Rend. Istit. Mat. Univ. Trieste
Volume 41 (2009), 1–25 (electronic).

A note on non-homogeneous hyperbolic operators with low regularity coefficients

Ferruccio Colombini, Francesco Fanelli

Abstract. *In this paper we obtain an energy estimate for a complete strictly hyperbolic operator with second order coefficients satisfying a log-Zygmund-continuity condition with respect to t, uniformly with respect to x, and a log-Lipschitz-continuity condition with respect to x, uniformly with respect to t.*

Keywords: Hyperbolic equations, log-Zygmund regularity, loss of derivatives
MS Classification 2000: 35L15

1. Introduction

Let us consider the second order operator

$$P \,=\, \partial_t^2 \,-\, \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(t)\partial_{x_j}) \tag{1}$$

and suppose that P is strictly hyperbolic, i.e. there exist two positive constants $\lambda_0 \leq \Lambda_0$ such that

$$\lambda_0\,|\xi|^2 \,\leq\, \sum_{i,j=1}^{n} a_{ij}(t)\,\xi_i\xi_j \,\leq\, \Lambda_0\,|\xi|^2 \tag{2}$$

for all $t \in \mathbb{R}$ and all $\xi \in \mathbb{R}^n$.

It is well-known (see e.g. [5] and [8]) that, if the coefficients a_{ij} are Lipschitz-continuous, then the following energy estimate holds for the operator P: for all $s \in \mathbb{R}$, there exists $C_s > 0$ such that

$$\sup_{t\in[0,T]} \left(\|u(t,\cdot)\|_{H^{s+1}} \,+\, \|\partial_t u(t,\cdot)\|_{H^s}\right) \,\leq \tag{3}$$

$$\leq\, C_s\,\left(\|u(0,\cdot)\|_{H^{s+1}} \,+\, \|\partial_t u(0,\cdot)\|_{H^s} \,+\, \int_0^T \|Pu(t,\cdot)\|_{H^s}\,dt\right)$$

for every function $u \in \mathcal{C}^2([0,T];H^\infty(\mathbb{R}^n))$.
In particular, the previous energy estimate implies that the Cauchy problem for (1) is well-posed in the space H^∞, with no loss of derivatives.

On the contrary, if the coefficients a_{ij} are not Lipschitz-continuous, then (3) is no more true in general, as it is shown by an example given by Colombini, De Giorgi and Spagnolo in the paper [2]. Nevertheless, under suitable weaker regularity assumptions on the coefficients, one can recover the H^∞-well-posedness again, but this time from an energy estimate with loss of derivatives.

A first result of this type was obtained in the quoted paper [2]. The authors supposed that there was a constant $C>0$ such that, for all $\varepsilon\in]0,T]$,

$$\int_0^{T-\varepsilon}|a_{ij}(t+\varepsilon)-a_{ij}(t)|\,dt\;\leq\;C\,\varepsilon\,\log\left(1+\frac{1}{\varepsilon}\right)\,. \tag{4}$$

The Fourier trasform with respect to x of the equation, togheter with the new "approximate energy technique" (i.e. the approximation of the coefficients is different in different zones of the phase space), enabled them to obtain the following energy estimate: there exist strictly positive constants K (indipendent of s) and C_s such that

$$\sup_{t\in[0,T]}\left(\|u(t,\cdot)\|_{H^{s+1-K}}\;+\;\|\partial_t u(t,\cdot)\|_{H^{s-K}}\right)\;\leq \tag{5}$$

$$\leq\;C_s\left(\|u(0,\cdot)\|_{H^{s+1}}\;+\;\|\partial_t u(0,\cdot)\|_{H^s}\;+\;\int_0^T\|Pu(t,\cdot)\|_{H^s}\,dt\right)$$

for all $u\in\mathcal{C}^2([0,T];H^\infty(\mathbb{R}^n))$.

Considering again the case that the coefficients of P depend only on the time variable, in the recent paper [7] (see also [9]) Tarama has weakened the regularity hypotesis further, supposing a log-Zygmund type integral condition, i.e. that there exists a constant $C>0$ such that, for all $\varepsilon\in]0,T/2]$,

$$\int_\varepsilon^{T-\varepsilon}|a_{ij}(t+\varepsilon)+a_{ij}(t-\varepsilon)-2a_{ij}(t)|\,dt\;\leq\;C\,\varepsilon\,\log\left(1+\frac{1}{\varepsilon}\right)\,. \tag{6}$$

Nevertheless, he has been able to prove the well-posedness to the Cauchy problem for (1) in the space H^∞: the improvement with respect to [2] was obtained introducing a new type of approximate energy, which involves the second derivatives of the approximating coefficients.

Much more difficulties arise if the operator P has coefficients depending both on the time variable t and on the space variables x. This case was considered by Colombini and Lerner in the paper [4]. They supposed a pointwise log-Lipschitz regularity condition, i.e. that there exists $C>0$ such that, for all $\varepsilon\in]0,T]$,

$$\sup_{\substack{y,z\in[0,T]\times\mathbb{R}^n\\|z|=\varepsilon}}|a_{ij}(y+z)-a_{ij}(y)|\;\leq\;C\,\varepsilon\,\log\left(1+\frac{1}{\varepsilon}\right)\,. \tag{7}$$

Because the coefficients of the operator P depend also on the space variables, here the Littlewood-Paley dyadic decomposition with respect to x takes the place of the Fourier trasform, and it is, togheter with the approximate energy technique, the key tool to obtain the energy estimate: for all fixed $\theta \in]0, 1/4]$, there exist β, $C > 0$ and $T^* \in]0, T]$ such that

$$\sup_{t\in[0,T^*]} (\|u(t,\cdot)\|_{H^{-\theta+1-\beta t}} + \|\partial_t u(t,\cdot)\|_{H^{-\theta-\beta t}}) \leq \tag{8}$$
$$\leq C \left(\|u(0,\cdot)\|_{H^{-\theta+1}} + \|\partial_t u(0,\cdot)\|_{H^{-\theta}} + \int_0^T \|Pu(t,\cdot)\|_{H^{-\theta-\beta t}} \, dt \right)$$

for all $u \in \mathcal{C}^2([0,T^*]; H^\infty(\mathbb{R}^n))$.
In this case, the loss of derivatives gets worse with the increasing of time.

In a recent paper ([3]), Colombini and Del Santo considered the case of one space variable (i.e. $n = 1$) and studied again the case of the coefficient a depending both on t and x, but under a special regularity condition: they mixed condition (6) togheter with (7). In particular, they supposed a to be log-Zygmund-continuous with respect to t, uniformly with respect to x, and log-Lipschitz-continuous with respect to x, uniformly with respect to t. The dyadic decomposition technique and the Tarama's approximate energy enabled them to obtain an estimate similar to (8).

The reason why they focused on the case $n = 1$ is that the case of several space variables needs some different and new ideas in the definition of the microlocal energy: this point still remains as an open problem.

In the present note, we will consider the case of the non-homogeneous operator

$$Lu = \partial_t^2 u - \partial_x(a(t,x)\partial_x u) + b_0(t,x)\partial_t u + b_1(t,x)\partial_x u + c(t,x)u , \tag{9}$$

where the coefficient a satisfy the same regularity assumptions as in [3]. We will also suppose that b_0, $b_1 \in L^\infty(\mathbb{R}_t; \mathcal{C}^\omega(\mathbb{R}_x))$, $\omega > 0$, and c bounded on $\mathbb{R}_t \times \mathbb{R}_x$. We will apply the Littlewood-Paley decomposition and the Tarama's approximate energy again to obtain an energy estimate with a loss of derivatives that depends on t, as in (8).

One can find the estimate of the second order coefficient a in the paper [3], however, for reader's convenience, we will give here all the details.

2. Main result

Let $a : \mathbb{R}^2 \to \mathbb{R}$ be a function such that, for positive constants $\lambda_0 \leq \Lambda_0$ and C_0, one has, for all $(t,x) \in \mathbb{R}^2$ and all $\tau > 0$, $y > 0$,

$$\lambda_0 \leq a(t,x) \leq \Lambda_0 \tag{10}$$

$$\sup_{(t,x)} |a(t+\tau,x) + a(t-\tau,x) - 2a(t,x)| \leq C_0\, \tau \log\left(\tfrac{1}{\tau} + 1\right) \tag{11}$$

$$\sup_{(t,x)} |a(t,x+y) - a(t,x)| \leq C_0\, y \log\left(\tfrac{1}{y} + 1\right) \tag{12}$$

Moreover, let

$$b_0\,,\, b_1 \;\in\; L^\infty(\mathbb{R}_t; \mathcal{C}^\omega(\mathbb{R}_x))\,, \tag{13}$$

where $\omega > 0$, and

$$c \;\in\; L^\infty(\mathbb{R}_t \times \mathbb{R}_x)\,. \tag{14}$$

THEOREM 2.1. *Let us consider, on the whole space $\mathbb{R}^2$, the operator*

$$Lu \;=\; \partial_t^2 u \;-\; \partial_x(a(t,x)\partial_x u) \;+\; b_0(t,x)\partial_t u \;+\; b_1(t,x)\partial_x u \;+\; c(t,x)u\,, \tag{15}$$

where the coefficients a, b_0, b_1 and c satisfy the hypothesis (10)-(14).
Then, for all fixed

$$\theta \;\in\; \left]0, \min\left\{\frac{1}{2}, \frac{\omega}{1+\log 2}\right\}\right[\;,$$

there exist $\beta^* > 0$, a time $T \in \mathbb{R}$ and a constant $C > 0$ such that

$$\sup_{[0,T]} \left(\|u(t,\cdot)\|_{H^{1-\theta-\beta^* t}} + \|\partial_t u(t,\cdot)\|_{H^{-\theta-\beta^* t}}\right) \;\leq \tag{16}$$

$$\leq\; C\left(\|u(0,\cdot)\|_{H^{1-\theta}} + \|\partial_t u(0,\cdot)\|_{H^{-\theta}} + \int_0^T \|Lu(t,\cdot)\|_{H^{1-\theta-\beta^* t}} dt\right)$$

for all $u \in \mathcal{C}^2([0,T]; H^\infty(\mathbb{R}_x))$.

3. Proof of Theorem 2.1

3.1. Approximation of the coefficient $a(t,x)$

Let $\rho \in \mathcal{C}_0^\infty(\mathbb{R})$ be an even function such that:

1. $0 \leq \rho \leq 1$
2. $supp\,\rho \,\subset\, [-1,1]$
3. $\int \rho(s)ds = 1$
4. $|\rho'(s)| \leq 2$;

for all $0 < \varepsilon \leq 1$, we set $\rho_\varepsilon(s) = \frac{1}{\varepsilon}\,\rho\left(\frac{s}{\varepsilon}\right)$.

Then, for all $0 < \varepsilon \leq 1$, we define

$$a_\varepsilon(t,x) \;:=\; \int_{\mathbb{R}_t\times\mathbb{R}_x} \rho_\varepsilon(t-s)\,\rho_\varepsilon(x-y)\,a(s,y)\,dsdy\;. \tag{17}$$

LEMMA 3.1. *The following inequalities hold true:*

1. *for all* $\varepsilon\in]0,1]$, *for all* $(t,x)\in\mathbb{R}^2$, *one has*

$$\lambda_0 \;\leq\; a_\varepsilon(t,x) \;\leq\; \Lambda_0\;; \tag{18}$$

2. *for all* $\varepsilon\in]0,1]$, *one has*

$$\sup_{(t,x)}|a_\varepsilon(t,x)-a(t,x)| \;\leq\; \frac{3}{2}C_0\,\varepsilon\,\log\left(\frac{1}{\varepsilon}+1\right)\;; \tag{19}$$

3. *for all* $\sigma\in]0,1[$, *a constant* $c_\sigma>0$ *exists such that, for all* $\varepsilon\in]0,1]$,

$$\sup_{(t,x)}|\partial_t a_\varepsilon(t,x)| \;\leq\; c_\sigma(\Lambda_0+C_0)\,\varepsilon^{\sigma-1}\;; \tag{20}$$

4. *for all* $\varepsilon\in]0,1]$, *one has*

$$\sup_{(t,x)}|\partial_x a_\varepsilon(t,x)| \;\leq\; C_0\|\rho'\|_{L^1}\log\left(\frac{1}{\varepsilon}+1\right) \tag{21}$$

$$\sup_{(t,x)}\left|\partial_t^2 a_\varepsilon(t,x)\right| \;\leq\; \frac{C_0}{2}\|\rho''\|_{L^1}\frac{1}{\varepsilon}\log\left(\frac{1}{\varepsilon}+1\right) \tag{22}$$

$$\sup_{(t,x)}|\partial_t\partial_x a_\varepsilon(t,x)| \;\leq\; C_0\|\rho'\|^2_{L^1}\frac{1}{\varepsilon}\log\left(\frac{1}{\varepsilon}+1\right)\;. \tag{23}$$

Proof. Inequalities in (18) immediately follow from the fact that $|\rho|\leq 1$.

Relation (19), instead, follows from (11), after one has observed that

$$a_\varepsilon(t,x)-a(t,x) = \frac{1}{2}\int_{\mathbb{R}_t}\rho_\varepsilon(s)\int_{\mathbb{R}_y}\rho_\varepsilon(x-y)(a(t+s,y)+a(t-s,y)-2a(t,y))dyds\,,$$

where we have used the fact that ρ is an even function.

Moreover, one has

$$\partial_t^2 a_\varepsilon(t,x) = \frac{1}{2}\int\rho''_\varepsilon(s)\int\rho_\varepsilon(x-y)(a(t+s,y)+a(t-s,y)-2a(t,y))dyds\;,$$

from which one can deduce (22).

Inequalities (21) and (23) derive from (12) in a very similar way.

Finally, relation (20) is a consequence of the fact that (10) and (11) imply that for all $\sigma \in]0,1[$, a constant $c'_\sigma > 0$ exists such that, for all $\tau > 0$, one has

$$\sup_{(t,x)} |a(t+\tau,x) - a(t,x)| \;\leq\; c'_\sigma\,(\Lambda_0 + C_0)\,\tau^\sigma\;. \tag{24}$$

□

3.2. Littlewood-Paley decomposition

We collect here some well-known facts about dyadic decomposition, referring to [1], [4] and [6] for the details.

Let $\varphi_0 \in \mathcal{C}_0^\infty(\mathbb{R}_\xi)$ be an even function, decreasing on $[0,+\infty[$, such that $0 \leq \varphi_0 \leq 1$ and

$$\varphi_0(\xi) = 1 \;\text{ if }\; |\xi| \leq 1 \quad , \quad \varphi_0(\xi) = 0 \;\text{ if }\; |\xi| \geq 2.$$

We set $\varphi(\xi) = \varphi_0(\xi) - \varphi_0(2\xi)$ and, for $\nu \in \mathbb{N}\backslash\{0\}$, $\varphi_\nu(\xi) := \varphi(2^{-\nu}\xi)$.

For a tempered distribution $u \in H^{-\infty}(\mathbb{R})$, we define

$$u_\nu(x) \;:=\; \varphi_\nu(D_x)u(x) \;=\; \frac{1}{2\pi}\int e^{ix\xi}\varphi_\nu(\xi)\widehat{u}(\xi)d\xi = \frac{1}{2\pi}\int \widehat{\varphi_\nu}(y)u(x-y)dy\,;$$

for all ν, u_ν is an entire analytic function belonging to L^2.
Moreover, for all $s \in \mathbb{R}$ there exists a constant $C_s > 0$ such that

$$\frac{1}{C_s}\sum_{\nu=0}^{+\infty} 2^{2\nu s}\|u_\nu\|^2_{L^2} \;\leq\; \|u\|^2_{H^s} \;\leq\; C_s\sum_{\nu=0}^{+\infty} 2^{2\nu s}\|u_\nu\|^2_{L^2} \tag{25}$$

and the following inequalities (called "Bernstein's inequalities") hold:

$$\|\partial_x u_\nu\|_{L^2} \;\leq\; 2^{\nu+1}\,\|u_\nu\|_{L^2} \qquad \text{for all } \nu \geq 0 \tag{26}$$

$$\|u_\nu\|_{L^2} \;\leq\; 2^{-(\nu-1)}\,\|\partial_x u_\nu\|_{L^2} \qquad \text{for all } \nu \geq 1\,. \tag{27}$$

We end this subsection quoting a result which will be useful in the following; for its proof, see [4].
We denote with $[A,B]$ the commutator between two linear operators A and B and with $\mathcal{L}(L^2)$ the space of bounded linear operators from L^2 to L^2.

LEMMA 3.2. *1. There exist $C > 0$, $\nu_0 \in \mathbb{N}$ such that, for all $a \in L^\infty(\mathbb{R})$ satisfying*

$$\sup_{x\in\mathbb{R}} |a(x+y) - a(x)| \;\leq\; C_0\, y\, \log\left(1 + \frac{1}{y}\right)$$

for all $y > 0$, one has, for all $\nu \geq \nu_0$,

$$\|\,[\varphi_\nu(D_x), a(x)]\,\|_{\mathcal{L}(L^2)} \;\leq\; C\,(\|a\|_{L^\infty} + C_0)\,2^{-\nu}\,\nu\,. \tag{28}$$

2. There exist $C > 0$, $\nu_0 \in \mathbb{N}$ such that, for all $b \in \mathcal{C}^\omega(\mathbb{R})$ and all $\nu \geq \nu_0$, one has

$$\| [\varphi_\nu(D_x), b(x)] \|_{\mathcal{L}(L^2)} \leq C \|b\|_{\mathcal{C}^\omega} 2^{-\nu} . \tag{29}$$

3.3. Approximate and total energy

Let $T_0 > 0$ and $u \in \mathcal{C}^2([0, T_0]; H^\infty(\mathbb{R}_x))$. If we set $u_\nu(t,x) = \varphi_\nu(D_x)u(t,x)$, we obtain

$$\begin{aligned} (Lu)_\nu &= \partial_t^2 u_\nu - \partial_x(a(t,x)\partial_x u_\nu) - \partial_x([\varphi_\nu(D_x), a]\partial_x u) \\ &+ b_0(t,x)\partial_t u_\nu + [\varphi_\nu(D_x), b_0]\partial_t u + b_1(t,x)\partial_x u_\nu + [\varphi_\nu(D_x), b_1]\partial_x u \\ &+ c(t,x)u_\nu + [\varphi_\nu(D_x), c]u . \end{aligned} \tag{30}$$

Now, we introduce the approximate energy of u_ν (see [3] and [7]), setting

$$e_{\nu,\varepsilon}(t) := \int_{\mathbb{R}} \left(\frac{1}{\sqrt{a_\varepsilon}} \left| \partial_t u_\nu + \frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}} u_\nu \right|^2 + \sqrt{a_\varepsilon} |\partial_x u_\nu|^2 + |u_\nu|^2 \right) dx , \tag{31}$$

and, taken θ as in the hypothesis of Theorem 2.1, we define the total energy of u:

$$E(t) := \sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t} 2^{-2\nu\theta} e_{\nu, 2^{-\nu}}(t) , \tag{32}$$

where $\beta > 0$ will be fixed later on.

REMARK 3.3. *From (25) and Bernstein's inequalities (26)-(27), it's easy to see that there exist two positive constants C_θ and C'_θ such that*

$$\begin{aligned} E(0) &\leq C_\theta \left(\|\partial_t u(0,\cdot)\|_{H^{-\theta}} + \|u(0,\cdot)\|_{H^{1-\theta}} \right) \\ E(t) &\geq C'_\theta \left(\|\partial_t u(t,\cdot)\|_{H^{-\theta-\beta^* t}} + \|u(t,\cdot)\|_{H^{1-\theta-\beta^* t}} \right) \end{aligned}$$

where we have set $\beta^* = \beta(\log 2)^{-1}$.

First, we derive $e_{\nu,\varepsilon}$, defined by (31), with respect to the time variable, and we obtain

$$
\begin{aligned}
\frac{d}{dt}e_{\nu,\varepsilon}(t) \quad &= \quad \int \frac{2}{\sqrt{a_\varepsilon}}\, \Re\left(\partial_t^2 u_\nu \,\cdot\, \overline{\left(\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right)}\right) dx \\
&+ \quad \int \frac{2}{\sqrt{a_\varepsilon}}\, \Re\left(R_\varepsilon u_\nu \,\cdot\, \overline{\left(\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right)}\right) dx \\
&+ \quad \int \partial_t\sqrt{a_\varepsilon}\, |\partial_x u_\nu|^2\, dx \\
&+ \quad \int 2\sqrt{a_\varepsilon}\, \Re(\partial_x u_\nu \,\cdot\, \overline{\partial_x\partial_t u_\nu})\, dx \\
&+ \quad \int 2\, \Re(u_\nu \,\cdot\, \overline{\partial_t u_\nu})\, dx\ ,
\end{aligned}
$$

where $R_\varepsilon v \;=\; \partial_t\left(\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}\right)v - \left(\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}\right)^2 v$. Now, we can put in the previous relation the value of $\partial_t^2 u_\nu$, given by (30).

Integrating by parts and taking advantage of the spectral localisation of u_ν, we have

$$
\begin{aligned}
\int \frac{2}{\sqrt{a_\varepsilon}}\, \Re\left(\partial_x(a\partial_x u_\nu) \,\cdot\, \overline{\left(\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right)}\right) dx \;=& \\
= \int 2\frac{\partial_x\sqrt{a_\varepsilon}}{a_\varepsilon}\, a\, \Re\left(\partial_x u_\nu \,\cdot\, \overline{\left(\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right)}\right) dx& \\
- \int \frac{\partial_t\sqrt{a_\varepsilon}}{a_\varepsilon}\, a\, |\partial_x u_\nu|^2\, dx& \\
- \int 2\frac{a}{\sqrt{a_\varepsilon}}\, \Re\left(\partial_x u_\nu \,\cdot\, \overline{\partial_x\partial_t u_\nu}\right)\, dx& \\
- \int \frac{a}{\sqrt{a_\varepsilon}}\, \partial_x\left(\frac{\partial_t\sqrt{a_\varepsilon}}{\sqrt{a_\varepsilon}}\right)\, \Re(\partial_x u_\nu \,\cdot\, \overline{u_\nu})\, dx\,;&
\end{aligned}
$$

taken care of the fact that b_0, b_1 and c are real-valued, finally we obtain the complete expression for the time derivative of the approximate energy:

$$
\begin{aligned}
\frac{d}{dt}e_{\nu,\varepsilon}(t) \;=\; & \int \frac{2}{\sqrt{a_\varepsilon}}\,\Re\left((Lu)_\nu\,\cdot\,\overline{\left(\partial_t u_\nu+\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right)}\right)dx \qquad (33)\\
&+\int \frac{2}{\sqrt{a_\varepsilon}}\,\Re\left(R_\varepsilon u_\nu\,\cdot\,\overline{\left(\partial_t u_\nu+\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right)}\right)dx\\
&+\int \partial_t\sqrt{a_\varepsilon}\left(1-\frac{a}{a_\varepsilon}\right)|\partial_x u_\nu|^2\,dx\\
&+\int 2\left(\sqrt{a_\varepsilon}-\frac{a}{\sqrt{a_\varepsilon}}\right)\Re(\partial_x u_\nu\,\cdot\,\overline{\partial_x\partial_t u_\nu})\,dx\\
&+\int 2\frac{\partial_x\sqrt{a_\varepsilon}}{a_\varepsilon}\,a\,\Re\left(\partial_x u_\nu\,\cdot\,\overline{\left(\partial_t u_\nu+\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right)}\right)dx\\
&-\int \frac{a}{\sqrt{a_\varepsilon}}\,\partial_x\left(\frac{\partial_t\sqrt{a_\varepsilon}}{\sqrt{a_\varepsilon}}\right)\Re(\partial_x u_\nu\,\cdot\,\overline{u_\nu})\,dx\\
&+\int 2\,\Re(u_\nu\,\cdot\,\overline{\partial_t u_\nu})\,dx\\
&+\int \frac{2}{\sqrt{a_\varepsilon}}\,\Re\left((\partial_x([\varphi_\nu(D_x),a]\partial_x u))\,\cdot\,\overline{\left(\partial_t u_\nu+\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right)}\right)dx\\
&-\int \frac{2}{\sqrt{a_\varepsilon}}\,b_0(t,x)\,\Re\left(\partial_t u_\nu\,\cdot\,\overline{\left(\partial_t u_\nu+\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right)}\right)dx\\
&-\int \frac{2}{\sqrt{a_\varepsilon}}\,\Re\left(([\varphi_\nu(D_x),b_0]\partial_t u)\,\cdot\,\overline{\left(\partial_t u_\nu+\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right)}\right)dx\\
&-\int \frac{2}{\sqrt{a_\varepsilon}}\,b_1(t,x)\,\Re\left(\partial_x u_\nu\,\cdot\,\overline{\left(\partial_t u_\nu+\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right)}\right)dx\\
&-\int \frac{2}{\sqrt{a_\varepsilon}}\,\Re\left(([\varphi_\nu(D_x),b_1]\partial_x u)\,\cdot\,\overline{\left(\partial_t u_\nu+\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right)}\right)dx\\
&-\int \frac{2}{\sqrt{a_\varepsilon}}\,c(t,x)\,\Re\left(u_\nu\,\cdot\,\overline{\left(\partial_t u_\nu+\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right)}\right)dx\\
&-\int \frac{2}{\sqrt{a_\varepsilon}}\,\Re\left(([\varphi_\nu(D_x),c]u)\,\cdot\,\overline{\left(\partial_t u_\nu+\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right)}\right)dx\,.
\end{aligned}
$$

3.4. Estimate for the approximate energy

We want to obtain an energy estimate; so, let us start to control each term of (33).
Through the rest of the proof, we will denote with C, C', C'' and $\widehat{C}$ constants depending only on λ_0, Λ_0, C_0 and on the norms of the coefficients of the lower order terms of the operator L in their respective functional spaces, and which are allowed to vary from line to line.

3.4.1. Terms with a and a_ε

Thanks to relations (10), (20) with $\sigma = 1/2$, (22) and Bernstein's inequalities, we deduce that there exists $C > 0$, depending only on λ_0, Λ_0 and C_0, such that, for all $\nu \in \mathbb{N}$,

$$\left|\int \frac{2}{\sqrt{a_\varepsilon}}\, \Re\left(R_\varepsilon u_\nu \cdot \overline{\left(\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}} u_\nu\right)}\right) dx\right| \leq C\frac{1}{\varepsilon}\,\log\left(\frac{1}{\varepsilon}+1\right) 2^{-\nu}\, e_{\nu,\varepsilon}(t)\,.$$

In the same way, from (10), (19) and (20), we have

$$\left|\int \partial_t\sqrt{a_\varepsilon}\left(1-\frac{a}{a_\varepsilon}\right)|\partial_x u_\nu|^2\, dx\right| \leq C\,\log\left(\frac{1}{\varepsilon}+1\right)\, e_{\nu,\varepsilon}(t)\,,$$

for a constant C depending again only on λ_0, Λ_0 and C_0.

Moreover, again from (10) and (19) and Bernstein's inequalities, we obtain

$$\begin{aligned}\left|\int 2\left(\sqrt{a_\varepsilon}-\frac{a}{\sqrt{a_\varepsilon}}\right)\Re(\partial_x u_\nu\cdot\overline{\partial_x\partial_t u_\nu})\,dx\right| &\leq \\ &\leq C\,\varepsilon\,\log\left(\frac{1}{\varepsilon}+1\right)\,\|\partial_x u_\nu\|_{L^2}\,\|\partial_x\partial_t u_\nu\|_{L^2} \\ &\leq C\,\varepsilon\,\log\left(\frac{1}{\varepsilon}+1\right)\,2^{\nu+1}\,\|\partial_x u_\nu\|_{L^2}\,\|\partial_t u_\nu\|_{L^2}\,;\end{aligned}$$

but we have

$$\|\partial_t u_\nu\|_{L^2} \leq \left\|\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right\|_{L^2} + \left\|\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right\|_{L^2}$$

and

$$\begin{aligned}\left\|\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_0\right\|_{L^2} &\leq C\,\varepsilon^{-1/2}\|u_0\|_{L^2} \\ \left\|\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right\|_{L^2} &\leq C\,\varepsilon^{-1/2}\,2^{-\nu}\|u_\nu\|_{L^2} \qquad (\nu\geq 1)\,,\end{aligned}$$

that give us the following:

$$\left|\int 2\left(\sqrt{a_\varepsilon}-\frac{a}{\sqrt{a_\varepsilon}}\right)\Re(\partial_x u_\nu\cdot\overline{\partial_x\partial_t u_\nu})dx\right|\leq C(\varepsilon\,2^\nu+1)\log\left(\frac{1}{\varepsilon}+1\right)e_{\nu,\varepsilon}(t)\,.$$

In a very similar way, from (21) one has

$$\left|\int 2\frac{\partial_x\sqrt{a_\varepsilon}}{a_\varepsilon}a\;\Re\left(\partial_x u_\nu\cdot\overline{\left(\partial_t u_\nu+\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right)}\right)dx\right|\leq C\log\left(\frac{1}{\varepsilon}+1\right)e_{\nu,\varepsilon}(t)\,;$$

moreover, from (20) with $\sigma=1/2$, (21) and (23) we deduce

$$\left|\int\frac{a}{\sqrt{a_\varepsilon}}\,\partial_x\left(\frac{\partial_t\sqrt{a_\varepsilon}}{\sqrt{a_\varepsilon}}\right)\,\Re(\partial_x u_\nu\,\cdot\,\overline{u_\nu})\,dx\right|\leq C\,\frac{1}{\varepsilon}\,\log\left(\frac{1}{\varepsilon}+1\right)\,2^{-\nu}\,e_{\nu,\varepsilon}(t)\;.$$

Finally, we have

$$\left|\int 2\,\Re(u_\nu\,\cdot\,\overline{\partial_t u_\nu})\,dx\right|\leq C\,\varepsilon^{-1/2}\,2^{-\nu}\,e_{\nu,\varepsilon}(t)\;.$$

3.4.2. Terms with b_0, b_1 and c

Thanks to the hypothesis (13)-(14), one has that there exist suitable constants, depending only on λ_0, Λ_0, C_0 and on the norms of b_0 and b_1 in the space

$L^\infty(\mathbb{R}_t;\mathcal{C}^\omega(\mathbb{R}_x))$ and of c in $L^\infty(\mathbb{R}_t\times\mathbb{R}_x)$, such that

$$
\begin{aligned}
\left|\int \frac{2}{\sqrt{a_\varepsilon}}\, b_0(t,x)\, \Re\left(\partial_t u_\nu \cdot \overline{\left(\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right)}\right)\, dx\right| \leq& \\
\leq\ C\, \|\partial_t u_\nu\|_{L^2}\, \left\|\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right\|_{L^2}& \\
\leq\ (C + C'\, 2^{-\nu}\varepsilon^{-1/2})\, e_{\nu,\varepsilon}(t)\ ;& \\
\left|\int \frac{2}{\sqrt{a_\varepsilon}}\, b_1(t,x)\, \Re\left(\partial_x u_\nu \cdot \overline{\left(\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right)}\right)\, dx\right| \leq& \\
\leq\ C\, \int \sqrt[4]{a_\varepsilon}|\partial_x u_\nu|\, \frac{1}{\sqrt[4]{a_\varepsilon}}\left|\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right|\, dx& \\
\leq\ 2C\, \int \sqrt{a_\varepsilon}|\partial_x u_\nu|^2\ +\ \frac{1}{\sqrt{a_\varepsilon}}\left|\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right|^2\, dx& \\
\leq\ 2\, e_{\nu,\varepsilon}(t)\ ;& \\
\left|\int \frac{2}{\sqrt{a_\varepsilon}}\, c(t,x)\, \Re\left(u_\nu \cdot \overline{\left(\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right)}\right)\, dx\right| \leq& \\
\leq\ C\, \int |u_\nu|\, \frac{1}{\sqrt[4]{a_\varepsilon}}\left|\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right|\, dx& \\
\leq\ 2C\, e_{\nu,\varepsilon}(t)\ ,&
\end{aligned}
$$

where we have delt with $\|\partial_t u_\nu\|_{L^2}$ as before.

Now, we join the approximation parameter ε with the dual variable ξ, setting

$$\varepsilon\ =\ 2^{-\nu}\ ;$$

so, from (33) and the previous inequalities, we obtain

$$
\begin{aligned}
\frac{d}{dt} e_{\nu,2^{-\nu}}(t) \;\leq\; & \widetilde{C}\,(\nu+1)\, e_{\nu,2^{-\nu}} \qquad (34)\\
& + \int \frac{2}{\sqrt{a_{2^{-\nu}}}}\, \Re\left((Lu)_\nu \cdot \overline{\left(\partial_t u_\nu + \frac{\partial_t \sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu\right)} \right) dx \\
& + \int \frac{2}{\sqrt{a_{2^{-\nu}}}}\, \Re\left((\partial_x([\varphi_\nu(D_x), a]\partial_x u)) \cdot \overline{\left(\partial_t u_\nu + \frac{\partial_t \sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu\right)} \right) dx \\
& - \int \frac{2}{\sqrt{a_{2^{-\nu}}}}\, \Re\left(([\varphi_\nu(D_x), b_0]\partial_t u) \cdot \overline{\left(\partial_t u_\nu + \frac{\partial_t \sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu\right)} \right) dx \\
& - \int \frac{2}{\sqrt{a_{2^{-\nu}}}}\, \Re\left(([\varphi_\nu(D_x), b_1]\partial_x u) \cdot \overline{\left(\partial_t u_\nu + \frac{\partial_t \sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu\right)} \right) dx \\
& - \int \frac{2}{\sqrt{a_{2^{-\nu}}}}\, \Re\left(([\varphi_\nu(D_x), c]u) \cdot \overline{\left(\partial_t u_\nu + \frac{\partial_t \sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu\right)} \right) dx\,,
\end{aligned}
$$

for a suitable constant $\widetilde{C}$, which depends only on λ_0, Λ_0, C_0 and on the norms of the coefficients of the operator L in their respective functional spaces.

3.5. Estimates for the commutator terms

Now, we have to deal with the commutator terms. As we will see, it's useful to consider immediately the sum over $\nu \in \mathbb{N}$.

First, we report an elementary lemma (see also [3]), which we will use very often in the next calculations.

LEMMA 3.4. *There exist two continuous, decreasing functions* α_1, $\alpha_2 :]0,1[\to]0,+\infty[$ *such that* $\lim_{c\to 0^+} \alpha_j(c) = +\infty$ *for* $j = 1\,,2$ *and such that, for all* $\delta \in]0,1]$ *and all* $n \geq 1$, *the following inequalities hold:*

$$
\sum_{j=1}^{n} e^{\delta j}\, j^{-1/2} \;\leq\; \alpha_1(\delta)\, e^{\delta n}\, n^{-1/2}\,, \qquad\qquad \sum_{j=n}^{+\infty} e^{-\delta j}\, j^{1/2} \;\leq\; \alpha_2(\delta)\, e^{-\delta n}\, n^{1/2}\,.
$$

Before going on, we take $\beta > 0$ and $T \in]0, T_0]$ such that

$$
\beta T \;=\; \frac{\theta}{2}\, \log 2\,.
$$

REMARK 3.5. *Notice that, thanks to the hypothesis of Theorem 2.1, this condition implies that*

$$
\beta T \;\leq\; \frac{\omega - \theta}{2}\,.
$$

Moreover, for all $t \in [0,T]$, we have:

$$\begin{aligned}
\beta t + \frac{\theta}{2}\log 2 &\geq \frac{\theta}{2}\log 2 \; > 0 \\
\beta t + \frac{\theta}{2}\log 2 &\leq \theta \log 2 \; \leq \frac{1}{2}\log 2 \; < 1 \\
(1-\theta)\log 2 - \beta t &\geq \left(1 - \frac{3}{2}\theta\right)\log 2 \; \geq \left(1 - \frac{3}{4}\right)\log 2 \; > 0 \\
(1-\theta)\log 2 - \beta t &\leq (1-\theta)\log 2 \; \leq \log 2 \; < 1 \, .
\end{aligned}$$

Finally, we set (with the same notations used in the subsection 3.2)

$$\psi_\mu \;=\; \varphi_{\mu-1} + \varphi_\mu + \varphi_{\mu+1} \qquad (\varphi_{-1} \equiv 0) \, .$$

As $\psi_\mu \equiv 1$ on the support of φ_μ, we can write

$$\partial_x u_\mu \;=\; \varphi_\mu(D_x)\partial_x u \;=\; \Psi_\mu(\varphi_\mu(D_x)\partial_x u) \;=\; \Psi_\mu \partial_x u_\mu \; ,$$

where Ψ_μ is the operator related to ψ_μ. So, given a generic function $f(t,x)$, one has

$$\begin{aligned}
[\varphi_\nu(D_x), f]\partial_x u &= [\varphi_\nu(D_x), f]\left(\sum_{\mu \geq 0} \partial_x u_\mu\right) \qquad (35) \\
&= \sum_{\mu \geq 0} ([\varphi_\nu(D_x), f]\Psi_\mu)\partial_x u_\mu \, .
\end{aligned}$$

After these preliminary remarks, we now can go on with commutators' estimates.

3.5.1. Term with $[\varphi_\nu(D_x), a]$

Due to Bernstein's inequalities, we have

$$\left\| \partial_x \left(\partial_t u_\nu + \frac{\partial_t \sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu \right) \right\|_{L^2} \;\leq\; C\, 2^\nu \, \left(e_{\nu,2^{-\nu}}(t)\right)^{1/2} \, .$$

So, using (35) and the fact that a_ε is real-valued, one has

$$\begin{aligned}
&\left| \int \frac{2}{\sqrt{a_{2^{-\nu}}}} \, \Re \left(\partial_x([\varphi_\nu(D_x), a]\partial_x u) \cdot \overline{\left(\partial_t u_\nu + \frac{\partial_t \sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu \right)} \right) dx \right| \leq \\
&\qquad \leq\; C \sum_\mu \|([\varphi_\nu(D_x), a]\Psi_\mu)\partial_x u_\mu\|_{L^2} \, 2^\nu \, \left(e_{\nu,2^{-\nu}}(t)\right)^{1/2} \\
&\qquad \leq\; C \sum_\mu \|[\varphi_\nu(D_x), a]\Psi_\mu\|_{\mathcal{L}(L^2)} \, \left(e_{\mu,2^{-\mu}}(t)\right)^{1/2} \, 2^\nu \, \left(e_{\nu,2^{-\nu}}(t)\right)^{1/2} \; ,
\end{aligned}$$

with the constant C which depends only on λ_0, Λ_0 and C_0.
Then,

$$\left|\sum_{\nu\geq 0} e^{-2\beta(\nu+1)t}2^{-2\nu\theta}\int \frac{2}{\sqrt{a_{2^{-\nu}}}}\Re\left(\partial_x([\varphi_\nu(D_x),a]\partial_x u)\overline{\left(\partial_t u_\nu+\frac{\partial_t\sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}}u_\nu\right)}\right)dx\right|$$
$$\leq C\sum_{\nu,\mu} k_{\nu\mu}(\nu+1)^{1/2}e^{-\beta(\nu+1)t}2^{-\nu\theta}\left(e_{\nu,2^{-\nu}}\right)^{1/2}(\mu+1)^{1/2}e^{-\beta(\mu+1)t}2^{-\mu\theta}\left(e_{\mu,2^{-\mu}}\right)^{1/2}$$

where we have set

$$k_{\nu\mu}=e^{-(\nu-\mu)\beta t}2^{-(\nu-\mu)\theta}2^{\nu}(\nu+1)^{-1/2}(\mu+1)^{-1/2}\|[\varphi_\nu(D_x),a]\Psi_\mu\|_{\mathcal{L}(L^2)}\,. \quad (36)$$

Observe that, if $|\nu-\mu|\geq 3$, then $\varphi_\nu\psi_\mu\equiv 0$, so $[\varphi_\nu(D_x),a]\Psi_\mu=\varphi_\nu(D_x)[a,\Psi_\mu]$.
Therefore, from lemma 3.2, in particular from (28), we deduce that

$$\|[\varphi_\nu(D_x),a(t,x)]\Psi_\mu\|_{\mathcal{L}(L^2)}\leq\begin{cases} C2^{-\nu}(\nu+1) & \text{if } |\nu-\mu|\leq 2\,,\\ C2^{-max\{\nu,\mu\}}(\max\{\nu,\mu\}+1) & \text{if } |\nu-\mu|\geq 3\,,\end{cases}$$

where the constant C depends only on Λ_0 and C_0.

Now our aim is to apply Schur's lemma, so to estimate the quantity

$$\sup_\mu\sum_\nu|k_{\nu\mu}|\,+\,\sup_\nu\sum_\mu|k_{\nu\mu}|\,. \quad (37)$$

To do this, we will use lemma 3.4 and the inequalities stated in remark (3.5).

1. Fix $\mu\leq 2$.

$$\begin{aligned}\sum_{\nu\geq 0}|k_{\nu\mu}| &\leq Ce^{(\mu+1)\beta t}\,2^{(\mu+1)\theta}(\mu+1)^{-1/2}\sum_\nu e^{-(\nu+1)\beta t}2^{-(\nu+1)\theta}(\nu+1)^{1/2}\\ &= Ce^{(\mu+1)\beta t}2^{(\mu+1)\theta}(\mu+1)^{-1/2}\sum_\nu e^{-(\nu+1)(\beta t+\theta\log 2)}(\nu+1)^{1/2}\\ &\leq C\,e^{3\beta t}\,2^{3\theta}\,\alpha_2(\beta t+\theta\log 2)\\ &\leq C\,2^{\frac{9}{2}\theta}\,\alpha_2(\theta\log 2)\,.\end{aligned}$$

2. Now, take $\mu\geq 3$ and first consider

$$
\begin{aligned}
&\sum_{\nu=0}^{\mu-3}|k_{\nu\mu}|\,\leq \\
&\leq Ce^{(\mu+1)\beta t}2^{-(\mu+1)(1-\theta)}(\mu+1)^{1/2}\sum_{\nu=0}^{\mu-3}e^{-(\nu+1)\beta t}2^{(\nu+1)(1-\theta)}(\nu+1)^{-1/2} \\
&\leq Ce^{(\mu+1)\beta t}2^{-(\mu+1)(1-\theta)}(\mu+1)^{1/2}\sum_{\nu=0}^{\mu-3}e^{(\nu+1)(-\beta t+(1-\theta)\log 2)}(\nu+1)^{-1/2} \\
&\leq C\,e^{(\mu+1)\beta t}\,2^{-(\mu+1)(1-\theta)}\,(\mu+1)^{1/2}\,\alpha_1(-\beta t+(1-\theta)\log 2)\,\cdot \\
&\qquad\qquad\cdot\,e^{(-\beta t+(1-\theta)\log 2)(\mu-2)}\,(\mu-2)^{-1/2} \\
&\leq C\,2^{\frac{9}{2}\theta}\,\alpha_1\left(\left(1-\frac{3}{2}\theta\right)\log 2\right)\,.
\end{aligned}
$$

For the second part of the sum, one has

$$
\begin{aligned}
&\sum_{\nu=\mu-2}^{+\infty}|k_{\nu\mu}|\,\leq \\
&\leq Ce^{(\mu+1)\beta t}2^{(\mu+1)\theta}(\mu+1)^{-1/2}\sum_{\nu=\mu-2}^{+\infty}e^{-(\nu+1)\beta t}2^{-(\nu+1)\theta}(\nu+1)^{1/2} \\
&\leq Ce^{(\mu+1)\beta t}2^{(\mu+1)\theta}(\mu+1)^{-1/2}\alpha_2(\beta t+\theta\log 2)\,\cdot \\
&\qquad\qquad\cdot\,e^{-(\beta t+\theta\log 2)(\mu-1)}(\mu-1)^{1/2} \\
&\leq C\,2^{\frac{7}{2}\theta}\,\alpha_2(\theta\log 2)\,.
\end{aligned}
$$

3. Fix now $\nu\geq 0$; we have

$$
\begin{aligned}
&\sum_{\mu=0}^{\nu+2}|k_{\nu\mu}|\,\leq \\
&\leq Ce^{-(\nu+1)\beta t}2^{-(\nu+1)\theta}(\nu+1)^{1/2}\sum_{\mu=0}^{\nu+2}e^{(\mu+1)\beta t}2^{(\mu+1)\theta}(\mu+1)^{-1/2} \\
&\leq Ce^{-(\nu+1)\beta t}2^{-(\nu+1)\theta}(\nu+1)^{1/2}\alpha_1(\beta t+\theta\log 2)\,\cdot \\
&\qquad\qquad\cdot\,e^{(\beta t+\theta\log 2)(\nu+3)}(\nu+3)^{-1/2} \\
&\leq C\,2^{\frac{7}{2}\theta}\,\alpha_1(\theta\log 2)\,.
\end{aligned}
$$

For the second part of the series, the following inequality holds:

$$\begin{aligned}
&\sum_{\mu=\nu+3}^{+\infty} |k_{\nu\mu}| \leq \\
&\leq C e^{-(\nu+1)\beta t} 2^{(\nu+1)(1-\theta)} (\nu+1)^{-\frac{1}{2}} \sum_{\mu=\nu+3}^{+\infty} e^{(\mu+1)\beta t} 2^{-(\mu+1)(1-\theta)} (\mu+1)^{\frac{1}{2}} \\
&\leq C e^{-(\nu+1)\beta t} 2^{(\nu+1)(1-\theta)} (\nu+1)^{-\frac{1}{2}} \alpha_2(-\beta t + (1-\theta)\log 2) \cdot \\
&\qquad\qquad \cdot e^{(-\beta t+(1-\theta)\log 2)(\nu+4)} (\nu+4)^{\frac{1}{2}} \\
&\leq C 2^{\frac{9}{2}\theta}\, \alpha_2\left(\left(1-\frac{3}{2}\theta\right)\log 2\right) .
\end{aligned}$$

In conclusion, there exists a positive function Π, with $\lim_{\theta\to 0^+} \Pi(\theta) = +\infty$, such that

$$\sup_\mu \sum_\nu |k_{\nu\mu}| + \sup_\nu \sum_\mu |k_{\nu\mu}| \leq C\,\Pi(\theta) ,$$

and so

$$\begin{aligned}
&\left| \sum_{\nu\geq 0} e^{-2\beta(\nu+1)t} 2^{-2\nu\theta} \int \frac{2}{\sqrt{a_{2^{-\nu}}}} \Re\left(\partial_x([\varphi_\nu(D_x), a]\partial_x u) \overline{\left(\partial_t u_\nu + \frac{\partial_t \sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu\right)}\right) dx \right| \\
&\qquad\qquad \leq C\,\Pi(\theta) \sum_{\nu=0}^{+\infty} (\nu+1) e^{-2\beta(\nu+1)t} 2^{-2\nu\theta} e_{\nu,2^{-\nu}}(t) .
\end{aligned}$$

3.5.2. Terms with $[\varphi_\nu(D_x), b_0]$ and $[\varphi_\nu(D_x), b_1]$

Now, let us consider

$$\begin{aligned}
&\left| \int \frac{2}{\sqrt{a_{2^{-\nu}}}} \Re\left([\varphi_\nu(D_x), b_0(t,x)]\partial_t u \overline{\left(\partial_t u_\nu + \frac{\partial_t \sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu\right)}\right) dx \right| \leq \\
&\qquad \leq 2 \left\| [\varphi_\nu(D_x), b_0(t,x)]\partial_t u \right\|_{L^2} \left\| \frac{1}{\sqrt{a_{2^{-\nu}}}} \left| \partial_t u_\nu + \frac{\partial_t \sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu \right| \right\|_{L^2} \\
&\qquad \leq 2 \left\| [\varphi_\nu(D_x), b_0(t,x)]\partial_t u \right\|_{L^2} \left(e_{\nu,2^{-\nu}}(t)\right)^{1/2} .
\end{aligned}$$

Thanks to relation (35), we have

$$\begin{aligned}
\left\| [\varphi_\nu(D_x), b_0(t,x)]\partial_t u \right\|_{L^2} &= \left\| [\varphi_\nu(D_x), b_0(t,x)] \sum_{\mu\geq 0} \Psi_\mu \partial_t u_\mu \right\|_{L^2} \\
&\leq \sum_{\mu\geq 0} \left\| [\varphi_\nu(D_x), b_0(t,x)]\Psi_\mu \right\|_{\mathcal{L}(L^2)} \left\| \partial_t u_\mu \right\|_{L^2} .
\end{aligned}$$

As we have done before, we have, for constants depending only on λ_0, Λ_0 and C_0,

$$\begin{aligned}\|\partial_t u_\mu\|_{L^2} &\leq \left(C\, e_{\mu,2^{-\mu}}(t) + C' \left\|\frac{\partial_t\sqrt{a_{2^{-\mu}}}}{2\sqrt{a_{2^{-\mu}}}}u_\mu\right\|_{L^2}\right)^{1/2} \\ &\leq \left(Ce_{\mu,2^{-\mu}}(t) \,+\, C'\frac{1}{2^{-\mu}}2^{-2\mu}\|\partial_x u_\mu\|_{L^2}\right)^{1/2} \\ &\leq (C + C'\,2^{-\mu})^{1/2}\left(e_{\mu,2^{-\mu}}\right)^{1/2} \\ &\leq C\,\left(e_{\mu,2^{-\mu}}\right)^{1/2}\,.\end{aligned}$$

Due to lemma 3.2, we obtain

$$\|[\varphi_\nu(D_x), b_0(t,x)]\Psi_\mu\|_{\mathcal{L}(L^2)} \leq \begin{cases} C\,2^{-\nu\omega} & \text{if } |\nu-\mu|\leq 2 \\ \\ C\,2^{-max\{\mu,\nu\}\omega} & \text{if } |\nu-\mu|\geq 3\end{cases}$$

where C is a constant depending only on $\|b_0\|_{L^\infty(\mathbb{R}_t;\,\mathcal{C}^\omega(\mathbb{R}_x))}$.
As a matter of fact, the kernel of operator $[\varphi_\nu(D_x), b_0]$ is

$$n(x,y) \,=\, \widehat{\varphi}(2^\nu(x-y))\,2^\nu\,(b_0(t,x)-b_0(t,y))\,;$$

so, to evaluate its norm we apply Schur's lemma and, thanks to the fact that b_0 is ω-hölder, we get the desired estimate.

Therefore,

$$\begin{aligned}&\left|\sum_{\nu\geq 0} e^{-2\beta t(\nu+1)}2^{-2\nu\theta}\int\frac{2}{\sqrt{a_{2^{-\nu}}}}\Re\left([\Delta_\nu, b_0]\partial_t u\overline{\left(\partial_t u_\nu + \frac{\partial_t\sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}}u_\nu\right)}\right)dx\right| \leq \\ &\qquad\leq \sum_{\nu,\mu\geq 0} e^{-\beta t(\nu+1)}\,2^{-\nu\theta}\,\left(e_{\nu,2^{-\nu}}\right)^{1/2}\,e^{-\beta t(\mu+1)}\,2^{-\mu\theta}\,\left(e_{\mu,2^{-\mu}}\right)^{1/2}\,l_{\nu\mu}\,,\end{aligned}$$

where we have defined

$$l_{\nu\mu} \,:=\, e^{-(\nu-\mu)\beta t}\,2^{-(\nu-\mu)\theta}\,\|[\varphi_\nu(D_x), b_0(t,x)]\Psi_\mu\|_{\mathcal{L}(L^2)}\,. \tag{38}$$

As made before, we are going to estimate $l_{\nu\mu}$ applying Schur's lemma.

1. Let us fix $\mu\leq 2$. Then

$$\begin{aligned}\sum_{\nu\geq 0}|l_{\nu\mu}| &\leq C\,e^{(\mu+1)\beta t}\,2^{(\mu+1)\theta}\sum_{\nu\geq 0}e^{-(\nu+1)\beta t}\,2^{-(\nu+1)\theta}\,2^{-\nu\omega} \\ &\leq C\,e^{3\beta t}\,2^{3\theta}\sum_{\nu\geq 0}e^{-(\nu+1)(\beta t+\theta\log 2)}\,(\nu+1)^{1/2} \\ &\leq C\,e^{3\beta t}\,2^{3\theta}\,\alpha_2(\beta t+\theta\log 2) \\ &\leq C\,2^{\frac{9}{2}\theta}\,\alpha_2(\theta\log 2)\,.\end{aligned}$$

2. Now, take $\mu \geq 3$ and consider first

$$\begin{aligned}\sum_{\nu=0}^{\mu-3} |l_{\nu\mu}| &\leq C\, e^{\mu\beta t}\, 2^{\mu\theta}\, 2^{-\mu\omega} \sum_{\nu=0}^{\mu-3} e^{-\nu\beta t}\, 2^{-\nu\theta} \\ &\leq C\, e^{\mu(\beta t-(\omega-\theta)\log 2)}\, (\mu-2) \\ &\leq C\, e^{-\mu(\omega-\frac{\theta}{2})\log 2}\, (\mu-2) \\ &\leq C\, M(\omega,\theta)\,,\end{aligned}$$

where $M(\omega,\theta)$ is the maximum of the function $z \mapsto e^{-\gamma z}\,(z-2)$, with $\gamma = \left(\omega - \frac{\theta}{2}\right)\log 2$.
For the second part of the sum, we have

$$\begin{aligned}\sum_{\nu=\mu-2}^{+\infty} |l_{\nu\mu}| &\leq Ce^{(\mu+1)\beta t} 2^{(\mu+1)\theta} \sum_{\nu=\mu-2}^{+\infty} e^{-(\nu+1)\beta t} 2^{-(\nu+1)\theta}\, 2^{-\nu\omega} \frac{(\nu+1)^{1/2}}{(\nu+1)^{1/2}} \\ &\leq C\, e^{(\mu+1)\beta t}\, 2^{(\mu+1)\theta}\, (\mu-1)^{-1/2}\, \alpha_2(\beta t+\theta\log 2)\,\cdot \\ &\qquad\qquad \cdot\, e^{-(\mu-1)\beta t}\, 2^{-(\mu-1)\theta}\, (\mu-1)^{1/2} \\ &\leq C\, e^{2(\beta t+\theta\log 2)}\, \alpha_2(\theta\log 2) \\ &\leq C\, 2^{\frac{9}{2}\theta}\, \alpha_2(\theta\log 2)\,.\end{aligned}$$

3. Fix now ν. Initially, we have

$$\begin{aligned}\sum_{\mu=0}^{\nu+2} |l_{\nu\mu}| &\leq C\, e^{-(\nu+1)\beta t}\, 2^{-(\nu+1)\theta}\, 2^{-\nu\omega} \sum_{\mu=0}^{\nu+2} e^{(\mu+1)(\beta t+\theta\log 2)}\, \frac{(\mu+1)^{1/2}}{(\mu+1)^{1/2}} \\ &\leq C\, e^{-(\nu+1)\beta t}\, 2^{-(\nu+1)\theta}\, 2^{-\nu\omega}\, (\nu+3)^{1/2}\, \alpha_1(\beta t+\theta\log 2)\,\cdot \\ &\qquad\qquad \cdot\, e^{(\nu+3)\beta t}\, 2^{(\nu+3)\theta}\, (\nu+3)^{-1/2} \\ &\leq C\, e^{2(\beta t+\theta\log 2)}\, \alpha_1(\theta\log 2) \\ &\leq C\, 2^{3\theta}\, \alpha_1(\theta\log 2)\,.\end{aligned}$$

Moreover,

$$\begin{aligned}\sum_{\mu=\nu+3}^{+\infty} |l_{\nu\mu}| &\leq C\, e^{-\nu\beta t}\, 2^{-\nu\theta} \sum_{\mu=\nu+3}^{+\infty} e^{-\mu(-\beta t+(\omega-\theta)\log 2)}\, \frac{\mu^{1/2}}{\mu^{1/2}} \\ &\leq C\, e^{-\nu\beta t}\, 2^{-\nu\theta}\, (\nu+3)^{-1/2}\, \alpha_2((\omega-\theta)\log 2-\beta t)\,\cdot \\ &\qquad\qquad \cdot\, e^{(\nu+3)\beta t}\, 2^{-(\nu+3)(\omega-\theta)}\, (\nu+3)^{1/2} \\ &\leq C\, e^{3\beta t}\, 2^{3\theta}\, 2^{-(\nu+3)\omega}\, \alpha_2\left(\left(\omega-\frac{3}{2}\theta\right)\log 2\right) \\ &\leq C\, 2^{\frac{9}{2}\theta}\, \alpha_2\left(\left(\omega-\frac{3}{2}\theta\right)\log 2\right)\,.\end{aligned}$$

From all these inequalities, thanks to Schur's lemma, one has that there exists a constant $\widetilde{M}(\omega,\theta)$, depending only on $\|b_0\|_{L^\infty(\mathbb{R}_t;\mathcal{C}^\omega(\mathbb{R}_x))}$ and on the parameter θ (which we have fixed at the beginning of the calculations), such that

$$\sup_\mu \sum_\nu |l_{\nu\mu}| + \sup_\nu \sum_\mu |l_{\nu\mu}| \leq C\,\widetilde{M}(\omega,\theta)\;;$$

from this relation, finally we get

$$\left|\sum_{\nu\geq 0} e^{-2\beta t(\nu+1)} 2^{-2\nu\theta} \int \frac{2}{\sqrt{a_{2^{-\nu}}}} \Re\left([\varphi_\nu(D_x),b_0]\partial_t u \overline{\left(\partial_t u_\nu + \frac{\partial_t\sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu\right)}\right) dx\right| \leq$$
$$\leq C\,\widetilde{M}(\omega,\theta) \sum_{\nu=0}^{+\infty} (\nu+1)\, e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, e_{\nu,2^{-\nu}}(t)\;.$$

With regard to the term with the commutator $[\varphi_\nu(D_x), b_1(t,x)]$, notice that, as we made before, one has, for constants depending only on λ_0, Λ_0 and C_0,

$$\left|\int \frac{2}{\sqrt{a_{2^{-\nu}}}} \Re\left([\varphi_\nu(D_x),b_1]\partial_x u \overline{\left(\partial_t u_\nu + \frac{\partial_t\sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu\right)}\right) dx\right| \leq$$
$$\leq C\sum_{\mu\geq 0} \|[\varphi_\nu(D_x),b_1]\Psi_\mu\|_{\mathcal{L}(L^2)} \|\sqrt[4]{a_{2^{-\nu}}}\,\partial_x u_\mu\|_{L^2} \left\|\frac{1}{\sqrt[4]{a_{2^{-\nu}}}}\left(\partial_t u_\nu + \frac{\partial_t\sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu\right)\right\|_{L^2}$$
$$\leq C\sum_{\mu\geq 0} \|[\varphi_\nu(D_x),b_1]\Psi_\mu\|_{\mathcal{L}(L^2)} \left(e_{\nu,2^{-\nu}}\right)^{1/2} \left(e_{\mu,2^{-\mu}}\right)^{1/2}.$$

Therefore,

$$\left|\sum_{\nu\geq 0} e^{-2\beta t(\nu+1)} 2^{-2\nu\theta} \int \frac{2}{\sqrt{a_{2^{-\nu}}}} \Re\left([\varphi_\nu(D_x),b_1]\partial_x u \overline{\left(\partial_t u_\nu + \frac{\partial_t\sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu\right)}\right) dx\right|$$
$$\leq C\sum_{\nu,\mu\geq 0} e^{-\beta(\nu+1)t} 2^{-\nu\theta} \left(e_{\nu,2^{-\nu}}\right)^{1/2} e^{-\beta(\mu+1)t} 2^{-\mu\theta} \left(e_{\mu,2^{-\mu}}\right)^{1/2} h_{\nu\mu},$$

where we have set again

$$h_{\nu\mu} = e^{-(\nu-\mu)\beta t}\, 2^{-(\nu-\mu)\theta}\, \|[\varphi_\nu(D_x),b_1]\Psi_\mu\|_{\mathcal{L}(L^2)}\;.$$

As b_0 and b_1 satisfy to the same hypothesis, the commutator $[\varphi_\nu(D_x),b_1]$ verifies the same inequalities as $[\varphi_\nu(D_x),b_0]$; so, if we repeat the same calculations, we obtain

$$\left|\sum_{\nu\geq 0} e^{-2\beta t(\nu+1)} 2^{-2\nu\theta} \int \frac{2}{\sqrt{a_{2^{-\nu}}}} \Re\left([\varphi_\nu(D_x), b_1]\partial_x u \overline{\left(\partial_t u_\nu + \frac{\partial_t \sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu\right)}\right) dx\right| \leq$$

$$\leq C\widetilde{M}(\omega,\theta) \sum_{\nu=0}^{+\infty} (\nu+1) e^{-2\beta(\nu+1)t} 2^{-2\nu\theta} e_{\nu,2^{-\nu}}(t).$$

3.5.3. Term with $[\varphi_\nu(D_x), c]$

Finally, we have to deal with the commutator $[\varphi_\nu(D_x), c(t,x)]$.

First, observe that there exist constants, depending only on λ_0, Λ_0 and C_0 as usual, such that one has

$$\left|\int \frac{2}{\sqrt{a_{2^{-\nu}}}} \Re\left([\varphi_\nu(D_x), c]\partial_x u \overline{\left(\partial_t u_\nu + \frac{\partial_t \sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu\right)}\right) dx\right| \leq$$

$$\begin{aligned}
&\leq C \left\|[\varphi_\nu(D_x), c]u\right\|_{L^2} \left\|\frac{1}{\sqrt[4]{a_{2^{-\nu}}}}\left(\partial_t u_\nu + \frac{\partial_t \sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu\right)\right\|_{L^2} \\
&\leq C \sum_{\mu\geq 0} \left\|[\varphi_\nu(D_x), c]\Psi_\mu\right\|_{\mathcal{L}(L^2)} \left\|u_\mu\right\|_{L^2} \left(e_{\nu,2^{-\nu}}(t)\right)^{1/2} \\
&\leq 2C \sum_{\mu\geq 0} \left\|[\varphi_\nu(D_x), c]\Psi_\mu\right\|_{\mathcal{L}(L^2)} 2^{-\mu} \left\|D_x u_\mu\right\|_{L^2} \left(e_{\nu,2^{-\nu}}(t)\right)^{1/2} \\
&\leq 2C \sum_{\mu\geq 0} \left\|[\varphi_\nu(D_x), c]\Psi_\mu\right\|_{\mathcal{L}(L^2)} 2^{-\mu} \left(e_{\mu,2^{-\mu}}(t)\right)^{1/2} \left(e_{\nu,2^{-\nu}}(t)\right)^{1/2} .
\end{aligned}$$

Thereby, we get the estimate

$$\left|\sum_{\nu\geq 0} e^{-2\beta t(\nu+1)} 2^{-2\nu\theta} \int \frac{2}{\sqrt{a_{2^{-\nu}}}} \Re\left([\varphi_\nu(D_x), c]\partial_x u \overline{\left(\partial_t u_\nu + \frac{\partial_t \sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}} u_\nu\right)}\right) dx\right|$$

$$\leq 2C \sum_{\nu,\mu\geq 0} e^{-\beta t(\nu+1)} 2^{-\nu\theta} \left(e_{\nu,2^{-\nu}}(t)\right)^{1/2} e^{-\beta t(\mu+1)} 2^{-\mu\theta} \left(e_{\mu,2^{-\mu}}(t)\right)^{1/2} m_{\nu\mu},$$

where we have defined

$$m_{\nu\mu} = e^{-(\nu-\mu)\beta t} \, 2^{-(\nu-\mu)\theta} \, 2^{-\mu} \left\|[\varphi_\nu(D_x), c]\Psi_\mu\right\|_{\mathcal{L}(L^2)} .$$

The kernel of the operator $[\varphi_\nu(D_x), c]$ is

$$n'(x,y) = \widehat{\psi}(2^\nu(x-y))\, 2^\nu \left(c(t,y) - c(t,x)\right);$$

so, remembering that c is bounded over $\mathbb{R}\times\mathbb{R}$, from Schur's lemma one gets

$$\|[\varphi_\nu(D_x),c]\|_{\mathcal{L}(L^2)}\,\leq\,C\qquad\qquad\forall\,\nu\geq 0\,,$$

where the constant C depends only on $\|c\|_{L^\infty(\mathbb{R}_t\times\mathbb{R}_x)}$.

Again, we are going to estimate the kernel $m_{\nu\mu}$ to apply Schur's lemma.

1. First, we take $\mu\leq 2$ and we have

$$\begin{aligned}\sum_{\nu\geq0}|m_{\nu\mu}| &\leq C\,e^{(\mu+1)\beta t}\,2^{(\mu+1)\theta}\,2^{-\mu}\sum_{\nu\geq0}e^{-(\nu+1)\beta t}\,2^{-(\nu+1)\theta}\\ &\leq C\,e^{3\beta t}\,2^{3\theta}\sum_{\nu\geq0}e^{-(\nu+1)(\beta t+\theta\log 2)}\,(\nu+1)^{1/2}\\ &\leq C\,e^{3\beta t}\,2^{3\theta}\,\alpha_2(\beta t+\theta\log 2)\\ &\leq C\,2^{\frac{9}{2}\theta}\,\alpha_2(\theta\log 2)\,.\end{aligned}$$

2. Now, we fix $\mu\geq 3$ and we consider the first part of the series:

$$\begin{aligned}&\sum_{\nu=0}^{\mu-3}|m_{\nu\mu}|\,\leq\\ &\leq Ce^{(\mu+1)\beta t}2^{-(\mu+1)(1-\theta)}2^{-\mu}\sum_{\nu=0}^{\mu-3}e^{-(\nu+1)\beta t}2^{(\nu+1)(1-\theta)}2^{\mu-\nu}\frac{(\nu+1)^{1/2}}{(\nu+1)^{1/2}}\\ &\leq Ce^{(\mu+1)\beta t}2^{-(\mu+1)(1-\theta)}(\mu-2)^{1/2}\sum_{\nu=0}^{\mu-3}e^{(\nu+1)((1-\theta)\log 2-\beta t)}(\nu+1)^{-1/2}\\ &\leq Ce^{(\mu+1)\beta t}2^{-(\mu+1)(1-\theta)}(\mu-2)^{1/2}\alpha_1((1-\theta)\log 2-\beta t)\;\cdot\\ &\qquad\qquad\cdot\,e^{-(\mu-2)\beta t}\,2^{(\mu-2)(1-\theta)}\,(\mu-2)^{-1/2}\\ &\leq C\,e^{3\beta t}\,2^{3\theta}\,\alpha_1\left(\left(1-\frac{3}{2}\theta\right)\log 2\right)\\ &\leq C\,2^{\frac{9}{2}\theta}\,\alpha_1\left(\left(1-\frac{3}{2}\theta\right)\log 2\right)\,.\end{aligned}$$

For the second part, one has:

$$\begin{aligned}&\sum_{\nu=\mu-2}^{+\infty}|m_{\nu\mu}|\\ &\leq Ce^{(\mu+1)\beta t}2^{(\mu+1)\theta}2^{-\mu}\sum_{\nu=\mu-2}^{+\infty}e^{-(\nu+1)\beta t}2^{-(\nu+1)\theta}\frac{(\nu+1)^{1/2}}{(\nu+1)^{1/2}}\\ &\leq Ce^{(\mu+1)\beta t}2^{(\mu+1)\theta}2^{-\mu}\alpha_2(\beta t+\theta\log 2)e^{-(\mu-1)\beta t}2^{-(\mu-1)\theta}\\ &\leq C\,2^{3\theta}\,\alpha_2(\theta\log 2)\,.\end{aligned}$$

3. Now, we fix $\nu \geq 0$. Initially, we consider

$$\begin{aligned}\sum_{\mu=0}^{\nu+2}|m_{\nu\mu}| &\leq C\,e^{-(\nu+1)\beta t}\,2^{-(\nu+1)\theta}\sum_{\mu=0}^{\nu+2}e^{(\mu+1)(\beta t+\theta\log 2)}\,2^{-\mu}\,\frac{(\mu+1)^{1/2}}{(\mu+1)^{1/2}}\\ &\leq C\,e^{-(\nu+1)\beta t}\,2^{-(\nu+1)\theta}\,(\nu+3)^{1/2}\,\alpha_1(\beta t+\theta\log 2)\,\cdot\\ &\qquad\qquad\cdot\,e^{(\nu+3)\beta t}\,2^{(\nu+3)\theta}\,(\nu+3)^{-1/2}\\ &\leq C\,e^{2(\beta t+\theta\log 2)}\,\alpha_1(\theta\log 2)\\ &\leq C\,2^{3\theta}\,\alpha_1(\theta\log 2)\,.\end{aligned}$$

The second part of the series, instead, can be treated as follow:

$$\begin{aligned}&\sum_{\mu=\nu+3}^{+\infty}|m_{\nu\mu}| \leq\\ &\leq Ce^{-(\nu+1)\beta t}2^{(\nu+1)(1-\theta)}\sum_{\mu=\nu+3}^{+\infty}e^{(\mu+1)\beta t}2^{-\mu}2^{-(\mu+1)(1-\theta)}2^{\mu-\nu}\frac{(\mu+1)^{1/2}}{(\mu+1)^{1/2}}\\ &\leq Ce^{-(\nu+1)\beta t}2^{(\nu+1)(1-\theta)}(\nu+4)^{-1/2}\sum_{\mu=\nu+3}^{+\infty}e^{-(\mu+1)((1-\theta)\log 2-\beta t)}(\mu+1)^{1/2}\\ &\leq C\,e^{-(\nu+1)\beta t}\,2^{(\nu+1)(1-\theta)}\,\alpha_2((1-\theta)\log 2-\beta t)\,e^{(\nu+4)\beta t}\,2^{-(\nu+4)(1-\theta)}\\ &\leq C\,2^{\frac{9}{2}\theta}\,\alpha_2\left(\left(1-\frac{3}{2}\theta\right)\log 2\right)\,.\end{aligned}$$

Finally, we obtain:

$$\begin{aligned}&\left|\sum_{\nu\geq 0}e^{-2\beta t(\nu+1)}\,2^{-2\nu\theta}\int\frac{2}{\sqrt{a_{2^{-\nu}}}}\,\Re\left([\varphi_\nu(D_x),c]\partial_x u\,\overline{\left(\partial_t u_\nu+\frac{\partial_t\sqrt{a_{2^{-\nu}}}}{2\sqrt{a_{2^{-\nu}}}}u_\nu\right)}\right)dx\right|\\ &\qquad\qquad\leq C\,\Pi(\theta)\sum_{\nu=0}^{+\infty}(\nu+1)\,e^{-2\beta(\nu+1)t}\,2^{-2\nu\theta}\,e_{\nu,2^{-\nu}}(t)\,,\end{aligned}$$

where the function Π is the same used in the estimate of the term $[\varphi_\nu(D_x),a]$.

3.6. End of the proof of theorem 2.1

Now we can complete the proof of theorem 2.1.

First, remembering the definition of total energy given by (32), we have that there exists a constant $C>0$, depending only on θ, such that

$$\begin{aligned}&\left|\sum_{\nu=0}^{+\infty}e^{-\beta(\nu+1)t}\,2^{-2\nu\theta}\int\frac{2}{\sqrt{a_\varepsilon}}\,\Re\left((Lu)_\nu\,\cdot\,\overline{\left(\partial_t u_\nu+\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right)}\right)dx\right|\leq\\ &\qquad\qquad\leq C\,(E(t))^{1/2}\,\|Lu\|_{H^{-\theta-\beta^* t}}\,.\end{aligned}$$

Therefore, if we set $\widetilde{\Pi}(\omega,\theta) = \max\{\widetilde{M}(\omega,\theta), \Pi(\theta)\}$, from relation (34) and from the estimates proved in the previous subsection, we have that, for suitable constants, depending only on λ_0, Λ_0, C_0 and on the norms of the coefficients of operator L in their respective functional spaces, the following inequality holds:

$$\begin{aligned} \frac{d}{dt}E(t) &\leq \left(C + C'\,\widetilde{\Pi}(\omega,\theta) - 2\beta\right) \sum_{\nu=0}^{+\infty} (\nu+1)\, e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, e_{\nu,2^{-\nu}}(t) \\ &+ C''\,(E(t))^{1/2}\, \|Lu\|_{H^{-\theta-\beta^* t}} . \end{aligned}$$

Now, let us fix β large enough, such that $C + C'\,\widetilde{\Pi}(\omega,\theta) - 2\beta \leq 0$: we can always do this, on condition that we take T small enough. With this choice, we have

$$\frac{d}{dt}E(t) \leq C''\,(E(t))^{1/2}\, \|Lu\|_{H^{-\theta-\beta^* t}} ;$$

now, the thesis of the theorem follows from Gronwall's lemma and remark 3.3.

References

[1] Jean-Michel Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann. Sci. École Norm. Sup. (4) **14** (1981), no. 2, 209–246.

[2] Ferruccio Colombini, Ennio De Giorgi, and Sergio Spagnolo, *Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temps*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **6** (1979), no. 3, 511–559.

[3] Ferruccio Colombini and Daniele Del Santo, *A note on hyperbolic operators with log-Zygmund coefficients*, J. Math. Sci. Univ. Tokyo **16** (2009), no. 1, 95–111.

[4] Ferruccio Colombini and Nicolas Lerner, *Hyperbolic operators with non-Lipschitz coefficients*, Duke Math. J. **77** (1995), no. 3, 657–698.

[5] Lars Hörmander, *Linear partial differential operators*, Die Grundlehren der mathematischen Wissenschaften, Bd. 116, Academic Press Inc., Publishers, New York, 1963.

[6] Guy Métivier, *Para-differential calculus and applications to the Cauchy problem for nonlinear systems*, Centro di Ricerca Matematica Ennio De Giorgi (CRM) Series, vol. 5, Edizioni della Normale, Pisa, 2008.

[7] Shigeo Tarama, *Energy estimate for wave equations with coefficients in some Besov type class*, Electron. J. Differential Equations (2007), No. 85, 12 pp. (electronic).

[8] François Trèves, *Basic linear partial differential equations*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975, Pure and Applied Mathematics, Vol. 62.

[9] Taeko Yamazaki, *On the $L^2(\mathbf{R}^n)$ well-posedness of some singular or degenerate partial differential equations of hyperbolic type*, Comm. Partial Differential Equations **15** (1990), no. 7, 1029–1078.

Authors' addresses:

Ferruccio Colombini
Dipartimento di Matematica, Università di Pisa
Largo B. Pontecorvo 5, 56127 Pisa, ITALY
E-mail: colombini@dm.unipi.it

Francesco Fanelli
SISSA
via Beirut 2/4, 34151 Trieste, ITALY
E-mail: francesco.fanelli@sissa.it

Received February dd, 2010
Revised Month dd, yyyy