Relaxation of multi-well energies in linearized elasticity and applications to nematic elastomers

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Abstract

The relaxation of a free-energy functional which describes the order-strain interaction in nematic elastomers is obtained explicitly. We work in the regime of small strains (linearized kinematics). Adopting the uniaxial order tensor theory (Frank model) to describe the liquid crystal order, we prove that the minima of the relaxed functional exhibit an effective biaxial nematic texture, as in the de Gennes order tensor model. In particular, this implies that, at a sufficiently macroscopic scale, the response of the material is soft even if the order of the system is assumed to be fixed at the microscopic scale. The relaxed energy density satisfies a solenoidal quasiconvexification formula.

Contents

1 Introduction

Formation of microstructure in complex materials is a very interesting phenomenon from both the physical and the mathematical point of view. A paradigmatic case is that of nematic elastomers, rubbery solids originating from the combination of rod-like liquid crystal molecules with an elastic medium [27]. To sketch the internal organization of such materials, consider that nematic molecules are linked to long polymeric chains, forming an anisotropic solid structure which has extraordinary properties of deformability. Rods can be directly attached to the backbone becoming part of the chain (main-chain polymers) or simply be pendant to it (side-chain polymers). It is well known ([5], [8], [9], [12], [13], [14]) that mechanical fields can deform the chains and re-arrange the local orientation of the mesogenic groups, modifying the optical properties of the material as well.

A general consensus on a satisfactory description of the coupling between mechanical fields and the local order of the liquid crystal in the sense of de Gennes [21] is still missing, both in the case of linear and non-linear elasticity ([7], [18], [20], [23]). Denoting with $\mathbf{Q}$ the order tensor and $\mathbf{F}$ the gradient of the displacement, a possible expression for the energy density describing nematic elastomers within the framework of linearized elasticity is [5]

$$f_{mec}(\mathbf{Q},\mathbf{F}) := \mu\Big|\Big(\frac{\mathbf{F}+\mathbf{F}^T}{2}\Big) - \gamma\mathbf{Q}\Big|^2 + \frac{\lambda}{2}(\operatorname{tr}\mathbf{F})^2. \tag{1.1}$$

The relationship between this expression and the one proposed in the Cambridge group [27] within the framework of finite elasticity is discussed in [15].

A further development consists in minimizing (1.1) with respect to $\mathbf{Q}$. In this new model the influence of the internal nematic variable $\mathbf{Q}$ is perceived only through its coupling to the strain. Depending on the choice of the set where $\mathbf{Q}$ is allowed to vary, different results are obtained:

$$f_X(\mathbf{F}) := \inf_{\mathcal{Q}_X} f_{mec}(\mathbf{Q},\mathbf{F}), \qquad \text{where } X \text{ stands either for } Fr, \text{ or } U, \text{ or } B. \tag{1.2}$$

The sets $\mathcal{Q}_{Fr} \subset \mathcal{Q}_U \subset \mathcal{Q}_B$ are defined later. For the sake of this discussion, it suffices to recall that $\mathcal{Q}_B$ is the set of order tensors in the de Gennes model. This is a convex set (this is a trivial fact, as highlighted e.g. in [5], see the comment after (1.1) therein) which contains biaxial matrices and the null element. Moreover, $\mathcal{Q}_U$ and $\mathcal{Q}_{Fr}$ are the non-convex sets of the uniaxial and Frank-like tensors respectively ([16], [17]). An important feature is that $\mathcal{Q}_U$ contains the null element, while $\mathcal{Q}_{Fr}$ does not. Therefore, in the case $X = Fr$ in (1.2), a state with biaxial or zero $\mathbf{Q}$ cannot be induced mechanically, while in the case $X = U$ the state $\mathbf{Q} = 0$ is reachable. On the other hand, in the case $X = B$ we can obtain all the biaxial states, including again $\mathbf{Q} = 0$.

It can happen, however, that biaxial states which may seem unattainable according to (1.2) in the case when we consider the energy density f_{Fr} and f_U, are allowed, in a suitable sense, by formation of microstructure. It turns out that f_{Fr} and f_U are non-convex energy densities, while f_B is convex. In fact, we show that in the cases where $X = Fr$ or $X = U$, microstructures are possible, and a biaxial order is obtained *effectively*, via relaxation even though biaxiality is excluded at the microscopic level. More precisely, using the language of the Calculus of Variations, we replace a minimization problem for a non-lower semicontinuous functional $\mathcal{F}$ with the problem of minimizing the largest lower semicontinuous functional not exceeding $\mathcal{F}$, namely, its relaxation.

The energy densities f_{Fr}, f_U, f_B can be respectively interpreted as the square of a distance function from the subsets $\mathcal{Q}_{Fr}, \mathcal{Q}_U, \mathcal{Q}_B$ (via the proportionality constant γ) in the space of symmetric matrices. Our relaxation result is based on the fact that the quasiconvexification of f_{Fr} and f_U is the square of the distance from the convex hull of $\gamma\mathcal{Q}_{Fr}$ and $\gamma\mathcal{Q}_U$, which is precisely $\gamma\mathcal{Q}_B$. Without specifying the setting and the technical assumptions, we anticipate here the main result of this paper. In the following result $\mathbf{u}$ represents the mechanical displacement.

Theorem. *Assume*

$$\mathcal{F}_X(\mathbf{u}) = \begin{cases} \displaystyle\int_\Omega f_X(\nabla \mathbf{u})dx & \textit{if } \operatorname{div}\mathbf{u} = 0, \\ +\infty & \textit{otherwise,} \end{cases}$$

where X stands either for Fr, or U, or B (Frank, Uniaxial, Biaxial) and f_X is as in (1.2). Then, the relaxation of $\mathcal{F}_X$ is

$$\mathcal{F}_B(\mathbf{u}) = \begin{cases} \displaystyle\int_\Omega f_B(\nabla \mathbf{u})dx & \textit{if } \operatorname{div}\mathbf{u} = 0, \\ +\infty & \textit{otherwise.} \end{cases}$$

The rest of the paper is organized as follows. In Section 2 we describe the model in detail and recall some mathematical tools in the theory of relaxation and Gamma-convergence. Then, in Section 3, we present our main relaxation result (Theorem 2). In Section 4 we discuss some applications.

2 Preliminaries

Notation and basics facts in linear algebra. We gather here the main symbols and the notation used throughout the paper. Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural and real numbers respectively. For any integer n, $\mathbb{R}^n$ is the space of n-dimensional vectors with canonical basis $\{\mathbf{i}_i\}, i = 1, \dots n$ and $\mathbb{M}^{n\times n}$ the space of square real matrices. The determinant, the trace and the transpose of the matrix $\mathbf{F}$ in $\mathbb{M}^{n\times n}$ are denoted by $\det \mathbf{F}$, $\operatorname{tr}\mathbf{F}$, $\mathbf{F}^T$ respectively. We endow $\mathbb{M}^{n\times n}$ with the usual inner product $\mathbf{F} : \mathbf{M} := \operatorname{tr}(\mathbf{F}\mathbf{M}^T) = \sum_{ij} F_{ij}M_{ij}$ and the corresponding norm $|\mathbf{F}| := (\mathbf{F} : \mathbf{F}^T)^{1/2}$. Here F_{ij}, M_{ij} are the cartesian components of $\mathbf{F}$ and $\mathbf{M}$. The identity in $\mathbb{M}^{n\times n}$ is denoted with $\mathbf{I}$. Focusing on the case $n = 3$, we denote with $\mathbb{M}^{3\times3}_0$, $\mathbb{M}^{3\times3}_{sym}$, $\mathbb{M}^{3\times3}_{0sym}$, $\mathbb{M}^{3\times3}_{skw}$ the subspaces of traceless, symmetric, traceless and symmetric (deviatoric) and skew-symmetric matrices respectively. We introduce the projections $\mathcal{E} : \mathbb{M}^{3\times3} \mapsto \mathbb{M}^{3\times3}_{sym}$, defined as $\mathcal{E}(\mathbf{F}) := (\mathbf{F} + \mathbf{F}^T)/2$, $\mathcal{E}_0 : \mathbb{M}^{3\times3} \mapsto \mathbb{M}^{3\times3}_{0sym}$, defined as $\mathcal{E}_0(\mathbf{F}) := \mathcal{E}(\mathbf{F}) - ((\operatorname{tr}\mathbf{F})/3)\mathbf{I}$. In this definition we can consider the trace of $\mathcal{E}(\mathbf{F})$ as well, since $\operatorname{tr}\mathbf{F} = \operatorname{tr}\mathcal{E}(\mathbf{F})$. This fact will be widely used in what follows. If we introduce $\mathbf{F}^{sk} := (\mathbf{F} - \mathbf{F}^T)/2$, we obtain the well known decomposition $\mathbf{F} = \mathcal{E}_0(\mathbf{F}) + \mathbf{F}^{sk} + \frac{(\operatorname{tr}\mathbf{F})}{3}\mathbf{I}$. We denote with $\mathbb{C} : \mathbb{M}^{3\times3}_{sym} \mapsto \mathbb{M}^{3\times3}_{sym}$ the fourth-order tensor of linearized isotropic elasticity, given as $\mathbb{C}(\mathbf{A}) = 2\mu\mathbf{A} + \lambda\operatorname{tr}(\mathbf{A})\mathbf{I}$, $\forall \mathbf{A} \in \mathbb{M}^{3\times3}_{sym}$, with positive μ, λ (Lamé constants). To emphasize the dependence on λ, we write $\mathbb{C}_\lambda$. We use the notation $\|\cdot\|_{\mathbb{C}_\lambda}$ for the norm on $\mathbb{M}^{3\times3}_{sym}$ induced by the metric $\mathbb{C}_\lambda$ (see formula (2.1) for an example of this notation).

2.1 The mechanical model

We recall a model introduced in [5] to describe the coupling between strain and order in nematic elastomers

$$f_{mec}(\mathbf{Q}, \mathbf{F}) := \frac{1}{2}\|\mathcal{E}(\mathbf{F}) - \gamma\mathbf{Q}\|^2_{\mathbb{C}_\lambda} = \mu\Big(|\mathcal{E}(\mathbf{F})|^2 + \gamma^2|\mathbf{Q}|^2 - 2\gamma\mathcal{E}(\mathbf{F}) : \mathbf{Q}\Big) + \frac{\lambda}{2}(\operatorname{tr}\mathbf{F})^2 \tag{2.1}$$

where γ is a positive constant. Labelling with $e_1(\mathbf{M}) \le e_2(\mathbf{M}) \le e_3(\mathbf{M})$ the ordered eigenvalues of the symmetric 3×3 matrix $\mathbf{M}$, the order tensor $\mathbf{Q}$ can be taken in different subsets of $\mathbb{M}^{3\times3}_{0sym}$ according to the following models (see [16], [21], [26]):

- the *biaxial theory*, namely, the one using as nematic state variable the order tensor $\mathbf{Q}$ of de Gennes

$$\mathcal{Q}_B := \Big\{\mathbf{M} \in \mathbb{M}^{3\times 3}_{0sym} : -\frac{1}{3} \leq e_i(\mathbf{M}) \leq \frac{2}{3}, i = 1, 2, 3\Big\}, \tag{2.2}$$

- the *uniaxial theory*, namely, the one obtained when the de Gennes order tensor is constrained to be uniaxial, i.e. two eigenvalues coincide

$$\mathcal{Q}_U := \Big\{\mathbf{M} \in \mathcal{Q}_B : e_3(\mathbf{M}) = -2e_1(\mathbf{M}) \quad \text{or} \quad e_1(\mathbf{M}) = -2e_3(\mathbf{M})\Big\}, \tag{2.3}$$

- *Frank theory*, namely, the one using as nematic state variable only the eigenframe of $\mathbf{Q}$ which is constrained to have eigenvalues $2/3, -1/3, -1/3$

$$\mathcal{Q}_{Fr} := \Big\{\mathbf{M} \in \mathcal{Q}_U : e_3(\mathbf{M}) = 2/3, e_2(\mathbf{M}) = e_1(\mathbf{M}) = -1/3\Big\}. \tag{2.4}$$

As explained in the introduction, we minimize (2.1) over the nematic state variables:

$$f_X(\mathbf{F}) := \inf_{\mathbf{Q}\in\mathcal{Q}_X} f_{mec}(\mathbf{Q}, \mathbf{F}) = \inf_{\mathbf{Q}\in\mathcal{Q}_X} \frac{1}{2}\|\mathcal{E}(\mathbf{F}) - \gamma\mathbf{Q}\|^2_{\mathbb{C}_\lambda}, \tag{2.5}$$

where X has to be replaced respectively by the labels Fr, U, B, with obvious meaning. It is clear that the *macroscopic* models thus obtained are the measure of the distance from the set $\gamma\mathcal{Q}_X$:

$$\inf_{\mathbf{Q}\in\mathcal{Q}_X} \|\mathcal{E}(\mathbf{F}) - \gamma\mathbf{Q}\|^2_{\mathbb{C}_\lambda} = \Big(\inf_{\mathbf{Q}\in\mathcal{Q}_X} \|\mathcal{E}(\mathbf{F}) - \gamma\mathbf{Q}\|_{\mathbb{C}_\lambda}\Big)^2 =: dist^2_{\mathbb{C}_\lambda}(\mathcal{E}(\mathbf{F}), \gamma\mathcal{Q}_X). \tag{2.6}$$

Here we write $dist_{\mathbb{C}_\lambda}$ to stress that this is the distance induced by the metric $\mathbb{C}_\lambda$. The standard euclidian distance can be obtained for $\lambda = 0$ and $\mu = 1/2$. There exists a unique element $\overline{\mathbf{Q}}$ which minimizes in $\mathcal{Q}_B$ the function $\|\mathcal{E}(\mathbf{F}) - \gamma\mathbf{Q}\|^2_{\mathbb{C}_\lambda}$. This is elementary, since the term proportional to the square of the trace is not subject to minimization and we are left with minimizing the square of the euclidian norm over the compact and convex set $\mathcal{Q}_B$:

$$\min_{\mathbf{Q}\in\mathcal{Q}_B} \frac{1}{2}\big\|\mathcal{E}(\mathbf{F}) - \gamma\mathbf{Q}\big\|^2_{\mathbb{C}_\lambda} = \Big(\min_{\mathbf{Q}\in\mathcal{Q}_B} \mu\big|\mathcal{E}(\mathbf{F}) - \gamma\mathbf{Q}\big|^2\Big) + \frac{\lambda}{2}(\operatorname{tr}\mathbf{F})^2. \tag{2.7}$$

The matrix $\overline{\mathbf{Q}}$ is also called the *projection* of $\mathcal{E}(\mathbf{F})/\gamma$ onto $\mathcal{Q}_B$ and referred as $\pi^{\mathcal{Q}_B}(\mathcal{E}(\mathbf{F})/\gamma)$. The projection of $\mathcal{E}(\mathbf{F})$ onto $\gamma\mathcal{Q}_B$ is $\pi^{\gamma\mathcal{Q}_B}(\mathcal{E}(\mathbf{F})) = \gamma\pi^{\mathcal{Q}_B}(\mathcal{E}(\mathbf{F})/\gamma)$. Since $\mathcal{Q}_{Fr} \subset \mathcal{Q}_U \subset \mathcal{Q}_B$, it follows that $f_B(\mathbf{F}) \leq f_U(\mathbf{F}) \leq f_{Fr}(\mathbf{F})$ and $f_B(\cdot)$ is a convex function. In Section 4 we report the explicit expressions of f_{Fr}, f_U, f_B.

In the following, we discuss the problem of relaxing the integral energies obtained from f_{Fr} and f_U considering the constraint of incompressibility.

2.2 Mathematical background

Let Ω be an open, bounded subset of $\mathbb{R}^n$. Letting $p \in [1, \infty)$, we introduce $L^p(\Omega)$, the space of measurable functions $u : \Omega \mapsto \mathbb{R}$ such that $\int_\Omega |u|^p dx < +\infty$, and moreover $L^p(\Omega, \mathbb{R}^n)$, $L^p(\Omega, \mathbb{M}^{n\times n})$, respectively the spaces of vectors or matrices with components in $L^p(\Omega)$. Analogously, $H^{1,p}(\Omega)$ is the space of scalar-valued L^p-functions whose gradient is in $L^p(\Omega, \mathbb{R}^n)$ and $H^{1,p}(\Omega, \mathbb{R}^n)$ is the space of vector-valued L^p-functions whose gradient is in $L^p(\Omega, \mathbb{M}^{n\times n})$. They are endowed with the usual norms. The space $H^{1,p}_o(\Omega, \mathbb{R}^n)$ is defined as the closure of

$C_c^\infty(\Omega,\mathbb{R}^n)$ in the topology of H^1. In the case $p=+\infty$, we obtain spaces of functions whose components are essentially bounded. If Ω is Lipschitz we label $H^{1,p}_{\Gamma_D}(\Omega,\mathbb{R}^n)$ the space of $H^{1,p}$-functions which vanish in the sense of traces (see [6, Thm 6.1-7]) on Γ_D, where Γ_D is a subset of $\partial\Omega$ of positive surface measure. We use the notation $C^k(\Omega)$, $C^k(\Omega,\mathbb{R}^n)$, $C^k_c(\Omega,\mathbb{R}^n)$, with $k\in\mathbb{N}\cup\{0,+\infty\}$ for the spaces of functions with continuous derivatives up to order k. Focusing on the case $n=3$, Ω is the *reference* domain occupied by a body. We denote with (x_1,x_2,x_3) the cartesian components of a point x in Ω. The system is described by the displacement $\mathbf{u}:\Omega\mapsto\mathbb{R}^3$ and the order tensor $\mathbf{Q}:\Omega\mapsto\mathcal{Q}_X$, where X stands either for Fr, or U, or B. We recall two inequalities which guarantee the coercivity of functionals even if the energy density does not control the full gradient of the displacement $\mathbf{u}$.

Korn's inequalities [6, Thms 6.3-3,6.3-4]. Let Ω be an open, bounded, connected subset of $\mathbb{R}^3$ with Lipschitz boundary. Let $\mathbf{z}\in H^{1,2}(\Omega,\mathbb{R}^3)$. Then, there exists a positive constant $C_1=C_1(\Omega)$ s.t.

$$\|\nabla\mathbf{z}\|^2_{L^2(\Omega,\mathbb{M}^{3\times3})}\le C_1(\Omega)\Big(\|\mathbf{z}\|^2_{L^2(\Omega,\mathbb{R}^3)}+\|\mathcal{E}(\nabla\mathbf{z})\|^2_{L^2(\Omega,\mathbb{M}^{3\times3})}\Big). \tag{2.8}$$

Let now $\mathbf{z}\in H^{1,2}_{\Gamma_D}(\Omega,\mathbb{R}^3)$. Then, there exists a positive constant $C_2=C_2(\Omega)$ s.t.

$$\|\nabla\mathbf{z}\|^2_{L^2(\Omega,\mathbb{M}^{3\times3})}\le C_2(\Omega)\Big(\|\mathcal{E}(\nabla\mathbf{z})\|^2_{L^2(\Omega,\mathbb{M}^{3\times3})}\Big). \tag{2.9}$$

In this paper we deal with requirements of partial convexity. For this subject our main references are [10], [24]. We recall that $f:\mathbb{M}^{n\times n}\mapsto\mathbb{R}\cup\{+\infty\}$ is rank-1 convex if by definition $f(s\xi_1+(1-s)\xi_2)\le sf(\xi_1)+(1-s)f(\xi_2)$ for every $s\in[0,1]$, $\xi_1,\xi_2\in\mathbb{M}^{3\times3}$ with $rank(\xi_1-\xi_2)\le1$. A function $f:\mathbb{M}^{n\times n}\mapsto\mathbb{R}\cup\{+\infty\}$ is said to be polyconvex, if there exists a convex function h such that $f(\mathbf{F})=h(M(\mathbf{F}))$, where $M(\mathbf{F})$ is the vector of all the minors of $\mathbf{F}$. We give the definition of quasiconvexity [1].

Definition 1. *A continuous function $f:\mathbb{M}^{n\times n}\mapsto\mathbb{R}$ is quasiconvex if and only if for every $\mathbf{Z}\in\mathbb{M}^{n\times n}$, ω open bounded subset of $\mathbb{R}^n$, $\mathbf{w}\in C^1_o(\omega,\mathbb{R}^n)$, we have*

$$f(\mathbf{Z})\le|\omega|^{-1}\int_\omega f\big(\mathbf{Z}+\nabla\mathbf{w}(y)\big)dy. \tag{2.10}$$

Remark 1. *We can take $\mathbf{w}\in C_c^\infty(\omega,\mathbb{R}^n)$ in (2.10) (see [1, Def. I.2] and comment below). If f is quasiconvex and satisfies growth conditions as (3.25), then (2.10) is true also for any $\mathbf{w}\in H^{1,2}_o(\omega,\mathbb{R}^n)$ (see [2, comment after Def. 2.2]) and in particular for any $\mathbf{w}\in H^{1,\infty}_o(\omega,\mathbb{R}^n)$.*

We define the convex envelope of a function f as $f^c(\xi):=\sup\{g(\xi):g\le f, g\,\text{convex}\}$. In the same way we define the poly-, quasi- and rank-one- convex envelopes, by requiring that the function g satisfies the corresponding requirement of partial convexity. In order to give a characterization for f^{rc}, which in the following is crucial, we follow [10, Sect. 6.4]. To start, we need some preliminary definitions (see [10, Sect. 5.2.5]).

Definition 2. *Let us write for any integer K*

$$\Lambda_K:=\Big\{\overline{\lambda}=(\lambda_1,\dots,\lambda_K):\lambda_i\ge0,\sum_i^K\lambda_i=1\Big\}. \tag{2.11}$$

Consider $\overline{\lambda}\in\Lambda_K$ and let $\xi_i\in\mathbb{M}^{n\times n}$, $1\le i\le K$. We say that $\{\lambda_i,\xi_i\}_{i=1}^K$ satisfy (H_K) if (by induction on the index i)

- *when $K=2$, then $rank(\xi_1-\xi_2)\le 1$;*
- *when $K>2$, then, up to a permutation, $rank(\xi_1-\xi_2)\le 1$ and if, for every $2\le i\le K-1$, we define*

$$\begin{cases} \mu_1=\lambda_1+\lambda_2 & \eta_1=\dfrac{\lambda_1\xi_1+\lambda_2\xi_2}{\lambda_1+\lambda_2} \\ \mu_i=\lambda_{i+1} & \eta_i=\xi_{i+1} \end{cases}$$

then $\{\mu_i,\eta_i\}_{i=1}^K$ satisfy (H_{K-1}).

Remark 2. *When $K=4,\overline{\lambda}\in\Lambda_4$, then $\{\lambda_i,\xi_i\}_{i=1}^4$ satisfy H_4 if, up to a permutation*

$$\begin{cases} rank(\xi_1-\xi_2)\le 1, \quad rank(\xi_3-\xi_4)\le 1 \\ rank(\eta_1-\eta_2)\le 1, \eta_1:=\dfrac{\lambda_1\xi_1+\lambda_2\xi_2}{\lambda_1+\lambda_2}, \eta_2:=\dfrac{\lambda_3\xi_3+\lambda_4\xi_4}{\lambda_3+\lambda_4} \end{cases}$$

holds.

Hence, for any $f:\mathbb{M}^{n\times n}\mapsto\mathbb{R}\cup\{+\infty\}$ one can characterize f^{rc} as [10, Thm 6.10]

$$f^{rc}(\xi)=\inf\Big\{\sum_i^K\lambda_i f(\xi_i):\overline{\lambda}\in\Lambda_K,\sum_i^K\lambda_i\xi_i=\xi,\ \{\lambda_i,\xi_i\}\text{ satisfy }(H_K)\Big\}. \tag{2.12}$$

If we restrict our attention to the case of real valued functions, the following chain of inequalities follows by definition (see [10], page 265)

$$f^c\le f^{pc}\le f^{qc}\le f^{rc}. \tag{2.13}$$

If $f:\mathbb{M}^{n\times n}\mapsto\mathbb{R}\cup\{+\infty\}$ the inequality $f^{qc}\le f^{rc}$ needs not hold.

We define some semi-convex hulls of sets. Given any set (not necessarily compact) $E\subset\mathbb{M}^{n\times n}$ we define E^c the smallest convex set containing E. It can be proved that

$$E^c=\Big\{\xi\in\mathbb{M}^{n\times n}:\xi=\sum_i^K\lambda_i\xi_i:\xi_i\in E,\overline{\lambda}\in\Lambda_K,K=1,2,3\dots\Big\}. \tag{2.14}$$

We define by induction E^{lc}, the lamination-convex envelope of E as

$$E^{lc}=\bigcup_{i=0}^{\infty}E^{(i)}, \tag{2.15}$$

where $E^{(0)}=E$,

$$E^{(1)}=\big\{\xi=s\xi_1+(1-s)\xi_2,\ \xi_1,\xi_2\in E,\ \text{rank}(\xi_1-\xi_2)\le 1,\ s\in[0,1]\big\} \tag{2.16}$$

that is the set of first order laminates of E and

$$E^{(i+1)}=E^{(i)}\cup\big\{\xi=s\xi_1+(1-s)\xi_2,\ \xi_1,\xi_2\in E^{(i)},\ \text{rank}(\xi_1-\xi_2)\le 1,\ s\in[0,1]\big\}. \tag{2.17}$$

Coherently with our definitions, we have this chain of inequalities:

$$E\subseteq E^{lc}\subseteq E^c. \tag{2.18}$$

The following proposition, which is due to Bogovskiĭ (see [19, Thm 3.1]), has an important rôle in order to treat the case of incompressible elastomers.

Proposition 1. *Consider $\mathbb{N} \ni n \geq 2$ and $p \in (1, \infty)$. Let Ω be an open, bounded, connected subset of $\mathbb{R}^n$ with Lipschitz boundary. Assume $\mathbf{z} \in H_o^{1,p}(\Omega, \mathbb{R}^n)$. Then, there exists at least one solution to the problem*

$$\begin{cases} \mathbf{w} \in H_o^{1,p}(\Omega, \mathbb{R}^n), \\ \operatorname{div} \mathbf{w} = \operatorname{div} \mathbf{z}, \\ \|\mathbf{w}\|_{H^{1,p}(\Omega, \mathbb{R}^n)} \leq C(\Omega, n, p) \|\operatorname{div} \mathbf{z}\|_{L^p(\Omega)}. \end{cases}$$

2.3 Gamma-convergence and relaxation

We follow the theory of Gamma-convergence in a topological space endowed with the weak topology as proposed in [2]. The general theory can be found in [11].

Definition 3. *Let $p \in (1, \infty)$. Let $\mathcal{F}$ be a functional defined on $H^{1,p}(\Omega, \mathbb{R}^n)$. We define the relaxation of $\mathcal{F}$ in the weak sequential (in brief w.s.) topology of $H^{1,p}(\Omega, \mathbb{R}^n)$*

$$\overline{\mathcal{F}} = \sup\{\mathcal{G} : \mathcal{G} \text{ is } H^{1,p}(\Omega, \mathbb{R}^n) \text{ w.s. lower semicontinuous}, \mathcal{G} \leq \mathcal{F}\}. \tag{2.19}$$

Definition 4. *Let $\{\mathcal{F}_h\}$ be a sequence of functionals defined on $H^{1,p}(\Omega, \mathbb{R}^n)$. We define for $\mathbf{u} \in H^{1,p}(\Omega, \mathbb{R}^n)$*

$$\Gamma\text{-}\liminf_{h \to +\infty} \mathcal{F}_h(\mathbf{u}) = \sup_{\mathcal{A} \in \mathcal{S}(\mathbf{u})} \liminf_{h \to +\infty} \inf\{\mathcal{F}_h(\mathbf{v}) : \mathbf{v} \in \mathcal{A}\},$$

$$\Gamma\text{-}\limsup_{h \to +\infty} \mathcal{F}_h(\mathbf{u}) = \sup_{\mathcal{A} \in \mathcal{S}(\mathbf{u})} \limsup_{h \to +\infty} \inf\{\mathcal{F}_h(\mathbf{v}) : \mathbf{v} \in \mathcal{A}\},$$

where $\mathcal{S}(\mathbf{u})$ is the family of all the open sets in the weak sequential topology of $H^{1,p}(\Omega, \mathbb{R}^n)$. If we have

$$\Gamma\text{-}\liminf_{h \to +\infty} \mathcal{F}_h(\mathbf{u}) = \Gamma\text{-}\limsup_{h \to +\infty} \mathcal{F}_h(\mathbf{u}),$$

then the common value is said to be the $\Gamma\text{-}\lim_{h \to +\infty} \mathcal{F}_h(\mathbf{u})$.

Proposition 2. *Let $\{\mathcal{F}_h\}$ be an increasing sequence of functionals defined on $H^{1,p}(\Omega, \mathbb{R}^n)$. Then, for every $\mathbf{u} \in H^{1,p}(\Omega, \mathbb{R}^n)$ there exists the*

$$\Gamma\text{-}\lim_{h \to +\infty} \mathcal{F}_h(\mathbf{u}) = \Gamma\text{-}\lim_{h \to +\infty} \overline{\mathcal{F}_h}(\mathbf{u}) = \sup_h \overline{\mathcal{F}_h}(\mathbf{u}),$$

where $\overline{\mathcal{F}_h}$ is the relaxation of $\mathcal{F}_h$.

Fundamental Theorem of Gamma-convergence

Let $\{\mathcal{F}_h\}$ be a sequence of functionals defined on $H^{1,p}(\Omega, \mathbb{R}^n)$. Suppose that:

- $\forall r \in \mathbb{R}$ there exists K_r a compact subset of $H^{1,p}(\Omega, \mathbb{R}^n)$ such that $\{\mathbf{u} \in H^{1,p}(\Omega, \mathbb{R}^n) : \mathcal{F}_h(\mathbf{u}) \leq r\} \subseteq K_r, \forall h$;
- $\forall \mathbf{u} \in H^{1,p}(\Omega, \mathbb{R}^n)$ there exits $\mathcal{F}(\mathbf{u}) = \Gamma\text{-}\lim_{h \to +\infty} \mathcal{F}_h(\mathbf{u})$.

Then we have

- $\lim_{h \to +\infty}(\inf \mathcal{F}_h) = \min \mathcal{F}$ (convergence of minima).
- Let $\{\mathbf{u}_h\} \subset H^{1,p}(\Omega, \mathbb{R}^n)$ be a minimizing sequence for $\{\mathcal{F}_h\}$ (i.e. $\lim_h \mathcal{F}_h(\mathbf{u}_h) = \lim_h \inf \mathcal{F}_h$). Then, up to subsequences, $\mathbf{u}_h \rightharpoonup \overline{\mathbf{u}}$ in $H^{1,p}$, where

$$\mathcal{F}(\overline{\mathbf{u}}) = \min \mathcal{F} \qquad \text{(convergence of minimum points).}$$

3 Relaxation theorems

As already observed, the energy densities f_{Fr}, f_U introduced in (2.5) are non-convex and, consequently, the associated integral functionals are not lower semicontinuous. We characterize the infima of the non-convex energies as the minima of the relaxed functionals. According to well known relaxation theorems (see Acerbi-Fusco Theorem [1] and [2, Theorem 2.3]), the relaxation coincides with the integral of the quasiconvex envelope of the original non-convex density. Hence, our goal is to obtain the quasiconvexification explicitly. This can be computed in practice by proving that the rank-1 convex envelopes of f_{Fr} and f_U coincide with their convex envelopes. In the case of finite-valued functions, this yields that the quasiconvex envelope coincides with the convex and rank-1 convex envelope, which are easier to compute.

On the other hand, experimental observations show that nematic elastomers are nearly incompressible (the bulk modulus is orders of magnitude larger than the shear modulus). The classical way to model such materials in linearized elasticity is to consider the limit ratio $\lambda/\mu = +\infty$, which is equivalent to restrict the admissible deformation gradients to the class of traceless matrices, and hence to define an energy functional in the presence of a linear constraint on the gradient of the displacement. We remark that a general strategy to treat such problems is to use the tools of A-quasiconvexification [4], the theory which studies the relaxation of non-convex functionals in the presence of linear (and constant-rank) constraints. In our particular case, however, an argument due to Braides [2] is sufficient to compute explicitly the relaxation of the three energies we consider. Indeed, it is possible to prove that the relaxation of the incompressible models coincides with the Gamma-limit of a sequence of relaxed models for compressible elastomers as the bulk modulus tends to $+\infty$. This fact is remarkable: it implies the convergence of minimizers (or almost minimizers) and minimum (or infimum) values of the energies of compressible rubbers to minimizers and minimum values of the Gamma-limit of the incompressible material.

We split the proof of our main result into several auxiliary propositions. In particular, we show that the projection of a constant strain onto $\gamma\mathcal{Q}_B$ can be obtained as a convex combination of elements which are compatible in the sense of (H_K) (see Definition 2 in Section 2.2) and whose deviators belong to $\gamma\mathcal{Q}_{Fr}$ or $\gamma\mathcal{Q}_U$. We define the sets of matrices

$$\mathcal{K}_X := \Big\{ \mathbf{M} \in \mathbb{M}_0^{3\times 3} : \mathcal{E}(\mathbf{M}) \in \mathcal{Q}_X \Big\} \tag{3.1}$$

where X stands either for Fr, or U, or B. The sets $\mathcal{K}_X$ inherit from $\mathcal{Q}_X$ some of their properties. In particular $\mathcal{K}_B$ is convex and $\mathcal{K}_{Fr} \subset \mathcal{K}_U \subset \mathcal{K}_B$. We start by showing that $\mathcal{K}_U^{lc} = \mathcal{K}_B$. As a corollary to the following proposition we show that $\mathcal{K}_B$ coincides with the set of first order laminates of $\mathcal{K}_U$ (see (2.16)).

Proposition 3. *Denote with $e_1(\mathbf{A}) \le e_2(\mathbf{A}) \le e_3(\mathbf{A})$ the ordered eigenvalues of the symmetric 3×3 matrix $\mathbf{A}$. Let (here $t \le 0$)*

$$\mathcal{M}_U^t := \Big\{ \mathbf{M} \in \mathbb{M}_0^{3\times 3} : e_1\big(\mathcal{E}(\mathbf{M})\big) = t, e_2\big(\mathcal{E}(\mathbf{M})\big), e_3\big(\mathcal{E}(\mathbf{M})\big) \in [t, -2t] \Big\}. \tag{3.2}$$

Then, the set $\mathcal{M}_U$ defined by

$$\mathcal{M}_U := \bigcup_{t\in[-1/3,0]} \mathcal{M}_U^t \tag{3.3}$$

is contained in $\mathcal{K}_U^{(1)}$, the set of first order laminates of $\mathcal{K}_U$.

Proof. Let $\mathbf{M} \in \mathcal{M}_U$. By the spectral theorem and up to re-labelling the axes, we may assume that the symmetric part of $\mathbf{M}$ is diagonal in the form $\overline{\mathbf{X}} = \text{diag}\,(t, \mu_2, \mu_3)$. There is nothing to prove if μ_2 or μ_3 are equal to t, because in that case the remaining eigenvalue is equal to $-2t$. We show that, for $\mu_2, \mu_3 \in (t, -2t)$, there exists a positive $\delta = \delta(\mu_2, \mu_3)$ and

$$\mathcal{K}_U \ni \overline{\mathbf{V}}^{\pm} = \text{diag}\,(t, \widehat{\mathbf{V}}^{\pm}) \quad \text{where} \quad \widehat{\mathbf{V}}^{\pm} = \begin{pmatrix} \mu_2 & \pm 2\delta \\ 0 & \mu_3 \end{pmatrix},$$

such that $\overline{\mathbf{V}}^{+} - \overline{\mathbf{V}}^{-} = 4\delta \mathbf{i}_2 \otimes \mathbf{i}_3$ and $\overline{\mathbf{X}} = \frac{1}{2}(\overline{\mathbf{V}}^{+} + \overline{\mathbf{V}}^{-}) \in \mathcal{K}_U^{(1)}$. We define

$$\widehat{\mathbf{X}}^{\pm} := \mathcal{E}(\widehat{\mathbf{V}}^{\pm}) = \begin{pmatrix} \mu_2 & \pm\delta \\ \pm\delta & \mu_3 \end{pmatrix},$$

$$\overline{\mathbf{X}}^{\pm} := \mathcal{E}(\overline{\mathbf{V}}^{\pm}). \tag{3.4}$$

The eigenvalues $\theta_{\alpha,\beta}$ of $\widehat{\mathbf{X}}^{\pm}$ are the solutions of $\det(\widehat{\mathbf{X}}^{\pm} - \theta \mathbf{I}) = 0$, namely

$$\theta_{\alpha,\beta} = \Big(\frac{\mu_2 + \mu_3}{2}\Big) \pm \sqrt{\frac{(\mu_3 - \mu_2)^2}{4} + \delta^2}. \tag{3.5}$$

By imposing θ_α to be equal to $-2t$ and recalling that $t + \mu_2 + \mu_3 = 0$, we obtain

$$\delta^2 = (2t + \mu_3)(2t + \mu_2) > 0 \quad \text{since} \quad -2t > \mu_2, \mu_3 > t. \tag{3.6}$$

By observing that $\theta_\alpha + \theta_\beta = \mu_2 + \mu_3$, this choice of δ yields $\theta_\beta = t$ and $\overline{\mathbf{V}}^{\pm} \in \mathcal{K}_U$. Now, define $\mathbf{M}^{sk} := (\mathbf{M} - \mathbf{M}^T)/2$. Hence, $\mathbf{M} = \frac{1}{2}((\overline{\mathbf{V}}^{+} + \mathbf{M}^{sk}) + (\overline{\mathbf{V}}^{-} + \mathbf{M}^{sk}))$, $rank((\overline{\mathbf{V}}^{+} + \mathbf{M}^{sk}) - (\overline{\mathbf{V}}^{-} + \mathbf{M}^{sk})) \leq 1$ and $\mathbf{M}$ is in $\mathcal{K}_U^{(1)}$. □

Corollary 1.

$$\mathcal{M}_U = \mathcal{K}_U^{(1)} = \mathcal{K}_U^{lc} = \mathcal{K}_U^{c} \equiv \mathcal{K}_B. \tag{3.7}$$

Proof. Since $\mathcal{K}_B$ is convex and $\mathcal{K}_U \subset \mathcal{K}_B$, then $\mathcal{K}_U^c \subseteq \mathcal{K}_B$. Then, it is straightforward to verify that $\mathcal{K}_B \subseteq \mathcal{M}_U$. □

Remark 3. As a by-product of Proposition 3, we deduce that $\mathcal{Q}_U^c = \mathcal{Q}_B$. Trivially, $\mathcal{Q}_U \subseteq \mathcal{Q}_B$ and $\mathcal{Q}_U^c \subseteq \mathcal{Q}_B$. To prove the opposite inclusion, notice that $\overline{\mathbf{X}}$ belongs to $\mathcal{Q}_B$ and it can be expressed as $\overline{\mathbf{X}} = \frac{1}{2}(\overline{\mathbf{X}}^{+} + \overline{\mathbf{X}}^{-})$, with $\overline{\mathbf{X}}^{\pm}$ in $\mathcal{Q}_U$.

With a similar argument we prove now that $\mathcal{K}_B$ coincides with the lamination-convex envelope of $\mathcal{K}_{Fr}$. In practice (see Corollary 2), it is enough to show that $\mathcal{K}_B$ coincides with the set of second order laminates of $\mathcal{K}_{Fr}$.

Proposition 4. *Denote with $e_1(\mathbf{A}) \leq e_2(\mathbf{A}) \leq e_3(\mathbf{A})$ the ordered eigenvalues of the symmetric 3×3 matrix $\mathbf{A}$. The set $\mathcal{M}_{Fr}$ defined by*

$$\mathcal{M}_{Fr} := \Big\{\mathbf{M} \in \mathbb{M}_0^{3\times 3} : e_1\big(\mathcal{E}(\mathbf{M})\big) = -\frac{1}{3}, e_2\big(\mathcal{E}(\mathbf{M})\big), e_3\big(\mathcal{E}(\mathbf{M})\big) \in \Big[-\frac{1}{3}, \frac{2}{3}\Big]\Big\} \tag{3.8}$$

is contained in $\mathcal{K}_{Fr}^{(1)}$, the se of first order laminates of $\mathcal{K}_{Fr}$.

Proof. The proof follows by taking t identically equal to $-1/3$ in the proof of Proposition 3. □

Corollary 2.

$$\mathcal{M}^{(1)}_{Fr} = \mathcal{K}^{(2)}_{Fr} = \mathcal{K}^{lc}_{Fr} = \mathcal{K}^{c}_{Fr} \equiv \mathcal{K}_B. \tag{3.9}$$

Proof. As above, $\mathcal{K}^c_{Fr} \subseteq \mathcal{K}_B$. Then, it is enough to prove that $\mathcal{K}_B \subseteq \mathcal{M}^{(1)}_{Fr}$. Take $\mathbf{G} \in \mathcal{K}_B$. Again, it is not restrictive to assume that $\overline{\mathbf{X}} := \mathcal{E}(\mathbf{G})$ is diagonal in the form $\overline{\mathbf{X}} = \mathrm{diag}\,(\mu_2, \mu_1, \mu_3)$ and $\mu_1 \leq \mu_2 \leq \mu_3$. If $\mu_1 = -1/3$ there is nothing to prove because this implies that $\mathbf{G} \in \mathcal{M}_{Fr}$. Similarly, we can assume $\mu_3 \neq 2/3$, otherwise the other two eigenvalues must be equal to $-1/3$. We show that, for $\mu_1, \mu_3 \in (-1/3, 2/3)$, there exists a positive $\delta = \delta(\mu_1, \mu_3)$ and

$$\mathcal{M}_{Fr} \ni \mathbf{G}^{\pm} = \mathrm{diag}\,(\mu_2, \widehat{\mathbf{G}}^{\pm}) \quad \text{where} \quad \widehat{\mathbf{G}}^{\pm} = \begin{pmatrix} \mu_1 & \pm 2\delta \\ 0 & \mu_3 \end{pmatrix},$$

such that $\mathbf{G}^+ - \mathbf{G}^- = 4\delta \mathbf{i}_2 \otimes \mathbf{i}_3$ and $\overline{\mathbf{X}} = \frac{1}{2}(\mathbf{G}^+ + \mathbf{G}^-) \in \mathcal{M}^{(1)}_{Fr}$. We define

$$\widehat{\mathbf{H}}^{\pm} := \mathcal{E}(\widehat{\mathbf{G}}^{\pm}) = \begin{pmatrix} \mu_1 & \pm\delta \\ \pm\delta & \mu_3 \end{pmatrix},$$

$$\mathbf{H}^{\pm} := \mathcal{E}(\mathbf{G}^{\pm}). \tag{3.10}$$

The eigenvalues $\theta_{\alpha,\beta}$ of $\widehat{\mathbf{H}}^{\pm}$ are the solutions of $\det(\widehat{\mathbf{H}}^{\pm} - \theta \mathbf{I}) = 0$, namely

$$\theta_{\alpha,\beta} = \Big(\frac{\mu_1 + \mu_3}{2}\Big) \pm \sqrt{\frac{(\mu_3 - \mu_1)^2}{4} + \delta^2}. \tag{3.11}$$

By imposing θ_α to be equal to $-1/3$ and recalling that $\mu_1 + \mu_2 + \mu_3 = 0$ we obtain

$$\delta^2 = \Big(\frac{1}{3} + \mu_1\Big)\Big(\frac{1}{3} + \mu_3\Big) > 0 \quad \text{since} \quad -\frac{1}{3} < \mu_1 \leq \mu_3. \tag{3.12}$$

By observing that $\theta_\alpha + \theta_\beta = \mu_1 + \mu_3$, this choice of δ yields $\theta_\beta = -\mu_2 + \frac{1}{3}$, and $\mathbf{G}^{\pm} \in \mathcal{M}_{Fr}$. Now, define $\mathbf{G}^{sk} := (\mathbf{G} - \mathbf{G}^T)/2$. Hence, $\mathbf{G} = \frac{1}{2}((\mathbf{G}^+ + \mathbf{G}^{sk}) + (\mathbf{G}^- + \mathbf{G}^{sk}))$, $rank((\mathbf{G}^+ + \mathbf{G}^{sk}) - (\mathbf{G}^- + \mathbf{G}^{sk})) \leq 1$ and $\mathbf{G}$ is in $\mathcal{M}^{(1)}_{Fr}$. □

Remark 4. We deduce from Corollary 2 that $\mathcal{Q}^c_{Fr} = \mathcal{Q}_B$. Trivially, $\mathcal{Q}^c_{Fr} \subseteq \mathcal{Q}_B$. To prove the converse implication, notice that $\overline{\mathbf{X}}$ by definition belongs to $\mathcal{Q}_B$ and is expressed as a convex combination of two symmetric matrices $\mathbf{H}^{\pm}$ in $\mathcal{M}_{Fr} \subseteq \mathcal{K}^{(1)}_{Fr}$ with coefficients equal to 1/2. Hence there exist $\mathbf{H}^{\pm}_{1,2} \in \mathcal{K}_{Fr}$ such that $\mathbf{H}^{\pm} = \frac{1}{2}(\mathbf{H}^{\pm}_1 + \mathbf{H}^{\pm}_2)$. Now, $\overline{\mathbf{X}}$ can be expressed as a convex combination of the symmetric matrices $\mathcal{E}(\mathbf{H}^{\pm}_{1,2})$ in $\mathcal{Q}_{Fr}$ with coefficients equal to 1/4.

The explicit constructions in Corollaries 1 and 2 are used to compute the quasiconvex envelope of the square of the distance from the sets $\gamma\mathcal{Q}_U, \gamma\mathcal{Q}_{Fr}$ (there is nothing to prove for the case $X = B$ because the energy density f_B is already convex). This is done in the next Lemma through a *lamination* construction. Here and in the following, we repeatedly adopt the notation $\big(f(\xi)\big)^{qc} \equiv f^{qc}(\xi)$, and similarly for the other envelopes.

Lemma 1. *Denote*

$$f_X(\xi) = \frac{1}{2} dist^2_{\mathbb{C}_\lambda}\Big(\mathcal{E}(\xi), \gamma\mathcal{Q}_X\Big), \tag{3.13}$$

where X stands either for Fr or U. Then,

$$\Big(f_X(\xi)\Big)^{qc} = f_B(\xi) = \frac{1}{2} dist^2_{\mathbb{C}_\lambda}\Big(\mathcal{E}(\xi), \gamma\mathcal{Q}_B\Big). \tag{3.14}$$

Proof. This is a consequence of (2.13) and of the chain of inequalities:

$$\big(dist^2_{\mathbb{C}_\lambda}(\mathcal{E}(\xi),\gamma\mathcal{Q}_X)\big)^{rc} \le dist^2_{\mathbb{C}_\lambda}(\mathcal{E}(\xi),\gamma\mathcal{Q}_B) \le \\ \big(dist^2_{\mathbb{C}_\lambda}(\mathcal{E}(\xi),\gamma\mathcal{Q}_X)\big)^{c} \le \big(dist^2_{\mathbb{C}_\lambda}(\mathcal{E}(\xi),\gamma\mathcal{Q}_X)\big)^{rc}, \tag{3.15}$$

where X stands either for Fr or U. The last inequality in (3.15) follows by definition. The second inequality is trivial if we consider that $dist^2_{\mathbb{C}_\lambda}(\mathcal{E}(\xi),\gamma\mathcal{Q}_B) \le dist^2_{\mathbb{C}_\lambda}(\mathcal{E}(\xi),\gamma\mathcal{Q}_X)$ and if we take the convex envelope on both sides. We are left to prove the first inequality. To this end, we apply (2.12) characterizing the rank-1 convex envelope of a function by exhibiting a family of matrices and positive coefficients along which the infimum is attained.

Fix $\varepsilon \in \mathbb{R}$ different from zero. For every $\xi \in \mathbb{M}^{3\times 3}$, $\mathbf{X} \in \mathcal{Q}_B$, $\mathbf{V} \in \mathcal{Q}_X$, a combination of the triangular and Young's inequalities yields

$$\|\mathcal{E}(\xi) - \gamma\mathbf{V}\|^2_{\mathbb{C}_\lambda} \le (1+\varepsilon^2)\|\mathcal{E}(\xi) - \gamma\mathbf{X}\|^2_{\mathbb{C}_\lambda} + \Big(1 + \frac{1}{\varepsilon^2}\Big)\|\gamma\mathbf{X} - \gamma\mathbf{V}\|^2_{\mathbb{C}_\lambda}. \tag{3.16}$$

Write $\overline{\mathbf{X}} = \pi^{\mathcal{Q}_B}(\mathcal{E}(\xi)/\gamma)$ instead of $\mathbf{X}$ in (3.16). Now, (3.16) reads

$$\|\mathcal{E}(\xi) - \gamma\mathbf{V}\|^2_{\mathbb{C}_\lambda} \le (1+\varepsilon^2) dist^2_{\mathbb{C}_\lambda}\big(\mathcal{E}(\xi),\gamma\mathcal{Q}_B\big) + \Big(1 + \frac{1}{\varepsilon^2}\Big)\|\gamma\overline{\mathbf{X}} - \gamma\mathbf{V}\|^2_{\mathbb{C}_\lambda}. \tag{3.17}$$

We have to distinguish two cases. Suppose $\overline{\mathbf{X}} \in \mathcal{Q}_X$. Taking $\inf_{\mathbf{V}\in\mathcal{Q}_X}$ on both sides of (3.17), we obtain

$$dist^2_{\mathbb{C}_\lambda}\big(\mathcal{E}(\xi),\gamma\mathcal{Q}_X\big) \le (1+\varepsilon^2) dist^2_{\mathbb{C}_\lambda}\big(\mathcal{E}(\xi),\gamma\mathcal{Q}_B\big) + 0, \tag{3.18}$$

and taking the limit $\varepsilon \to 0$ the claim follows. Assume now $\overline{\mathbf{X}} \in \mathcal{Q}_B \backslash \mathcal{Q}_X$ and notice that $\overline{\mathbf{X}} \in \mathcal{K}_B \backslash \mathcal{K}_X$ as well. Corollaries 1 and 2 show that $\mathcal{K}_B$ can be laminated in the sense that $\mathcal{K}_B = \mathcal{K}^{(1)}_U = \mathcal{K}^{(2)}_{Fr}$. Precisely, there exist families of coefficients and matrices $\{\lambda_i\}_{i=1}^K \times \{\overline{\mathbf{V}}_i\}_{i=1}^K \in [0,1] \times \mathcal{K}_X$, with $\{\lambda_i, \overline{\mathbf{V}}_i\}_{1=1}^K$ satisfying (H_K) with K finite ($K \le 2$ for $X = U$ and $K \le 4$ for $X = Fr$) such that $\overline{\mathbf{X}} = \sum_i \lambda_i \overline{\mathbf{V}}_i$. Define $\overline{\mathbf{X}}_i := \mathcal{E}(\overline{\mathbf{V}}_i) \in \mathcal{Q}_X$, $\xi^\perp := \xi - \gamma\overline{\mathbf{X}}$ and $\xi_i := \gamma\overline{\mathbf{V}}_i + \xi^\perp$ for any $i = 1, \ldots, K$. Trivially, $\{\lambda_i, \xi_i\}_{1=1}^K$ still satisfy (H_K) and $\xi = \sum_i \lambda_i \xi_i$.

We can repeat the construction in (3.16) writing ξ_i and $\overline{\mathbf{X}}_i$ instead of $\xi, \mathbf{X}$ and take $\inf_{\mathbf{V}\in\mathcal{Q}_X}$ on both sides, yielding for every $i = 1, \ldots, K$

$$dist^2_{\mathbb{C}_\lambda}\big(\mathcal{E}(\xi_i),\gamma\mathcal{Q}_X\big) \le (1+\varepsilon^2)\|\mathcal{E}(\xi^\perp)\|^2_{\mathbb{C}_\lambda}. \tag{3.19}$$

Here we use the fact that $\mathcal{E}(\xi_i) = \gamma\overline{\mathbf{X}}_i + \mathcal{E}(\xi^\perp)$. Let us multiply both sides of formula (3.19) by λ_i and sum up in i yielding

$$\sum_i^K \lambda_i dist^2_{\mathbb{C}_\lambda}\big(\mathcal{E}(\xi_i),\gamma\mathcal{Q}_X\big) \le (1+\varepsilon^2)\sum_i^K \lambda_i\|\mathcal{E}(\xi^\perp)\|^2_{\mathbb{C}_\lambda}. \tag{3.20}$$

In view of (2.12) we finally obtain

$$\big(dist^2_{\mathbb{C}_\lambda}(\mathcal{E}(\xi),\gamma\mathcal{Q}_X)\big)^{rc} \le (1+\varepsilon^2)\|\mathcal{E}(\xi^\perp)\|^2_{\mathbb{C}_\lambda} = (1+\varepsilon^2) dist^2_{\mathbb{C}_\lambda}\big(\mathcal{E}(\xi),\gamma\mathcal{Q}_B\big). \tag{3.21}$$

The claim is proved taking the limit $\varepsilon \to 0$. □

Remark 5. We summarize some of the properties of the energy density f_X, where X stands either for Fr, or U, or B.

$$f_X(\cdot) \text{ is continuous,} \tag{3.22}$$

$$0 \leq f_X(\mathbf{Z}), \tag{3.23}$$

$$c_1|\mathcal{E}(\mathbf{Z})|^2 - c_2 \leq f_X(\mathbf{Z}), \tag{3.24}$$

$$f_X(\mathbf{Z}) \leq c_3|\mathcal{E}(\mathbf{Z})|^2 + c_4, \tag{3.25}$$

$$|f_X(\mathbf{Z}_1) - f_X(\mathbf{Z}_2)| \leq c_5\big(c_6 + |\mathcal{E}(\mathbf{Z}_1)| + |\mathcal{E}(\mathbf{Z}_2)|\big)\,\big|\mathcal{E}(\mathbf{Z}_1) - \mathcal{E}(\mathbf{Z}_2)\big|, \tag{3.26}$$

for every $\mathbf{Z}, \mathbf{Z}_1, \mathbf{Z}_2 \in \mathbb{M}^{3\times 3}$ and were c_i with $i = 1, \ldots, 6$ are suitable positive constants. To prove (3.22) and (3.26) we recall that the distance is a Lipschitz function. Then, (3.23 − 3.25) are trivial.

Proposition 5 ([1], [2]-Thm. 2.3)**.** *Let Ω be any open, bounded subset of $\mathbb{R}^3$. Let $f : \mathbb{M}^{3\times 3} \mapsto [0, \infty[$ verify* (3.22)*,* (3.24) *and* (3.25)*. Define on $H^{1,2}(\Omega, \mathbb{R}^3)$*

$$\mathcal{F}(\mathbf{u}) := \int_\Omega f(\nabla\mathbf{u})dx, \qquad \mathcal{F}^o(\mathbf{u}) := \begin{cases} \mathcal{F}(\mathbf{u}) & \text{on } H^{1,2}_o(\Omega, \mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

Then, the relaxation of $\mathcal{F}$ and $\mathcal{F}^o$ is

$$\overline{\mathcal{F}}(\mathbf{u}) = \int_\Omega f^{qc}(\nabla\mathbf{u})dx, \qquad \overline{\mathcal{F}^o}(\mathbf{u}) = \begin{cases} \overline{\mathcal{F}}(\mathbf{u}) & \text{on } H^{1,2}_o(\Omega, \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

respectively. Moreover

$$\inf_{H^{1,2}(\Omega,\mathbb{R}^3)} \mathcal{F}^o(\mathbf{u}) = \min_{H^{1,2}(\Omega,\mathbb{R}^3)} \overline{\mathcal{F}^o}(\mathbf{u}). \tag{3.27}$$

After these preparations, we are in a position to discuss our relaxation theorems.

3.1 The case of compressible elastomers

As a by-product of Lemma 1 and Proposition 5, we obtain the relaxation of the non-convex mechanical energy in the case of compressible materials, for which λ in (2.5) is finite.

Theorem 1. *Let Ω be an open, bounded, connected subset of $\mathbb{R}^3$ with Lipschitz boundary and denote with Γ_D an open subset of $\partial\Omega$ of positive surface measure. Take $f_X(\cdot)$ as defined in* (2.5)*, where X stands either for Fr, or U, or B and take some function $\mathbf{g}(x) \in H^{1,2}(\Omega, \mathbb{R}^3)$. Let us define on $H^{1,2}(\Omega, \mathbb{R}^3)$*

$$\mathcal{I}_X(\mathbf{u}) = \int_\Omega f_X(\nabla\mathbf{u})dx. \tag{3.28}$$

Then, the relaxation of $\mathcal{I}_X$ is

$$\mathcal{I}_B(\mathbf{u}) = \int_\Omega f_B(\nabla\mathbf{u})dx. \tag{3.29}$$

Moreover, if we define $\mathcal{I}_X^{\Gamma_D,\mathbf{g}}$ by setting $\mathcal{I}_X^{\Gamma_D,\mathbf{g}} = \mathcal{I}_X$ on $\mathbf{g} + H^{1,2}_{\Gamma_D}(\Omega, \mathbb{R}^3)$ and $+\infty$ outside, the relaxation of $\mathcal{I}_X^{\Gamma_D,\mathbf{g}}$ is equal to $\mathcal{I}_B$ on $\mathbf{g} + H^{1,2}_{\Gamma_D}(\Omega, \mathbb{R}^3)$ and $+\infty$ outside.

Proof. For $X = B$ these results are trivial. Define (here X stands either for Fr, or U, or B)

$$\mathcal{I}_X^o(\mathbf{u}) = \begin{cases} \mathcal{I}_X(\mathbf{u}) & \text{if } \mathbf{u} \in H_o^{1,2}(\Omega, \mathbb{R}^3), \\ +\infty & \text{otherwise in } H^{1,2}(\Omega, \mathbb{R}^3). \end{cases}$$

In the cases where $X = Fr$ or $X = U$, the quasiconvex envelope of f_X is f_B (Lemma 1) and hence the relaxation of $\mathcal{I}_X$, $\mathcal{I}_X^o$ are $\mathcal{I}_B$, $\mathcal{I}_B^o$ respectively (see Proposition 5). In particular, the quasiconvexification formula can be expressed as ([22, Remark 5.3, pages 157-158], [3, Paragr. 6.2, pages 55-59])

$$\big(f_X\big)^{qc}(\mathbf{Z}) = \inf\Big\{|\omega|^{-1}\int_\omega f_X\big(\mathbf{Z} + \nabla\mathbf{w}(y)\big)dy : \mathbf{w} \in H_o^{1,2}(\omega, \mathbb{R}^3)\Big\}, \tag{3.30}$$

where ω is any bounded open subset of $\mathbb{R}^3$ with $|\partial\omega| = 0$. The infimum in (3.30) can be taken in $C_c^\infty(\omega, \mathbb{R}^3)$. This is due to the density of C_c^∞ and the continuity of $\mathcal{I}_X$ in the strong convergence of $H^{1,2}$. Also, since $C_c^\infty(\omega, \mathbb{R}^3) \subset H_o^{1,\infty}(\omega, \mathbb{R}^3) \subset H_o^{1,2}(\omega, \mathbb{R}^3)$, we can simply consider test functions in $H_o^{1,\infty}(\omega, \mathbb{R}^3)$.

Now, we can extend the proof to more general boundary conditions. The relaxation result is true also if we choose some function $\mathbf{g}(x) \in H^{1,2}(\Omega, \mathbb{R}^3)$ and we define a new functional $\mathcal{I}_X^{o,\mathbf{g}}$ equal to $\int_\Omega f_X(\nabla\mathbf{u})dx$ only on the set $\{\mathbf{u} \in \mathbf{g} + H_o^{1,2}(\Omega, \mathbb{R}^3)\}$, and $+\infty$ outside. In fact, introduce $\mathbf{v} := \mathbf{u} - \mathbf{g}$ and $g_X(\nabla\mathbf{v}) := f_X(\nabla\mathbf{v} + \nabla\mathbf{g})$. Defining now (here X stands either for Fr, or U, or B)

$$\mathcal{G}_X^o(\mathbf{v}) = \begin{cases} \displaystyle\int_\Omega g_X(\nabla\mathbf{v})dx & \text{if } \mathbf{v} \in H_o^{1,2}(\Omega, \mathbb{R}^3), \\ +\infty & \text{otherwise in } H^{1,2}(\Omega, \mathbb{R}^3), \end{cases}$$

we obtain that the relaxation of $\mathcal{G}_X^o$ is $\mathcal{G}_B^o$. Then, we can define $\mathcal{I}_X^{\Gamma_D,\mathbf{g}}$ equal to $\int_\Omega f_X(\nabla\mathbf{u})dx$ only on the set $\{\mathbf{u} \in \mathbf{g} + H_{\Gamma_D}^{1,2}(\Omega, \mathbb{R}^3)\}$ and $+\infty$ otherwise in $H^1(\Omega, \mathbb{R}^3)$. It is immediate to see that the relaxation of $\mathcal{I}_X^{\Gamma_D,\mathbf{g}}$ is $+\infty$ outside $\{\mathbf{u} \in \mathbf{g} + H_{\Gamma_D}^{1,2}(\Omega, \mathbb{R}^3)\}$ because this set is weakly closed. Fix $\mathbf{u}$ in $\mathbf{g} + H_{\Gamma_D}^{1,2}(\Omega, \mathbb{R}^3)$. Recalling that $\overline{\mathcal{I}_X^{\Gamma_D,\mathbf{g}}} = \Gamma\text{-}\lim \mathcal{I}_X^{\Gamma_D,\mathbf{g}}$, we can write

$$\begin{aligned} \int_\Omega f_B(\nabla\mathbf{u})dx = \Gamma\text{-}\liminf \mathcal{I}_X(\mathbf{u}) \le \Gamma\text{-}\liminf \mathcal{I}_X^{\Gamma_D,\mathbf{g}}(\mathbf{u}) = \\ \Gamma\text{-}\liminf \mathcal{I}_X^{\Gamma_D,\widehat{\mathbf{g}}}(\mathbf{u}) \le \Gamma\text{-}\liminf \mathcal{I}_X^{o,\widehat{\mathbf{g}}}(\mathbf{u}) = \int_\Omega f_B(\nabla\mathbf{u})dx, \end{aligned} \tag{3.31}$$

where $\widehat{\mathbf{g}} := \mathbf{u}$. The same inequality holds for the $\Gamma\text{-}\limsup$, proving the claim. □

3.2 The case of incompressible elastomers

Theorem 2. *Let Ω be an open, bounded, connected subset of $\mathbb{R}^3$ with Lipschitz boundary and denote with Γ_D an open subset of $\partial\Omega$ of positive surface measure. Take $f_X(\cdot)$ as in (2.5) (where X stands either for Fr, or U, or B) and define on $H^{1,2}(\Omega, \mathbb{R}^3)$*

$$\mathcal{F}_X(\mathbf{u}) = \begin{cases} \displaystyle\int_\Omega f_X(\nabla\mathbf{u})dx & \text{if } \operatorname{div}\mathbf{u} = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Then, the relaxation of $\mathcal{F}_X$ is

$$\mathcal{F}_B(\mathbf{u}) = \begin{cases} \displaystyle\int_\Omega f_B(\nabla\mathbf{u})dx & \text{if } \operatorname{div}\mathbf{u} = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, take $\mathbf{g}(x) \in H^{1,2}(\Omega,\mathbb{R}^3)$ *with* $\operatorname{div}\mathbf{g} = 0$ *a.e. in* Ω *and define* $\mathcal{F}_X^{\Gamma_D,\mathbf{g}}$ *by setting* $\mathcal{F}_X^{\Gamma_D,\mathbf{g}} = \mathcal{F}_X$ *on* $\mathbf{g} + H^{1,2}_{\Gamma_D}(\Omega,\mathbb{R}^3)$ *and* $+\infty$ *outside. Then the relaxation of* $\mathcal{F}_X^{\Gamma_D,\mathbf{g}}$ *is equal to* $\mathcal{F}_B$ *on* $\mathbf{g}+H^{1,2}_{\Gamma_D}(\Omega,\mathbb{R}^3)$ *and* $+\infty$ *outside. Finally,* f_B *satisfies a solenoidal quasiconvexification formula, namely*

$$f_B(\mathbf{Z}) = \inf\Big\{|\omega|^{-1}\int_\omega f_X\big(\mathbf{Z}+\nabla\mathbf{w}(y)\big)dy : \mathbf{w}\in H^{1,2}_o(\omega,\mathbb{R}^3), \operatorname{div}\mathbf{w}=0\Big\}\ \forall \mathbf{Z}\in\mathbb{M}^{3\times 3}_0, \quad (3.32)$$

where ω *is any open, bounded, connected, nonempty subset of* $\mathbb{R}^3$ *with Lipschitz boundary.*

Remark 6. *Formula* (3.32) *holds under the hypotheses of Theorem* 2 *even if we replace* $\mathbf{w}\in H^{1,2}_o(\omega,\mathbb{R}^3)$ *with* $\mathbf{w}\in C^\infty_c(\omega,\mathbb{R}^3)$*, since the closure of* $\{\mathbf{w}\in C^\infty_c(\omega,\mathbb{R}^3), \operatorname{div}\mathbf{w}=0\}$ *in* $H^{1,2}$ *is precisely* $\{\mathbf{w}\in H^{1,2}_o(\omega,\mathbb{R}^3), \operatorname{div}\mathbf{w}=0\}$ *(see [25, Thm 1.6], [19, Sect. III.4] and [2]) and* $\mathcal{F}_X$ *is continuous in the strong topology of* $H^{1,2}$ *on the set* $\{\mathbf{u}\in H^{1,2}_o(\omega,\mathbb{R}^3), \operatorname{div}\mathbf{u}=0\}$.

Proof of Theorem 2. This relaxation result follows essentially from an idea of Braides (see [2]). For the reader's convenience, we give here the main lines of the proof when the functional is defined on $H^{1,2}(\Omega,\mathbb{R}^3)$.

First of all, in the case when $X = B$, there is nothing to prove because the energy density $f_B(\cdot)$ is convex. In particular, since $\mathcal{F}_B$ is lower semicontinuous, then $\overline{\mathcal{F}_X}(\mathbf{u}) \geq \mathcal{F}_B(\mathbf{u})$ (here X stands either for Fr or U). We are left to prove the reverse inequality, that is

$$\overline{\mathcal{F}_X}(\mathbf{u}) \leq \mathcal{F}_B(\mathbf{u}). \quad (3.33)$$

To this end, let us introduce for any $h\in\mathbb{N}$

$$\mathcal{F}^h_X(\mathbf{u}) = \int_\Omega \big(f_X(\nabla\mathbf{u}) + h(\operatorname{div}\mathbf{u})^2\big)dx, \quad \text{where } X \text{ stands either for } Fr, \text{ or } U, \text{ or } B. \quad (3.34)$$

We notice that the energy density appearing in (3.34) is still in the form of the square of the distance from the set $\gamma\mathcal{Q}_X$:

$$f_X(\mathbf{F}) + h(\operatorname{tr}\mathbf{F})^2 = \frac{1}{2}dist^2_{\mathbb{C}_{\lambda+2h}}\big(\mathcal{E}(\mathbf{F}),\gamma\mathcal{Q}_X\big). \quad (3.35)$$

The following chain of equalities has a crucial rôle in order to prove (3.33):

$$\Gamma\text{-}\lim_{h\to+\infty}\mathcal{F}^h_X = \Gamma\text{-}\lim_{h\to+\infty}\overline{\mathcal{F}^h_X} = \sup_h \overline{\mathcal{F}^h_X} = \mathcal{F}_B. \quad (3.36)$$

Line (3.36) follows by Proposition 2. The only point which needs a proof is the last equality. Theorem 1 applies to $\mathcal{F}^h_X$ for any $h\in\mathbb{N}$, and hence the relaxation of $\mathcal{F}^h_X$ on $H^{1,2}(\Omega,\mathbb{R}^3)$ is the integral of the quasiconvex envelope of the energy density. The result of Lemma 1 applies to $f_X(\mathbf{F})$ if we replace λ with $\lambda' := \lambda+2h$ in formula (3.13) yielding

$$\Big(f^h_X(\mathbf{F})\Big)^{qc} = \frac{1}{2}dist^2_{\mathbb{C}_{\lambda+2h}}\Big(\mathcal{E}(\mathbf{F}),\gamma\mathcal{Q}_B\Big), \quad \text{where } X \text{ stands either for } Fr \text{ or } U,$$

and $\overline{\mathcal{F}^h_X} = \mathcal{F}^h_B$. Now, we take the limit of $\mathcal{F}^h_B$ as $h\to+\infty$. By Beppo-Levi's Theorem on monotone convergence, the supremum of a family of increasing integrals coincides with the integral of the pointwise limit of the energy densities

$$\lim_{h\to+\infty}\Big[\big(f^h_X(\mathbf{F})\big)^{qc}\Big] = \sup_h\Big[\inf_{\mathcal{Q}_B}\mu|\mathcal{E}(\mathbf{F})-\gamma\mathbf{Q}|^2 + \Big(\frac{\lambda}{2}+h\Big)(\operatorname{tr}\mathbf{F})^2\Big] = \begin{cases} f_B(\mathbf{F}) & \text{if } \operatorname{tr}\mathbf{F}=0,\\ +\infty & \text{otherwise,}\end{cases}$$

and hence $\sup_h \overline{\mathcal{F}_X^h} = \mathcal{F}_B$. With this, (3.36) is proved.

Now, $\overline{\mathcal{F}_X}(\mathbf{u}) \le \mathcal{F}_B(\mathbf{u})$ is trivial if $\operatorname{div}\mathbf{u} \neq 0$ and in the rest of the proof we suppose $\operatorname{div}\mathbf{u} = 0$. Thanks to the coercivity condition (3.24) it is sufficient to prove that, for any sequence $\mathbf{u}_h \rightharpoonup \mathbf{u}$ in $H^{1,2}(\Omega,\mathbb{R}^3)$, there exists a sequence $\{\mathbf{z}_h\}$ with $\operatorname{div}\mathbf{z}_h = 0$ and $\mathbf{z}_h \rightharpoonup \mathbf{u}$ in $H^{1,2}(\Omega,\mathbb{R}^3)$ such that

$$\liminf_{h\to+\infty} \mathcal{F}_X(\mathbf{z}_h) \le \liminf_{h\to+\infty} \mathcal{F}_X^h(\mathbf{u}_h). \tag{3.37}$$

In view of (3.36), taking the infimum over all the sequences $\{\mathbf{u}_h\}$ weakly converging to $\mathbf{u}$, we obtain $\mathcal{F}_B$ on the right hand side of (3.37). Moreover, thanks to Theorem 1, we may restrict ourselves to sequences $\{\mathbf{u}_h\}$ such that $\mathbf{u}_h - \mathbf{u} \in H_o^{1,2}(\Omega,\mathbb{R}^3)$. We apply Proposition 1 with $p = 2, n = 3$. For every $h \in \mathbb{N}$, let $\mathbf{w}_h \in H_o^{1,2}(\Omega,\mathbb{R}^3)$ such that

$$\begin{cases} \operatorname{div}\mathbf{w}_h = \operatorname{div}(\mathbf{u}_h - \mathbf{u}) = \operatorname{div}\mathbf{u}_h, \\ \|\mathbf{w}_h\|_{H^{1,2}(\Omega,\mathbb{R}^3)} \le C\|\operatorname{div}\mathbf{u}_h\|_{L^2(\Omega)}. \end{cases} \tag{3.38}$$

Since $\mathcal{F}_B(\mathbf{u}) < +\infty$, we can suppose that $\mathcal{F}_X^h(\mathbf{u}_h) \le Const$ for every h so that

$$\|\operatorname{div}\mathbf{u}_h\|^2_{L^2(\Omega)} \le Const/h \tag{3.39}$$

and, by Proposition 1, we have that $\mathbf{w}_h \to 0$ strongly in $H^{1,2}(\Omega,\mathbb{R}^3)$ as $h \to +\infty$. If we define

$$\mathbf{z}_h := \mathbf{u}_h - \mathbf{w}_h, \tag{3.40}$$

we have that $\mathbf{z}_h \rightharpoonup \mathbf{u}$ in $H^{1,2}(\Omega,\mathbb{R}^3)$, $\mathbf{z}_h - \mathbf{u}_h \in H_o^{1,2}(\Omega,\mathbb{R}^3)$ and $\operatorname{div}\mathbf{z}_h = 0$. Now, by (3.26) and Hölder's inequality, we have

$$\Big|\int_\Omega f_X(\nabla\mathbf{u}_h)dx - \int_\Omega f_X(\nabla\mathbf{z}_h)dx\Big| \le Const\|\mathcal{E}(\nabla\mathbf{w}_h)\|_{L^2(\Omega,\mathbb{M}^{3\times3})} \tag{3.41}$$

and, in conclusion,

$$\begin{aligned} \overline{\mathcal{F}_X}(\mathbf{u}) \le \liminf_{h\to+\infty} \mathcal{F}_X(\mathbf{z}_h) \le \liminf_{h\to+\infty} \int_\Omega f_X(\mathbf{u}_h)dx + \lim_{h\to+\infty}\Big|\int_\Omega f_X(\mathbf{u}_h)dx - \mathcal{F}_X(\mathbf{z}_h)\Big| \\ \le \liminf_{h\to+\infty} \mathcal{F}_X^h(\mathbf{u}_h) + 0. \end{aligned} \tag{3.42}$$

The relaxation result above holds also if we define a functional $\mathcal{F}_X^o(\mathbf{u})$ equal to $\mathcal{F}_X(\mathbf{u})$ on $H_o^{1,2}(\Omega,\mathbb{R}^3)$ and equal to $+\infty$ otherwise in $H^{1,2}(\Omega,\mathbb{R}^3)$. By proceeding as above, but taking all functions in $H_o^{1,2}(\Omega,\mathbb{R}^3)$ we obtain that $\overline{\mathcal{F}_X^o} = \mathcal{F}_B^o$. Moreover, we can extend the proof to more general boundary conditions. Indeed, the relaxation result is true also if we choose some function $\mathbf{g}(x) \in H^{1,2}(\Omega,\mathbb{R}^3)$ with $\operatorname{div}\mathbf{g} = 0$ and we define a new functional $\mathcal{F}_X^{o,\mathbf{g}}$ equal to $\int_\Omega f_X(\nabla\mathbf{u})dx$ only on the set $\{\mathbf{u} \in \mathbf{g} + H_o^{1,2}(\Omega,\mathbb{R}^3), \operatorname{div}\mathbf{u} = 0\}$ and $+\infty$ otherwise in $H^{1,2}(\Omega,\mathbb{R}^3)$. In fact, we can introduce $\mathbf{v} := \mathbf{u} - \mathbf{g}$ and $g_X(\nabla\mathbf{v}) := f_X(\nabla\mathbf{v} + \nabla\mathbf{g})$. Defining now

$$\mathbf{v} \mapsto \begin{cases} \displaystyle\int_\Omega g_X(\nabla\mathbf{v})dx & \text{if } \mathbf{v} \in H_o^{1,2}(\Omega,\mathbb{R}^3), \operatorname{div}\mathbf{v} = 0, \\ +\infty & \text{otherwise in } H^{1,2}(\Omega,\mathbb{R}^3), \end{cases}$$

we can proceed exactly as in the proof of Theorem 1 where an analogous case is treated.

Finally, formula (3.32) follows as in [2, Proposition 3.4].

□

4 Discussion

In this section we apply our relaxation result to some concrete examples and we discuss some physical implications of our analysis.

4.1 Physical implications

In the case when we consider the energy density f_{Fr}, we accept a direct coupling between strain and local orientation of the liquid crystal molecules. Experimental results show that a uniaxial stretch aligns the molecules along the direction of maximal stretch. Whether a macroscopic deformation may alter also the local order of the molecules and not only the local direction is a debated problem. In particular, in the case when we consider the energy density f_U, we admit the possibility to enforce the melting of the order of the system ($\mathbf{Q} = 0$). Even more optimistically, with f_B we allow the whole class of biaxial states. We recall the main consequences of our relaxation result. In the following corollary, X stands either for Fr, or U, or B.

Corollary 3. *Under the hypotheses of Theorem* 1

$$\inf_{H^{1,2}(\Omega,\mathbb{R}^3)} \mathcal{I}_X^{\Gamma_D,\mathbf{g}}(\mathbf{u}) = \min_{H^{1,2}(\Omega,\mathbb{R}^3)} \mathcal{I}_B^{\Gamma_D,\mathbf{g}}(\mathbf{u}). \tag{4.1}$$

Moreover, under the hypotheses of Theorem 2

$$\inf_{H^{1,2}(\Omega,\mathbb{R}^3)} \mathcal{F}_X^{\Gamma_D,\mathbf{g}}(\mathbf{u}) = \min_{H^{1,2}(\Omega,\mathbb{R}^3)} \mathcal{F}_B^{\Gamma_D,\mathbf{g}}(\mathbf{u}). \tag{4.2}$$

Proof. This is a property of the relaxation (see Proposition 5). The right hand side in (4.1) and (4.2) is a minimum thanks to Korn's inequality (2.9) and Poincaré inequality. □

Corollary 3 finds an application in traction problems. Let us assume $\Omega = (-X_1, X_1) \times (-X_2, X_2) \times (-X_3, X_3)$, $\Gamma_D = \{-X_1\} \times (-X_2, X_2) \times (-X_3, X_3) \cup \{X_1\} \times (-X_2, X_2) \times (-X_3, X_3)$ for some $X_1, X_2, X_3 > 0$, $\mathbf{g}(x) = \mathbf{F}\,\mathbf{x}$ where $\mathbf{F}$ is a constant matrix with $\operatorname{tr}\mathbf{F} = 0$ and $\mathbf{x} := x - O$, where O is the origin. Then, the equilibrium solution to problem (4.2)-left is characterized by a biaxial tensor field. This is true not only if the elastomer is modelled in the frame of the de Gennes theory, but also in the case of the uniaxial tensors by developing an *effective* biaxial microstructure.

For completeness we report here the expressions of the energy density in the three cases (see [5] for more details). We denote with $\mathbf{E}$ the symmetric part of $\mathbf{F}$, decomposed in spherical and deviatoric part $\mathbf{E} = \frac{1}{3}(\operatorname{tr}\mathbf{E})\mathbf{I} + \Delta\mathbf{E}$, where $\Delta\mathbf{E} := (\mathbf{E} - \frac{1}{3}(\operatorname{tr}\mathbf{E})\mathbf{I})$ and $\Delta e_i, i = 1, 2, 3$ are the eigenvalues of $\Delta\mathbf{E}$, which satisfy $\Delta e_1 \le \Delta e_2 \le \Delta e_3$ and $\Delta e_1 + \Delta e_2 + \Delta e_3 = 0$.

$$\frac{1}{\mu}\Big(f_{Fr}(\mathbf{F}) - \Big(\frac{\lambda}{2} + \frac{\mu}{3}\Big)(\operatorname{tr}\mathbf{F})^2\Big) = \frac{3}{2}\Big(\Delta e_1 + \frac{\gamma}{3}\Big)^2 + \frac{1}{2}\Big(\Delta e_1 + 2\Delta e_3 - \gamma\Big)^2$$
$$\text{on } -\frac{1}{2}\Delta e_1 \le \Delta e_3 \le -2\Delta e_1,$$

$$\frac{1}{\mu}\Big(f_U(\mathbf{F}) - \Big(\frac{\lambda}{2} + \frac{\mu}{3}\Big)(\operatorname{tr}\mathbf{F})^2\Big) = \begin{cases} \frac{1}{2}(\Delta e_3 + 2\Delta e_1)^2 & \text{on } R_1 \\ \frac{1}{2}(\Delta e_1 + 2\Delta e_3)^2 & \text{on } N_1 \\ \frac{3}{2}(\Delta e_1 + \frac{\gamma}{3})^2 + & \\ \frac{1}{2}(\Delta e_1 + 2\Delta e_3 - \gamma)^2 & \text{on } R_2, \\ \frac{3}{2}(\Delta e_3 - \frac{\gamma}{6})^2 + & \\ \frac{1}{2}(\Delta e_3 + 2\Delta e_1 + \frac{\gamma}{2})^2 & \text{on } N_2, \end{cases}$$

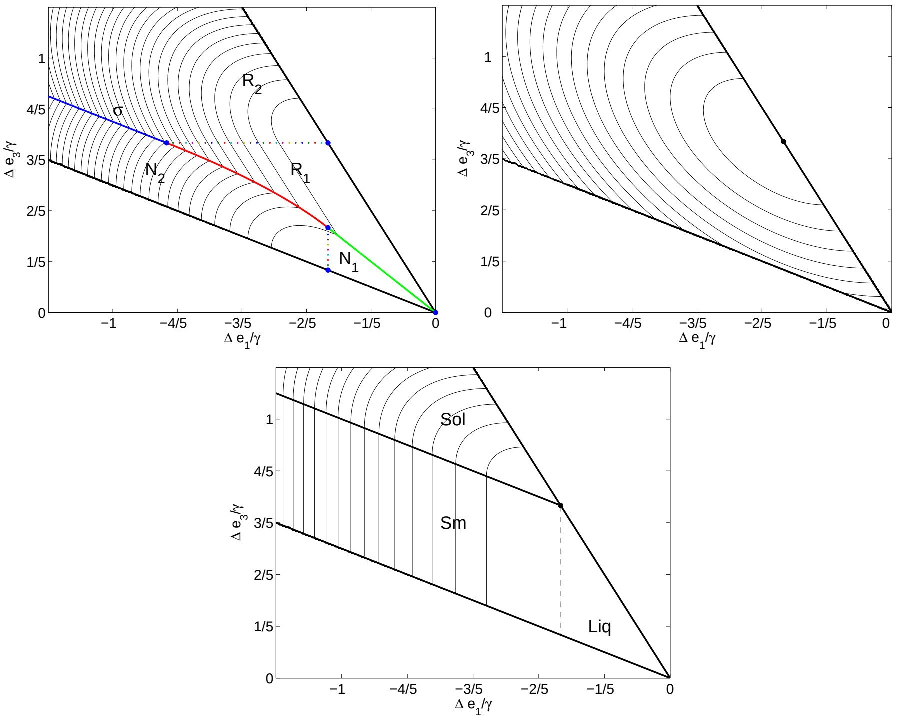

Figure 1: LEFT: Phase diagram and level curves, uniaxial model. RIGHT: level curves, Frank model. BELOW: Phase diagram and level curves, biaxial model.

$$\frac{1}{\mu}\Big(f_B(\mathbf{F}) - \Big(\frac{\lambda}{2}+\frac{\mu}{3}\Big)(\mathrm{tr}\,\mathbf{F})^2\Big) = \begin{cases} 0 & \text{on} \quad \mathrm{Liq} \\ \frac{3}{2}(\Delta e_1 + \frac{\gamma}{3})^2 & \text{on} \quad \mathrm{Sm} \\ \frac{3}{2}(\Delta e_1 + \frac{\gamma}{3})^2 + & \\ \frac{1}{2}(\Delta e_1 + 2\Delta e_3 - \gamma)^2 & \text{on} \quad \mathrm{Sol}, \end{cases}$$

where

$$\mathrm{R}_1 := \Big\{\Big(\frac{\Delta e_1}{\gamma}, \frac{\Delta e_3}{\gamma}\Big) : \sigma\Big(\frac{\Delta e_3}{\gamma}\Big) \le \frac{\Delta e_1}{\gamma} \le -\frac{\Delta e_3}{2\gamma}, \frac{\Delta e_3}{\gamma} < \frac{2}{3}\Big\}$$

$$\mathrm{N}_1 := \Big\{\Big(\frac{\Delta e_1}{\gamma}, \frac{\Delta e_3}{\gamma}\Big) : -\frac{2\Delta e_3}{\gamma} \le \frac{\Delta e_1}{\gamma} < \sigma\Big(\frac{\Delta e_3}{\gamma}\Big), \frac{\Delta e_1}{\gamma} > -\frac{1}{3}\Big\}$$

$$\mathrm{R}_2 := \Big\{\Big(\frac{\Delta e_1}{\gamma}, \frac{\Delta e_3}{\gamma}\Big) : \sigma\Big(\frac{\Delta e_3}{\gamma}\Big) \le \frac{\Delta e_1}{\gamma} \le -\frac{\Delta e_3}{2\gamma}, \frac{\Delta e_3}{\gamma} \ge \frac{2}{3}\Big\}$$

$$\mathrm{N}_2 := \Big\{\Big(\frac{\Delta e_1}{\gamma}, \frac{\Delta e_3}{\gamma}\Big) : -\frac{2\Delta e_3}{\gamma} \le \frac{\Delta e_1}{\gamma} < \sigma\Big(\frac{\Delta e_3}{\gamma}\Big), \frac{\Delta e_1}{\gamma} \le -\frac{1}{3}\Big\},$$

$$\sigma(t) := \begin{cases} -t & \text{for} \quad t \in [0, 1/3) \\ -1.5t^2 - 1/6 & \text{for} \quad t \in [1/3, 2/3) \\ -2t + 1/2 & \text{for} \quad t \ge 2/3, \end{cases}$$

$$\text{Liq} := \Big\{ \Big(\frac{\Delta e_1}{\gamma}, \frac{\Delta e_3}{\gamma} \Big) : -\frac{2\Delta e_3}{\gamma} \leq \frac{\Delta e_1}{\gamma} \leq -\frac{\Delta e_3}{2\gamma}, \frac{\Delta e_1}{\gamma} > -\frac{1}{3} \Big\}$$
$$\text{Sm} := \Big\{ \Big(\frac{\Delta e_1}{\gamma}, \frac{\Delta e_3}{\gamma} \Big) : -\frac{2\Delta e_3}{\gamma} \leq \frac{\Delta e_1}{\gamma} < -\frac{2\Delta e_3}{\gamma} + 1, \frac{\Delta e_1}{\gamma} \leq -\frac{1}{3} \Big\}$$
$$\text{Sol} := \Big\{ \Big(\frac{\Delta e_1}{\gamma}, \frac{\Delta e_3}{\gamma} \Big) : -\frac{2\Delta e_3}{\gamma} + 1 \leq \frac{\Delta e_1}{\gamma} \leq -\frac{\Delta e_3}{2\gamma}, \frac{\Delta e_1}{\gamma} \leq -\frac{1}{3} \Big\}.$$

Remark 7. Let ξ be any matrix in $\mathbb{M}^{3\times 3}$. Following [14], if $\pi^{\mathcal{Q}_B}(\mathcal{E}(\xi)/\gamma)$ belongs to $\mathcal{Q}_{Fr}$, we say that ξ belongs to the *solid* regime of the material. If $\pi^{\mathcal{Q}_B}(\mathcal{E}(\xi)/\gamma)$ belongs to $\mathcal{Q}_B \backslash \mathcal{Q}_{Fr}$, we say that ξ belongs to the *smectic* regime of the material if only one order of laminations is required to relax the energy, or to the *liquid* regime if two order of laminations are required (see Figure 1-below).

Remark 8. Another by-product of Theorem 2 is implicitly given by (3.36). Notice that this formula holds trivially when $X = B$ since $\mathcal{F}_B^h \equiv \overline{\mathcal{F}_B^h}$. This proves that the functional $\mathcal{F}_B$ of an incompressible material can be approximated in the sense of Gamma-convergence by a sequence of energies with increasing bulk moduli.

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