\left\{x \in \Omega :\, |x_N| \le \frac{\varepsilon_n}{2\epsilon_0}\right\}\right), \tag{4.40}$$

which tends to 0 as $n \to +\infty$.

Let $\Omega' := \{x' \in \mathbb{R}^{N-1} : (x', 0) \in \Omega\}$. By (4.37) we have $\{x \in \Omega : |x_N| \le \varepsilon_n/(2\epsilon_0)\} = \Omega' \times [-\varepsilon_n/(2\epsilon_0), \varepsilon_n/(2\epsilon_0)]$ for n large enough. Setting $t := x_N \epsilon_0/\varepsilon_n$, by $(H1)$, (4.38), and Fubini's Theorem, we have

$$\begin{aligned} F_{\varepsilon_n}(w_n, \Omega) &= \int_{\left\{x\in\Omega:\, |x_N| \le \varepsilon_n/(2\epsilon_0)\right\}} \frac{1}{\varepsilon_n} \left[W(w) - q\epsilon_0^2 |\nabla w|^2 + \epsilon_0^4 |\nabla^2 w|^2\right] \left(\frac{\epsilon_0 x}{\varepsilon_n}\right) dx \\ &= \int_{\Omega'} \int_{-1/2}^{1/2} \left[\frac{1}{\epsilon_0} W(w) - q\epsilon_0 |\nabla w|^2 + \epsilon_0^3 |\nabla^2 w|^2\right] \left(\frac{\epsilon_0 x'}{\varepsilon_n}, t\right) dt\, dx' \end{aligned} \tag{4.41}$$

for n large enough. Since w is periodic in the first $N - 1$ variables, the functions

$$x' \mapsto \int_{-1/2}^{1/2} W(w(x', t))\, dt, \qquad x' \mapsto \int_{-1/2}^{1/2} |\nabla w(x', t)|^2\, dt, \qquad x' \mapsto \int_{-1/2}^{1/2} |\nabla^2 w(x', t)|^2\, dt$$

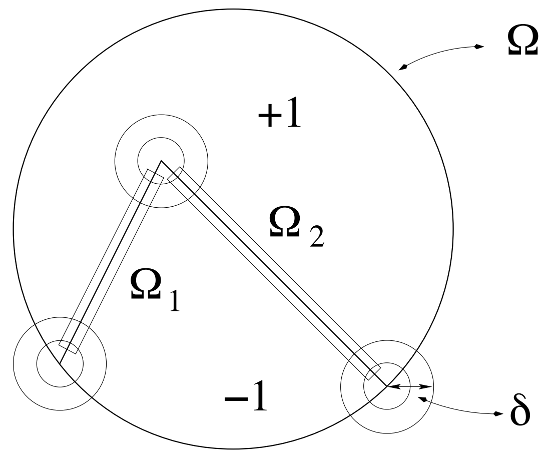

FIGURE 4.2. Construction in Substep 2B. Here $\Omega \cap \partial P = H_1 \cup H_2 \cup F$ is just an angle and F is its vertex.

are periodic, and, by Fubini's Theorem, belong to $L^1_{\mathrm{loc}}\left(\mathbb{R}^{N-1}\right)$. Therefore we can apply the Riemann-Lebesgue Lemma, obtaining from (4.41) and from Fubini's Theorem that

$$\begin{aligned}\lim_{n\to+\infty} F_{\varepsilon_n}(w_n,\Omega) &= \mathcal{L}^{N-1}(\Omega') \int_Q \left[\frac{1}{\epsilon_0} W(w) - q\epsilon_0 |\nabla w|^2 + \epsilon_0^3 |\nabla^2 w|^2\right] dx \\ &\le \big(\mathbf{m}_N + \rho\big)\mathcal{H}^{N-1}\big(\Omega' \times \{0\}\big) = \big(\mathbf{m}_N + \rho\big) \operatorname{Per}_\Omega(\{u=1\}).\end{aligned} \tag{4.42}$$

where we use also (4.34). This concludes the proof of (4.35).

Substep 2B. Consider now the case in which u has a polyhedral interface, i.e., the set E_0 in (4.33) has the form $E_0 = P \cap \Omega$, with P polyhedral. This means that $\partial P = H_1 \cup H_2 \cup \cdots \cup H_L \cup F$, where the sets H_i are pairwise disjoint (relatively open) convex polyhedra of dimension $N-1$, while the set F is the union of a finite number of convex polyhedra of dimension $N-2$. In particular, each set H_i is contained in a hyperplane, i.e.,

$$H_i \subset \{x \in \mathbb{R}^N : (x - x_i)\cdot \nu_i = 0\}, \tag{4.43}$$

for some $x_i \in \mathbb{R}^N$ and $\nu_i \in \mathbb{S}^{N-1}$. We assume that ν_i is the inner unit normal to ∂P on H_i. To simplify the proof, we assume also that

$$\partial\Omega \cap \partial P \text{ is the union of a finite number of } C^1 \text{ manifolds of dimension } N-2. \tag{4.44}$$

Fix $0 < \delta < 1$ small and let (see Figure 4.2)

$$U_\delta := \Big\{x \in \Omega : \operatorname{dist}\Big(x, F \cup (\partial\Omega \cap \partial P)\Big) \le \delta\Big\}. \tag{4.45}$$

We can find a finite family $H'_1, H'_2, \ldots, H'_L$ of relatively open subsets of $H_1, H_2, \ldots, H_L$, with $(N-2)$-dimensional boundary of class C^∞, such that

$$\Big\{x \in H_i \cap \Omega : \operatorname{dist}\big(x, \partial\Omega \cup F\big) \ge \frac{\delta}{2}\Big\} \subset H'_i \subset \overline{H'_i} \subset H_i \cap \Omega. \tag{4.46}$$

Fix $0 < \eta < \delta/2$. For every $i = 1, 2, \ldots, L$ let

$$\Omega_i := \{x + t\nu_i : x \in H'_i,\ |t| < \eta\}, \tag{4.47}$$

where ν_i is given by (4.43). We assume also that η is so small that the sets $\Omega_1, \Omega_2, \ldots, \Omega_L$ are pairwise disjoint.

Since Ω_i satisfies (4.37) for every $i = 1, 2, \dots, L$, we can now apply the construction of Substep A, with Ω replaced by Ω_i, and we obtain a sequence $w_n^i \in H^2(\Omega_i)$ such that

$$w_n^i \to u \quad \text{in } L^2(\Omega_i) \text{ as } n \to +\infty \tag{4.48}$$

and

$$\lim_{n\to+\infty} F_{\varepsilon_n}(w_n^i, \Omega_i) \le \big(\mathbf{m}_N + \rho\big)\mathcal{H}^{N-1}\big(H_i \cap \Omega_i\big). \tag{4.49}$$

By construction,

$$w_n^i(x) = u(x) \quad \text{for } x \in \overline{\Omega}_i \text{ and } \operatorname{dist}(x, H_i) \ge \frac{\varepsilon_n}{2\epsilon_0}. \tag{4.50}$$

In order to define u_n in Ω, we extend u to $\mathbb{R}^N$ by setting $u(x) := \chi_P(x) - \chi_{\mathbb{R}^N\setminus P}(x)$ for every $x \in \mathbb{R}^N$ and define $\widetilde{u}_n := u * \Psi_{\varepsilon_n}$ (see (2.3)). We choose cut-off functions $\varphi_\delta \in C_c^\infty(\mathbb{R}^N; [0, 1])$ such that

$$\varphi_\delta = 0 \text{ in } U_\delta, \quad \varphi_\delta = 1 \text{ in } \mathbb{R}^N \setminus U_{2\delta}, \quad \|\nabla\varphi_\delta\|_\infty \le C/\delta, \quad \|\nabla^2\varphi_\delta\|_\infty \le C/\delta^2. \tag{4.51}$$

We define u_n in Ω by setting

$$u_n := \begin{cases} \varphi_\delta w_n^i + (1-\varphi_\delta)\widetilde{u}_n & \text{in each } \overline{\Omega}_i,\ i = 1, \dots, L, \\ \widetilde{u}_n & \text{in } A := \Omega \setminus (\overline{\Omega}_1 \cup \dots \cup \overline{\Omega}_L). \end{cases} \tag{4.52}$$

We claim that $u_n \in H^2(\Omega)$. By (4.50) and (4.52), for n large enough we have $u_n = \widetilde{u}_n$ in a neighborhood of $\{x \in \partial\Omega_i : \operatorname{dist}(x, H_i) = \eta\}$ (the part of $\partial\Omega_i$ parallel to H_i). Note that, by (4.45), (4.46), and (4.47), the part of $\partial\Omega_i$ orthogonal to H_i,

$$D_i := \{x \in \partial\Omega_i : \operatorname{dist}(x, H_i) < \eta\}$$

is contained in the interior of U_δ. Therefore by (4.51), $u_n = \widetilde{u}_n$ in a neighborhood of D_i. This shows that $u_n \in H^2(\Omega)$.

By (4.48) and the fact that $\widetilde{u}_n \to u$ in $L^2(\Omega)$, we have that $u_n \to u$ in $L^2(\Omega)$ as $n \to +\infty$.

Note that by (4.6) and (4.7), u_n is different from ± 1 only in the region

$$\mathcal{R}_n := \big\{x \in \Omega : \operatorname{dist}(x, \partial P) \le \max\{\varepsilon_n/(2\epsilon_0), \varepsilon_n\}\big\}$$

and so, since $A \cap \mathcal{R}_n \subset U_\delta$ for n large enough and $\mathcal{H}^{N-1}(\partial P \cap U_\delta) \le C\delta$, we get

$$F_{\varepsilon_n}(u_n, A) \le \int_{A\cap\mathcal{R}_n} \left[\frac{1}{\varepsilon_n}W(\widetilde{u}_n) + |q|\,\varepsilon_n|\nabla\widetilde{u}_n|^2 + \varepsilon_n^3|\nabla^2\widetilde{u}_n|^2\right] dx \le C\delta. \tag{4.53}$$

Moreover,

$$\begin{aligned} F_{\varepsilon_n}(u_n, \Omega_i) \le &\int_{\Omega_i\cap U_\delta} \left[\frac{1}{\varepsilon_n}W(\widetilde{u}_n) + |q|\,\varepsilon_n|\nabla\widetilde{u}_n|^2 + \varepsilon_n^3|\nabla^2\widetilde{u}_n|^2\right] dx \\ &+ \int_{\Omega_i\cap(U_{2\delta}\setminus U_\delta)} \left[\frac{1}{\varepsilon_n}W(u_n) + |q|\,\varepsilon_n|\nabla u_n|^2 + \varepsilon_n^3|\nabla^2 u_n|^2\right] dx \\ &+ \int_{\Omega_i\setminus U_{2\delta}} \left[\frac{1}{\varepsilon_n}W(w_n^i) - q\varepsilon_n|\nabla w_n^i|^2 + \varepsilon_n^3|\nabla^2 w_n^i|^2\right] dx =: \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3. \end{aligned} \tag{4.54}$$

Since $F \cup (\partial\Omega \cap \partial P)$ has dimension $N-2$, we have $\mathcal{H}^{N-1}(H_i \cap U_\delta) \le C\delta$. Therefore, reasoning as in (4.16), we obtain

$$\mathcal{K}_1 \le C\delta. \tag{4.55}$$

Next we estimate $\mathcal{K}_2$ and $\mathcal{K}_3$. By (4.17) in Ω_i we have

$$W(u_n) \le CW(w_n^i) + C, \tag{4.56}$$

$$\nabla u_n = \varphi_\delta\nabla w_n^i + (1-\varphi_\delta)\nabla\widetilde{u}_n + (w_n^i - \widetilde{u}_n)\nabla\varphi_\delta, \tag{4.57}$$

$$\nabla^2 u_n = \varphi_\delta\nabla^2 w_n^i + (1-\varphi_\delta)\nabla^2\widetilde{u}_n + 2\nabla\varphi_\delta \odot (\nabla w_n^i - \nabla\widetilde{u}_n) + (w_n^i - \widetilde{u}_n)\nabla^2\varphi_\delta. \tag{4.58}$$

Using $(H1)$, Young's inequality, (4.51) and (4.56), we obtain

$$\begin{aligned}\mathcal{K}_2 \leq C \int_{\Omega_i\cap(U_{2\delta}\setminus U_\delta)\cap\mathcal{R}_n} &\left[\frac{1}{\varepsilon_n} W(w_n^i) + \frac{1}{\varepsilon_n} + \left(|q|\varepsilon_n + \frac{\varepsilon_n^3}{\delta^2}\right)\left|\nabla w_n^i\right|^2 + \varepsilon_n^3\left|\nabla^2 w_n^i\right|^2\right] dx \\ + C \int_{\Omega_i\cap(U_{2\delta}\setminus U_\delta)\cap\mathcal{R}_n} &\left[\left(|q|\varepsilon_n + \frac{\varepsilon_n^3}{\delta^2}\right)|\nabla \widetilde{u}_n|^2 + \varepsilon_n^3|\nabla^2 \widetilde{u}_n|^2 + \left(|q|\frac{\varepsilon_n}{\delta^2} + \frac{\varepsilon_n^3}{\delta^4}\right)\left|w_n^i - \widetilde{u}_n\right|^2\right] dx\end{aligned} \tag{4.59}$$

Using (4.38) and arguing as in (4.41) and (4.42), we obtain

$$\begin{aligned}\lim_{n\to+\infty} \int_{\Omega_i\cap(U_{2\delta}\setminus U_\delta)\cap\mathcal{R}_n} &\left[\frac{1}{\varepsilon_n} W(w_n^i) + \frac{1}{\varepsilon_n} + \left(|q|\varepsilon_n + \frac{\varepsilon_n^3}{\delta^2}\right)\left|\nabla w_n^i\right|^2 + \varepsilon_n^3\left|\nabla^2 w_n^i\right|^2\right] dx \\ &\leq C\mathcal{H}^{N-1}\big(H_i \cap (U_{2\delta}\setminus U_\delta)\big) \leq C\delta,\end{aligned} \tag{4.60}$$

where we used the fact that $w \in H^2(Q)$ and the constant C depends on w, and, in turn, on ρ.

Using (2.4) and (4.39) we also get

$$\begin{aligned}\int_{\Omega_i\cap(U_{2\delta}\setminus U_\delta)\cap\mathcal{R}_n} &\left[\left(|q|\varepsilon_n + \frac{\varepsilon_n^3}{\delta^2}\right)|\nabla \widetilde{u}_n|^2 + \varepsilon_n^3|\nabla^2 \widetilde{u}_n|^2 + \left(|q|\frac{\varepsilon_n}{\delta^2} + \frac{\varepsilon_n^3}{\delta^4}\right)\left|w_n^i - \widetilde{u}_n\right|^2\right] dx \\ &\leq \frac{C}{\varepsilon_n}\left(1 + \frac{\varepsilon_n^2}{\delta^2} + \frac{\varepsilon_n^4}{\delta^4}\right)\mathcal{L}^N(\Omega_i \cap (U_{2\delta}\setminus U_\delta)\cap\mathcal{R}_n) + C\left(\frac{\varepsilon_n^3}{\delta^2} + \frac{\varepsilon_n^5}{\delta^4}\right)\|w\|^2_{L^2(Q)} \\ &\leq C\left(1 + \frac{\varepsilon_n^2}{\delta^2} + \frac{\varepsilon_n^4}{\delta^4}\right)\delta + C\left(\frac{\varepsilon_n^3}{\delta^2} + \frac{\varepsilon_n^5}{\delta^4}\right)\|w\|^2_{L^2(Q)},\end{aligned} \tag{4.61}$$

where we have also used the change of variables that leads to (4.40). Combining (4.59), (4.60), and (4.61) we obtain

$$\mathcal{K}_2 \leq C\delta + C\frac{\varepsilon_n^2}{\delta^4} \tag{4.62}$$

for $\varepsilon_n < 1$.

Using the change of variables $y = \epsilon_0 R_i^T(x - x_i)/\varepsilon_n$, where R_i and x_i are the rotation and the vector that appear in (4.38), the periodicity of w_n^i with respect to the variables tangential to H_i, and reasoning as in (4.41) and (4.42), we get

$$\lim_{n\to+\infty} \int_{\Omega_i\cap U_{2\delta}} \varepsilon_n|\nabla w_n^i|^2\, dx \leq C\delta \int_Q |\nabla w|^2\, dx.$$

Thus,

$$\begin{aligned}\mathcal{K}_3 &\leq \int_{\Omega_i}\left[\frac{1}{\varepsilon_n}W(w_n^i) - q\varepsilon_n|\nabla w_n^i|^2 + \varepsilon_n^3|\nabla^2 w_n^i|^2\right] dx + \int_{\Omega_i\cap U_{2\delta}} |q|\,\varepsilon_n|\nabla w_n^i|^2\, dx \\ &\leq \int_{\Omega_i}\left[\frac{1}{\varepsilon_n}W(w_n^i) - q\varepsilon_n|\nabla w_n^i|^2 + \varepsilon_n^3|\nabla^2 w_n^i|^2\right] dx + C\delta.\end{aligned}$$

Combining the previous inequality with (4.49), (4.54), (4.55), (4.53), and (4.62), we obtain

$$\begin{aligned}\limsup_{n\to+\infty} F_{\varepsilon_n}(u_n, \Omega) &\leq \limsup_{n\to+\infty} \sum_{i=1}^{L} F_{\varepsilon_n}(u_n, \Omega_i) + \limsup_{n\to+\infty} F_{\varepsilon_n}(u_n, A) \\ &\leq \big(\mathbf{m}_N + \rho\big) \sum_{i=1}^{L} \mathcal{H}^{N-1}\big(\Omega_i \cap H_i\big) + C\delta \\ &\leq \big(\mathbf{m}_N + \rho\big)\, \mathcal{H}^{N-1}\big(\Omega \cap \partial P\big) + C\delta.\end{aligned}$$

By fixing δ sufficiently small, we have that (1.9) holds.

Substep 2C. We now consider the case in which the set E_0 in (4.33) is an arbitrary set of finite perimeter in Ω. Since Ω is bounded and has C^1 boundary, by first approximating E_0 with smooth sets (see Remark 3.43 in [1]) and then with polyhedral sets, we may find sets $E_k \subset \Omega$ of the form $E_k = P_k \cap \Omega$, where P_k is a polyhedral set satisfying (4.44), such that $\mathcal{H}^{N-1}(\partial E_k \cap \partial\Omega) = 0$, $\chi_{E_k} \to \chi_{E_0}$ in $L^2(\Omega)$, and $\mathrm{Per}_\Omega(E_k) \to \mathrm{Per}_\Omega(E_0)$ as $k \to +\infty$. By applying Substep 2B to each function $u_k := \chi_{E_k} - \chi_{\Omega\setminus E_k}$ the result follows by the lower semicontinuity of the Γ-upper limit (see, e.g., Proposition 6.8 in [5]). □

5. The One-Dimensional Cell Problem

This section is devoted to the study of the property of the constant $\mathbf{m}_1$, defined in (1.4) with $N = 1$, which is the effective interface energy density in dimension one. In Subsection 5.1, we prove that the infimum in $\mathbf{m}_1$ is realized when $\varepsilon \to 0^+$. Precisely, we show that $\mathbf{m}_1$ coincides with the constant

$$\mathbf{m}^* := \inf\left\{\int_{\mathbb{R}} \left[W(u) - q(u')^2 + (u'')^2\right] dx : \, u \in H^2_{\mathrm{loc}}(\mathbb{R}), \lim_{x\to\pm\infty} u(x) = \pm 1\right\}. \tag{5.1}$$

We also prove that the infimum in (5.1) is attained and that $\mathbf{m}^*$ is continuous as a function of q (see Theorem 5.5). This implies, in particular, that when $q \to 0$ the constant $\mathbf{m}^*$ reduces to the effective energy density per unit area obtained in [7]. In Subsection 5.2, we prove that, under the additional assumption that the double well potential W is even, minimizers of (5.1) have only one zero.

5.1. Properties of $\mathbf{m}_1$.

Theorem 5.1. *Let $q < q^*$ and assume (H1) and (H2). Then $\mathbf{m}_1 = \mathbf{m}^*$.*

We begin with a preliminary result.

Lemma 5.2. *Let $q < q^*$ and assume (H1) and (H2). For every $\eta > 0$ there exists $\delta > 0$ such that*

$$\int_{-\ell-1}^{-\ell} \left[W(v) - q(v')^2 + (v'')^2\right] dx + \int_{\ell}^{\ell+1} \left[W(w) - q(w')^2 + (w'')^2\right] dx < \eta \tag{5.2}$$

for every $\ell \geq 1/2$ and every $a_0, a_1, b_0, b_1 \in \mathbb{R}$ with

$$|a_0 + 1| < \delta, \quad |a_1| < \delta, \quad |b_0 - 1| < \delta, \quad |b_1| < \delta, \tag{5.3}$$

where v and w are the polynomials of third degree such that $v(-\ell-1) = -1$, $v'(-\ell-1) = 0$, $v(-\ell) = a_0$, $v'(-\ell) = a_1$, $w(\ell) = b_0$, $w'(\ell) = b_1$, $w(\ell+1) = 1$, and $w'(\ell+1) = 0$.

Proof. Since W is continuous and $W(\pm 1) = 0$, the result follows from a straightforward computation. We omit the details. □

Remark 5.3. Fix $\sigma \in (0,1)$ so small that $(q+\sigma)/(1-\sigma) < q^*$. Reasoning as in the proof of Theorem 1.1, but using Theorem 3.4 (i) in place of Theorem 1.2, we have that

$$\int_I \left[W(u) - q(u')^2 + (u'')^2\right] dx \geq \sigma \int_I \left[W(u) + (u')^2 + (u'')^2\right] dx \tag{5.4}$$

for every interval I with $|I| \geq 1$ and every $u \in H^2_{\mathrm{loc}}(I)$.

We now turn to the proof of Theorem 5.1.

Proof of Theorem 5.1. Setting $\ell = 1/(2\varepsilon)$, in view of (1.4), (1.5), and (3.3), we may write

$$\mathbf{m}_1 = \inf_{\ell\geq 1/2} \inf_{u\in\mathcal{A}^\ell} \int_{-\ell}^{\ell} \left[W(u) - q(u')^2 + (u'')^2\right] dx,$$

where

$$\mathcal{A}^\ell := \left\{ u \in H^2_{\rm loc}(\mathbb{R}) : \, u(x) = -1 \text{ near } x = -\ell, \;\; u(x) = 1 \text{ near } x = \ell \right\}. \tag{5.5}$$

Given any $u \in \mathcal{A}^\ell$, set $\hat{u}(x) := -1$ for $x \in (-\infty, \ell)$, $\hat{u}(x) := u(x)$ for $x \in [-\ell, \ell]$, and $\hat{u}(x) := 1$ for $x \in (\ell, +\infty)$. Since $\hat{u} \in H^2_{\rm loc}(\mathbb{R})$, it is admissible in the minimum problem for $\mathbf{m}^*$, hence

$$\mathbf{m}^* \leq \int_{\mathbb{R}} \left[W(\hat{u}) - q(\hat{u}')^2 + (\hat{u}'')^2 \right] dx = \int_{-\ell}^{\ell} \left[W(u) - q(u')^2 + (u'')^2 \right] \, dx.$$

Taking the infimum over all $u \in \mathcal{A}^\ell$ and $\ell \geq 1/2$ gives $\mathbf{m}^* \leq \mathbf{m}_1$.

To prove the converse inequality, fix $\eta > 0$ and let $u \in H^2_{\rm loc}(\mathbb{R})$ be such that $u(x) \to \pm 1$ as $x \to \pm\infty$ and

$$\int_{\mathbb{R}} \left[W(u) - q(u')^2 + (u'')^2 \right] dx \leq \mathbf{m}^* + \eta. \tag{5.6}$$

Fix $\sigma \in (0,1)$ so small that $(q+\sigma)/(1-\sigma) < q^*$ as in Remark 5.3. Then, by (5.4),

$$\int_{\mathbb{R}} \left[W(u) + (u')^2 + (u'')^2 \right] dx \leq \frac{\mathbf{m}^* + \eta}{\sigma},$$

which implies that $u' \in H^1(\mathbb{R})$, since $W \geq 0$. By Morrey's Theorem (see [13, Theorem 11.34]), it follows that $u'(x) \to 0$ as $x \to \pm\infty$. Let $\delta > 0$ be the number given in Lemma 5.2. Since $u(x) \to \pm 1$ and $u'(x) \to 0$ as $x \to \pm\infty$, there exists $\ell \geq 1/2$ satisfying

$$|u(-\ell) + 1| < \delta, \quad |u'(-\ell)| < \delta, \quad |u(\ell) - 1| < \delta, \quad |u'(\ell)| < \delta.$$

Let v and w be the polynomials of third degree such that $v(-\ell-1) = -1$, $v'(-\ell-1) = 0$, $v(-\ell) = u(-\ell)$, $v'(-\ell) = u'(-\ell)$, $w(\ell) = u(\ell)$, $w'(\ell) = u'(\ell)$, $w(\ell+1) = 1$, and $w'(\ell+1) = 0$. Define

$$\overline{u}\,(x) := \begin{cases} -1 & \text{for } x \leq -\ell - 1, \\ v\,(x) & \text{for } -\ell - 1 < x < -\ell, \\ u\,(x) & \text{for } -\ell \leq x \leq \ell, \\ w\,(x) & \text{for } \ell < x < \ell + 1, \\ +1 & \text{for } x \geq \ell + 1. \end{cases}$$

Since $\overline{u} \in H^2_{\rm loc}(\mathbb{R})$, it belongs to $\mathcal{A}^{\ell+2}$, and so

$$\begin{aligned} \mathbf{m}_1 &\leq \int_{-\ell-1}^{\ell+1} \left[W(\overline{u}) - q(\overline{u}')^2 + (\overline{u}'')^2 \right] \, dx = \int_{-\ell}^{\ell} \left[W(u) - q(u')^2 + (u'')^2 \right] dx \\ &+ \int_{-\ell-1}^{-\ell} \left[W(v) - q(v')^2 + (v'')^2 \right] dx + \int_{\ell}^{\ell+1} \left[W(w) - q(w')^2 + (w'')^2 \right] dx \\ &\leq \int_{-\ell}^{\ell} \left[W(u) - q(u')^2 + (u'')^2 \right] dx + \eta, \end{aligned}$$

where in the last inequality we have used Lemma 5.2. Since by Theorem 3.4 (i),

$$\left(\int_{-\infty}^{-\ell} + \int_{\ell}^{+\infty} \right) \left[W(u) - q(u')^2 + (u'')^2 \right] dx \geq 0,$$

we have that

$$\mathbf{m}_1 \leq \int_{\mathbb{R}} \left[W(u) - q(u')^2 + (u'')^2 \right] dx + \eta \leq \mathbf{m}^* + 2\eta,$$

by (5.6). As $\eta \to 0^+$, we get $\mathbf{m}_1 \leq \mathbf{m}^*$. $\square$

Remark 5.4. It follows from the proof of the previous theorem that

$$\mathbf{m}^* = \inf \left\{ \int_{\mathbb{R}} \left[W(u) - q(u')^2 + (u'')^2 \right] dx \, : \, u \in H^2_{\rm loc}(\mathbb{R}), \, \lim_{x \to \pm\infty} u(x) = \pm 1, \, \lim_{x \to \pm\infty} u'(x) = 0 \right\}.$$

To highlight the dependence on q, in what follows we write $\mathbf{m}^*_q$ for the constant defined in (5.1).

Theorem 5.5. *Assume* $(H1)$ *and* $(H2)$. *Then for every* $q < q^*$ *the minimum problem* (5.1) *defining* $\mathbf{m}_q^*$ *has a solution and the function* $q \mapsto \mathbf{m}_q^*$ *is continuous from* $(-\infty, q^*)$ *into* $(0, +\infty)$.

Proof. We recall that $\mathbf{m}_q^* > 0$ for every $q < q^*$ by Proposition 4.1 and Theorem 5.1. Fix $q \in (-\infty, q^*)$ and a sequence $\{q_n\} \subset (-\infty, q^*)$ such that $q_n \to q$.

Step 1. We show that

$$\limsup_{n\to+\infty} \mathbf{m}_{q_n}^* \leq \mathbf{m}_q^*. \tag{5.7}$$

Let $\eta > 0$ and let $u \in H^2_{\mathrm{loc}}(\mathbb{R})$ be such that $u(x) \to \pm 1$ as $x \to \pm\infty$ and

$$\int_{\mathbb{R}} \Big[W(u) - q(u')^2 + (u'')^2\Big]\, dx \leq \mathbf{m}_q^* + \eta.$$

Since u is also admissible for the minimum problem defining $\mathbf{m}_{q_n}^*$, for every n we get

$$\mathbf{m}_{q_n}^* \leq \int_{\mathbb{R}} \Big[W(u) - q_n(u')^2 + (u'')^2\Big]\, dx.$$

Taking the limit as $n \to +\infty$, and using the fact that $u' \in L^2(\mathbb{R})$ (see (5.6)), we deduce that

$$\limsup_{n\to+\infty} \mathbf{m}_{q_n}^* \leq \int_{\mathbb{R}} \Big[W(u) - q(u')^2 + (u'')^2\Big]\, dx \leq \mathbf{m}_q^* + \eta.$$

It now suffices to let $\eta \to 0^+$.

Step 2. It remains to show that

$$\liminf_{n\to+\infty} \mathbf{m}_{q_n}^* \geq \mathbf{m}_q^* \tag{5.8}$$

and that the minimum problem (5.1) defining $\mathbf{m}_q^*$ has a solution. For every n let $u_n \in H^2_{\mathrm{loc}}(\mathbb{R})$ be such that $u_n(x) \to \pm 1$ as $x \to \pm\infty$ and

$$\int_{\mathbb{R}} \big[W(u_n) - q_n(u_n')^2 + (u_n'')^2\big]\, dx \leq \mathbf{m}_{q_n}^* + \frac{1}{n}. \tag{5.9}$$

Since $u_n(x) \to \pm 1$ as $x \to \pm\infty$, each function u_n must vanish at some point. By translation invariance, we may assume that $u_n(0) = 0$ for every n.

We claim that there exist a subsequence of $\{u_n\}$, not relabeled, and a function u in $H^2_{\mathrm{loc}}(\mathbb{R})$ such that $\{u_n\}$ converges weakly to u in $H^2_{\mathrm{loc}}(\mathbb{R})$ and

$$\int_{\mathbb{R}} \big[W(u) - q(u')^2 + (u'')^2\big]\, dx \leq \liminf_{n\to+\infty} \mathbf{m}_{q_n}^*. \tag{5.10}$$

Fix $\sigma \in (0,1)$ so small that $(q+\sigma)/(1-\sigma) < q^*$. Since $q_n \to q$, for all n sufficiently large we have that $(q_n+\sigma)/(1-\sigma) < q^*$. Hence, by (5.4), (5.9) and (5.7), in this order, we have that

$$\int_{\mathbb{R}} \big[W(u_n) + (u_n')^2 + (u_n'')^2\big]\, dx \leq \frac{\mathbf{m}_q^* + 1}{\sigma} \tag{5.11}$$

for all n sufficiently large. Since $W \geq 0$, by extracting a subsequence, not relabeled, we may assume that $\{u_n'\}$ converges weakly in $H^1(\mathbb{R})$ to some function w.

Moreover, in view of (5.11), $(H2)$, and the Rellich Compactness Theorem, by using a diagonal argument we may find a subsequence of $\{u_n\}_n$, not relabeled, converging weakly in $H^2_{\mathrm{loc}}(\mathbb{R})$ to a function $u \in H^2_{\mathrm{loc}}(\mathbb{R})$, with $u' = w$ $\mathcal{L}^1$-a.e. in $\mathbb{R}$. In particular, u' belongs to $H^1(\mathbb{R})$. We may also assume that $(q_n+\sigma)/(1-\sigma) < q^*$ for all n.

Using the facts that $q_n \to q$ and that, for every fixed k, $\{u_n\}_n$ converges to u weakly in $H^2((-k,k))$ and strongly in $H^1((-k,k))$, we have

$$\int_{-k}^{k} \big[W(u) - q(u')^2 + (u'')^2\big]\, dx \leq \liminf_{n\to+\infty} \int_{-k}^{k} \big[W(u_n) - q_n(u_n')^2 + (u_n'')^2\big]\, dx. \tag{5.12}$$

Since by Theorem 3.4 (i),

$$\left(\int_{-\infty}^{-k}+\int_{k}^{+\infty}\right)\left[W(u_n)-q_n(u_n')^2+(u_n'')^2\right]dx\geq 0,$$

from (5.7), (5.9), and (5.12), we deduce that

$$\int_{-k}^{k}\left[W(u)-q(u')^2+(u'')^2\right]dx\leq\liminf_{n\to+\infty}\int_{\mathbb{R}}\left[W(u_n)-q_n(u_n')^2+(u_n'')^2\right]dx=\liminf_{n\to+\infty}\mathbf{m}^*_{q_n}.$$

Using the facts that $W\geq 0$ and u', $u''\in L^2(\mathbb{R})$, and applying Lebesgue's Monotone Convergence Theorem to the sequence $\left\{\chi_{(-k,k)}W(u)\right\}$ and Lebesgue's Dominated Convergence Theorem to the sequences $\left\{\chi_{(-k,k)}(u')^2\right\}$ and $\left\{\chi_{(-k,k)}(u'')^2\right\}$, we may let $k\to+\infty$ in the previous inequality to obtain (5.10).

We claim that $u(x)\to\pm1$ as $x\to\pm\infty$. Fix $0<\rho<\frac{1}{2}$ and let

$$\begin{aligned}x_n&:=\sup\left\{x<0:\,|u_n+1|\leq\rho\text{ in }(-\infty,x)\right\},\\ y_n&:=\inf\left\{x>0:\,|u_n-1|\leq\rho\text{ in }(x,+\infty)\right\}.\end{aligned}$$

Note that, since $u_n(0)=0$ and $u_n(x)\to\pm1$ as $x\to\pm\infty$, then $-\infty<x_n<0<y_n<+\infty$. We claim that

$$\{y_n-x_n\}\text{ is bounded.}\tag{5.13}$$

If not, then there exists a subsequence, not relabeled, satisfying $y_n-x_n\to+\infty$. Using (5.4), we obtain

$$\begin{aligned}\int_{-\infty}^{x_n}\left[W(u_n)-q_n(u_n')^2+(u_n'')^2\right]dx&\geq\sigma\int_{-\infty}^{x_n}\left[W(u_n)+(u_n')^2\right]dx\\ &\geq\frac{\sigma}{2}\int_{-\infty}^{x_n}\sqrt{W(u_n)}|u_n'|\,dx\geq\frac{\sigma}{2}\min\left\{\int_{-1}^{-1+\rho}\sqrt{W(s)}\,ds,\int_{-1-\rho}^{-1}\sqrt{W\,(s)}\,ds\right\}.\end{aligned}\tag{5.14}$$

Similarly, we get

$$\int_{y_n}^{+\infty}\left[W(u_n)-q_n(u_n')^2+(u_n'')^2\right]dx\geq\frac{\sigma}{2}\min\left\{\int_{1-\rho}^{1}\sqrt{W(s)}\,ds,\int_{1}^{1+\rho}\sqrt{W(s)}\,ds\right\}.\tag{5.15}$$

Let $K>0$ be the minimum between the two numbers in the right-hand side of (5.14) and (5.15).

Fix $\eta\in(0,K)$ and let $\delta>0$ be as in Lemma 5.2. For every n we define

$$\begin{aligned}A_n&:=\{x\in[x_n,y_n]:\,|u_n(x)-1|\geq\delta,\ |u_n(x)+1|\geq\delta\},\\ B_n&:=\{x\in[x_n,y_n]:\,|u_n'(x)|\geq\delta\}.\end{aligned}$$

By (5.11) we obtain

$$\mathcal{L}^1(A_n)\leq\frac{\mathbf{m}^*_q+1}{\sigma\omega_\delta}\quad\text{and}\quad\mathcal{L}^1(B_n)\leq\frac{\mathbf{m}^*_q+1}{\sigma\delta^2},$$

where $\omega_\delta:=\min\{W(s):\,|s-1|\geq\delta,\ |s+1|\geq\delta\}$ (note that $\omega_\delta>0$ by $(H1)$, $(H2)$, and by the continuity of W). Since $y_n-x_n\to+\infty$, for n large enough there exists $z_n\in(x_n+1,y_n-1)\setminus(A_n\cup B_n)$. Without loss of generality, we may assume that $|u_n(z_n)-1|<\delta$ and $|u_n'(z_n)|<\delta$. By Lemma 5.2 and the fact that $q_n\to q$, we obtain a function v_n such that $v_n\in H^2_{\text{loc}}(\mathbb{R})$, $v_n=u_n$ in $(-\infty,z_n)$, v_n is a polynomial of third degree in $[z_n,z_n+1]$, $v_n=1$ in $(z_n+1,+\infty)$, and

$$\begin{aligned}&\int_{\mathbb{R}}\left[W(v_n)-q_n(v_n')^2+(v_n'')^2\right]dx\\ &=\int_{-\infty}^{z_n}\left[W(u_n)-q_n(u_n')^2+(u_n'')^2\right]dx+\int_{z_n}^{z_n+1}\left[W(v_n)-q_n(v_n')^2+(v_n'')^2\right]dx\\ &\leq\int_{-\infty}^{z_n}\left[W(u_n)-q_n(u_n')^2+(u_n'')^2\right]dx+\eta\end{aligned}$$

for all n sufficiently large. Since v_n is admissible for $\mathbf{m}^*_{q_n}$, in view of the previous inequality,

$$\int_{-\infty}^{z_n} \left[W(u_n) - q_n(u_n')^2 + (u_n'')^2\right] dx \geq \mathbf{m}^*_{q_n} - \eta. \tag{5.16}$$

We write

$$\int_{\mathbb{R}} \left[W(u_n) - q_n(u_n')^2 + (u_n'')^2\right] dx = \int_{-\infty}^{z_n} \left[W(u_n) - q_n(u_n')^2 + (u_n'')^2\right] dx$$
$$+ \int_{z_n}^{y_n} \left[W(u_n) - q_n(u_n')^2 + (u_n'')^2\right] dx + \int_{y_n}^{+\infty} \left[W(u_n) - q_n(u_n')^2 + (u_n'')^2\right] dx.$$

Since $y_n - z_n \geq 1$, by Theorem 3.4 (i) the second term in the right-hand side of the previous inequality is nonnegative. Therefore, using (5.15) and (5.16), we deduce

$$\int_{\mathbb{R}} \left[W(u_n) - q_n(u_n')^2 + (u_n'')^2\right] dx \geq \mathbf{m}^*_{q_n} - \eta + K$$

for all n sufficiently large. Since $\eta < K$, this contradict (5.9) for all n sufficiently large, and concludes the proof of (5.13).

Since $x_n \leq 0 \leq y_n$, there exists a constant $0 < \ell_\rho < +\infty$ such that $-\ell_\rho < x_n \leq 0 \leq y_n < \ell_\rho$ for every n. It follows from the definition of y_n that $|u_n(x) - 1| \leq \rho$ for every $x > \ell_\rho$ and for every n. Letting $n \to \infty$ and using the fact that $\{u_n\}$ converges strongly to u in $H^1_{\mathrm{loc}}(\mathbb{R})$, we have that $|u(x) - 1| \leq \rho$ for every $x > \ell_\rho$. Since this result holds for every $0 < \rho < \frac{1}{2}$, we have shown that $u(x) \to 1$ as $x \to +\infty$. In the same way we prove that $u(x) \to -1$ as $x \to -\infty$.

Hence, u is admissible for $\mathbf{m}^*_q$, which, together with (5.7) and (5.10), implies that

$$\mathbf{m}^*_q \leq \int_{\mathbb{R}} \left[W(u) - \tilde{q}(u')^2 + (u'')^2\right] dx \leq \liminf_{n\to+\infty} \mathbf{m}^*_{q_n} \leq \limsup_{n\to+\infty} \mathbf{m}^*_{q_n} \leq \mathbf{m}^*_q.$$

This shows that the function $q \mapsto \mathbf{m}^*_q$ is continuous at q and that u is a solution of the minimum problem (5.1) defining $\mathbf{m}^*_q$. □

Remark 5.6. It follows from the previous theorem that

$$\lim_{q\to 0} \mathbf{m}^*_q = \mathbf{m}^*_0,$$

where

$$\mathbf{m}^*_0 = \inf\left\{ \int_{\mathbb{R}} \left[W(u) + (u'')^2\right] dx : \, u \in H^2_{\mathrm{loc}}(\mathbb{R}), \, \lim_{x\to\pm\infty} u(x) = \pm 1 \right\}$$

is the effective energy density per unit area obtained in [7].

5.2. **The One-Dimensional Case with W Even.** In this section, we assume that $W\colon \mathbb{R} \to [0, +\infty)$ is an even continuous function satisfying $(H1)$ and $(H2)$. Define

$$\mathbf{p}_+ := \inf\left\{ \int_0^{+\infty} \left[W(u) - q(u')^2 + (u'')^2\right] dx : \, u \in H^2_{\mathrm{loc}}([0, +\infty)), \, \lim_{x\to+\infty} u(x) = 1, \right.$$
$$\left. u(0) = 0, \, u(x) \geq 0 \text{ for all } x \geq 0 \right\}, \tag{5.17}$$

and

$$\mathbf{p} := \inf\left\{ \int_0^{+\infty} \left[W(u) - q(u')^2 + (u'')^2\right] dx : \, u \in H^2_{\mathrm{loc}}([0, +\infty)), \right.$$
$$\left. \lim_{x\to+\infty} u(x) = 1, \, u(0) = 0 \right\}, \tag{5.18}$$

where $H^2_{\rm loc}([0,+\infty))$ is the space of all functions $u\colon [0,+\infty) \to \mathbb{R}$ such that $u \in H^2((0,T))$ for every $T > 0$. We have the following characterization.

We have the following characterization.

Proposition 5.7. *Let $q < q^*$ and let $W\colon \mathbb{R} \to [0,+\infty)$ be an even continuous function satisfying $(H1)$ and $(H2)$. Then the minimum problems* (5.17) *and* (5.18) *have a solution and*

$$2\mathbf{p}_+ = 2\mathbf{p} = \mathbf{m}^*. \tag{5.19}$$

Proof. Fix $\sigma \in (0,1)$ so small that $(q+\sigma)/(1-\sigma) < q^*$. Then, for every admissible u for (5.17), by (5.4), $(H2)$, and the fact that $u \geq 0$,

$$\int_0^{+\infty} \left[W(u) - q(u')^2 + (u'')^2\right] dx \geq \sigma \int_0^{+\infty} \left[(u-1)^2 + (u')^2 + (u'')^2\right] dx,$$

therefore $u - 1 \in H^2(0,\infty)$. Reasoning as in Step 2 of the proof of Theorem 5.5, it follows if $\{u_n\}$ is a minimizing sequence of the infimum problem in (5.17), then it admits a subsequence converging in $H^2_{\rm loc}(0,\infty)$ to a minimizer u of the same problem.

We claim now that $2\mathbf{p}_+ \geq 2\mathbf{p} \geq \mathbf{m}^*$. The first inequality is a direct consequence of (5.17) and (5.18). In order to verify that $2\mathbf{p} \geq \mathbf{m}^*$, let $\eta > 0$ and let u be an admissible function for (5.18) such that

$$\int_0^{+\infty} \left[W(u) - q(u')^2 + (u'')^2\right] dx \leq \mathbf{p} + \eta.$$

We define

$$w(x) := \begin{cases} -u(-x) & \text{if } x < 0, \\ u(x) & \text{if } x \geq 0. \end{cases}$$

Then w is an admissible function for (5.1), and so.

$$\mathbf{m}^* \leq 2\mathbf{p} + 2\eta.$$

Letting $\eta \to 0^+$ we conclude that $\mathbf{m}^* \leq 2\mathbf{p}$.

It suffices to prove that $\mathbf{m}^* \geq 2\mathbf{p}_+$. Let $\eta > 0$ and let u be an admissible function for (5.1) such that

$$\int_{\mathbb{R}} \left[W(u) - q(u')^2 + (u'')^2\right] dx \leq \mathbf{m}^* + \eta. \tag{5.20}$$

Since $u(x) \to \pm 1$ as $x \to \pm\infty$, the function u must vanish at some point. By translation invariance, we may assume that $u(0) = 0$. Let x_1 and x_2 be the smallest and the largest zero of u, respectively. Then, $x_1 \leq 0 \leq x_2$, and by Proposition 3.4 (ii),

$$\int_{x_1}^{x_2} \left[W(u) - q(u')^2 + (u'')^2\right] dx \geq 0 \tag{5.21}$$

Define $w(x) := -u(-x + x_1)$ and $v\,(x) := u(x + x_2)$ for $x \geq 0$. Since v and w are admissible functions for $\mathbf{p}_+$ and W is even, we obtain from (5.20) and (5.21)

$$\begin{aligned} \mathbf{m}^* + \eta &\geq \int_{-\infty}^{x_1} \left[W(u) - q(u')^2 + (u'')^2\right] dx + \int_{x_2}^{+\infty} \left[W(u) - q(u')^2 + (u'')^2\right] dx \\ &= \int_0^{+\infty} \left[W(w) - q(w')^2 + (w'')^2\right] dx + \int_0^{+\infty} \left[W(v) - q(v')^2 + (v'')^2\right] dx \geq 2\mathbf{p}_+. \end{aligned}$$

Letting $\eta \to 0^+$ we deduce that $\mathbf{m}^* \geq 2\mathbf{p}_+$.

Since $\mathbf{p}_+ = \mathbf{p}$ and the minimum problem (5.17) for $\mathbf{p}_+$ has a solution, so does problem (5.18) for $\mathbf{p}$. $\square$

Proposition 5.8. *Let $q < q^*$, let $W\colon \mathbb{R} \to [0,+\infty)$ be an even continuous function satisfying $(H1)$ and $(H2)$, and let u be a minimizer of* (5.18)*. Then $u > 0$ in $(0,+\infty)$.*

Proof. We argue by contradiction. Assume that u in not strictly positive in $(0,+\infty)$. Let x_1 the last zero of u. Define $w(x) := u(x+x_1)$ for $x \geq 0$. Since w is admissible for the infimum problem defining $\mathbf{p}$, we obtain

$$\begin{aligned}
\mathbf{p} &= \int_0^{+\infty} \left[W(u) - q(u')^2 + (u'')^2\right] dx \\
&= \int_0^{x_1} \left[W(u) - q(u')^2 + (u'')^2\right] dx + \int_0^{+\infty} \left[W(w) - q(w')^2 + (w'')^2\right] dx \\
&\geq \int_0^{x_1} \left[W(u) - q(u')^2 + (u'')^2\right] dx + \mathbf{p}.
\end{aligned}$$

Since $u(0) = u(x_1) = 0$, we are in a position to apply Proposition 3.4 (ii) to get that

$$\int_0^{x_1} \left[W(u) - q(u')^2 + (u'')^2\right] dx = 0. \tag{5.22}$$

If $\sigma \in (0,1)$ is such that $(q+\sigma)/(1-\sigma) < q^*$, then by (4.1),

$$(1-\sigma)\int_0^{x_1} \left[W(u) - \frac{q+\sigma}{1-\sigma}(u')^2 + (u'')^2\right] dx + \sigma \int_0^{x_1} \left[W(u) + (u')^2 + (u'')^2\right] dx = 0.$$

Since the first term in the left-hand side of the previous equality is nonnegative by Proposition 3.4 (ii), both terms should be zero. This means that u should be constantly equal to 1 or -1 in $(0, x_1)$, which contradicts the fact that $u(0) = u(x_1) = 0$. □

As consequence of the previous proposition we conclude that minimizers of (5.1) have exactly one zero.

Proposition 5.9. *Let $q < q^*$, let $W \colon \mathbb{R} \to [0,+\infty)$ be an even continuous function satisfying $(H1)$ and $(H2)$, and let u be a minimizer of* (5.1). *Then there exists $x_0 \in \mathbb{R}$ such that $u > 0$ in $(x_0,+\infty)$, $u < 0$ in $(-\infty, x_0)$, and $u(x_0) = 0$.*

Proof. Since $u(x) \to \pm 1$ as $x \to \pm\infty$, the function u vanishes at some point. By translation invariance, we may assume that $u(0) = 0$. Let $w(x) := -u(-x)$ for $x \geq 0$. Since both w and u are admissible functions for $\mathbf{p}$ and W is even, we have

$$\mathbf{m}^* = \int_0^{+\infty} \left[W(w) - q(w')^2 + (w'')^2\right] dx + \int_0^{+\infty} \left[W(u) - q(u')^2 + (u'')^2\right] dx\,,$$

$$\int_0^{+\infty} \left[W(w) - q(w')^2 + (w'')^2\right] dx \geq \mathbf{p}, \qquad \int_0^{+\infty} \left[W(u) - q(u')^2 + (u'')^2\right] dx \geq \mathbf{p}.$$

Since, by Proposition 5.7, $2\mathbf{p} = \mathbf{m}^*$, we must have

$$\int_0^{+\infty} \left[W(w) - q(w')^2 + (w'')^2\right] dx = \mathbf{p} = \int_0^{+\infty} \left[W(w) - q(w')^2 + (w'')^2\right] dx,$$

which implies that both u and w are minimizers for $\mathbf{p}$. By Proposition 5.8, we get $u > 0$ in $(0,+\infty)$, and $w > 0$ in $(0,+\infty)$, that is, $u < 0$ in $(-\infty, 0)$. □

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