SECOND ORDER APPROXIMATIONS OF QUASISTATIC EVOLUTION PROBLEMS IN FINITE DIMENSION

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Abstract. In this paper, we study the limit, as ε goes to zero, of a particular solution of the equation $\varepsilon^2 A\ddot{u}^\varepsilon(t) + \varepsilon B\dot{u}^\varepsilon(t) + \nabla_x f(t, u^\varepsilon(t)) = 0$, where $f(t,x)$ is a potential satisfying suitable coerciveness conditions. The limit $u(t)$ of $u^\varepsilon(t)$ is piece-wise continuous and verifies $\nabla_x f(t, u(t)) = 0$. Moreover, certain jump conditions characterize the behaviour of $u(t)$ at the discontinuity times. The same limit behaviour is obtained by considering a different approximation scheme based on time discretization and on the solutions of suitable autonomous systems.

1. Introduction

The problem of finding a function $t \mapsto u(t)$ satisfying

$$\nabla_x f(t, u(t)) = 0 \quad \text{and} \quad \nabla_x^2 f(t, u(t)) > 0 \tag{1.1}$$

appears in many areas of applied mathematics. Usually, the real-valued function $f(t,x)$ represents a time-dependent energy, defined for $t \in [0,T]$ and $x \in \mathbb{R}^n$. The symbol ∇_x denotes the gradient with respect to x, while ∇_x^2 is the corresponding Hessian. The inequality in (1.1) means that the matrix $\nabla_x^2 f(t, u(t))$ is positive definite. Therefore, (1.1) says that, for every t, the state $u(t)$ is a stable equilibrium point for the potential $f(t, \cdot)$.

If we look for a continuous solution $t \mapsto u(t)$, defined only in a neighbourhood of a prescribed time, the problem is solved by the Implicit Function Theorem. In many applications, however, we want to obtain a piece-wise continuous solution $t \mapsto u(t)$ on the whole interval $[0,T]$. The main problem is, therefore, to extend the solution beyond its maximal interval of continuity. A first possibility is to select, for every t, a global minimizer $u(t)$ of $f(t, \cdot)$. This choice exhibits some drawbacks, as we shall explain later. Different extension criteria can be proposed, motivated by different interpretations of the problem.

Problem (1.1) can be considered, for instance, as describing the limiting case of a system governed by an overdamped dynamics, as the relaxation time tends to 0. Indeed, one can prove that, when the relaxation time is very small, the state $u(t)$ of the system is always close to a stable equilibrium for the potential $f(t, \cdot)$, which, in general, is not a global minimizer of $f(t, \cdot)$. The first general result in this direction was obtained by Zanini (see [15]), who considers (1.1) as limit of the viscous dynamics governed by the gradient flow

$$\varepsilon\dot{u}^\varepsilon(t) + \nabla_x f(t, u^\varepsilon(t)) = 0. \tag{1.2}$$

She proves that the limit $u(t)$ of the solution $u^\varepsilon(t)$ of problem (1.2) is a piece-wise continuous function satisfying (1.1), and describes the trajectories followed by the system at the jump times. Under different and stronger hypotheses, similar vanishing viscosity limits have been studied in finite dimension in [6], [11], [5], [13], and [12], and even in infinite dimension in [1], [2], [14], [10], [3], and [4].

Simple examples show that the solution $u(t)$ found in [15] is, in general, different from the global minimizer. We note that the global minimizer may exhibit abrupt discontinuities at times where it must jump from a potential well to another one with the same energy level. This jump cannot be justified if we interpret (1.1) as limit of a dynamic problem, since the state should overcome a potential barrier during the jump.

In this paper we consider (1.1) as the limiting case of a sequence of second order evolution problems, describing the underlying dynamics and depending on a small parameter $\varepsilon > 0$, namely

$$\varepsilon^2 A\ddot{u}^\varepsilon(t) + \varepsilon B\dot{u}^\varepsilon(t) + \nabla_x f(t, u^\varepsilon(t)) = 0, \tag{1.3}$$

where A and B are positive definite and symmetric matrices. This describes the evolution of a mechanical system where both inertia and friction are taken into account, encoded in A and B, respectively. Under suitable assumptions on f, we prove that the solution u^ε of (1.3) is such that $(u^\varepsilon, \varepsilon B\dot{u}^\varepsilon)$ tends to $(u, 0)$, where u is piece-wise continuous and satisfies (1.1). Moreover, the trajectories of the system at the jump times are described through suitable autonomous second order systems related to A, B, and $\nabla_x f$.

Let us explain, in more details, the content of Sections 2-5. In Section 2 we construct a suitable piece-wise continuous solution u of problem (1.1), and in Section 3 we show that the solutions $u^\varepsilon(t)$ of (1.3), with the same initial conditions, converge to $u(t)$ at every continuity time t.

The function u is defined in the following way. We begin with a point $u(0)$ such that $\nabla_x f(0, u(0)) = 0$ and $\nabla_x^2 f(0, u(0)) > 0$, and, by the Implicit Function Theorem, we find a continuous solution u of (1.1) up to $t_1 \leq T$ such that $\nabla_x^2 f(t_1, u(t_1^-))$ has only one zero eigenvalue. In a "generic" situation (see Assumption 3 and Remark 2.3), what occurs at t_1 is a saddle-node bifurcation of the vector field $F(t, \cdot)$ corresponding to the first order autonomous system equivalent to

$$A\ddot{w}(s) + B\dot{w}(s) + \nabla_x f(t, w(s)) = 0. \tag{1.4}$$

For (t, x) close enough to $(t_1, u(t_1^-))$, if t is on the left of t_1, $F(t, \cdot)$ has two zeros, a saddle and a node, while, for t on the right of t_1, there are no zeros of this vector field. Under these conditions, it is also possible to prove (see Lemma 2.4 and Section 5) existence and uniqueness, up to time-translations, of the non constant solution of system (1.4) such that

$$\lim_{s\to-\infty}(w(s), \dot{w}(s)) = (u(t_1^-), 0). \tag{1.5}$$

Moreover, the limit

$$\lim_{s\to+\infty}(w(s), \dot{w}(s)) = (x_1^r, 0)$$

exists, and x_1^r is another zero of $\nabla_x f(t_1, \cdot)$. If $t_1 < T$, we make the "generic" assumption that $\nabla_x^2 f(t_1, x_1^r)$ is positive definite (see Assumption 4), so that we restart the procedure and, in turn, find a solution of (1.1) on $[t_1, t_2)$, for a certain $t_2 \leq T$, and so on. In this way, we find a piece-wise continuous solution u of (1.1), with certain discontinuity times $t_1, ..., t_{m-1}$, and, for $j = 1, ..., m-1$, a heteroclinic solution w_j of (1.5) with $t = t_j$, which connects a degenerate critical point of $f(t_j, \cdot)$ at $s = -\infty$ to a non-degenerate critical point at $s = +\infty$ (see Proposition 2.6).

In Section 3 we prove that, if $(u^\varepsilon(0), \varepsilon\dot{u}^\varepsilon(0)) \to (u(0), 0)$, then $(u^\varepsilon, \varepsilon B\dot{u}^\varepsilon)$ converges to $(u, 0)$ uniformly on compact subsets of $[0, T]\setminus\{t_1, ..., t_{m-1}\}$, while a proper rescaling v_j^ε of u^ε is such that $(v_j^\varepsilon, \dot{v}_j^\varepsilon)$ converges uniformly to $(w_j, \dot{w}_j)$ on compact subsets of $\mathbb{R}$ (see Theorem 3.1 and Remark 3.9). This shows that (1.4) governs the fast dynamics of the system at the jump times.

Theorem 3.2 summarizes these convergences in a more geometric statement involving the Hausdorff distance.

In Section 4 we show that the solution u of (1.1) introduced in Section 2 can be obtained as limit of a discrete time approximation, which uses only autonomous systems. Let $\tau_i^k = \frac{i}{k}T$. For every k, let u_i^k be defined by $u_0^k = u(0)$ and, for $i = 1, ..., k$, by

$$u_i^k := \lim_{\sigma \to +\infty} v_i^k(\sigma), \tag{1.6}$$

where v_i^k is the solution of the autonomous system

$$A\ddot{v}_i^k(\sigma) + B\dot{v}_i^k(\sigma) + \nabla_x f(\tau_i^k, v_i^k(\sigma)) = 0, \tag{1.7}$$

with initial conditions $(v_i^k(0), \dot{v}_i^k(0)) = (u_{i-1}^k, 0)$. The existence of the limit in (1.6) is a property of the autonomous system, ensured by Lemma 2.4.

We prove that $u_i^k = u(\tau_i^k)$, unless τ_i^k is close to the discontinuity times $t_1, ..., t_{m-1}$ of u. More precisely, given an arbitrary neighbourhood U of the set $\{t_1, ..., t_{m-1}\}$, we prove that $u_i^k = u(\tau_i^k)$ whenever k is sufficiently large and $\tau_i^k \notin U$ (see Lemma 4.3 and Lemma 4.4). This implies that the piece-wise constant and the piece-wise affine interpolations of the values u_i^k's converge uniformly to u on compact subsets of $[0, T] \setminus \{t_1, ..., t_{m-1}\}$.

In order to obtain the convergence to the heteroclines w_j's near the jump times, as well as the convergence of the velocity (Proposition 4.8 and Theorem 4.1), we introduce a suitable interpolation of u_i^k based on the solution v_i^k of (1.7) (see (4.8)).

Section 5 is an appendix which contains the proof of the existence and uniqueness of the heteroclinic solution of a first order autonomous system when certain transversality conditions at the zeros of the vector field are satisfied.

2. Setting of the problem and preliminaries

Assumption 1. $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is a C^3-function satisfying, for every $(t, x) \in [0, T] \times \mathbb{R}^n$, the properties:

(i) $\nabla_x f(t, x) \cdot x \geq b|x|^2 - a$, for some $a \geq 0$ and $b > 0$;
(ii) $f_t(t, x) \leq d|x|^2 + c$, for some d, $c \geq 0$.

We use the following terminology: $x \in \mathbb{R}^n$ is a *critical* point of $f(t, \cdot)$ if $\nabla_x f(t, x) = 0$. A critical point x of $f(t, \cdot)$ is *degenerate* if $\det \nabla_x^2 f(t, x) = 0$. Observe that, from Assumption 1 (i), it descends that there exist $\tilde{a} \geq 0$ and $\tilde{b} > 0$ such that

$$f(t, x) \geq \tilde{b}|x|^2 - \tilde{a}, \text{ for every } (t, x) \in [0, T] \times \mathbb{R}^n. \tag{2.1}$$

Moreover, Assumption 1 implies that all critical points of $f(t, \cdot)$ belong to the closed ball $\overline{B}$ centered in zero and with radius $\sqrt{\frac{a}{b}}$. Since $f(t, \cdot)$ has minimum and maximum on $\overline{B}$, we can state that, for every $t \in [0, T]$, $f(t, \cdot)$ has at least one critical point and it belongs to $\overline{B}$.

Assumption 2. The set of all pairs (t, ξ) such that ξ is a degenerate critical point of $f(t, \cdot)$ is discrete and there are no degenerate critical points corresponding to $t = 0$ or $t = T$.

Remark 2.1. Assumptions 1-2 imply that, for every $t \in [0, T]$, the set of critical points of $f(t, \cdot)$ is discrete. Indeed, by Assumption 2, the set of degenerate critical points of $f(t, \cdot)$ is discrete, while the set of nondegenerate critical points of $f(t, \cdot)$ is discrete by the Implicit Function Theorem.

Definition 2.2. *We say that* $(\tau, \xi) \in [0, T] \times \mathbb{R}^n$ *is a* degenerate approximable critical pair *if* ξ *is a degenerate critical point of* $f(\tau, \cdot)$ *and there exist sequences* $\{t_n\}$ *and* $\{\xi_n\}$ *converging to* τ *from the left and to* ξ*, respectively, with* $\nabla_x f(t_n, \xi_n) = 0$ *and* $\nabla_x^2 f(t_n, \xi_n)$ *positive definite.*

Observe that if (τ,ξ) is a degenerate approximable critical pair, then $\nabla_x^2 f(\tau,\xi)$ is positive semidefinite. From now on, A and B will be two given symmetric and positive definite matrices in $\mathbb{R}^{n\times n}$, unless differently specified. λ_{min}^A and λ_{min}^B will denote the minimum eigenvalue of A and B, respectively.

Assumption 3. If $(\tau,\xi)\in[0,T]\times\mathbb{R}^n$ is a degenerate approximable critical pair, then there exists $l\in\mathbb{R}^n\setminus\{0\}$ such that:

(i) $\ker\nabla_x^2 f(\tau,\xi)=\mathrm{span}(l)$;

(ii) $(A^{-T}Bl)\cdot\nabla_x f_t(\tau,\xi)\neq 0$, where f_t denotes the partial derivative of f with respect to t;

(iii) $(A^{-T}Bl)\cdot\nabla_x^3 f(\tau,\xi)[l,l]\neq 0$.

Remark 2.3. Set $\eta=\begin{bmatrix}\xi\\0\end{bmatrix}\in\mathbb{R}^{2n}$ and define

$$F\left(t,\begin{bmatrix}x\\y\end{bmatrix}\right):=\begin{bmatrix}B^{-1}y\\-BA^{-1}\left(y+\nabla_x f(t,x)\right)\end{bmatrix},\quad t\in[0,T],\quad x,y\in\mathbb{R}^n. \tag{2.2}$$

Let Assumption 3 hold for some (τ,ξ) degenerate approximable critical pair and observe, first, that

$$F(\tau,\eta)=0.$$

Since

$$\nabla_\eta F(\tau,\eta)=\begin{bmatrix}0 & B^{-1}\\-BA^{-1}\nabla_x^2 f(\tau,\xi) & -BA^{-1}\end{bmatrix},$$

where ∇_η denotes $\frac{\partial}{\partial(x,y)}$, by setting

$$\omega=\begin{bmatrix}l\\0\end{bmatrix},\qquad \nu=\begin{bmatrix}B^2A^{-1}l\\l\end{bmatrix},$$

it turns out, from Assumption 3 (i), that

$$\ker\nabla_\eta F(\tau,\eta)=\mathrm{span}(\omega),\qquad \ker\nabla_\eta F(\tau,\eta)^T=\mathrm{span}(\nu). \tag{2.3}$$

Moreover, simple calculations give that

$$F_t(\tau,\eta)=\begin{bmatrix}0\\-BA^{-1}\nabla_x f_t(\tau,\xi)\end{bmatrix},\quad \nabla_\eta^2 F(\tau,\eta)[\omega,\omega]=\begin{bmatrix}0\\-BA^{-1}\nabla_x^3 f(\tau,\xi)[l,l]\end{bmatrix},$$

so that, from Assumption 3 (ii) and (iii), we obtain that

$$\nu\cdot\nabla_\eta^2 F(\tau,\eta)[\omega,\omega]\neq 0,\qquad \nu\cdot F_t(\tau,\eta)\neq 0. \tag{2.4}$$

Observe that $\lambda\in\mathbb{C}$ is an eigenvalue of $\nabla_\eta F(\tau,\eta)$ if and only if there exists $\begin{bmatrix}x\\y\end{bmatrix}\neq 0$ such that

$$\begin{cases} y=\lambda Bx,\\ \nabla_x^2 f(\tau,\xi)x=-\lambda(B+\lambda A)x.\end{cases} \tag{2.5}$$

Let us show that the algebraic multiplicity of the null eigenvalue of $\nabla_\eta F(\tau,\eta)$ is

$$m_a(0)=1. \tag{2.6}$$

Recall that $m_a(0)$ corresponds to the dimension of the generalized eigenspace associated to the null eigenvalue, that is $\ker(\nabla_\eta F(\tau,\eta))^k$, where k is the smallest integer k such that $\ker(\nabla_\eta F(\tau,\eta))^k=\ker(\nabla_\eta F(\tau,\eta))^{k+1}$. Thus, in order to prove that $m_a(0)=1$, it is enough to show that $\ker(\nabla_\eta F(\tau,\eta))^2\subseteq\ker(\nabla_\eta F(\tau,\eta))$, because the other inclusion is trivial. If $\begin{bmatrix}x\\y\end{bmatrix}\in\ker(\nabla_\eta F(\tau,\eta))^2$, then, in view of (2.3), we have that

$$\nabla_\eta F(\tau,\eta)\begin{bmatrix}x\\y\end{bmatrix}=\alpha\begin{bmatrix}l\\0\end{bmatrix}, \tag{2.7}$$

for some $\alpha \in \mathbb{C}$. Therefore, if $\alpha = 0$, then $\begin{bmatrix} x \\ y \end{bmatrix} \in \ker(\nabla_\eta F(\tau,\eta))$, while, if $\alpha \neq 0$, we find, from (2.7), that

$$\begin{cases} y = \alpha B l, \\ \nabla_x^2 f(\tau,\xi)x = y, \end{cases}$$

and, in turn, that $0 = x\cdot\left(\nabla_x^2 f(\tau,\xi)l\right) = \alpha Bl\cdot l \neq 0$, which is an absurd.
Now, we want to show that every eigenvalue λ of $\nabla_\eta F(\tau,\eta)$ is such that:

$$\text{if } \lambda \neq 0, \quad \text{then} \quad \operatorname{Re}(\lambda) < 0. \tag{2.8}$$

Let $\begin{bmatrix} x \\ y \end{bmatrix}$ be an eigenvalue associated to the eigenvalue $\lambda \neq 0$ and write $x \in \mathbb{C}^n \setminus \{0\}$ as $x = a + ib$, for some a, $b \in \mathbb{R}^n$. In the case a, $b \in \operatorname{span}(l)$, from the second equation of (2.5) we obtain that $(B + \lambda A)l = 0$. The scalar product of this equality with l gives

$$\lambda = -\frac{Bl\cdot l}{Al\cdot l} \leq -\frac{\lambda^B_{min}}{|A|} < 0.$$

In the case $\{a,b\} \nsubseteq \operatorname{span}(l)$, we consider the hermitian product of the second equation of (2.5) with x, which gives

$$C = -\lambda(C_A\lambda + C_B), \tag{2.9}$$

where

$$C := \left(\nabla_x^2 f(\tau,\xi)a\cdot a + \nabla_x^2 f(\tau,\xi)b\cdot b\right) \in \mathbb{R},$$
$$C_A := Aa\cdot a + Ab\cdot b, \qquad\qquad C_B := Ba\cdot a + Bb\cdot b.$$

Now, by setting $\lambda = \lambda_1 + i\lambda_2$ for some λ_1, $\lambda_2 \in \mathbb{R}$, from (2.9) we obtain

$$\lambda_2(C_B + 2C_A\lambda_1) = 0, \tag{2.10}$$

and

$$C_A\lambda_1^2 + C_B\lambda_1 - C_A\lambda_2^2 + C = 0. \tag{2.11}$$

We want to prove that $\lambda_1 < 0$. If $\lambda_2 \neq 0$, from (2.10) it is easy to deduce that

$$\lambda_1 \leq -\frac{\lambda^B_{min}}{2|A|} < 0.$$

In the case $\lambda_2 = 0$, we can suppose $b = 0$. From (2.11) and from the fact that λ_1 is real we obtain that $C_B^2 - 4CC_A \geq 0$ and that

$$\lambda_1 \leq \frac{-Ba\cdot a + \sqrt{(Ba\cdot a)^2 - 4(\nabla_x^2 f(\tau,\xi)a\cdot a)(Aa\cdot a)}}{2Aa\cdot a}.$$

Since $a \notin \operatorname{span}(l) = \ker \nabla_x^2 f(\tau,\xi)$ and $\nabla_x^2 f(\tau,\xi) \geq 0$, we have that $\nabla_x^2 f(\tau,\xi)a\cdot a \geq \lambda_\tau |a|^2$, where $\lambda_\tau > 0$ is the smallest eigenvalue of $\nabla_x^2 f(\tau,\xi)$ different from 0. By using this fact, together with the hypotheses on A and B, we can easily prove, by rationalization, that

$$\frac{-Ba\cdot a + \sqrt{(Ba\cdot a)^2 - 4(\nabla_x^2 f(\tau,\xi)a\cdot a)(Aa\cdot a)}}{2Aa\cdot a} \leq -\frac{\lambda_\tau\lambda^A_{min}}{|A||B|}.$$

This concludes the proof of (2.8).
Let us collect together (2.3), (2.4), (2.6) and (2.8), which descend from Assumption 3. We obtain that $F : [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, defined as is (2.2), is a C^2 function such that $F(\tau,\eta) = 0$ and satisfies the following properties:

(TC1) 0 is an eigenvalue of $\nabla_\eta F(\tau,\eta)$ with $m_a(0) = 1$, $\operatorname{Re}(\lambda) < 0$ for every eigenvalue $\lambda \neq 0$, and there exist ω, $\nu \in \mathbb{R}^m$ such that $\omega\cdot\nu \neq 0$ and $\ker \nabla_\eta F(\tau,\eta) = \operatorname{span}(\omega)$, $\ker \nabla_\eta F(\tau,\eta)^T = \operatorname{span}(\nu)$;
(TC2) $\nu\cdot F_t(\tau,\eta) \neq 0$;
(TC3) $\nu\cdot\nabla_\eta^2 F(\tau,\eta)[\omega,\omega] \neq 0$.

Such *transversality conditions* ensure (see [7, Theorem 3.4.1]) the existence a smooth curve of equilibria $\left(t(\cdot), \begin{bmatrix} x \\ y \end{bmatrix}(\cdot)\right)$ passing through (τ, η), tangent to the hyperplane $\{\tau\} \times \mathbb{R}^{2n}$. If conditions (TC2) and (TC3) have the same sign, for every $t < \tau$ close to τ there are two solutions of $F(t, \cdot) = 0$, a saddle and a node, while for every $t > \tau$ (close to τ) there are no solutions. If conditions (TC2) and (TC3) have opposite sign, the reverse is true. The set of vector fields satisfying (TC1)-(TC3) is open and dense in the space of C^∞ one-parameter families of vector fields with an equilibrium at (τ, ξ) with a zero eigenvalue.

With the next lemma we introduce the heterocline which will allow us to connect, at a specific time τ, a degenerate critical point of $f(\tau, \cdot)$ to another suitable critical point of $f(\tau, \cdot)$.

Lemma 2.4. *Let $(\tau, \xi) \in [0, T] \times \mathbb{R}^n$ be a degenerate approximable critical pair. Suppose that Assumption 1 and 2 and Assumption 3 (i) and (iii) hold. Then, there exists a unique, up to time-translations, heteroclinic solution of*

$$\begin{cases} A\ddot{w}(s) + B\dot{w}(s) + \nabla_x f(\tau, w(s)) = 0, \quad s \in (-\infty, 0] \\ \lim_{s\to-\infty} w(s) = \xi, \\ \lim_{s\to-\infty} \dot{w}(s) = 0. \end{cases} \tag{2.12}$$

This means that such a solution w is defined on all $\mathbb{R}$, there exists $\lim_{s\to+\infty} w(s) :=$ $\zeta \in \mathbb{R}^n$*, with ζ critical point of $f(\tau, \cdot)$ different from ξ, and there exists* $\lim_{s\to+\infty} \dot{w}(s) = 0$.

By Remark 2.3, existence and uniqueness (up to time-translations) of the solution of (2.12), different from the constant solution ξ, follow from Proposition (5.1) with $\begin{bmatrix} x \\ y \end{bmatrix}$ in place of x and $F(\tau, \cdot)$ in place of F. The proof of Lemma (2.4) can be concluded in view of the following lemma.

Lemma 2.5. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function such that*

$$g(x) \geq C_1|x|^2 - C_2, \quad \text{for every } x \in \mathbb{R}^n, \tag{2.13}$$

for some constants $C_1 > 0$ and $C_2 \geq 0$. Suppose that the set of critical points of g is discrete. Let w be the (unique) solution of the Cauchy problem associated to

$$A\ddot{w} + B\dot{w} + \nabla g(w) = 0, \tag{2.14}$$

with initial conditions at some $s_0 \in \mathbb{R}$.

Then, $(w, \dot{w})$ is bounded and defined on $[s_0, +\infty)$ and there exists the limit

$$\lim_{s\to+\infty} (w(s), \dot{w}(s)) = (\zeta, 0), \tag{2.15}$$

where ζ is a critical point of g. Moreover, if $(w, \dot{w})$ is bounded on its maximal interval of existence, then $(w, \dot{w})$ is bounded and defined on all $\mathbb{R}$ and there exists the limit

$$\lim_{s\to-\infty} (w(s), \dot{w}(s)) = (\xi, 0),$$

where ξ is a critical point of g.

Proof. Let us denote by (s_0^-, s_0^+) the maximal interval of existence of w. Consider, for $x, y \in \mathbb{R}^n$, the function

$$V\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) := \frac{1}{2}Ay\cdot y + g(x),$$

and observe that, by multiplying (2.14) by $\dot{w}$, we obtain that

$$\frac{d}{ds}V\left(\begin{bmatrix} w(s) \\ \dot{w}(s) \end{bmatrix}\right) = -B\dot{w}(s)\cdot\dot{w}(s) \leq 0.$$

Thus, for every $s \in [s_0, s_0^+)$, we have that

$$\frac{1}{2}\lambda^A_{min}|\dot{w}(s)|^2 + g(w(s)) \leq \frac{1}{2}A\dot{w}(s)\cdot\dot{w}(s) + g(w(s)) \leq \frac{1}{2}A\dot{w}(s_0)\cdot\dot{w}(s_0) + g(w(s_0)).$$

Therefore, by using (2.13), we deduce that the positive semiorbit of $(w, \dot{w})$ is bounded and therefore defined on $[s_0, +\infty)$. This fact, together with the monotonicity of $V\left(\left[\begin{smallmatrix} w \\ \dot{w} \end{smallmatrix}\right]\right)$ on $[s_0, +\infty)$, implies that there exists the limit

$$\lim_{s\to+\infty} V\left(\begin{bmatrix} w(s) \\ \dot{w}(s) \end{bmatrix}\right) := L \in \mathbb{R}. \tag{2.16}$$

Let $\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}$ be a point of the ω-limit set associated to $(w, \dot{w})$ (which is nonempty because of the boundedness of the positive semiorbit of $(w, \dot{w})$), and consider the solution φ of the problem

$$\begin{cases} A\ddot{w}(s) + B\dot{w}(s) + \nabla g(w(s)) = 0, & s \in [s_0, +\infty) \\ w(s_0) = \overline{x}, \\ \dot{w}(s_0) = \overline{y}. \end{cases}$$

Since, from (2.16), $V\left(\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}\right) = L$, and the ω-limit sets are invariant sets, we obtain that $V(\varphi(s)) = L$ for every $s \geq s_0$. Thus, we have that

$$\frac{d}{ds}V(\varphi(s)) = -B\dot{\varphi}(s)\cdot\dot{\varphi}(s) = 0, \quad \text{for every } s \geq s_0,$$

and, in turn, that $\overline{y} = 0$ and $\ddot{\varphi}(s) = 0$ for every $s \geq s_0$. Moreover, considering also (2.14), it turns out that $\nabla g(\overline{x}) = 0$. In this way, we have proved that the ω-limit set is contained in the set $Z := \{(\zeta, 0) \in \mathbb{R}^n \mid \zeta \text{ is a critical point of } g\}$, which is, by assumption, discrete. Therefore, the ω-limit set, that is connected, is reduced to one point of Z, and this proves (2.15). The proof of the rest part of the lemma can be done in a similar way, by using the boundedness of $(w, \dot{w})$ on $(s_0^-, +\infty)$ and again the monotonicity of $V\left(\left[\begin{smallmatrix} w \\ \dot{w} \end{smallmatrix}\right]\right)$. □

Assumption 4. For every degenerate approximable critical pair $(\tau, \xi) \in [0, T] \times \mathbb{R}^n$, let w be the unique solution (up to time-translation) of (2.12). We assume that

$$\nabla^2_x f(\tau, w(+\infty)) \text{ is positive definite.}$$

With the following proposition and definition, we construct a suitable piece-wise continuous solution of problem (1.1).

Proposition 2.6. *Under Assumptions 1-4, let $x_0^r \in \mathbb{R}^n$ be such that $\nabla_x f(0, x_0^r) = 0$ and $\nabla^2_x f(0, x_0^r)$ is positive definite.*

There exists a partition $0 = t_0 < ... < t_m = T$ of the interval $[0, T]$ and, for every $j \in \{1, ..., m-1\}$, two distinct points x_j^r, $x_j^s \in \mathbb{R}^n$ with the following properties:

(1) *for every $j \in \{1, ..., m\}$, there exists a unique function $u_j : [t_{j-1}, t_j) \to \mathbb{R}^n$ of class C^2 such that*

$$\nabla_x f(\cdot, u_1(\cdot)) \equiv 0 \text{ and } \nabla^2_x f(\cdot, u_j(\cdot)) \text{ is positive definite on } [t_{j-1}, t_j),$$

and

$$u_j(t_{j-1}) = x_{j-1}^r;$$

(2) *for every $j \in \{1, ..., m-1\}$,*

$$x_j^s = \lim_{t\to t_j^-} u_j(t),$$

(t_j, x_j^s) is a degenerate approximable critical pair and there exists a unique (up to time-translation) function $w_j : \mathbb{R} \to \mathbb{R}^n$ of class C^2 such that

$$A\ddot{w}_j(s) + B\dot{w}_j(s) + \nabla_x f(t_j, w_j(s)) = 0, \quad s \in \mathbb{R}, \tag{2.17}$$

and

$$\lim_{s\to-\infty} w_j(s) = x_j^s, \qquad \lim_{s\to+\infty} w_j(s) = x_j^r.$$

The proof of Proposition 2.6 is similar to [15, Proposition 1]. The only difference is the choice of the heteroclinic solutions: in [15], those are solutions of equations of the type $\dot{w}_j(s) = \nabla_x f(t_j, w_j(s))$; here, equations (2.17) are taken into account. The procedure to select a solution can be summarized in the following way: beginning from x_0^r, we find a unique function u_1 solution of problem (1.1) on the maximal interval of existence $[0, t_1)$, and such that $u_1(0) = x_0^r$. If $t_1 < T$, it turns out that there exists $x_1^s := \lim_{t\to t_1^-} u_1(t)$ (the index s stands for "singular") and (t_1, x_1^s) is a degenerate approximable critical pair. Thus, Assumption 3 holds for (t_1, x_1^s). In particular, Lemma 2.4 tells us that Assumption 3 (i) and (iii) (together with Assumption 1 and 2) ensures the existence and uniqueness, up to time-translations, of the solution w_1 of (2.17) with $j = 1$, satisfying $w_1(-\infty) = x_1^s$. Moreover, there exists $\lim_{s\to+\infty} w_1(s) =: x_1^r$ (the index r stands for "regular") and x_1^r is a critical point of $f(t_1, \cdot)$. At this point, we use Assumption 4, so that $\nabla_x^2 f(t_1, x_1^r) > 0$, and begin again the procedure with (t_1, x_1^r) in place of $(0, x_0^r)$, to find the solution u_2 of (1.1), defined on the maximal (on the right) interval of existence $[t_1, t_2)$, and such that $u_2(t_1) = x_1^r$, and so on. Observe that, by Assumption 2, $\nabla_x^2 f(T, u_m(T^-))$ is positive definite. The functions $u_1, ..., u_m$, on their respective intervals of existence, give us the so selected solution u, as the next definition set.

Definition 2.7. *Under Assumptions 1-4, we define $u : [0, T] \to \mathbb{R}^n$ by:*

$$u(t) := u_j(t), \quad \textit{if } t \in [t_{j-1}, t_j) \ \textit{for some } j \in \{1, ..., m\},$$
$$u(T) := \lim_{t\to T} u_m(t),$$

where u_j, for j=1,...,m, is the function obtained in Proposition 2.6.

Note that, since (t_j, x_j^s) is an approximable critical pair for every $j \in \{1, ..., m-1\}$, Assumption 3 implies that the transversality conditions listed in Remark 2.3 hold for F (see (2.2) for a definition) at $\left(t_j, \begin{bmatrix} x_j^s \\ 0 \end{bmatrix}\right)$ for $j = 1, ..., m-1$, as shown in Remark 2.3. Moreover, conditions (TC2) and (TC3) have the same sign, otherwise we couldn't have found a solution of $\nabla_x f(t, \cdot) = 0$ on the left of t_j. Thus, there are two regular branches of solutions of $F(t, \cdot) = 0$ in a left neighbourhood of t_j. This is equivalent to say that there are two regular branches of solutions of $\nabla_x f(t, \cdot) = 0$ in a left neighbourhood of t_j. One of these branches is the already defined u_j. For $j = 1, ..., m-1$, we denote the other branch, which is the saddles' branch, by $\overline{u}_j$, and it is

$$\overline{u}_j : [t_j^*, t_j) \to \mathbb{R}^n, \quad \text{for some } t_j^* \in (t_{j-1}, t_j). \tag{2.18}$$

Note that, for $\delta > 0$ sufficiently small, it is possible to find t_j^δ and t_j^{**} such that

$$t_{j-1} < t_j^* < t_j^\delta < t_j < t_j^{**}, \tag{2.19}$$

and the following properties hold:

$$\text{for every } t \in [t_j^\delta, t_j), \ x \in \overline{B}\left(x_j^s, \frac{\delta}{4}\right) \text{ and } \nabla_x f(t, x) = 0$$
$$\text{if and only if } x \in \{u_j(t), \overline{u}_j(t)\}, \tag{2.20}$$

$$x \in \overline{B}(x_j^s, \delta) \text{ satisfies } \nabla_x f(t_j, x) = 0 \text{ if and only if } x = x_j^s, \tag{2.21}$$

$$|\nabla_x f(\cdot, \cdot)| > 0 \ \text{ on } (t_j, t_j^{**}] \times \overline{B}(x_j^s, \delta). \tag{2.22}$$

Throughout the following two sections, we denote by ω_{u_1} the modulus of continuity of u_1 on a compact easily deducible from the context.

3. Approximating by singular perturbations

We consider the equation

$$\varepsilon^2 A\ddot{u}^\varepsilon(t) + \varepsilon B\dot{u}^\varepsilon(t) + \nabla_x f(t, u^\varepsilon(t)) = 0, \quad t \in [0,T]. \tag{3.1}$$

In both the present approximation method and the one presented in Section 4, we take into account the following objects. Let $x_0^r \in \mathbb{R}^n$ be such that $\nabla_x f(0, x_0^r) = 0$ and $\nabla_x^2 f(0, x_0^r)$ is positive definite. We consider a point $(x_0, y_0) \in \mathbb{R}^{2n}$ such that v_0 is the solution of the autonomous problem

$$\begin{cases} A\ddot{v}_0(\sigma) + B\dot{v}_0(\sigma) + \nabla_x f(0, v_0(\sigma)) = 0, & \sigma \in [0, +\infty) \\ v_0(0) = x_0, \\ \dot{v}_0(0) = y_0, \end{cases} \tag{3.2}$$

and

$$\lim_{\sigma \to +\infty} v_0(\sigma) = x_0^r. \tag{3.3}$$

Under Assumptions 1 and 2, Lemma 2.5 ensures the existence of the solution of problem (3.2) and of the limit in (3.3). Also, it tells us that $v_0(+\infty)$ is a critical point of $f(0,\cdot)$ and that $\dot{v}_0(+\infty) = 0$. The main results of this section are given by the following two theorems, which describe how the function u of Definition 2.7 and the trajectories of the heteroclines w_j's at the jump times t_j's are approximated by suitable solutions u^ε of (3.1).

Theorem 3.1. *Under Assumptions 1-4, let $x_0^r \in \mathbb{R}^n$ be such that $\nabla_x f(0, x_0^r) = 0$ and $\nabla_x^2 f(0, x_0^r)$ is positive definite. Let $u : [0,T] \to \mathbb{R}^n$, with $u(0) = x_0^r$, be given by Definition 2.7 and $u^\varepsilon : [0,T] \to \mathbb{R}^n$ a solution of (3.1) such that*

$$(u^\varepsilon(0), \varepsilon\dot{u}^\varepsilon(0)) \to (x_0, y_0), \tag{3.4}$$

where (x_0, y_0) satisfies (3.2) and (3.3). Then, we have that
(1) $(u^\varepsilon, \varepsilon B\dot{u}^\varepsilon)$ converges uniformly to $(u, 0)$ on compact subsets of $(0,T]\backslash\{t_1, ..., t_{m-1}\}$;
(2) for every $j \in \{1, ..., m-1\}$, there exists a sequence $\{a_j^\varepsilon\}$, with $a_j^\varepsilon \to t_j$, and a heteroclinic solution w_j of

$$\begin{cases} A\ddot{w}_j(s) + B\dot{w}_j(s) + \nabla_x f(t_j, w_j(s)) = 0, \\ \lim_{s\to-\infty} w_j(s) = x_j^s, \\ \lim_{s\to-\infty} \dot{w}_j(s) = 0, \end{cases} \tag{3.5}$$

such that

$$(v_j^\varepsilon, \dot{v}_j^\varepsilon) \to (w_j, \dot{w}_j) \quad \textit{uniformly on compact subsets of } \mathbb{R},$$

where

$$v_j^\varepsilon(s) := u^\varepsilon(a_j^\varepsilon + \varepsilon s), \quad s \in \left[-\frac{a_j^\varepsilon}{\varepsilon}, \frac{T - a_j^\varepsilon}{\varepsilon}\right].$$

The next theorem can be viewed as a corollary of Theorem 3.1 and gives a geometric interpretation of how $(u^\varepsilon, \varepsilon B\dot{u}^\varepsilon)$ approximates $(u, 0)$ and the trajectory of $(w_j, B\dot{w}_j)$ for $j = 1, ..., m-1$. It deals with the following sets. Recall the heteroclines given by Proposition (2.6) and the function v_0 previously introduced. We define

$$\mathscr{I}_0 := \{(v_0(s), B\dot{v}_0(s)), \ \ s \geq 0\} \quad \text{and} \quad \mathscr{I}_j := \{(w_j(s), B\dot{w}_j(s)), \ \ s \in \mathbb{R}\}, \tag{3.6}$$

for $j = 1, ..., m-1$, and set

$$\Gamma^\varepsilon := \{(t, u^\varepsilon(t), \varepsilon B\dot{u}^\varepsilon(t)) : \ t \in [0,T]\}, \quad \Gamma := \Gamma_{reg} \cup \Gamma_{sing}, \tag{3.7}$$

where

$$\Gamma_{reg} := \{(t, u(t), 0) : \ t \in [0,T]\}, \tag{3.8}$$

and

$$\Gamma_{sing} := [\{0\} \times \mathscr{I}_0] \cup \bigcup_{j=1}^{m-1} \{t_j\} \times [\mathscr{I}_j \cup \{(x_j^s, 0)\}]. \tag{3.9}$$

Observe that the set Γ_{sing} does not change if we replace some w_j's by some of their time-translated. Here and in what follows, $d(\cdot,\cdot)$ denotes the euclidean distance either between two points or between a point and a set. We denote by d_H the *Haudorff distance*. Recall that if K_1 and K_2 are two compact subsets of a compact metric space, the Hausdorff distance between K_1 and K_2 is defined as

$$d_H(K_1, K_2) := \sup_{x\in K_1} d(x, K_2) + \sup_{x\in K_2} d(x, K_1).$$

Theorem 3.2. *Under the hypotheses of Theorem 3.1, we have that*

$$d_H(\Gamma^\varepsilon, \Gamma) \to 0, \quad \textit{as } \varepsilon \to 0^+.$$

In order to prove Theorem 3.1 and Theorem 3.2, we need some preliminary results. First, we state a property of uniform boundedness of the solutions of equation (3.1).

Lemma 3.3. *Let Assumption 1 hold and let t^ε be a sequence converging to some $\tilde{t} \in [0,T]$. Then, there exists a unique u^ε:$[t^\varepsilon, T] \to \mathbb{R}^n$ of class C^2, solution of the Dirichlet problem associated to (3.1) with initial condition at t^ε. Moreover, if $u^\varepsilon(t^\varepsilon)$ and $\varepsilon\dot{u}^\varepsilon(t^\varepsilon)$ are uniformly bounded as $\varepsilon \to 0^+$, then $u^\varepsilon(t)$ and $\varepsilon\dot{u}^\varepsilon(t)$ are uniformly bounded with respect to $t \in [t^\varepsilon, T]$ and ε sufficiently small.*

Proof. The standard theory of ordinary differential equations tells us that there exists locally a unique solution u^ε of the Cauchy problem associated to (3.1). Multiplying equation (3.1) by $\dot{u}^\varepsilon(t)$, it turns out the equation

$$\frac{\varepsilon^2}{2}\frac{d}{dt}A\dot{u}^\varepsilon\cdot\dot{u}^\varepsilon + \varepsilon B\dot{u}^\varepsilon\cdot\dot{u}^\varepsilon + \frac{d}{dt}f(t,u^\varepsilon) - f_t(t,u^\varepsilon) = 0,$$

which, by integration between t^ε and $t \in [t^\varepsilon, T]$ and by the positive definiteness of A and B, gives

$$\frac{\varepsilon^2}{2}\lambda^A_{min}|\dot{u}^\varepsilon(t)|^2 + f(t,u^\varepsilon(t)) \le \frac{\varepsilon^2}{2}A\dot{u}^\varepsilon(t^\varepsilon)\cdot\dot{u}^\varepsilon(t^\varepsilon) + f(t^\varepsilon, u^\varepsilon(t^\varepsilon)) + \int_{t^\varepsilon}^t f_t(\tau, u^\varepsilon(\tau))d\tau. \tag{3.10}$$

Then, by using Assumption 1 and (2.1), we have that

$$|u^\varepsilon(t)|^2 \le K_1^\varepsilon + K_2\int_0^t |u^\varepsilon(\tau)|^2 d\tau, \quad \text{for every} \quad t \in [0,T],$$

where

$$K_1^\varepsilon = \frac{1}{\tilde{b}}\left[\frac{\varepsilon^2}{2}A\dot{u}^\varepsilon(t^\varepsilon)\cdot\dot{u}^\varepsilon(t^\varepsilon) + f(t^\varepsilon, u^\varepsilon(t^\varepsilon)) + c(T-t^\varepsilon) + \tilde{a}\right], \qquad K_2 := \frac{d}{\tilde{b}}. \tag{3.11}$$

By differential inequalities (see, e. g., [8]), we obtain that

$$|u^\varepsilon(t)|^2 \le K_1^\varepsilon e^{K_2(T-t^\varepsilon)}, \quad \text{for every} \quad t \in [0,T],$$

so that, by hypothesis and by (3.11), $u^\varepsilon(t)$ is uniformly bounded with respect to $t \in [t^\varepsilon, T]$ and ε sufficiently small. This fact, together with (3.10), gives that also $\varepsilon\dot{u}^\varepsilon$ is uniformly bounded with respect to $t \in [t^\varepsilon, T]$ and ε small enough. This in particular implies that u^ε and $\dot{u}^\varepsilon$ are defined on $[t^\varepsilon, T]$ and completes the proof. □

The following proposition will play a crucial role in the proof of the main results of this section. In order to better handle equation (3.1), we use the function $F : [0,T]\times\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ defined in (2.2), so that (3.1) is equivalent to

$$\varepsilon\begin{bmatrix}\dot u^\varepsilon\\ \dot v^\varepsilon\end{bmatrix} = F\left(t,\begin{bmatrix}u^\varepsilon\\ v^\varepsilon\end{bmatrix}\right).$$

Next, we make use of Lemma 3.3 in the following way: if $u^\varepsilon(t^\varepsilon)$ and $\varepsilon\dot u^\varepsilon(t^\varepsilon)$ are uniformly bounded as $\varepsilon\to 0^+$, for a certain sequence t^ε converging to $\tilde t\in[0,T]$, then

$$\{(su^\varepsilon(t)+(1-s)u(t),\varepsilon sB\dot u^\varepsilon(t)) : (s,t)\in[0,1]\times[t^\varepsilon,T]\} \tag{3.12}$$

is uniformly bounded as $\varepsilon\to 0^+$. We denote by ω the modulus of continuity of $\nabla_\eta F(t,\cdot)$ on a compact which contains the set (3.12) for every ε small enough, ω uniform with respect to $t\in[0,T]$.

Proposition 3.4. *By referring to the previous paragraph for the notation, let $f:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ be a C^2 function, $0\le\bar t<\hat t\le T$ and $u\in C([\bar t,\hat t],\mathbb{R}^n)$ such that (1.1) is satisfied on $[\bar t,\hat t]$. Then, there exists $\alpha>0$ such that*

$$\nabla_x^2 f(t,u(t))\ge\alpha>0,\quad \textit{for every } t\in[\bar t,\hat t]. \tag{3.13}$$

Let $t^\varepsilon\in[\bar t,\hat t)$ be such that

$$t^\varepsilon\to\tilde t,\quad \textit{for some } \tilde t\in[\bar t,\hat t),$$

and consider $u^\varepsilon\in C^2([t^\varepsilon,T],\mathbb{R}^n)$ a solution of (3.1) on $[t^\varepsilon,T]$ such that $u^\varepsilon(t^\varepsilon)$ and $\varepsilon\dot u^\varepsilon(t^\varepsilon)$ are uniformly bounded as $\varepsilon\to 0^+$.

There exists a positive constant $C=C(f,u)$ such that, if $r\in(0,C)$ and

$$\limsup_{\varepsilon\to 0^+}|(u^\varepsilon(t^\varepsilon)-u(\tilde t),\varepsilon B\dot u^\varepsilon(t^\varepsilon))|<\min\{r,r\omega(2r)\}, \tag{3.14}$$

then

$$\limsup_{\varepsilon\to 0^+}||(u^\varepsilon-u,\varepsilon B\dot u^\varepsilon)||_{\infty,[t^\varepsilon,\hat t]}\le r. \tag{3.15}$$

The proof of Proposition 3.4 requires two lemmas.

Lemma 3.5. *Let $A\in\mathbb{R}^{n\times n}$ be such that*

$$\mathrm{Re}(\lambda)\le-\alpha<0,$$

for every λ eigenvalue of A, for a certain $\alpha>0$. Then, there exists a constant C_A, depending on A, such that

$$\left|e^{tA}\right|\le C_A e^{-\frac{\alpha}{2}t},\quad \textit{for every } t\ge 0.$$

The proof of Lemma 3.5 is straightforward, once A is written in Jordan canonical form. In the appendix, we recall more general estimates of this kind (see (5.2)-(5.3)). By the following remark, we underline the fact that the constant C_A of the previous lemma is not universal, but generally depending on A.

Remark 3.6. For $a\in\mathbb{R}$, consider the matrix $A=\begin{bmatrix}-1 & a\\ 0 & -1\end{bmatrix}$, whose spectrum is $\{-1\}$. Since A is the sum of the matrices $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$ and $\begin{bmatrix}0 & a\\ 0 & 0\end{bmatrix}$, which commute, it is easy to compute

$$e^{tA}=e^{-t}\begin{bmatrix}1 & at\\ 0 & 1\end{bmatrix}.$$

The norm of e^{tA} is $e^{-t}\sqrt{2+a^2t^2}$. Therefore, a constant C not depending on A and such that $\left|e^{tA}\right|\le Ce^{-\frac{t}{2}}$ should satisfy $\sqrt{2+a^2t^2}\le Ce^{\frac{t}{2}}$ for every $a\in\mathbb{R}$; but this is impossible.

Lemma 3.7. *Let $A \in \mathbb{R}^{n\times n}$ be such that*

$$\left|e^{tA}\right| \le Ce^{-\gamma t}, \quad \textit{for every } t \ge 0,$$

for some constants C, $\gamma > 0$. There exist two positive constants δ and b, depending only on C and γ, such that, if $B \in \mathbb{R}^{n\times n}$ and $|B| \le \delta$, then

$$\left|e^{t(A+B)}\right| \le be^{-\frac{\gamma}{2}t}, \quad \textit{for every } t \ge 0.$$

Proof. Observe that in the case in which A and B commute the proof is straightforward. Otherwise, for $x \in \mathbb{R}^n$, let us consider the solution v^x of the problem

$$\begin{cases} \dot{v}(t) = (A+B)v(t), & t > 0 \\ v(0) = x. \end{cases} \tag{3.16}$$

Since

$$\left|e^{t(A+B)}\right| = \sup_{x\in\mathbb{R}^n\setminus\{0\}} \frac{|v^x(t)|}{|x|},$$

the thesis follows if we prove that there exist δ, $b > 0$, depending only on C and γ, such that, if $|B| \le \delta$, then

$$|v^x(t)| \le be^{-\frac{\gamma}{2}t}|x|, \quad \text{for every } t \ge 0 \text{ and } x \in \mathbb{R}^n. \tag{3.17}$$

For certain constants δ, $b > 0$ to be chosen later, let us fix a function $z \in C([0,+\infty),\mathbb{R}^n)$ such that $|z(t)| \le be^{-\frac{\gamma}{2}t}|x|$ for all $t \ge 0$, and consider, for $|B| \le \delta$, the problem

$$\begin{cases} \dot{v}(t) = Av(t) + Bz(t), & t > 0 \\ z(0) = x. \end{cases} \tag{3.18}$$

The solution of 3.18 can be represented by the variation of constants formula, so that we can estimate it in the following way.

$$|v(t)| \le C\left\{e^{-\gamma t}|x| + \int_0^t e^{-\gamma(t-s)}|B||z(s)|ds\right\} \le C|x|e^{-\frac{\gamma}{2}t}\left\{1 + \frac{2b\delta}{\gamma}\right\}. \tag{3.19}$$

In order to obtain (3.17), we want $C\left(1 + \frac{2b\delta}{\gamma}\right) \le b$, therefore we choose

$$\delta < \frac{\gamma}{2C}, \qquad b \ge \frac{\gamma C}{\gamma - 2\delta C}. \tag{3.20}$$

Now, we define the space

$$X := \left\{v \in C([0,+\infty),\mathbb{R}^n) \;:\; v(0) = x \text{ and } \sup_{t\in[0,+\infty)} v(t)e^{\frac{\gamma}{2}t} < \infty\right\},$$

which is a Banach space endowed with the norm $\|v\|_X := \sup_{t\in[0,+\infty)} v(t)e^{\frac{\gamma}{2}t}$, and the subset

$$\Omega := \{v \in X \;:\; \|v\|_X \le |x|b\}.$$

From (3.19) and thanks to the choice (3.20), we have obtained that the operator

$$G : \Omega \to \Omega,$$

that to each $z \in \Omega$ associates the solution of (3.18), is well-defined. If we prove that G is a contraction from Ω to Ω, we will prove that the solution v of (3.16) satisfies (3.17), that is our aim. Let z_1, $z_2 \in \Omega$ and suppose $|B| \le \delta$. Then, we have that

$$\|G(z_1) - G(z_2)\|_X = \sup_{t\ge 0} e^{\frac{\gamma}{2}t}\left|\int_0^t e^{(t-s)A}B[z_1(s) - z_2(s)]ds\right| \le \frac{2C\delta}{\gamma}\|z_1 - z_2\|_X.$$

From (3.20), it descends that $\frac{2C\delta}{\gamma} < 1$, so that G is a contraction from Ω to Ω. $\square$

Proof of Proposition 3.4. Let u^ε be a solution of (3.1) and define $W_\varepsilon := \begin{bmatrix} u^\varepsilon \\ v^\varepsilon \end{bmatrix} - W$, where $W := \begin{bmatrix} u \\ 0 \end{bmatrix}$, so that equation (3.1) is equivalent to

$$\varepsilon \dot W_\varepsilon = F(t, W + W_\varepsilon) - \varepsilon \dot W, \tag{3.21}$$

with F defined as in (2.2). Observe that $W_\varepsilon = \begin{bmatrix} u^\varepsilon - u \\ \varepsilon B \dot u^\varepsilon \end{bmatrix}$. Set

$$M(t) := \nabla_\eta F(t, W(t)), \qquad t \in [\bar t, \hat t],$$

and notice that, from the regularity assumptions on f and u, it descends that $M \in C([\bar t, \hat t])$. First, let us explain how we find the constant C of the statement. Since (3.13) holds, we can prove, as done in Remark 2.3, that there exists $\beta > 0$ such that $\mathrm{Re}(\lambda) \le -\beta < 0$ for every λ eigenvalue of $M(s)$, for every $s \in [\bar t, \hat t]$. Therefore, from Lemma 3.5 and Lemma 3.7, it turns out that there exists $b > 0$ such that

$$\left| e^{tM(s)} \right| \le b e^{-\frac{\beta}{4} t}, \quad \text{for every } t \ge 0 \text{ and } s \in [\bar t, \hat t]. \tag{3.22}$$

Indeed, from Lemma 3.5, we have that, for every $t \ge 0$,

$$\left| e^{tM(s)} \right| \le C_{M(s)} e^{-\frac{\beta}{2} t}, \tag{3.23}$$

with $C_{M(s)} > 0$ a constant depending on $M(s)$ for every $s \in [\bar t, \hat t]$. Considering (3.23) for a certain $s_0 \in [\bar t, \hat t]$, let δ_0, $b_0 > 0$, depending on $C_{M(s_0)}$ and $\frac{\beta}{2}$, be given by Lemma 3.7. By the uniform continuity of M on $[\bar t, \hat t]$, there exists $\sigma_0 > 0$ and a finite number of s_i in $[\bar t, \hat t]$ such that, if $s \in [\bar t, \hat t]$, then $|s - s_i| < \sigma_0$ for some i and $|M(s) - M(s_i)| \le \delta_0$, so that, by Lemma 3.7,

$$\left| e^{tM(s)} \right| = \left| e^{t(M(s_i) + M(s) - M(s_i))} \right| \le b_0 e^{-\frac{\beta}{4} t}, \quad \text{for every } t \ge 0.$$

Now, let $C > 0$ be a constant (depending on b and β and , in turn, on f and u) such that, if $0 < r < C$, then

$$\omega(2r) \le \frac{1}{2b} \left(1 + \frac{10}{\beta} \max\{1, 2b\} \right)^{-1}. \tag{3.24}$$

The reason why the estimate (3.24) is needed will be clear at the end of the proof. By now, let $0 < r < C$ and suppose that (3.14) holds true for a certain $t^\varepsilon \to \tilde t \in [\bar t, \hat t)$. Then, there exists $\varepsilon_r > 0$ such that

$$|(u^\varepsilon(t^\varepsilon) - u(\tilde t), \varepsilon B \dot u^\varepsilon(t^\varepsilon))| \le \min\{r, r\omega(2r)\}, \quad \text{for every } \varepsilon \in (0, \varepsilon_r). \tag{3.25}$$

Since $t^\varepsilon \to \tilde t$, it is easy to check that (3.25) implies, up to a smaller ε_r, that

$$|W_\varepsilon(t^\varepsilon)| \le 2 \min\{r, r\omega(2r)\}, \quad \text{for every } \varepsilon \in (0, \varepsilon_r). \tag{3.26}$$

Therefore, it makes sense to define, for $\varepsilon \in (0, \varepsilon_r)$,

$$\hat t^\varepsilon := \inf\{t \in [t^\varepsilon, \hat t] \ : \ |W_\varepsilon(t)| > 2r\},$$

with the convention $\inf \emptyset = \hat t$, so that $\|W_\varepsilon\|_{\infty, [t^\varepsilon, \hat t^\varepsilon]} \le 2r$ for every $\varepsilon \in (0, \varepsilon_r)$.
Claim. There exists $\tilde\varepsilon_r \in (0, \varepsilon_r]$ such that

$$\|W_\varepsilon\|_{\infty, [t^\varepsilon, \hat t^\varepsilon]} \le r, \quad \text{for every } \varepsilon \in (0, \tilde\varepsilon_r).$$

Observe that the claim implies that $\hat t^\varepsilon = \hat t$ and, in turn, that $\|W_\varepsilon\|_{\infty, [t^\varepsilon, \hat t]} \le r$ for every $\varepsilon \in (0, \tilde\varepsilon_r)$, that is (3.15).
Proof of the claim. Using again the uniform continuity of M on $[\bar t, \hat t]$, let $\sigma > 0$ be such that $||M(t) - M(s)|| \le \omega(2r)$ if $|s - t| < \sigma$, and define

$$\tau_i = \tau_i(\varepsilon) := t^\varepsilon + i\sigma, \quad \text{for} \quad i = 0, ..., k_\varepsilon, \quad \text{where} \quad k_\varepsilon := \left\lfloor \frac{\hat t^\varepsilon - t^\varepsilon}{\sigma} \right\rfloor.$$

and

$$M_\varepsilon(t) := \begin{cases} M(t^\varepsilon), & t \in [t^\varepsilon, \tau_1) \\ M(\tau_1), & t \in [\tau_1 \tau_2) \\ \vdots \\ M(\tau_{k_\varepsilon}), & t \in [\tau_{k_\varepsilon}, \hat{t}^\varepsilon]. \end{cases}$$

Observe that $M_\varepsilon(t) = M\left(t^\varepsilon + \lfloor \frac{t-t^\varepsilon}{\sigma} \rfloor\right)$. With such definitions, we obtain that

$$||M_\varepsilon - M||_{\infty,[t^\varepsilon,\hat{t}^\varepsilon]} \leq \omega(2r). \tag{3.27}$$

Let us write equation (3.21) on $[t^\varepsilon, \hat{t}^\varepsilon]$ in the following equivalent way:

$$\varepsilon \dot{W}_\varepsilon = M_\varepsilon W_\varepsilon + H_\varepsilon,$$

where

$$H_\varepsilon := (M - M_\varepsilon)W_\varepsilon + [F(t, W + W_\varepsilon) - MW_\varepsilon] - \varepsilon \dot{W}.$$

Clearly, there exists $\tilde{\varepsilon}_r \in (0, \varepsilon_r]$ such that

$$||\varepsilon \dot{W}||_{\infty,[t^\varepsilon,\hat{t}^\varepsilon]} \leq r\omega(2r), \quad \text{for every } \varepsilon \in (0, \tilde{\varepsilon}_r). \tag{3.28}$$

Noticing that $F(\cdot, W(\cdot)) \equiv 0$ on $[\bar{t}, \hat{t}]$, it turns out that

$$\begin{aligned} &||F(t, W + W_\varepsilon) - MW_\varepsilon||_{\infty,[t^\varepsilon,\hat{t}^\varepsilon]} \leq \\ &\quad 2r \sup\left\{|\nabla_\eta F(t, W(t) + sW_\varepsilon(t)) - M(t)| : (s,t) \in [0,1] \times [t^\varepsilon, \hat{t}^\varepsilon]\right\} \leq 2r\omega(2r). \end{aligned} \tag{3.29}$$

(3.27), (3.28) and (3.29) imply that

$$||H_\varepsilon||_{\infty,[t^\varepsilon,\hat{t}^\varepsilon]} \leq 5r\omega(2r). \tag{3.30}$$

By setting $Z_\varepsilon(t) := W_\varepsilon(\varepsilon t)$, let us consider another equation equivalent to (3.21) on $[t^\varepsilon, \hat{t}^\varepsilon]$:

$$\dot{Z}_\varepsilon = M_\varepsilon(\varepsilon t) Z_\varepsilon + H_\varepsilon(\varepsilon t), \qquad t \in \left[\frac{t^\varepsilon}{\varepsilon}, \frac{\hat{t}^\varepsilon}{\varepsilon}\right]. \tag{3.31}$$

If $k_\varepsilon = 0$, that is $\hat{t}^\varepsilon - t^\varepsilon < \sigma$, the solution of (3.31) is

$$Z_\varepsilon(t) = e^{\left(t - \frac{t^\varepsilon}{\varepsilon}\right) M(t^\varepsilon)} Z_\varepsilon\left(\frac{t^\varepsilon}{\varepsilon}\right) + \int_{\frac{t^\varepsilon}{\varepsilon}}^{t} e^{(t-\tau)M(t^\varepsilon)} H_\varepsilon(\varepsilon\tau) d\tau.$$

Then, by using (3.22) and (3.30), we have that

$$||W_\varepsilon||_{\infty,[t^\varepsilon,\hat{t}^\varepsilon]} = ||Z_\varepsilon||_{\infty,[\frac{t^\varepsilon}{\varepsilon},\frac{\hat{t}^\varepsilon}{\varepsilon}]} \leq b\left[|W_\varepsilon(t^\varepsilon)| + \frac{20}{\beta} r\omega(2r)\right] \leq 2b\left(1 + \frac{10}{\beta}\right) r\omega(2r),$$

where the last inequality is due to (3.26). Then, the thesis follows from (3.24).
If $k_\varepsilon \neq 0$, we define Z_ε^0 as the solution of equation (3.31) in $[0, \frac{\tau_1}{\varepsilon})$ and, for $i = 1, ..., k_\varepsilon$, we define Z_ε^i as the solution of equation (3.31) in $\left[\frac{\tau_i}{\varepsilon}, \min\left\{\frac{\tau_{i+1}}{\varepsilon}, \frac{\hat{t}^\varepsilon}{\varepsilon}\right\}\right)$ with $Z_\varepsilon^{(i-1)}\left(\frac{\tau_i}{\varepsilon}-\right)$ as initial condition at $\frac{\tau_i}{\varepsilon}$. By using the variation of constants formula, it turns out that

$$|Z_\varepsilon^0| \leq R_\varepsilon^0, \qquad \text{on } \left[\frac{t^\varepsilon}{\varepsilon}, \frac{\tau_1}{\varepsilon}\right), \tag{3.32}$$

where

$$R_\varepsilon^0(t) := b\left(e^{-\frac{\beta}{4}\left(t - \frac{t^\varepsilon}{\varepsilon}\right)} |W_\varepsilon(t^\varepsilon)| + \frac{20}{\beta} r\omega(2r)\right), \quad t \in \left[\frac{t^\varepsilon}{\varepsilon}, \frac{\tau_1}{\varepsilon}\right],$$

and

$$|Z_\varepsilon^i| \leq R_\varepsilon^i, \quad \text{on } \left[\frac{\tau_i}{\varepsilon}, \frac{\tau_{i+1}}{\varepsilon}\right), \ i = 1, ..., k_\varepsilon - 1, \tag{3.33}$$

$$|Z_\varepsilon^{k_\varepsilon}| \leq R_\varepsilon^{k_\varepsilon}, \quad \text{on } \left[\frac{\tau_{k_\varepsilon}}{\varepsilon}, \frac{t_\varepsilon}{\varepsilon}\right], \tag{3.34}$$

where

$$R^i_\varepsilon(t) := b\left[e^{-\frac{\beta}{4}\left(t-\frac{\tau_i}{\varepsilon}\right)}R^{i-1}_\varepsilon\left(\frac{\tau_i}{\varepsilon}\right)+\frac{4}{\beta}\omega(r)r\right],\quad i=1,...,k_\varepsilon,\quad t\in\left[\frac{\tau_i}{\varepsilon},\frac{\tau_{i+1}}{\varepsilon}\right].$$

It is easy to check that, up to a smaller $\tilde{\varepsilon}_r$ such that $b\exp\left(-\frac{\beta\sigma}{4\varepsilon}\right)\leq\frac{1}{2}$ for every $\varepsilon\in(0,\tilde{\varepsilon}_r)$, $R^i_\varepsilon\left(\frac{\tau_{i+1}}{\varepsilon}\right)\leq 2r\omega(2r)\left(1+\frac{20b}{\beta}\right)$, for $i=0,...,k_\varepsilon-1$ and thus that

$$R^i_\varepsilon\left(\frac{\tau_i}{\varepsilon}\right)\leq 2br\omega(2r)\left(11+\frac{20b}{\beta}\right),$$

for $i=0,...,k_\varepsilon$. Hence, from the choice made in (3.24), we have that $R^i_\varepsilon\left(\frac{\tau_i}{\varepsilon}\right)\leq r$ for $i=0,...,k_\varepsilon$, and therefore, since R^i_ε is decreasing in t, from (3.32)-(3.34) we obtain that

$$\begin{aligned}||W_\varepsilon||_{\infty,[t^\varepsilon,\hat{t}^\varepsilon]} &\leq \max\left\{\max_{i\in\{0,\dots,k_\varepsilon-1\}}||Z^i_\varepsilon||_{\infty,\left[\frac{\tau_i}{\varepsilon},\frac{\tau_{i+1}}{\varepsilon}\right)},||Z^{k_\varepsilon}_\varepsilon||_{\infty,\left[\frac{\tau_{k_\varepsilon}}{\varepsilon},\frac{t_\varepsilon}{\varepsilon}\right]}\right\}\\ &\leq \max_{i\in\{0,\dots,k_\varepsilon\}}R^i_\varepsilon\left(\frac{\tau_i}{\varepsilon}\right)\leq r,\quad\text{for every }\varepsilon\in(0,\tilde{\varepsilon}_r).\end{aligned}$$

□

Proposition 3.4 allows us to prove a first part of Theorem 3.1.

Proof of Theorem 3.1 restricted to $(0,t_1)$. We begin the proof of Theorem 3.1 by showing that

$$(u^\varepsilon,\varepsilon B\dot{u}^\varepsilon)\to(u,0)\quad\text{uniformly on compact subsets of }\ (0,t_1). \tag{3.35}$$

Consider $[t^*,\hat{t}]\subseteq(0,t_1)$ and let $\delta>0$ be sufficiently small, in order to apply Proposition 3.4. Observe that the function

$$v^\varepsilon_0(s) := u^\varepsilon(\varepsilon s),\quad s\in\left[0,\frac{T}{\varepsilon}\right], \tag{3.36}$$

satisfies the problem

$$\begin{cases}A\ddot{v}^\varepsilon_0(s)+B\dot{v}^\varepsilon_0(s)+\nabla_x f(\varepsilon s,v^\varepsilon_0(s))=0,\quad s\in\left[0,\frac{T}{\varepsilon}\right]\\ v^\varepsilon_0(0)=u^\varepsilon(0),\\ \dot{v}^\varepsilon_0(0)=\varepsilon\dot{u}^\varepsilon(0),\end{cases}$$

so that, by (3.4),

$$(v^\varepsilon_0,\dot{v}^\varepsilon_0)\to(v_0,\dot{v}_0)\quad\text{uniformly on compact subsets of }[0,+\infty), \tag{3.37}$$

where v_0 satisfies (3.2) and (3.3). This convergence, the limit in (3.3) and the fact that $\dot{v}_0(+\infty)=0$ imply that there exists $s^\delta_0>0$ such that

$$|(v_0(s)-x^r_0,B\dot{v}_0(s))|\leq\frac{1}{2}\min\{\delta,\delta\omega(2\delta)\},\quad\text{for every }s\geq s^\delta_0, \tag{3.38}$$

and

$$\limsup_{\varepsilon\to0^+}|(u^\varepsilon(\varepsilon s^\delta_0)-x^r_0,\varepsilon B\dot{u}^\varepsilon(\varepsilon s^\delta_0))|<\min\{\delta,\delta\omega(2\delta)\}, \tag{3.39}$$

where ω is defined in Proposition 3.4. Then, by using Proposition 3.4 with $\bar{t}=\tilde{t}=0$ and u_1 in place of u, so that $u(\tilde{t})=x^r_0$, and

$$b^\varepsilon_0 := \varepsilon s^\delta_0 \tag{3.40}$$

in place of t^ε, we obtain that

$$\limsup_{\varepsilon\to0^+}||(u^\varepsilon-u,\varepsilon B\dot{u}^\varepsilon)||_{\infty,[t^*,\hat{t}]}\leq\limsup_{\varepsilon\to0^+}||(u^\varepsilon-u,\varepsilon B\dot{u}^\varepsilon)||_{\infty,[b^\varepsilon_0,\hat{t}]}\leq\delta, \tag{3.41}$$

and, in turn, (3.35). □

Statement (3.35), together with the fact that $\lim_{t\to t_1^-} u(t) = x_1^s$ and the definition of $t_1^\delta < t_1$ (see (2.20)), implies that

$$|(u^\varepsilon(t_1^\delta) - x_1^s, \varepsilon B\dot{u}^\varepsilon(t_1^\delta))| \leq \frac{\delta}{2},$$

for every ε sufficiently small. Then, we consider, in dependence on δ, the first time larger that t_1^δ in which $(u^\varepsilon(t), \varepsilon B\dot{u}^\varepsilon(t))$ escapes from $\overline{B}((x_1^s, 0), \delta)$:

$$a_1^\varepsilon := \max\{\bar{t} \in [t_1^\delta, t_1^{**}] : (u^\varepsilon(t), \varepsilon B\dot{u}^\varepsilon(t)) \in \overline{B}((x_1^s, 0), \delta) \text{ for every } t \in [t_1^\delta, \bar{t}]\}, \tag{3.42}$$

where $t_1^{**} > t_1$ is defined in (2.22). Observe that, for every ε small enough, a_1^ε is well-defined, since the maximum is taken over a nonempty set. Notice that, if $a_1^\varepsilon < t_1^{**}$, it turns out that $(u^\varepsilon(a_1^\varepsilon), \varepsilon B\dot{u}^\varepsilon(a_1^\varepsilon)) \in \partial B((x_1^s, 0), \delta)$.

Lemma 3.8. *For every $\delta > 0$ small enough, we have that*

$$a_1^\varepsilon \to t_1.$$

Proof. We divide the proof in two steps. Fix $\delta > 0$ sufficiently small.
(i) Let $\tau_k \geq t_1^\delta$ be a sequence approaching t_1 from the left, as $k \to +\infty$. From (3.35) we have that, for every k, there exists ε_k such that $||(u^\varepsilon - u, \varepsilon B\dot{u}^\varepsilon)||_{\infty,[t_1^\delta,\tau_k]} \leq \frac{\delta}{2}$ for all $\varepsilon \in (0, \varepsilon_k)$. Thus, also in view of the definition of t_1^δ, we obtain that

$$(u^\varepsilon(t), \varepsilon B\dot{u}^\varepsilon(t)) \in \overline{B}((x_1^s, 0), \delta), \quad \text{for every } t \in [t_1^\delta, \tau_k] \text{ and } \varepsilon \in (0, \varepsilon_k),$$

and, in turn, from the definition of a_1^ε, that $a_1^\varepsilon \geq \tau_k$ for every $\varepsilon \in (0, \varepsilon_k)$ and every k, so that

$$\liminf_{\varepsilon\to 0} a_1^\varepsilon \geq t_1.$$

(ii) Here, we want to prove that

$$\limsup_{\varepsilon\to 0} a_1^\varepsilon \leq t_1.$$

Suppose, by contradiction, that there exists a sequence $\{\varepsilon_k\}$, with $\varepsilon_k \to 0$ as $k \to +\infty$, and a certain $\hat{t} > t_1$ such that $\{a_1^{\varepsilon_k}\} \subseteq [\hat{t}, t_1^{**}]$. Then, up to a subsequence, we have that

$$a_1^{\varepsilon_k} \to \tilde{t} \in [\hat{t}, t_1^{**}]. \tag{3.43}$$

Now, observe that the function $v_1^\varepsilon := u^\varepsilon(a_1^\varepsilon + \varepsilon s)$ satisfies the problem

$$\begin{cases} A\ddot{v}_1^\varepsilon(s) + B\dot{v}_1^\varepsilon(s) + \nabla_x f(a_1^\varepsilon + \varepsilon s, v_1^\varepsilon(s)) = 0, & s \in \left[-\frac{a_1^\varepsilon}{\varepsilon}, \frac{T - a_1^\varepsilon}{\varepsilon}\right] \\ v_1^\varepsilon(0) = u^\varepsilon(a_1^\varepsilon), \\ \dot{v}_1^\varepsilon(0) = \varepsilon\dot{u}^\varepsilon(a_1^\varepsilon). \end{cases}$$

From the definition of a_1^ε, we have that $(v_1^\varepsilon(0), B\dot{v}_1^\varepsilon(0)) \in \overline{B}((x_1^s, 0), \delta)$, and, in turn, up to a further subsequence, that

$$(v_1^{\varepsilon_k}(0), B\dot{v}_1^{\varepsilon_k}(0)) \to (z, \dot{z}) \in \overline{B}((x_1^s, 0), \delta). \tag{3.44}$$

(3.43) and (3.44) imply that

$$(v_1^{\varepsilon_k}, \dot{v}_1^{\varepsilon_k}) \to (w, \dot{w}), \tag{3.45}$$

uniformly on compact subsets of a common interval of existence, where w is the solution of the problem

$$\begin{cases} A\ddot{w}(s) + B\dot{w}(s) + \nabla_x f(\tilde{t}, w(s)) = 0, \\ w(0) = z, \\ B\dot{w}(0) = \dot{z}. \end{cases} \tag{3.46}$$

It is easy to check, by using (3.45), Lemma 2.5 and the definition of a_1^ε, that w and $\dot w$ are defined on all $\mathbb{R}$ and that $(w(s), B\dot w(s)) \in \overline{B}((x_1^s, 0), \delta)$ for every $s \in (-\infty, 0]$. Moreover, by Lemma 2.5, there exists the limit

$$\lim_{s\to-\infty}(s) := w(-\infty) \in \overline{B}(x_1^s, \delta), \tag{3.47}$$

and it satisfies

$$\nabla_x f(\tilde t, w(-\infty)) = 0. \tag{3.48}$$

(3.47) and (3.48) contradict the fact that $\tilde t \in (t_1, t_1^{**}]$, since, by definition of t_1^{**} (recall that δ has to be small enough), it must be $|\nabla_x f(\tilde t, \cdot)| > 0$ in $\overline{B}(x_1^s, \delta)$. □

By Lemma 2.4, any solution of problem (3.5) differs from any other solution by time translation, so that the trajectories $\mathscr{I}_j$'s (defined in (3.6)) are uniquely defined. By using Morse-Sard Theorem (see, e. g., [9, Theorem 1.3 ch. 3]) applied to the function $t \mapsto |(w_j(t) - x_j^s, B\dot w_j(t))|^2$, it is easy to check that the set

$$E_j := \{\delta > 0 \mid \mathscr{I}_j \text{ is tangent to } \partial B((x_j^s, 0), \delta) \text{ at a point of intersection}\} \tag{3.49}$$

has zero measure. The reason why we introduce the sets E_j, $j = 1, ..., m-1$, will be clear in the next proof.

Proof of Theorem 3.1, complete. Let δ be sufficiently small. First, let us prove statement (2) in the case $j = 1$. Consider an arbitrary sequence $\varepsilon_k \to 0$ and the function

$$v_1^\varepsilon(s) := u^\varepsilon(a_1^\varepsilon + \varepsilon s), \qquad s \in \left[-\frac{a_1^\varepsilon}{\varepsilon}, \frac{T - a_1^\varepsilon}{\varepsilon}\right], \tag{3.50}$$

with a_1^ε given by (3.42). Observe that v_1^ε depends on δ. By using Lemma 3.8 and arguing similarly to its proof, we can show that, up to a subsequence,

$$(v_1^{\varepsilon_k}(0), B\dot v_1^{\varepsilon_k}(0)) \to (z, \dot z) \in \partial B((x_1^s, 0), \delta),$$

and that $(v_1^{\varepsilon_k}, \dot v_1^{\varepsilon_k}) \to (w_1, \dot w_1)$ uniformly on compact subsets of $\mathbb{R}$, where w_1 is the solution of problem (3.46), with t_1 in place of $\tilde t$, and satisfies

$$w_1(-\infty) = x_1^s, \quad \dot w_1(-\infty) = 0. \tag{3.51}$$

The first condition in (3.51) is due to the fact that $w_1(-\infty) \in \overline{B}(x_1^s, \delta)$ must be a critical point of $f(t_1, \cdot)$ and, since we are supposing δ small enough, the unique critical point of $f(t_1, \cdot)$ in $\overline{B}(x_1^s, \delta)$ is x_1^s (see (2.21)). Observe that w_1 depends on δ. To conclude the proof, it remains to show that, given any other sequence $\varepsilon_h \to 0$, $(v_{\varepsilon_h}, \dot v_{\varepsilon_h})$ converges (up to a subsequence) to $(w, \dot w)$, as $(v_1^{\varepsilon_k}, \dot v_1^{\varepsilon_k})$ does. By repeating the same arguments above, we have that, up to a subsequence, $(v_1^{\varepsilon_h}, \dot v_1^{\varepsilon_h}) \to (\tilde w_1, \dot{\tilde w}_1)$ uniformly on compact subsets of $\mathbb{R}$, where $(\tilde w_1, \dot{\tilde w}_1)$ satisfies the same system that $(w_1, \dot w_1)$ satisfies, and the conditions in (3.51). Therefore, by Lemma 2.4, we have that

$$\tilde w_1(s) = w_1(s + s_0), \quad s \in \mathbb{R}, \tag{3.52}$$

for a certain constant s_0, which we can assume to be nonnegative. Let us suppose, by contradiction, that $s_0 > 0$. By (3.52) and the definition of a_1^ε, we have, on one hand, that

$$(w_1(s), B\dot w_1(s)) \in \overline{B}((x_1^s, 0), \delta), \quad \text{for every } s \le s_0; \tag{3.53}$$

on the other hand, since E_1 has measure 0 (see (3.49)), it is not restrictive to assume $\delta \notin E_1$, so that there exists $\sigma > 0$ such that $(w_1(s), B\dot w_1(s)) \notin \overline{B}((x_1^s, 0), \delta)$ for every $s \in (0, \sigma)$, against (3.53). Therefore, it has to be $s_0 = 0$ and, in turn, $w_1 = \tilde w_1$. Thus, we have proved that

$$(v_1^\varepsilon, \dot v_1^\varepsilon) \to (w_1, \dot w_1) \text{ uniformly on compact subsets of } \mathbb{R}, \tag{3.54}$$

where, among the solutions of the problem

$$\begin{cases} A\ddot{w}(s) + B\dot{w}(s) + \nabla_x f(t_1, w(s)) = 0, \\ \lim_{s\to-\infty} w(s) = x_1^s, \end{cases}$$

w_1 is the one such that $(w_1(0), B\dot{w}_1(0)) = (z, \dot{z})$ (where $(u^\varepsilon(a_1^\varepsilon), \varepsilon B\dot{u}^\varepsilon(a_1^\varepsilon)) \to (z, \dot{z}) \in \partial B((x_1^s, 0), \delta))$. Moreover,

$$(w_1(s), B\dot{w}_1(s)) \in \overline{B}((x_1^s, 0), \delta), \quad \text{for every } s \leq 0. \tag{3.55}$$

Now, recall that, by Proposition 2.6, the behaviour of w_1 at $+\infty$ selects a point which allows us to find, as done for $[0, t_1)$, a solution u_2 of $\nabla_x f(t, \cdot) = 0$ on $[t_1, t_2)$. More precisely:

$$\lim_{s\to+\infty} (w_1(s), \dot{w}_1(s)) = (x_1^r, 0), \qquad u_2(t_1) = x_1^r,$$

$$\nabla_x f(\cdot, u_2(\cdot)) \equiv 0 \ \text{ and } \ \nabla_x f(\cdot, u_2(\cdot)) \text{ is positive definite on } [t_1, t_2).$$

In particular, there exists $s_1^\delta > 0$ such that

$$|(w_1(s) - x_1^r, B\dot{w}_1(s))| \leq \frac{\delta}{2} \ \text{ for every } s \geq s_1^\delta. \tag{3.56}$$

Moreover, due to (3.54) and to the definition of v_1^ε, there exists $\varepsilon_\delta > 0$ such that

$$|(u^\varepsilon(b_1^\varepsilon) - w_1(s_1^\delta), \varepsilon B\dot{u}^\varepsilon(b_1^\varepsilon) - B\dot{w}_1(s_1^\delta))| < \frac{\delta}{2}, \quad \text{for every } \varepsilon \in (0, \varepsilon_\delta), \tag{3.57}$$

where

$$b_1^\varepsilon = b_1^\varepsilon(\delta) := a_1^\varepsilon + s_1^\delta \varepsilon,$$

so that

$$|(u^\varepsilon(b_1^\varepsilon) - x_1^r, \varepsilon B\dot{u}^\varepsilon(b_1^\varepsilon)| < \delta \quad \text{for every } \varepsilon \in (0, \varepsilon_\delta). \tag{3.58}$$

By using (3.58) and Proposition 3.4 with $\tilde{t} = t_1$, b_1^ε in place of t^ε, u_2 in place of u (since it can be $b_1^\varepsilon < t_1$, note that u_2 is defined in a left neighbourhood of t_1, also) and δ in place of $\min\{r, r\omega(2r)\}$, we can prove the first statement of the theorem restricted to (t_1, t_2). This fact allows us to extend the definition of a_1^ε and b_1^ε to the cases $j = 2, ..., m-1$ and to repeat the same arguments used until here to complete the proof of the theorem. □

Remark 3.9. By looking at the hypotheses of Theorem 3.1, observe that in the case in which

$$(u^\varepsilon(0), \varepsilon\dot{u}^\varepsilon(0)) \to (x_0^r, 0),$$

we have that

$$(u^\varepsilon, \varepsilon B\dot{u}^\varepsilon) \to (u, 0) \ \text{ on compact subsets of } \ [0, T] \setminus \{t_1, ..., t_{m-1}\}.$$

In order to check this on a compact $[0, \hat{t}]$ of $(0, t_1)$ (the rest part of the proof is the same of Theorem 3.1), it is enough to apply Proposition 3.4 with $\bar{t} = \tilde{t} = 0$ and $t^\varepsilon = 0$.

In the proof of Theorem 3.1, we have defined some special times which we collect in the following definition.

Definition 3.10. *Let $\delta > 0$ be sufficiently small. For $j = 1, ..., m-1$, we define*

$$a_j^\varepsilon := \max\{\bar{t} \in [t_j^\delta, t_j^{**}] : (u^\varepsilon(t), \varepsilon B\dot{u}^\varepsilon(t)) \in \overline{B}((x_j^s, 0), \delta) \text{ for every } t \in [t_j^\delta, \bar{t}]\},$$

where $t_j^\delta \in (t_{j-1}, t_j)$ is defined in (2.20) and satisfies

$$|(u^\varepsilon(t_j^\delta) - x_j^s, \varepsilon B\dot{u}^\varepsilon(t_j^\delta))| \leq \frac{\delta}{2},$$

for every ε small enough. Moreover, we set

$$b_j^\varepsilon = b_j^\varepsilon(\delta) := a_j^\varepsilon + s_j^\delta \varepsilon, \quad j = 1, ..., m-1,$$

where $s_j^\delta > 0$ *is such that*

$$|(w_j(s) - x_j^r, B\dot{w}_j(s))| \leq \frac{\delta}{2}, \quad \textit{for every } s \geq s_j^\delta.$$

We are now in position to prove the last result of this section.

Proof of Theorem 3.2. See (3.6)-(3.9) for the definitions of Γ and Γ^ε. Chosen $\delta > 0$ small enough and such that

$$\delta \notin \bigcup_{j=1}^{m-1} E_j,$$

where E_j, for $j = 1, ..., m-1$, is defined in (3.49) (recall that $\bigcup_{j=1}^{m-1} E_j$ has zero measure), we suppose to work with the particular heteroclinic solutions depending on δ found in the proof of Theorem 3.1 (see (3.55)). Due to the definition of the Hausdorff distance, we divide the proof in two parts.
(a) Here, we show that there exists $\varepsilon_\delta > 0$ such that

$$\sup_{\Gamma^\varepsilon} d(\cdot, \Gamma) \leq 2\delta, \quad \text{for every } \varepsilon \in (0, \varepsilon_\delta). \tag{3.59}$$

Set

$$d_\varepsilon(t) := d((t, u^\varepsilon(t), \varepsilon B\dot{u}^\varepsilon(t)), \Gamma), \qquad t \in [0, T].$$

By referring to (2.19)-(2.20), to (3.40) and Definition 3.10 for the notation, and in view of the fact that

$$b_0^\varepsilon \to 0, \qquad a_j^\varepsilon, b_j^\varepsilon \to t_j, \quad \text{for } j = 1, ..., m-1, \tag{3.60}$$

consider, for every ε small enough, the partition

$$0 < b_0^\varepsilon < t_1^\delta < a_1^\varepsilon < b_1^\varepsilon < ... < b_{m-1}^\varepsilon < T.$$

In order to prove (3.59), it is enough to give a proper estimate of d_ε on $[0, b_1^\varepsilon)$, since we can proceed in a similar way on the remaining part of the interval $[0, T]$. By looking at the definition of v_0^ε (see (3.36)), observe that

$$\begin{aligned} \sup_{t \in [0, b_0^\varepsilon)} d_\varepsilon(t) &\leq \sup_{s \in [0, s_0^\delta]} [\varepsilon s + d((v_0^\varepsilon(s), B\dot{v}_0^\varepsilon(s)), \mathscr{I}_0)] \\ &\leq b_0^\varepsilon + ||(v_0^\varepsilon - v_0, B\dot{v}_0^\varepsilon - B\dot{v}_0)||_{\infty, [0, s_0^\delta]}, \end{aligned} \tag{3.61}$$

while, by using (3.41) with t_1^δ in place of $\hat{t}$, it turns out that

$$\sup_{t \in [b_0^\varepsilon, t_1^\delta)} d_\varepsilon(t) \leq ||(u^\varepsilon - u, \varepsilon B\dot{u}^\varepsilon)||_{\infty, [b_0^\varepsilon, t_1^\delta]} \leq \delta. \tag{3.62}$$

Now, observe that we can suppose

$$t_1 - t_1^\delta \leq \frac{\delta}{2}. \tag{3.63}$$

This fact, together with the definition of a_1^ε, implies that

$$\sup_{t \in [t_1^\delta, a_1^\varepsilon)} d_\varepsilon(t) \leq \sup_{t \in [t_1^\delta, a_1^\varepsilon)} \{|t - t_1| + |(u^\varepsilon(t) - x_1^s, \varepsilon B\dot{u}^\varepsilon(t))|\} \leq \max\left\{|t_1 - a_1^\varepsilon|, \frac{\delta}{2}\right\} + \delta \tag{3.64}$$

Finally, consider that

$$\begin{aligned} \sup_{t \in [a_1^\varepsilon, b_1^\varepsilon)} d_\varepsilon(t) &\leq \sup_{t \in [a_1^\varepsilon, b_1^\varepsilon)} \{|t - t_1| + d((u^\varepsilon(t), \varepsilon B\dot{u}^\varepsilon(t)), \mathscr{I}_1)\} \\ &\leq \varepsilon s_1^\delta + |t_1 - a_1^\varepsilon| + ||(v_1^\varepsilon - w_1, B\dot{v}_1^\varepsilon - B\dot{w}_1)||_{\infty, [0, s_1^\delta]}. \end{aligned} \tag{3.65}$$

(3.61)-(3.62) and (3.64)-(3.65), together with (3.37), Theorem 3.1 (2), the convergences in (3.60) and the convergence of εs_1^δ to 0, imply that there exists $\varepsilon_\delta > 0$ such that

$$\sup_{t\in[0,b_1^\varepsilon)} d_\varepsilon(t) \leq 2\delta, \quad \text{for every } \varepsilon \in (0,\varepsilon_\delta),$$

and, in turn, imply (3.59).
(b) Here, we show that there exists $\tilde{\varepsilon}_\delta > 0$ such that

$$\sup_{\Gamma} d(\cdot,\Gamma^\varepsilon) \leq 2\delta, \quad \text{for every } \varepsilon \in (0,\tilde{\varepsilon}_\delta). \tag{3.66}$$

By the definition of Γ and by the fact that $(x_1^s, 0) \in \overline{\mathscr{I}}_1$, it is sufficient to analyze

$$\sup_{\{0\}\times\mathscr{I}_0} d(\cdot,\Gamma^\varepsilon), \qquad \sup_{t\in[0,t_1)} d((t,u_1(t),0),\Gamma^\varepsilon), \qquad \sup_{\{t_1\}\times\mathscr{I}_1} d(\cdot,\Gamma^\varepsilon),$$

since the other cases can be treated in similar way. Let us consider separately $s \in [0,s_0^\delta]$ and $s > s_0^\delta$, and write

$$\begin{aligned}\sup_{s\in[0,s_0^\delta]} d((0,v_0(s),B\dot{v}_0(s)),\Gamma^\varepsilon) &\leq \sup_{s\in[0,s_0^\delta]} [\varepsilon s + d((v_0(s),B\dot{v}_0(s)),(u^\varepsilon(\varepsilon s),\varepsilon B\dot{u}^\varepsilon(\varepsilon s)))]\\ &\leq b_0^\varepsilon + ||(v_0^\varepsilon - v_0, B\dot{v}_0^\varepsilon - B\dot{v}_0)||_{\infty,[0,s_0^\delta]},\end{aligned} \tag{3.67}$$

and, in view of (3.38),

$$\begin{aligned}\sup_{s>s_0^\delta} d((0,v_0(s),\dot{v}_0(s)),\Gamma^\varepsilon) &\leq b_0^\varepsilon + \sup_{s>s_0^\delta} d((v_0(s),B\dot{v}_0(s)),(u^\varepsilon(b_0^\varepsilon),\varepsilon B\dot{u}^\varepsilon(b_0^\varepsilon)))\\ &\leq b_0^\varepsilon + \frac{\delta}{2} + |(u^\varepsilon(b_0^\varepsilon) - x_0^r, \varepsilon B\dot{u}^\varepsilon(b_0^\varepsilon))|.\end{aligned} \tag{3.68}$$

Now, to carry out a proper estimate $\sup_{t\in[0,t_1)} d((t,u_1(t),0),\Gamma^\varepsilon)$, we divide $[0,t_1)$ in $[0,b_0^\varepsilon)$, $[b_0^\varepsilon,t_1^\delta)$ and $[t_1^\delta,t_1)$. It turns out that

$$\begin{aligned}\sup_{t\in[0,b_0^\varepsilon)} d((t,u_1(t),0),\Gamma^\varepsilon) &\leq b_0^\varepsilon + \sup_{t\in[0,b_0^\varepsilon)} d((u_1(t),0),(u^\varepsilon(b_0^\varepsilon),\varepsilon B\dot{u}^\varepsilon(b_0^\varepsilon)))\\ &\leq b_0^\varepsilon + \omega_{u_1}(b_0^\varepsilon) + |(u^\varepsilon(b_0^\varepsilon) - x_0^r, \varepsilon B\dot{u}^\varepsilon(b_0^\varepsilon))|.\end{aligned} \tag{3.69}$$

Moreover, we have that

$$\sup_{t\in[b_0^\varepsilon,t_1^\delta)} d((t,u_1(t),0),\Gamma^\varepsilon) \leq ||u^\varepsilon - u_1, \varepsilon B\dot{u}^\varepsilon||_{\infty,[b_0^\varepsilon,t_1^\delta]}, \tag{3.70}$$

and, in view of (3.63) and (2.20), that

$$\begin{aligned}\sup_{[t_\delta^1,t_1)} d((t,u_1(t),0),\Gamma^\varepsilon) &\leq \sup_{[t_\delta^1,t_1)} d((t,u_1(t),0),(t_1^\delta,u^\varepsilon(t_1^\delta),\varepsilon B\dot{u}^\varepsilon(t_1^\delta)))\\ &\leq \frac{\delta}{2} + |(u^\varepsilon(t_1^\delta) - u_1(t_1^\delta),\varepsilon B\dot{u}^\varepsilon(t_1^\delta))| + \sup_{[t_1^\delta,t_1)} |u_1(t) - u_1(t_\delta^1)|\\ &\leq \delta + |(u^\varepsilon(t_1^\delta) - u_1(t_1^\delta),\varepsilon B\dot{u}^\varepsilon(t_1^\delta))|.\end{aligned} \tag{3.71}$$

Finally, consider $\sup_{\{t_1\}\times\mathscr{I}_1} d(\cdot,\Gamma^\varepsilon)$. Observe that

$$\begin{aligned}d((t_1,w_1(s),B\dot{w}_1(s)),\Gamma^\varepsilon) \leq |t_1 - b_1^\varepsilon| &+ |(w_1(s) - w_1(s_1^\delta), B\dot{w}_1(s) - B\dot{w}_1(s_1^\delta))|\\ &+ |(u^\varepsilon(b_1^\varepsilon) - w_1(s_1^\delta), \varepsilon B\dot{u}^\varepsilon(b_1^\varepsilon) - B\dot{w}_1(s_1^\delta))|,\end{aligned}$$

so that, from (3.56)-(3.57), we obtain that

$$\sup_{s>s_1^\delta} d((t_1,w_1(s),B\dot{w}_1(s)),\Gamma^\varepsilon) \leq |t_1 - b_1^\varepsilon| + \frac{3}{2}\delta. \tag{3.72}$$

Now, similarly to what is done in (3.56)-(3.58), we can define c_1^ε in the following way. Since $(w_1(-\infty), \dot w_1(-\infty)) = (x_1^s, 0)$, there exists $\overline{s}_1^\delta < 0$ such that

$$|(w_1(s) - x_1^s, B\dot w_1(s))| \leq \frac{\delta}{2}, \quad \text{for every } s \leq \overline{s}_1^\delta. \tag{3.73}$$

Moreover, due to (3.50) and (3.54), there exists $\tilde\varepsilon_\delta > 0$ such that

$$|(u^\varepsilon(a_1^\varepsilon + \overline{s}_1^\delta\varepsilon) - w_1(s_1^\delta), \varepsilon B\dot u^\varepsilon(a_1^\varepsilon + \overline{s}_1^\delta\varepsilon) - B\dot w_1(s_1^\delta))| \leq \frac{\delta}{2}, \quad \text{for every } \varepsilon \in (0, \tilde\varepsilon_\delta). \tag{3.74}$$

Let us define

$$c_1^\varepsilon = c_1^\varepsilon(\delta) := a_1^\varepsilon + \overline{s}_1^\delta\varepsilon < a_1^\varepsilon,$$

and observe that $c_1^\varepsilon \to t_1$. We have that

$$\sup_{s \in [\overline{s}_1^\delta, s_1^\delta]} d((t_1, w_1(s), B\dot w_1(s)), \Gamma^\varepsilon) \leq |t_1 - c_1^\varepsilon| + ||(v_1^\varepsilon - w_1, B\dot v_1^\varepsilon - B\dot w_1)||_{\infty, [\overline{s}_1^\delta, s_1^\delta]}, \tag{3.75}$$

while, since

$$\sup_{s < \overline{s}_1^\delta} d((t_1, w_1(s), B\dot w_1(s)), \Gamma^\varepsilon) \leq |t_1 - c_1^\varepsilon| + |(w_1(s) - w_1(s_1^\delta), B\dot w_1(s) - B\dot w_1(s_1^\delta))|$$
$$+ |(u^\varepsilon(c_1^\varepsilon) - w_1(s_1^\delta), \varepsilon B\dot u^\varepsilon(c_1^\varepsilon) - B\dot w_1(s_1^\delta))|,$$

from (3.73) and (3.74) it turns out that

$$\sup_{s < \overline{s}_1^\delta} d((t_1, w_1(s), B\dot w_1(s)), \Gamma^\varepsilon) \leq |t_1 - c_1^\varepsilon| + \frac{3}{2}\delta. \tag{3.76}$$

(3.67)-(3.72) and (3.75)-(3.76), together with (3.37), (3.39), (3.35) and (3.54), give that, up to a smaller $\tilde\varepsilon_\delta$,

$$\sup_{\{0\}\times\mathscr{I}_0} d(\cdot, \Gamma^\varepsilon), \ \sup_{t \in [0, t_1)} d((t, u_1(t), 0), \Gamma^\varepsilon), \ \sup_{\{t_1\}\times\mathscr{I}_1} d(\cdot, \Gamma^\varepsilon) \leq 2\delta,$$

for every $\varepsilon \in (0, \tilde\varepsilon_\delta)$, and, in turn, give (3.66). □

4. Approximating by time discretization

In this section, we study a discrete-time approximation of the same limit problem constructed in Section 2 and approximated in Section 3 by singular perturbations. The present approximation process is modelled on the following idea. We consider a partition $0 = \tau_0^k < \tau_1^k < ... < \tau_{k-1}^k < \tau_k^k = T$ of the interval $[0, T]$ such that

$$\rho_k := \max_{0 \leq i \leq k-1} (\tau_{i+1}^k - \tau_i^k) \to 0, \quad \text{as } k \to +\infty, \tag{4.1}$$

and suppose to have defined u_{i-1}^k as the approximation of the function u given by Definition 2.7 on the interval $[\tau_{i-1}^k, \tau_i^k)$. Since $u(\tau_i^k)$ is a critical point of $f(\tau_i^k, \cdot)$, we find the next approximating point u_i^k by considering the solution v_i^k of the autonomous problem

$$\begin{cases} A\ddot v_i^k(\sigma) + B\dot v_i^k(\sigma) + \nabla_x f(\tau_i^k, v_i^k(\sigma)) = 0, & \sigma \in [0, +\infty) \\ v_i^k(0) = u_{i-1}^k, \\ \dot v_i^k(0) = 0, \end{cases} \tag{4.2}$$

and setting

$$u_i^k := \lim_{\sigma \to +\infty} v_i^k(\sigma), \quad i = 2, ..., k. \tag{4.3}$$

Consider a point $x_0^r \in \mathbb{R}^n$ such that $\nabla_x f(0, x_0^r) = 0$ and $\nabla_x^2 f(0, x_0^r)$ is positive definite. Clearly, the first approximating point of this process could be defined as the limit at $+\infty$ of the solution of (4.2) with τ_1^k and $x_0^r = u(0)$ in place of τ_i^k and u_{i-1}^k, respectively. Actually, it does not cost much more effort to define

$$u_1^k := \lim_{\sigma\to+\infty} v_1^k(\sigma), \tag{4.4}$$

and v_1^k as the solution of

$$\begin{cases} A\ddot{v}_1^k(\sigma) + B\dot{v}_1^k(\sigma) + \nabla_x f(\tau_1^k, v_1^k(\sigma)) = 0, & \sigma \in [0,+\infty) \\ v_1^k(0) = x_k, \\ \dot{v}_1^k(0) = y_k. \end{cases} \tag{4.5}$$

Here,

$$(x_k, y_k) \to (x_0, y_0), \quad \text{as } k \to +\infty,$$

and (x_0, y_0) lies in the basin of attraction of $(x_0^r, 0)$ for the autonomous problem at time 0, that is (x_0, y_0) satisfies (3.2) and (3.3). In order to uniform the notation, we set $u_0^k = x_k$. Note that Lemma 2.5 ensures the existence of the solutions of problems (4.2), (4.5), (3.2) and of the limits in (4.3), (4.4), (3.3). Also, Lemma 2.5 tells us that u_i^k is a critical point of $f(\tau_i^k, \cdot)$ and that $\dot{v}_0(+\infty) = \dot{v}_k^i(+\infty) = 0$, for $i = 1, ..., k$.

Let Γ be the same set defined in (3.7)-(3.9). In order to define a suitable set Γ^k approximating Γ, we choose arbitrarily some

$$\alpha_i^k \in (\tau_{i-1}^k, \tau_i^k) \ \text{ for } i = 1, ..., k,$$

and introduce a function u^k which has, on every $[\tau_{i-1}^k, \tau_i^k]$, the following features. On $[\tau_{i-1}^k, \alpha_i^k]$, it is a suitable reparametrization of v_i^k from a certain big interval $[0, a_i^k]$ to $[\tau_{i-1}^k, \alpha_i^k]$, and, on $[\alpha_i^k, \tau_i^k]$, it is a convex combination of $v_i^k(a_i^k)$ taken in α_i^k and u_i^k taken in τ_i^k. More precisely, we fix a sequence $\delta_k \to 0$ and a constant $C > 0$, and, for $i = 1, ..., k$, we consider a value $a_i^k > 0$ with the following properties:

$$\min_i a_i^k \to +\infty, \quad \text{as } k \to +\infty, \tag{4.6}$$

and, for every k,

$$\frac{|v_i^k(a_i^k) - u_i^k|}{\tau_i^k - \alpha_i^k} \le C, \qquad |\dot{v}_i^k(a_i^k))| \le \delta_k, \quad \text{uniformly with respect to } i. \tag{4.7}$$

It is clear that such values exist, by Lemma 2.5. Then, we define the function $u^k \in C([0,T], \mathbb{R}^n)$ by

$$u^k(t) := \begin{cases} v_i^k\left(\frac{t-\tau_{i-1}^k}{\alpha_i^k - \tau_{i-1}^k} a_i^k\right), & t \in [\tau_{i-1}^k, \alpha_i^k], \\ \frac{(\tau_i^k - t)v_i^k(a_i^k) + (t-\alpha_i^k)u_i^k}{\tau_i^k - \alpha_i^k}, & t \in [\alpha_i^k, \tau_i^k]. \end{cases} \tag{4.8}$$

Observe that

$$u^k(0) = v_1^k(0) = x_k, \quad u^k(\tau_{i-1}^k) = v_i^k(0) = u_{i-1}^k, \quad \text{for } i = 2, ..., k,$$

$$u^k(\alpha_i^k) = v_i^k(a_i^k), \quad \text{for } i = 1, ..., k,$$

and that

$$\{u^k(t) : \ t \in [\tau_{i-1}^k, \alpha_i^k]\} = \{v_i^k(\sigma) : \ \sigma \in [0, a_i^k]\},$$

while, on $[\alpha_i^k, \tau_i^k]$, $u^k(t)$ is an affine function connecting $v_i^k(a_i^k)$ to u_i^k. Moreover, u^k can be not differentiable at $\alpha_1^k, \tau_1^k, \alpha_2^k, \tau_2^k, ..., \alpha_k^k$. Therefore, with abuse of notation, we define

$$\dot{u}^k(\tau_i^k) := \lim_{\tau\to(\tau_i^k)^+} \dot{u}^k(\tau), \quad \text{for } \ i = 0, ..., k-1,$$

$$\dot{u}^k(\alpha_i^k) := \lim_{\tau\to(\alpha_i^k)^+} \dot{u}^k(\tau), \quad \text{for } \ i = 1, ..., k,$$

and $\dot{u}^k(T) := \lim_{\tau\to T^-} \dot{u}^k(\tau)$, so that

$$\dot{u}^k(t) := \begin{cases} \frac{a_i^k}{\alpha_i^k-\tau_{i-1}^k}\dot{v}_i^k\left(\frac{t-\tau_{i-1}^k}{\alpha_i^k-\tau_{i-1}^k}a_i^k\right), & t\in[\tau_{i-1}^k,\alpha_i^k), \\ \frac{u_i^k-v_i^k(a_i^k)}{\tau_i^k-\alpha_i^k}, & t\in[\alpha_i^k,\tau_i^k), \end{cases} \tag{4.9}$$

and $\dot{u}^k(T) := \frac{u_k^k-v_k^k(a_k^k)}{T-\alpha_k^k}$. Note that, for $i=2,...,k$, $\dot{u}^k(\tau_{i-1}^k) = \frac{a_i^k}{\alpha_i^k-\tau_{i-1}^k}\dot{v}_i^k(0)=0$, while $\dot{u}^k(0)=\frac{a_1^k}{\alpha_1^k}y_k$. Finally, we need some coefficients which have, in the present analysis, the same role played by ε in Section 3. To this aim, we define

$$h_k(t) := \sum_{i=1}^k \frac{\alpha_i^k-\tau_{i-1}^k}{a_i^k}\chi_{[\tau_{i-1}^k,\tau_i^k)}(t), \quad t\in[0,T), \tag{4.10}$$

with $h_k(T)=\frac{a_k^k-\tau_{k-1}^k}{a_k^k}$, and, in turn,

$$\Gamma^k := \overline{\{(t,u^k(t),h_k(t)B\dot{u}^k(t)):\ t\in[0,T]\}}. \tag{4.11}$$

By referring to Section 2 for the Assumptions 1-4, we are in position to state the main result of this section.

Theorem 4.1. *Under the hypotheses of Theorem 3.1, we have that*

$$d_H(\Gamma^k,\Gamma)\to 0, \quad \textit{as} \quad k\to+\infty.$$

To prove Theorem 4.1, we need some preliminary results. Under Assumptions 1 and 2, fix $\tau\in[0,T]$ and let $\tilde{x}$, $\tilde{y}\in\mathbb{R}^n$ be such that, if v is the solution of the problem

$$\begin{cases} A\ddot{v}(\sigma)+B\dot{v}(\sigma)+\nabla_x f(\tau,v(\sigma))=0, & \sigma\in[0,+\infty) \\ v(0)=\tilde{x}, \\ \dot{v}(0)=\tilde{y}, \end{cases}$$

and $v_\infty := \lim_{\sigma\to+\infty} v(\sigma)$, then $\nabla_x^2 f(\tau,v_\infty)$ is positive definite. By the Implicit Function Theorem, there exist a connected neighbourhood U of τ in [0,T], a neighbourhood V of v_∞ in $\mathbb{R}^n$ and a C^2 function $u:U\to\mathbb{R}^n$ such that $u(\tau)=v_\infty$ and, if $(t,x)\in U\times V$, then $\nabla_x f(t,x)=0$ if and only if $x=u(t)$. Moreover, $\nabla_x^2 f(t,u(t))$ is positive definite on U.

Consider three sequences $x_k\to\tilde{x}$, $y_k\to\tilde{y}$ and $\tau_k\in[0,T]$ such that $\tau_k\to\tau$, and denote by v_k the solution of the problem

$$\begin{cases} A\ddot{v}_k(\sigma)+B\dot{v}_k(\sigma)+\nabla_x f(\tau_k,v_k(\sigma))=0, & \sigma\in[0,+\infty) \\ v_k(0)=x_k, \\ \dot{v}_k(0)=y_k. \end{cases}$$

By continuous dependence, we have that $(v_k,\dot{v}_k)\to(v,\dot{v})$ uniformly on compact subsets of $[0,+\infty)$, and, by Lemma 2.5, we know that $v_k(+\infty)$ is a critical point of $f(\tau_k,\cdot)$ and $\dot{v}_k(+\infty)=0$. The following lemma tells us that, if k is sufficiently large, $v_k(+\infty)=u(\tau_k)$. Moreover, this convergence is uniform with respect to k.

Lemma 4.2. *Under Assumptions 1 and 2, let u and v_k be as defined above. Then, there exists k_0 such that*

$$\lim_{\sigma\to+\infty}(v_k(\sigma),\dot{v}_k(\sigma))=(u(\tau_k),0), \quad \textit{for every } k\geq k_0. \tag{4.12}$$

Moreover, for every $\delta>0$, there exists k_δ, $\sigma_\delta>0$ such that

$$(v_k(\sigma),B\dot{v}_k(\sigma))\in\overline{B}((u(\tau_k),0),\delta), \quad \textit{for every } \sigma\geq\sigma_\delta \textit{ and } k\geq k_\delta. \tag{4.13}$$

Proof. Let us refer to the previous paragraph for the notation. For every $t \in U$ and every $x \in \mathbb{R}^n$, there exists $\alpha \in [0,1]$, depending on x and $u(t)$, such that

$$f(t,x) = f(t,u(t)) + \nabla_x^2 f(t,u(t)+\alpha(x-u(t)))(x-u(t),x-u(t)). \tag{4.14}$$

Let $\overline{\tau} < \tau < \hat{\tau}$ be such that $[\overline{\tau},\hat{\tau}] \subseteq U$. Since $\nabla_x^2 f(\cdot,u(\cdot))$ is positive definite on $[\overline{\tau},\hat{\tau}]$, there exists $R>0$, depending on $[\overline{\tau},\hat{\tau}]$, such that, if $\delta \in (0,R)$, then

$$\min\left\{\lambda : \lambda \text{ is an eigenvalue of } \nabla_x^2 f(t,u(t)+z),\ |z|\le\delta,\ t\in[\overline{\tau},\hat{\tau}]\right\} := \beta_{2\delta} > 0. \tag{4.15}$$

Choose $\delta \in (0,R)$. From (4.14) and (4.15), we obtain that

$$\min_{|x-u(t)|=\frac{\delta}{2}} f(t,x) \ge f(t,u(t)) + \beta_\delta \frac{\delta^2}{4}, \quad \text{for every } t\in[\overline{\tau},\hat{\tau}], \tag{4.16}$$

while the uniform continuity of $f(\cdot,u(\cdot))$ on $[\overline{\tau},\hat{\tau}]$ implies that

$$\max_{|x-u(t)|\le r} f(t,x) \le f(t,u(t)) + \beta_\delta \frac{\delta^2}{8}, \quad \text{for every } t\in[\overline{\tau},\hat{\tau}], \tag{4.17}$$

for a certain $r \in \left(0,\frac{\delta}{2}\right)$. Since $(v(\sigma),\dot{v}(\sigma)) \to (v_\infty,0)$, as $\sigma\to+\infty$, we can find $\sigma_\delta>0$ such that

$$|v(\sigma)-v_\infty| \le \frac{r}{3}, \quad \text{and} \quad |\dot{v}(\sigma)| \le \frac{\delta}{2}\min\left\{\frac{1}{2}\sqrt{\frac{\beta_\delta}{2|A|}},\frac{1}{|B|}\right\}, \quad \text{for every } \sigma\ge\sigma_\delta. \tag{4.18}$$

By the uniform continuity of $(v_k,\dot{v}_k)$ to $(v,\dot{v})$ on compact subsets of $[0,+\infty)$, there exists k_δ such that, for every $k\ge k_\delta$,

$$|v_k(\sigma_\delta)-v(\sigma_\delta)| \le \frac{r}{3}, \quad |\dot{v}_k(\sigma_\delta)-\dot{v}(\sigma_\delta)| \le \frac{\delta}{4}\sqrt{\frac{\beta_\delta}{2|A|}}. \tag{4.19}$$

Also, we can suppose that

$$|u(\tau_k)-v_\infty| \le \frac{r}{3}, \quad \text{for every } k\ge k_\delta. \tag{4.20}$$

Let $\sigma\ge\sigma_\delta$ and $k\ge k_\delta$. By arguing as in the proof of Lemma 2.5, we obtain that

$$f(\tau_k,v_k(\sigma)) \le \frac{1}{2}A\dot{v}_k(\sigma_\delta)\cdot\dot{v}_k(\sigma_\delta) + f(\tau_k,v_k(\sigma_\delta)), \tag{4.21}$$

and, by using (4.18)-(4.20), we have that

$$|v_k(\sigma_\delta)-u(\tau_k)| \le |v_k(\sigma_\delta)-v(\sigma_\delta)| + |v(\sigma_\delta)-v_\infty| + |v_\infty-u(\tau_k)| \le r. \tag{4.22}$$

Then, noticing that $\tau_k\in[\overline{\tau},\hat{\tau}]$ for every k sufficiently large, (4.17), (4.21) and (4.22) imply that

$$\begin{aligned} f(\tau_k,v_k(\sigma)) &\le \frac{|A|}{2}|\dot{v}_k(\sigma_\delta)|^2 + f(\tau_k,u(\tau_k)) + \beta_\delta\frac{\delta^2}{8} \\ &\le f(\tau_k,u(\tau_k)) + \frac{3}{16}\beta_\delta\delta^2, \end{aligned} \tag{4.23}$$

where in the last inequality we have used also the second estimate in (4.18) and (4.19). From (4.16) and (4.23), we obtain that $v_k(\sigma)\in B\left(u(\tau_k),\frac{\delta}{2}\right)$ for all $\sigma\ge\sigma_\delta$ and $k\ge k_\delta$. This fact, together with the second estimate of (4.18), gives (4.13). In particular, let us fix $\delta_0>0$ such that $\overline{B}\left(u(\tau_k),\frac{\delta_0}{2}\right)\subseteq V$ for every $k\ge k_0$, for a certain $k_0>0$. Then, by Lemma 2.5 and by the fact that the unique critical point of $f(\tau_k,\cdot)$ in $\overline{B}\left(u(\tau_k),\frac{\delta_0}{2}\right)$ is $u(\tau_k)$, (4.12) is proved. □

For the following lemma, observe that, for $j = 1, ..., m-1$, the function u_{j+1}, defined in Proposition 2.6, is more generally defined on $[\bar{t}_j, t_{j+1})$, for a certain $\bar{t}_j < t_j$ sufficiently close to t_j such that

$$\nabla_x f(\cdot, u_{j+1}(\cdot)) \equiv 0 \text{ and } \nabla_x^2 f(\cdot, u_{j+1}(\cdot)) \text{ is positive definite on } [\bar{t}_j, t_{j+1}). \quad (4.24)$$

Since the notation is unavoidably heavy, be careful to distinguish the functions u_j's from the functions u^k's defined in (4.8) proceeding from the points u_i^k's, defined in (4.3) and (4.4). The next lemma tells us essentially that, for k large enough, the points u_i^k are indeed values approximating u_1 on compact subsets of $(0, t_1)$.

Lemma 4.3. *Choose $\hat{t} \in (0, t_1)$ and $\delta > 0$. There exists $\hat{k}_\delta$, $\sigma_\delta > 0$ such that, for every $k \geq \hat{k}_\delta$, we have that*

$$(v_1^k(\sigma), B\dot{v}_1^k(\sigma)) \in \overline{B}((u_1(\tau_1^k), 0), \delta), \quad \textit{for every } \sigma \geq \sigma_\delta, \quad (4.25)$$

and, if $\tau_i^k \in [\tau_2^k, \hat{t}]$, then

$$(v_i^k(\sigma), B\dot{v}_i^k(\sigma)) \in \overline{B}((u_1(\tau_i^k), 0), \delta), \quad \textit{for every } \sigma \geq 0. \quad (4.26)$$

In particular, there exists $\hat{k}$ such that

$$u_i^k = u_1(\tau_i^k), \quad \textit{for every } \tau_i^k \in [\tau_1^k, \hat{t}] \quad \textit{and} \quad k \geq \hat{k}. \quad (4.27)$$

To show that (4.25) and (4.27) hold for $i = 1$, we can use Lemma 4.2, with $\tau = 0$, $\tilde{x} = x_0$, $\tilde{y} = y_0$, v_0 in place of v, u_1 in place of u, $v_\infty = x_0^r$, and τ_1^k, v_1^k in place of τ_k, v_k, respectively. The proof of the remaining part of Lemma 4.3 can be done by induction and by using essentially the same arguments of the proof of Lemma 4.2.

While Lemma 4.3 takes into account the approximating points u_i^k on compact subsets of $(0, t_1)$, the following lemma, whose proof is similar to the previous one, deals with $[\bar{t}_j, t_{j+1})$, which is a slight modification of $[t_j, t_{j+1})$ in the sense of (4.24), for $j = 1, ..., m-1$.

Lemma 4.4. *For $j = 1, ..., m-1$, let $\bar{t}_j < t_j$ be sufficiently close to t_j so that (4.24) holds . For $j = 1, ..., m-2$, choose $\hat{t}_j \in [t_j, t_{j+1})$, and set $\hat{t}_{m-1} = T$. For every $\delta > 0$, there exists $\hat{k}_\delta > 0$ such that, if*

$$\tau_l^k, \tau_{l+1}^k \in [\bar{t}_j, \hat{t}_j] \quad \textit{and} \quad u_l^k = u_{j+1}(\tau_l^k),$$

for some $j \in \{1, ..., m-1\}$, then

$$(v_i^k(\sigma), B\dot{v}_i^k(\sigma)) \in \overline{B}((u_{j+1}(\tau_i^k), 0), \delta), \quad \textit{for every } \sigma \geq 0,$$

for every $\tau_i^k \in [\tau_{l+1}^k, \hat{t}_j]$ and $k \geq \hat{k}_\delta$. In particular, there exists $\hat{k} > 0$ such that

$$u_i^k = u_{j+1}(\tau_i^k), \quad \textit{for every} \quad \tau_i^k \in [\tau_{l+1}^k, \hat{t}_j] \quad \textit{and} \quad k \geq \hat{k}.$$

In order to prove Theorem 4.1, we need to select some special indices among $i = 0, ..., k$ and show certain properties of those. Lemma (4.3) and (4.4) suggest that we can expect that there exist some indices o_k^j which mark a transition around t_j from the approximation of u_j to the approximation of u_{j+1}, that is $u_i^k = u_j(\tau_i^k)$ for every $\tau^k_{o_k^j} < i \leq \tau^k_{o_k^{j+1}}$. Unluckily, it is not really like this, since, as we will see, it may happen that, if $\tau_i^k \leq t_j$ is "too much close" to t_j, $u_i^k \in \{u_j(\tau_i^k), \overline{u}_j(\tau_i^k)\}$ (see (2.18) for a definition). We will show later that the indices introduced by the following definition, which depends on a small parameter δ estimating the distance from x_j^s, are those responsible for the transition.

Definition 4.5. *Let $\delta > 0$ be small enough. For every $j \in \{1, ..., m-1\}$, we define*

$$o_k^j = o_k^j(\delta) := \min A_k^j,$$

where $A_k^j = A_k^j(\delta)$ *is the set of the indices* $i \in \{0, ..., k-1\}$ *such that*

$$\tau_i^k \le t_j, \qquad u_i^k \in \overline{B}\left(x_j^s, \frac{\delta}{2}\right),$$

and

$$(v_{i+1}^k(\sigma), B\dot{v}_{i+1}^k(\sigma)) \in \partial B((x_j^s, 0), \delta), \quad \textit{for some } \sigma > 0,$$

where x_j^s, *for* $j = 1, ..., m-1$, *is defined in Proposition 2.6.*

Remark 4.6. Observe that, for k sufficiently large, the definition of o_k^j is well-posed, since $A_k^j \neq \emptyset$. Let us check this fact in the case $j = 1$. For $j = 2, ..., m-1$, the proof can be conducted in a similar way, by using also the next lemma. Recall (2.20) and choose $\tilde{t} \in (t_1^\delta, t_1)$. By Lemma 4.3, for every k sufficiently large, there exists al least one index i such that $\tau_i^k \in [t_1^\delta, \tilde{t}]$ and $u_i^k = u_1(\tau_i^k) \in \overline{B}\left(x_1^s, \frac{\delta}{4}\right)$. Now, there are two possibilities:
(1) $\tau_{i+1}^k > t_1$: in this case, we can suppose, up to bigger k's, that $\tau_{i+1}^k \le t_1^{**}$ (see (2.19) and (2.22) for the notation). Thus, by recalling that $u_{i+1}^k = v_{i+1}^k(+\infty)$ is a critical point of $f(\tau_{i+1}^k, \cdot)$, we have that $u_{i+1}^k \notin \overline{B}(x_1^s, \delta)$. Then, since $u_i^k = v_{i+1}^k(0) \in \overline{B}\left(x_1^s, \frac{\delta}{4}\right)$ and $\dot{v}_{i+1}^k(0) = 0$, it turns out that $(v_{i+1}^k(\sigma), B\dot{v}_{i+1}^k(\sigma)) \in \partial B((x_1^s, 0), \delta)$ for some $\sigma > 0$ and therefore $i \in A_k^1$;
(2) $\tau_{i+1}^k \le t_1$: in this case, if $(v_{i+1}^k(\sigma), B\dot{v}_{i+1}^k(\sigma)) \in \partial B((x_1^s, 0), \delta)$ for some $\sigma > 0$, then $i \in A_k^1$; otherwise, $\lim_{\sigma\to+\infty}(v_{i+1}^k(\sigma), B\dot{v}_{i+1}^k(\sigma)) = (u_{i+1}^k, 0) \in \overline{B}((x_1^s, 0), \delta)$. But $u_{i+1}^k \in \{u_j(\tau_{i+1}^k), \overline{u}_j(\tau_{i+1}^k)\}$ and $t_1^\delta \le \tau_i^k < \tau_{i+1}^k \le t_1$, therefore $u_{i+1}^k \in \overline{B}\left(x_1^s, \frac{\delta}{4}\right)$. At this point, we begin again by considering τ_{i+2}^k and, in turn, case (1) or (2).
By this procedure, in a finite number of steps we find some $i \in A_k^1$.

It is useful to underline two facts which emerge from Remark 4.6:

$$u_{o_k^j}^k \in \left\{u_j\left(\tau_{o_k^j}^k\right), \overline{u}_j\left(\tau_{o_k^j}^k\right)\right\}, \tag{4.28}$$

and we cannot determine whether $\tau_{o_k^j+1}^k > t_j$ or $\tau_{o_k^j+1}^k \le t_j$. The following lemma will be useful to prove the main result of this section and tells us (see point (3)) that the index o_k^j marks the transition from the branches u_j and $\overline{u}_j$ to u_{j+1}, as it was expected.

Lemma 4.7. *For every* $j \in \{1, ..., m-1\}$ *and* $\delta > 0$ *small enough, the following properties hold:*

1. $\tau_{o_k^j}^k \to t_j^-$;
2. $u_{o_k^j}^k \to x_j^s$;
3. *for every* k *large enough,* $u_{o_k^j+1}^k = u_{j+1}(\tau_{o_k^j+1}^k)$, *hence* $u_{o_k^j+1}^k \to x_j^r$.

Proof. Let us begin with the case $j = 1$ and write, in order to simplify the notation, $o_k = o_k(\delta)$ in place of o_k^1.
(1) Observe that, from Definition 4.5, $\tau_{o_k}^k \le t_1$. Let $\bar{t} < t_1$ be arbitrarily close to t_1. We want to show that there exists $\bar{k}$ such that $\tau_{o_k}^k \in (\bar{t}, t_1]$, for every $k \ge \bar{k}$. We can suppose that $\bar{t} \ge t_1^\delta$ (see 2.20). Observe that, if x, $y \in \mathbb{R}^n$ vary in a compact, by uniform continuity there exists $\rho = \rho(\delta) > 0$ such that

$$|f(t,x) - f(t,y)| < \frac{\delta^2 \lambda_{min}^A}{32|B|^2}, \quad \text{for every } t \in [0,T] \text{ and } |x-y| \le \rho. \tag{4.29}$$

Choose $\tilde{t} \in (\bar{t}, t_1)$ and set $\tilde{\delta} := \frac{1}{2}\min\{\rho, \delta\}$. Lemma 4.3 tells us that, for every k large enough (depending on $\tilde{\delta}$ and on $[\bar{t}, \tilde{t}]$), there exists an index $i \ge 1$ such that

$$t_1^\delta \le \bar{t} \le \tau_i^k < \tau_{i+1}^k \le \tilde{t}, \tag{4.30}$$

$$u_i^k = u_1(\tau_i^k), \tag{4.31}$$

and

$$v_{i+1}^k(\sigma) \in \overline{B}(u_1(\tau_{i+1}^k), \tilde{\delta}), \qquad \text{for every } \sigma \geq 0. \tag{4.32}$$

Thus, from (4.32) and the definition of $\tilde{\delta}$ and of t_1^δ, we obtain that

$$|v_{i+1}^k(\sigma) - x_1^s| \leq |v_{i+1}^k(\sigma) - u_1(\tau_{i+1}^k)| + |u_1(\tau_{i+1}^k) - x_1^s| \leq \frac{3}{4}\delta. \tag{4.33}$$

Also, recall that

$$\begin{aligned}\frac{\lambda_{min}^A}{2}|\dot{v}_{i+1}^k(\sigma)|^2 + f(\tau_{i+1}^k, v_{i+1}^k(\sigma)) \quad &\leq \quad \frac{1}{2}A\dot{v}_{i+1}^k(\sigma)\cdot\dot{v}_{i+1}^k(\sigma) + f(\tau_{i+1}^k, v_{i+1}^k(\sigma)) \\ &\leq \quad f(\tau_{i+1}^k, u_i^k),\end{aligned} \tag{4.34}$$

for every $\sigma \geq 0$, and observe that, by (4.1) and (4.31)-(4.32), we can suppose, up to greater k's, that

$$\begin{aligned}|u_i^k - v_{i+1}^k(\sigma)| &= |u_1(\tau_i^k) - v_{i+1}^k(\sigma)| \\ &\leq |u_1(\tau_i^k) - u_1(\tau_{i+1}^k)| + |u_1(\tau_{i+1}^k) - v_{i+1}^k(\sigma)| \leq \rho,\end{aligned}$$

for every $\sigma \geq 0$. Thus, from (4.29) and (4.34), it descends that

$$|B\dot{v}_{i+1}^k(\sigma)| \leq \sqrt{\frac{2}{\lambda_{min}^A}[f(\tau_{i+1}^k, u_i^k) - f(\tau_{i+1}^k, v_{i+1}^k(\sigma))]^{\frac{1}{2}}} < \frac{\delta}{4}, \quad \text{for every } \sigma \geq 0. \tag{4.35}$$

(4.30), (4.31), (4.33) and (4.35) give that

$$u_i^k \in \overline{B}\left(x_1^s, \frac{\delta}{2}\right) \quad \text{and} \quad (v_{i+1}^k(\sigma), B\dot{v}_{i+1}^k(\sigma)) \in B((x_1^s, 0), \delta), \quad \text{for every } \sigma \geq 0. \tag{4.36}$$

By the same arguments just used, we can prove that, whenever $l < i$ is such that $u_l^k \in \overline{B}\left(x_1^s, \frac{\delta}{2}\right)$, we have that $(v_k^{l+1}(\sigma), B\dot{v}_k^{l+1}(\sigma)) \in B((x_1^s, 0), \delta)$ for every $\sigma \geq 0$. This fact, together with (4.36) and the definition of o_k, implies that $o_k > i$ and therefore $\tau_{o_k}^k > \tau_i^k \geq \bar{t}$.

(2) This limit follows from property (1) and from (4.28).

(3) To further simplify the notation, let us write v_k instead of $v_{o_k+1}^k$, so that v_k is the solution of

$$\begin{cases} A\ddot{v}_k(\sigma) + B\dot{v}_k(\sigma) + \nabla_x f(\tau_{o_k+1}^k, v_k(\sigma)) = 0, \quad \sigma \in [0, +\infty) \\ v_k(0) = u_{o_k}^k, \\ \dot{v}_k(0) = 0. \end{cases}$$

By Definition 4.5, the following parameter is well-defined for every k sufficiently large:

$$\sigma_k := \min\{\sigma > 0 : \ (v_k(\sigma), B\dot{v}_k(\sigma)) \in \partial B((x_1^s, 0), \delta)\}. \tag{4.37}$$

The compactness of $\partial B((x_1^s, 0), \delta)$ implies that, up to a subsequence,

$$(v_k(\sigma_k), B\dot{v}_k(\sigma_k)) \to (z, \dot{z}), \tag{4.38}$$

for a certain $(z, \dot{z}) \in \partial B((x_1^s, 0), \delta)$. We claim that

$$\sigma_k \to +\infty, \qquad \text{as } k \to +\infty. \tag{4.39}$$

Indeed, if it was (up to a subsequence) $\sigma_k \in [0, M]$ and $\sigma_k \to \hat{\sigma}$, for some $M > 0$ and $\hat{\sigma} \in [0, M]$, we would find the following contradiction: by points (1) and (2), it is $(v_k, \dot{v}_k)$ uniformly convergent to $(v, \dot{v})$ on $[0, M]$, where v is the solution of

$$\begin{cases} A\ddot{v}(\sigma) + B\dot{v}(\sigma) + \nabla_x f(t_1, v(\sigma)) = 0, \quad \sigma \in [0, +\infty) \\ v(0) = x_1^s, \\ \dot{v}(0) = 0. \end{cases}$$

In particular, it is $(v_k(\sigma_k), \dot v_k(\sigma_k)) \to (v(\hat\sigma), \dot v(\hat\sigma))$ and, in turn, by (4.37), $(v(\hat\sigma), B\dot v(\hat\sigma)) \in \partial B((x_1^s, 0), \delta)$. This is an absurd, because $v \equiv x_1^s$.
Now, let us define

$$\tilde v_k(\sigma) := v_k(\sigma + \sigma_k), \quad \sigma \in [-\sigma_k, +\infty),$$

which satisfies the system $A\ddot{\tilde v}_k(\sigma) + B\dot{\tilde v}_k(\sigma) + \nabla_x f(\tau^k_{o_k+1}, \tilde v_k(\sigma)) = 0$, for $\sigma \in [-\sigma_k, +\infty)$ and the conditions $\tilde v_k(0) = v_k(\sigma_k)$, $\dot{\tilde v}_k(0) = \dot v_k(\sigma_k)$, and let w_1 be the solution of

$$\begin{cases} A\ddot w(\sigma) + B\dot w(\sigma) + \nabla_x f(t_1, w(\sigma)) = 0, & \sigma \in [0, +\infty) \\ w(0) = z, \\ B\dot w(0) = \dot z. \end{cases}$$

By point (1), (4.38) and (4.39), we have that $(\tilde v_k, \dot{\tilde v}_k) \to (w_1, \dot w_1)$ uniformly on compacts subsets of any common interval of existence. By using this fact together with the definition of σ_k, it is easy to show that

$$\{(w_1(\sigma), B\dot w_1(\sigma)) : \ \sigma \le 0\} \subseteq \overline{B}((x_1^s, 0), \delta). \tag{4.40}$$

Thus, by Lemma 2.4 together with (2.21) and Proposition 2.6, we have that

$$\lim_{s\to-\infty} (w_1(s), \dot w_1(s)) = (x_1^s, 0), \quad \lim_{s\to+\infty} (w_1(s), \dot w_1(s)) = (x_1^r, 0), \tag{4.41}$$

and that

$$(\tilde v_k, \dot{\tilde v}_k) \to (w_1, \dot w_1), \quad \text{uniformly on compact subsets of } \mathbb{R}. \tag{4.42}$$

Observing that, by definition, $u^k_{o_k+1} = \lim_{\sigma\to+\infty} \tilde v_k(\sigma)$, it is enough to apply Lemma 4.2 with t_1, z, $B^{-1}\dot z$, x_1^r, u_2, $v_k(\sigma_k)$, $\dot v_k(\sigma_k)$, $\tau^k_{o_k+1}$ and $\tilde v_k$ in place of τ, $\tilde x$, $\tilde y$, v_∞, u, x_k, y_k, τ_k and v_k, respectively, to conclude that $u^k_{o_k+1} = u_2(\tau^k_{o_k+1})$ for every k large enough, and, in turn, that $u^k_{o_k+1} \to x_1^r$.
The proof of the cases $j = 2, ..., m-1$ can be done in a similar way, by using, more, the case $j = 1$. □

Lemma 4.3 and Lemma 4.7 allow us to state and prove a result of approximation of u on compact subsets of $(0, T] \setminus \{t_1, ..., t_{m-1}\}$. Since the jump times t_j's are not so far considered, the heteroclines w_j's appear in the statement just because they are involved in the definition of u through their limit points x_j^s and x_j^r at $-\infty$ and $+\infty$, respectively (see Definition 2.7). Notice that Proposition 4.8, by including the uniform convergence to 0 of the "modified" velocity $h_k B\dot u^k$ on compact subsets of $(0, T] \setminus \{t_1, ..., t_{m-1}\}$, recovers all the information collected in Theorem 3.1 (1) by using a different approach. We refer the reader to (4.6)-(4.10) for the notation.

Proposition 4.8. *Under the hypotheses of Theorem 3.1, we have that*

$$(u^k, h_k B\dot u^k) \to (u, 0), \ \textit{uniformly on compact subsets of } (0, T] \setminus \{t_1, ..., t_{m-1}\}.$$

Proof. Let us consider the interval $(0, t_1)$. The proof for the other intervals can be done in a similar way, by using, more, Lemma 4.7 (3). Choose $\bar t$ and $\tilde t$ such that $0 < \bar t < \tilde t < t_1$ and $\delta > 0$ arbitrarily small. Observe that, for k sufficiently large, there exits i and m such that $i - m \ge 2$ and

$$\tau^k_{i-m-1} \le \bar t < \alpha^k_{i-m} < ... < \alpha^k_i \le \tilde t < \tau^k_i \le \frac{\bar t + \tilde t}{2},$$

so that it is sufficient to analyze the following two model cases.
(i) If $t \in [\bar t, \alpha^k_{i-m})$, then $u^k(t) = v^k_{i-m}(\sigma)$ and $h_k(t)\dot u^k(t) = \dot v^k_{i-m}(\sigma)$ for some $\sigma \in [0, a^k_{i-m})$. Thus, since

$$|(u^k(t) - u_1(t), h_k(t)B\dot u^k(t))| \le |(v^k_{i-m}(\sigma) - u_1(\tau^k_{i-m}), B\dot v^k_{i-m}(\sigma))| + |u_1(\tau^k_{i-m}) - u_1(t)|,$$

in view of Lemma 4.3, we have that, for every k large enough,

$$||(u^k - u_1, h_k B\dot{u}^k)||_{\infty,[\bar{t},\alpha^k_{i-m})} \leq \delta + \omega_{u_1}(\rho_k).$$

(ii) If $t \in [\alpha^k_i, \tilde{t}]$ then $u^k(t) = \frac{v^k_i(a^k_i)(\tau^k_i - t) + (t-\alpha^k_i)u^k_i}{\tau^k_i - \alpha^k_i}$, so that, by using Lemma 4.3 ($u^k_i = u_1(\tau^k_i)$ for every k large enough, since $\tau^k_i \leq \frac{\tilde{t}+t_1}{2}$),

$$|(u^k(t) - u_1(t), h_k(t)B\dot{u}^k(t))| \leq$$
$$|v^k_i(a^k_i) - u^k_i| + |B| \left| \frac{\alpha^k_i - \tau^k_{i-1}}{a^k_i} \frac{u^k_i - v^k_i(a^k_i)}{\tau^k_i - \alpha^k_i} \right| + |u_1(\tau^k_i) - u_1(t)|. \quad (4.43)$$

(4.7) and (4.43) give that

$$||(u^k - u_1, h_k B\dot{u}^k)||_{\infty,[\alpha^k_i,\tilde{t}]} \leq C(\tau^k_i - \alpha^k_i) + |B| C \frac{\alpha^k_i - \tau^k_{i-1}}{a^k_i} + \omega_{u_1}(\rho_k).$$

Since $\rho_k \to 0$, cases (i) and (ii) tell us that $||(u^k - u_1, h_k B\dot{u}^k)||_{\infty,[\bar{t},\tilde{t}]} \leq 2\delta$ for every k large enough. □

From Proposition 4.8, one can easily deduce an approximation resul related to the piece-wise constant and the piece-wise affine interpolations of the points u^k_i, seen as the piece-wise constant and the piece-wise affine approximations of u^k, respectively. To be precise, let us set

$$\tilde{u}^k(t) := \begin{cases} x_k, & t \in [0, \tau^k_1) \\ u^k_1, & t \in [\tau^k_1 \tau^k_2) \\ \vdots \\ u^k_{k-1}, & t \in [\tau^k_{k-1}, T], \end{cases} \qquad \hat{u}^k(t) := \begin{cases} \frac{(\tau^k_1 - t)x_k + t u^k_1}{\tau^k_1}, & t \in [0, \tau^k_1) \\ \frac{(\tau^k_2 - t)u^k_1 + (t - \tau^k_1)u^k_2}{\tau^k_2 - \tau^k_1}, & t \in [\tau^k_1 \tau^k_2) \\ \vdots \\ \frac{(T-t)u^k_{k-1} + (t - \tau^k_{k-1})u^k_k}{T - \tau^k_{k-1}}, & t \in [\tau^k_{k-1}, T]. \end{cases}$$

We have that

$$\tilde{u}^k, \hat{u}^k \to u, \text{ uniformly on compact subsets of } (0, T] \setminus \{t_1, ..., t_{m-1}\}.$$

This fact requires, for its proof, much less than what we have needed to prove Proposition 4.8, since it does not take into account the velocity.

Remark 4.9. Observe that, if $x_0 = x^r_0$ and $y_0 = 0$ (recall (3.2) and (3.3)), we obtain that $v_0 \equiv x^r_0$. In this case, it turns out that

$$(u^k, h_k B\dot{u}^k) \to (u, 0), \text{ uniformly on compact subsets of } [0, T] \setminus \{t_1, ..., t_{m-1}\}.$$

To see this, choose $\hat{t} \in (0, t_1)$ and $\delta > 0$. In view of the proof of Proposition 4.8, it is enough to consider the case $t \in [0, \alpha^k_1) \subseteq [0, \hat{t}]$, so that

$$|(u^k(t) - u_1(t), h_k(t)B\dot{u}^k(t))| = |(v^k_1(\sigma), B\dot{v}^k_1(\sigma))|, \quad \text{for a certain } \sigma \in [0, a^k_1). \quad (4.44)$$

Let σ_δ and $\hat{k}_\delta$ be given by Lemma 4.3. Then, from (4.44), we have that, if $\sigma \geq \sigma_\delta$ and $k \geq \hat{k}_\delta$,

$$\begin{aligned} |(u^k(t) - u_1(t), h_k(t)B\dot{u}^k(t))| &\leq |(v^k_1(\sigma) - u_1(\tau^k_1), B\dot{v}^k_1(\sigma))| + |u_1(\tau^k_1) - u_1(t)| \\ &\leq \delta + \omega_{u_1}(\rho_k); \end{aligned}$$

if $\sigma \in [0, \sigma_\delta)$,

$$\begin{aligned} |(u^k(t) - u_1(t), h_k(t)B\dot{u}^k(t))| &\leq |(v^k_1(\sigma) - x^r_0, B\dot{v}^k_1(\sigma))| + |x^r_0 - u_1(t)| \\ &\leq ||(v^k_1 - x^r_0, B\dot{v}^k_1)||_{\infty,[0,\sigma_\delta]} + \omega_{u_1}(\rho_k), \end{aligned}$$

and we can conclude by using the fact that $(v^k_1, \dot{v}^k_1) \to (x^r_0, 0)$ uniformly on compact subsets of $[0, +\infty)$ and $\rho_k \to 0$.

What it remains to do now is an accurate study at time 0 and at the jump times t_j's. This is done in the proof of the main result of this section. We refer to (3.6)-(3.9) and (4.11) for the definitions of the sets Γ and Γ_k.

Proof of Theorem 4.1. We follow the position already used in the proof of Lemma 4.7: we write o_k and v_k in place of o_k^1 and $v^k_{o_k+1}$, respectively. In the sequel, whenever $\delta > 0$ is arbitrarily chosen, it will be implicit that the following objects, which depend on δ and have been defined in the proof of Lemma 4.7, are involved: the sequence $\{\sigma_k\}$ such that $\sigma_k \to +\infty$ and functions $\tilde{v}_k(\sigma) := v_k(\sigma + \sigma_k)$ and w_1 such that (4.40)-(4.42) hold and

$$\{(\tilde{v}_k(\sigma), B\dot{\tilde{v}}_k(\sigma)) : \sigma \in [-\sigma_k, 0]\} \subseteq \overline{B}((x_1^s, 0), \delta).$$

Choose $\delta > 0$ arbitrarily small. In order to prove the theorem, we are going to show that, for every k large enough,

$$d_H(\Gamma^k, \Gamma) = \sup_{\Gamma^k} d(\cdot, \Gamma) + \sup_{\Gamma} d(\cdot, \Gamma^k) \leq 2\delta. \tag{4.45}$$

Recall t_1^δ from (2.20) and observe that we can suppose

$$t_1 - t_1^\delta < \delta. \tag{4.46}$$

Moreover, by Lemma 4.3, there exists k_δ such that, for every $k \geq k_\delta$,

$$\tau^k_{o_k} \in \left(\frac{t_1^\delta + t_1}{2}, t_1\right] \quad \text{and} \quad u_i^k = u_1(\tau_i^k) \text{ for every } \tau_i^k \in \left[\tau_1^k, \frac{t_1^\delta + t_1}{2}\right]. \tag{4.47}$$

We divide the proof in two parts, in view of the definition of the Hausdorff distance.
(a) Here, consider $\sup_{\Gamma^k} d(\cdot, \Gamma)$ and set

$$d_k(t) := d((t, u^k(t), h_k(t)B\dot{u}^k(t)), \Gamma), \quad t \in [0, T].$$

By considering the partition $0 < \tau^k_{o_k} < \tau^k_{o_k+1} < \ldots < T$, which depends on δ (see Definition 4.5), it is clear that it is sufficient to analyze d_k in the model cases $t \in \left[0, \tau^k_{o_k}\right)$ and $t \in \left[\tau^k_{o_k}, \tau^k_{o_k+1}\right)$, since the case $j \neq 1$ can be treated in a similar way. Let us divide part (a) in two subparts.
(a1) Consider first $d_k(t)$ for $t \in [\tau^k_{o_k}, \tau^k_{o_k+1})$. From Lemma 4.2, from (4.42) and from the fact that $w_1(+\infty) = x_1^r$ with $\nabla_x f(t_1, x_1^r) = 0$ and $\nabla_x^2 f(t_1, x_1^r)$ is positive definite, it descends that there exist $\tilde{\sigma}_\delta$ such that, up to a greater k_δ,

$$(\tilde{v}_k(\sigma), B\dot{\tilde{v}}_k(\sigma)) \in \overline{B}((u_2(\tau^k_{o_k+1}), 0), \delta), \quad \text{for every } \sigma \geq \tilde{\sigma}_\delta \text{ and } k \geq k_\delta. \tag{4.48}$$

If $t \in [\tau^k_{o_k}, \alpha^k_{o_k+1})$, we have that

$$d_k(t) = d((t, v_k(\sigma), B\dot{v}_k(\sigma)), \Gamma), \qquad \text{for some } \sigma \in [0, a^k_{o_k+1}).$$

Recall that $\tilde{v}_k(\sigma) = v_k(\sigma + \sigma_k)$, with $\sigma_k \to +\infty$. Therefore, if $\sigma - \sigma_k \geq \tilde{\sigma}_\delta$, we use (4.48) to obtain that

$$\begin{aligned} d_k(t) &\leq |t - t_1| + d((\tilde{v}_k(\sigma - \sigma_k), B\dot{\tilde{v}}_k(\sigma - \sigma_k)), (x_1^r, 0)) + |u_2(\tau^k_{o_k+1}) - x_1^r| \\ &\leq (t_1 - \tau^k_{o_k}) + \rho_k + \delta + |u_2(\tau^k_{o_k+1}) - x_1^r|, \end{aligned} \tag{4.49}$$

for every $k \geq k_\delta$. If $0 \leq \sigma - \sigma_k < \tilde{\sigma}_\delta$, then

$$\begin{aligned} d_k(t) &\leq |t - t_1| + d((\tilde{v}_k(\sigma - \sigma_k), B\dot{\tilde{v}}_k(\sigma - \sigma_k)), \mathscr{I}_1) \\ &\leq (t_1 - \tau^k_{o_k}) + \rho_k + ||(\tilde{v}_k - w_1, B\dot{\tilde{v}}_k - B\dot{w}_1)||_{\infty, [0, \tilde{\sigma}_\delta]}. \end{aligned} \tag{4.50}$$

In the case $\sigma < \sigma_k$, by the definition of σ_k we have that $(v_k(\sigma), B\dot{v}_k(\sigma)) \in \overline{B}((x_1^s, 0), \delta)$. This fact, together with (4.49), (4.50), (4.42) and Lemma 4.7, gives that

$$d_k(t) \leq 2\delta, \qquad \text{for every } t \in [\tau^k_{o_k}, \alpha^k_{o_k+1}), \tag{4.51}$$

for every k sufficiently large.

In the remaining case $t \in [\alpha^k_{o_k+1}, \tau^k_{o_k+1})$, we use (4.7) and Lemma 4.7 (c), so that

$$d_k(t) \leq (t_1 - \tau^k_{o_k}) + \rho_k + C|\tau^k_{o_k+1} - \alpha^k_{o_k+1}| + |u_2(\tau^k_{o_k+1}) - x^r_1| + |B|C\frac{\alpha^k_{o_k+1} - \tau^k_{o_k}}{a^k_{o_k+1}}. \tag{4.52}$$

(4.51), (4.52) and again Lemma 4.7 give that, for k large enough,

$$\sup_{\left[\tau^k_{o_k}, \tau^k_{o_k+1}\right)} d_k \leq 2\delta. \tag{4.53}$$

(a2) Now, let us take into account $d_k(t)$ for $t \in [0, \tau^k_{o_k})$. We have to distinguish the case $t \in \left[0, \frac{t^\delta_1+t_1}{2}\right)$ from $t \in \left[\frac{t^\delta_1+t_1}{2}, \tau^k_{o_k}\right)$. Suppose $t \in [\tau^k_i, \tau^k_{i+1})$ for some $i \geq 1$. The case $t \in [0, \tau^k_1)$ can be handled similarly to the case (a1), but more easier, since in this case we have to use the uniform convergence on compacts of $(v^k_1, \dot{v}^k_1)$ to $(v_0, \dot{v}_0)$, instead of the one of $(\tilde{v}_k, \dot{\tilde{v}}_k)$ to $(w_1, \dot{w}_1)$.
If $t \in \left[0, \frac{t^\delta_1+t_1}{2}\right) \cap [\tau^k_i, \alpha^k_{i+1})$, we have that $d_k(t) = d((t, v^k_{i+1}(\sigma), B\dot{v}^k_{i+1}(\sigma)), \Gamma)$ for some $\sigma \in [0, a^k_{i+1})$. Thus, by using (4.26) in Lemma 4.3, up to a bigger k_δ, we obtain that

$$\tau^k_{i+1} \leq \frac{t^\delta_1 + t_1}{2}, \quad \text{for every} \quad k \geq k_\delta,$$

so that

$$u^k_{i+1} = u_1(\tau^k_{i+1}), \tag{4.54}$$

and

$$d_k(t) \leq (\tau^k_{i+1} - t) + |(v^k_{i+1}(\sigma) - u_1(\tau^k_{i+1}), B\dot{v}^k_{i+1}(\sigma))| \leq \rho_k + \delta, \quad \text{for every } \sigma \geq 0. \tag{4.55}$$

If $t \in \left[0, \frac{t^\delta_1+t_1}{2}\right) \cap [\alpha^k_{i+1}, \tau^k_{i+1})$, again in view of (4.54) it turns out that

$$\begin{aligned} d_k(t) &\leq (t - \alpha^k_{i+1}) + |v^k_{i+1}(a^k_{i+1}) - u_1(\tau^k_{i+1})| + |B|\frac{\alpha^k_{i+1} - \tau^k_i}{a^k_{i+1}}\frac{|u_1(\tau^k_{i+1}) - v^k_{i+1}(a^k_{i+1})|}{\tau^k_{i+1} - \alpha^k_{i+1}} \\ &\leq \rho_k + C|\tau^k_{i+1} - \alpha^k_{i+1}| + |B|C\frac{\alpha^k_{i+1} - \tau^k_i}{a^k_{i+1}}, \end{aligned} \tag{4.56}$$

where the last inequality is due to (4.7).
In the case $t \in \left[\frac{t^\delta_1+t_1}{2}, \tau^k_{o_k}\right) \cap [\tau^k_i, \tau^k_{i+1})$, observe first that we can suppose, for larger k's, that $\tau^k_i \geq t^\delta_1$. Thus, since $u^k_i \in \{u_1(\tau^k_i), \overline{u}_1(\tau^k_i)\}$, we have that

$$u^k_i \in \overline{B}\left(\left(x^s_1, \frac{\delta}{4}\right)\right).$$

This fact, together with the fact that $\tau^k_i < \tau^k_{o_k}$ and the definition of o_k, gives that

$$(v^k_{i+1}(\sigma), B\dot{v}^k_{i+1}(\sigma)) \in B((x^s_1, 0), \delta), \quad \text{for every } \sigma \geq 0. \tag{4.57}$$

Thus, if $t \in [\tau^k_i, \alpha^k_{i+1})$, so that $d_k(t) = d((t, v^k_{i+1}(\sigma), B\dot{v}^k_{i+1}(\sigma)), \Gamma)$ for some $\sigma \in [0, a^k_{i+1})$, from (4.46) and (4.57) it turns out that

$$\begin{aligned} d_k(t) &\leq d((t, v^k_{i+1}(\sigma), B\dot{v}^k_{i+1}(\sigma)), (t_1, x^s_1, 0)) \\ &\leq \frac{t_1 - t^\delta_1}{2} + |(v^k_{i+1}(\sigma) - x^s_1, B\dot{v}^k_{i+1}(\sigma))| \leq 2\delta. \end{aligned} \tag{4.58}$$

Otherwise, if $t \in [\alpha^k_{i+1}, \tau^k_{i+1})$, we have

$$\begin{aligned} d_k(t) &\leq \rho_k + |v^k_{i+1}(a^k_{i+1}) - u_1(\tau^k_{i+1})| + |B| \frac{\alpha^k_{i+1} - \tau^k_i}{a^k_{i+1}} \frac{|u^k_{i+1} - v^k_{i+1}(a^k_{i+1})|}{\tau^k_{i+1} - \alpha^k_{i+1}} \\ &\leq \rho_k + C|\tau^k_{i+1} - \alpha^k_{i+1}| + |u^k_{i+1} - x^s_1| + |x^s_1 - u_1(\tau^k_{i+1})| + |B| C \frac{\alpha^k_{i+1} - \tau^k_i}{a^k_{i+1}}, \end{aligned}$$

where, in the last inequality, we have used (4.7). Then, by using (4.1), the definition of t^δ_1 and (4.57) (which gives $u^k_{i+1} \in \overline{B}(x^s_1, \delta)$), we have that, for every k large enough,

$$d_k(t) \leq 2\delta, \quad \text{for every } t \in \left[\frac{t^\delta_1 + t_1}{2}, \tau^k_{o_k}\right) \cap [\alpha^k_{i+1}, \tau^k_{i+1}). \tag{4.59}$$

(4.55), (4.56), (4.58) and (4.59) imply that, for every k large enough,

$$\sup_{[0, \tau^k_{o_k})} d_k \leq 2\delta. \tag{4.60}$$

(b) Here, we consider $\sup_\Gamma d(\cdot, \Gamma^k)$. By the definition of Γ and by the fact that $(x^s_1, 0) \in \overline{\mathscr{I}}_1$, it is sufficient to consider the cases

$$\sup_{t \in [0, t_1)} d((t, u_1(t), 0), \Gamma^k), \quad \sup_{\{0\} \times \mathscr{I}_0} d(\cdot, \Gamma^k), \quad \sup_{\{t_1\} \times \mathscr{I}_1} d(\cdot, \Gamma^k).$$

Let us divide part (b) in three subparts.
(b1) If $t \in [0, t^\delta_1)$, suppose $t \in [\tau^k_i, \tau^k_{i+1})$ for a certain index i, so that, up to a bigger k_δ, $\tau^k_{i+1} \leq t^\delta_1$ and , in turn, $u^k_{i+1} = u_1(\tau^k_{i+1})$ for $k \geq k_\delta$. Therefore, by recalling that $u^k(\tau^k_{i+1}) = u^k_{i+1}$ and $\dot{u}^k(\tau^k_{i+1}) = 0$, we obtain that

$$d((t, u_1(t), 0), \Gamma^k) \leq (\tau^k_{i+1} - t) + |u_1(t) - u_1(\tau^k_{i+1})| \leq \rho_k + \omega_{u_1}(\rho_k), \tag{4.61}$$

for every $k \geq k_\delta$. For $t \in [t^\delta_1, t_1)$, we write, in view of (4.46) and (4.47),

$$\begin{aligned} d((t, u_1(t), 0), \Gamma^k) &\leq d((t, u_1(t), 0), (\tau^k_{o_k}, u^k(\tau^k_{o_k}), h_k(\tau^k_{o_k}) B \dot{u}^k(\tau^k_{o_k})) \\ &\leq (t_1 - t^\delta_1) + |u_1(t) - x^s_1| + |x^s_1 - u^k_{o_k}| < 2\delta, \end{aligned} \tag{4.62}$$

for every $k \geq k_\delta$. (4.61) and (4.62), together with (4.1), give, up to a bigger k_δ,

$$\sup_{t \in [0, t_1)} d((t, u_1(t), 0), \Gamma^k) < 2\delta, \quad \text{for every } \; k \geq k_\delta. \tag{4.63}$$

(b2) Consider $(0, v_0(\sigma), B\dot{v}_0(\sigma)) \in \{0\} \times \mathscr{I}_0$ and let $s^\delta_0 > 0$ be defined as in (3.38). Since $a^k_1 > s^\delta_0$ for every k large enough, it turns out that

$$d((0, v_0(\sigma), B\dot{v}_0(\sigma)), \Gamma^k) \leq \alpha^k_1 + \min_{s \in [0, \sigma_\delta]} d((v_0(\sigma), B\dot{v}_0(\sigma)), (v^k_1(s), B\dot{v}^k_1(s))).$$

Then, if $\sigma \in [0, s^\delta_0]$, we have that

$$d((0, v_0(\sigma), B\dot{v}_0(\sigma)), \Gamma^k) \leq \alpha^k_1 + ||(v_0 - v^k_1, B\dot{v}_0 - B\dot{v}^k_1)||_{\infty, [0, \sigma_\delta]}, \tag{4.64}$$

while, if $\sigma > s^\delta_0$,

$$\begin{aligned} d((0, v_0(\sigma), B\dot{v}_0(\sigma)), \Gamma^k) &\leq d((v_0(\sigma), B\dot{v}_0(\sigma)), (u^k(\tau^k_1), h_k(\tau^k_1) B\dot{u}^k(\tau^k_1))) \\ &\leq d((v_0(\sigma), B\dot{v}_0(\sigma)), (x^r_0, 0)) + |x^r_0 - u_1(\tau^k_1)|. \end{aligned} \tag{4.65}$$

(4.64) and (4.65), together with (4.1) and the uniform convergence of $(v^k_1, \dot{v}^k_1)$ to $(v_0, \dot{v}_0)$ on compact subsets of $[0, +\infty)$, give

$$\sup_{\{0\} \times \mathscr{I}_0} d(\cdot, \Gamma^k) \leq \delta, \quad \text{for every } k \text{ large enough.} \tag{4.66}$$

(b3) Finally, let us take into account $\sup_{\{t_1\}\times\mathscr{I}_1} d(\cdot,\Gamma^k)$. By recalling (4.40), we obtain that

$$\begin{aligned} d((t_1,w_1(s),B\dot{w}_1(s)),\Gamma^k) &\leq (t_1-\tau^k_{o_k})+d((w_1(s),B\dot{w}_1(s)),(u^k_{o_k},0)) \\ &\leq (t_1-\tau^k_{o_k})+\delta+|x^s_1-u^k_{o_k}|, \end{aligned} \tag{4.67}$$

for every $s<0$. Similarly, if $s^\delta_1>0$ is defined as in (3.56), for every $s>s^\delta_1$ we can write

$$\begin{aligned} d((t_1,w_1(s),B\dot{w}_1(s)),\Gamma^k) &\leq |t_1-\tau^k_{o_k+1}|+d((w_1(s),B\dot{w}_1(s)),(u^k_{o_k+1},0))| \\ &\leq |t_1-\tau^k_{o_k+1}|+\frac{\delta}{2}+|x^r_1-u^k_{o_k+1}|. \end{aligned} \tag{4.68}$$

For the rest part of the proof we need the following claim, whose proof is postponed at the end.

Claim. For every $\hat{s}\geq 0$ and $\delta>0$ sufficiently small, there exists $k_{\hat{s},\delta}>0$ such that

$$a^k_{o_k+1}-\sigma_k>\hat{s}, \quad \text{for every} \quad k\geq k_{\hat{s},\delta}.$$

It remains to consider $s\in[0,s^\delta_1]$. In this case,

$$\begin{aligned} d((t_1,w_1(s),B\dot{w}_1(s)),\Gamma^k) &\leq \inf_{t\in[\tau^k_{o_k},\alpha^k_{o_k+1})} d((t_1,w_1(s),B\dot{w}_1(s)),(t,u^k(t),h_k(t)B\dot{u}^k(t))) \\ &\leq (t_1-\tau^k_{o_k})+\rho_k+\inf_{\sigma\in[0,a^k_{o_k+1})} d((w_1(s),B\dot{w}_1(s)),(v_k(\sigma),B\dot{v}_k(\sigma))) \\ &= (t_1-\tau^k_{o_k})+\rho_k+\inf_{\sigma\in[-\sigma_k,a^k_{o_k+1}-\sigma_k)} d((w_1(s),B\dot{w}_1(s)),(\tilde{v}_k(\sigma),B\dot{\tilde{v}}_k(\sigma))). \end{aligned}$$

Thus, since $\sigma_k\to+\infty$, in view of Lemma 4.7 (1), of the claim and of (4.42) we have that, up to a bigger k_δ,

$$\begin{aligned} d((t_1,w_1(s),B\dot{w}_1(s)),\Gamma^k) &\leq (t_1-\tau^k_{o_k})+\rho_k+\min_{\sigma\in[0,s^\delta_1]} d((w_1(s),B\dot{w}_1(s)),(\tilde{v}_k(\sigma),B\dot{\tilde{v}}_k(\sigma))) \\ &\leq (t_1-\tau^k_{o_k})+\rho_k+||(w_1-\tilde{v}_k,B\dot{w}_1-B\dot{\tilde{v}}_k)||_{\infty,[0,s^\delta_1]} \\ &\leq \delta+\rho_k, \end{aligned} \tag{4.69}$$

for every $k\geq k_\delta$. (4.67), (4.68) and (4.69), together with (4.1), imply that, up to a bigger k_δ,

$$\sup_{\{t_1\}\times\mathscr{I}_1} d(\cdot,\Gamma^k)\leq 2\delta, \quad \text{for every} \quad k\geq k_\delta. \tag{4.70}$$

By considering together the estimates in (4.53), (4.60), (4.63), (4.66) and (4.70), which hold also for generic j's in place of $j=1$, we obtain (4.45). □

Proof of the claim. Suppose, by contradiction, that, for a certain $\hat{s}\geq 0$, $\delta>0$ and up to a subsequence, $a^k_{o_k+1}\leq\hat{s}+\sigma_k$ for every k. Then, by the definition of σ_k (see (4.37)),

$$(v_k(a^k_{o_k+1}-\hat{s}),B\dot{v}_k(a^k_{o_k+1}-\hat{s}))\in\overline{B}((x^s_1,0),\delta)$$

so that, up to a subsequence, $(v_k(a^k_{o_k+1}-\hat{s}),B\dot{v}_k(a^k_{o_k+1}-\hat{s}))\to(p,\dot{p})$, for some $(p,\dot{p})\in\overline{B}((x^s_1,0),\delta)$. Consider

$$\hat{v}_k(\sigma):=v_k(\sigma+a^k_{o_k+1}-\hat{s}), \quad \text{for } \sigma\geq\hat{s}-a^k_{o_k+1}.$$

From Lemma 4.7, from the definition of v_k and from the fact that $a^k_{o_k+1}\to+\infty$, it is clear that $(\hat{v}_k,\dot{\hat{v}}_k)$ converges uniformly on compact subsets of $\mathbb{R}$ to $(\hat{w}_1,\dot{\hat{w}}_1)$, where $\hat{w}_1$ is the solution of

$$\begin{cases} A\ddot{\hat{w}}_1+B\dot{\hat{w}}_1+\nabla_x f(t_1,\hat{w}_1)=0, \\ \hat{w}_1(0)=p, \\ B\dot{\hat{w}}_1(0)=\dot{p}. \end{cases}$$

Observe that

$$(v_k(a^k_{o_k+1}), B\dot v_k(a^k_{o_k+1})) = (\hat v_k(\hat s), B\dot{\hat v}_k(\hat s)) \to (\hat w_1(\hat s), B\dot{\hat w}_1(\hat s)),$$

and $(\hat w_1(\hat s), B\dot{\hat w}_1(\hat s)) \neq (x^r_1, 0)$, otherwise it would be $(\hat w_1, B\dot{\hat w}_1) \equiv (x^r_1, 0)$, so that $(x^r_1, 0) = (\hat w_1(0), B\dot{\hat w}_1(0)) = (p, \dot p) \in \overline{B}((x^s_1, 0), \delta)$, which is not true if δ is small enough. This is a contradiction, since, by Lemma 4.7 (3) and by (4.7), we obtain that, for every k large enough,

$$\begin{aligned}|(v_k(a^k_{o_k+1}) - x^r_1, B\dot v_k(a^k_{o_k+1}))| &\le |(v_k(a^k_{o_k+1}) - u^k_{o_k+1}, B\dot v_k(a^k_{o_k+1}))| + |u^k_{o_k+1} - x^r_1| \\ &\le C(\tau^k_{o_k+1} - \alpha^k_{o_k+1}) + |B|\delta_k + |u^k_{o_k+1} - x^r_1|,\end{aligned}$$

so that

$$(\hat v_k(\hat s), B\dot{\hat v}_k(\hat s)) = (v_k(a^k_{o_k+1}), B\dot v_k(a^k_{o_k+1})) \to (x^r_1, 0).$$

□

5. Appendix: existence and uniqueness of the heteroclinic solution

For sake of completeness and since we could not find in the literature a satisfying proof, we state and prove here a result of existence and uniqueness, up to translations, of the solution of a first order autonomous system, issuing from a zero of the vector field where suitable transversality conditions are satisfied.

Proposition 5.1. *Let $F : \mathbb{R}^m \to \mathbb{R}^m$ be a C^2 function such that $F(\eta) = 0$. Let the following two conditions be satisfied:*

(i) 0 *is an eigenvalue of $\nabla F(\eta)$ with $m_a(0) = 1$ and $\mathrm{Re}(\lambda) < 0$ for every eigenvalue $\lambda \neq 0$. This implies that there exist ω, $\nu \in \mathbb{R}^m$ such that $\omega\cdot\nu \neq 0$ and $\ker \nabla F(\eta) = \mathrm{span}(\omega)$, $\ker \nabla F(\eta)^T = \mathrm{span}(\nu)$;*

(ii) $\nu\cdot\nabla^2 F(\eta)[\omega, \omega] \neq 0$.

Excluding the constant solution η, there are infinitely many solutions of the problem

$$\begin{cases} \dot x(t) = F(x(t)), & t \in (-\infty, 0] \\ \lim_{t\to-\infty} x(t) = \eta, \end{cases} \tag{5.1}$$

and they differ from each other by time-translations.

From assumption (i) of Proposition 5.1 it descends that $\mathbb{R}^m$ can be decomposed as

$$\mathbb{R}^m = X_1 \oplus X_2, \quad \text{with} \quad X_1 := \mathrm{span}(\omega) \ \text{ and } \ X_2 := \{\nu\}^\perp.$$

We denote by π_i the projection on X_i, $i = 1, 2$, so that every $x \in \mathbb{R}^m$ can be uniquely written as $x = x_1 + x_2$, where $x_i = \pi_i(x)$. Observe that

$$\pi_1(x) = x_\omega \omega, \quad \text{where} \quad x_\omega := \frac{x\cdot\nu}{\omega\cdot\nu}.$$

For every $\mathbb{R}^m$-valued function g, we use the notation

$$g_\omega := (g(\cdot))_\omega, \qquad g_i := (g(\cdot))_i, \quad i = 1, 2.$$

To further simplify the notation, we write A in place of $\nabla F(\eta)$ and denote by β the *spectral gap* of A, that is

$$\beta := \min\{|\mathrm{Re}(\lambda)| \ : \ \lambda \text{ is eigenvalue of } A \text{ and } \mathrm{Re}(\lambda) \neq 0\}.$$

It is well-known that for every $\varepsilon \in (0, \beta)$, there exists $C_\varepsilon > 0$ (also depending on A) such that the following fundamental estimates hold:

$$\left\|e^{tA} \circ \pi_1\right\| \le C_\varepsilon e^{\varepsilon|t|}, \quad \text{for every } t \in \mathbb{R}, \tag{5.2}$$

$$\left\|e^{tA} \circ \pi_2\right\| \le C_\varepsilon e^{-(\beta-\varepsilon)t}, \quad \text{for every } t \ge 0. \tag{5.3}$$

Remember that both π_1 and π_2 commute with A and hence with e^{tA}. The proof of Proposition 5.1 requires the following lemma.

Lemma 5.2. *Under the same assumptions of Proposition 5.1, for every $a > 0$ sufficiently large there exists a unique solution of the problem*

$$\begin{cases} \dot{x}(t) = F(x(t)), \quad t \in (-\infty, 0] \\ x_\omega(0) = \eta_\omega + \frac{1}{a}, \\ \lim_{t\to-\infty} x(t) = \eta, \end{cases} \tag{5.4}$$

in the space

$$Y^a := \{y : (-\infty, 0] \to \mathbb{R}^m \ : \ \|y_1\|_{Y_1^a} < \infty, \ \|y_2\|_{Y_2^a} < \infty\},$$

where

$$\|y_1\|_{Y_1^a} := \sup_{t\le 0} |t-a||y_1(t)|, \quad \|y_2\|_{Y_2^a} := \sup_{t\le 0} |t-a|^{\frac{3}{2}}|y_2(t)|.$$

Proof. First, observe that Y^a is a Banach space with the norm

$$\|y\|_{Y^a} := \|y_1\|_{Y_1^a} + \|y_2\|_{Y_2^a}.$$

Note that we can suppose $\eta = 0$ and $|\omega|$, $|\nu| = 1$. Also, we can suppose

$$\nu\cdot\nabla^2 F(0)[\omega,\omega] = 2(\omega\cdot\nu). \tag{5.5}$$

If we write

$$F(x(t)) = \nabla F(0)x(t) + \frac{1}{2}\nabla^2 F(0)[x(t)]^2 + o(|x(t)|^2),$$

observe that, by assumptions (i) and (ii) and by (5.5), $F_\omega(x(t))$ has the following expression:

$$F_\omega(x(t)) = x_\omega^2(t) + \frac{\nu}{\omega\cdot\nu}\left\{\nabla^2 F(0)[x_1(t), x_2(t)] + \frac{1}{2}\nabla^2 F(0)[x_2(t)]^2 + o(|x(t)|^2)\right\}; \tag{5.6}$$

while, by assumption (i) and by noticing that $\mathrm{Rank}(\nabla F(0)) \subseteq X_2$, we have that

$$F_2(x(t)) = \nabla F(0)x_2(t) + \pi_2\left\{\frac{1}{2}\nabla^2 F(0)[x(t)]^2 + o(|x(t)|^2\right\}. \tag{5.7}$$

For $y \in Y^a$, with $a > 0$ to be chosen, we define on $(-\infty, 0]$ the following functions:

$$h_1^y(\cdot) \ := \ F_\omega(y(\cdot)) - y_\omega^2(\cdot), \tag{5.8}$$

$$h_2^y(\cdot) \ := \ F_2(y(\cdot)) - \nabla F(0)y_2(\cdot). \tag{5.9}$$

Let y vary in $B_R := \{y \in Y^a \ : \ \|y\|_{Y^a} \le R\}$ for a certain $R > 0$ to be chosen later and observe that, from the definition of $\|\cdot\|_{Y^a}$, easily follows that $\|y\|_{Y^a} \ge \min\{a, a^{\frac{3}{2}}\}\|y\|_{\infty,(-\infty,0]}$. Therefore, for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that, if

$$\|y\|_{\infty,(-\infty,0]} \le \frac{R}{\min\{a, a^{\frac{3}{2}}\}} \le \delta(\varepsilon), \tag{5.10}$$

the following estimates, which descend from (5.6) and (5.8) and from (5.7) and (5.9), respectively, hold for every $t \le 0$:

$$\begin{aligned} |h_1^y(t)| &\le \frac{1}{|\omega\cdot\nu|}\left[\|\nabla^2 F(0)\|\left(|y_1(t)||y_2(t)| + \frac{|y_2(t)|^2}{2}\right) + \varepsilon|y(t)|^2\right] \\ &\le \frac{R^2}{(t-a)^2}\frac{1}{|\omega\cdot\nu|}\left[\|\nabla^2 F(0)\|\left(\frac{1}{\sqrt{a}} + \frac{1}{2a}\right) + 2\varepsilon\left(1 + \frac{1}{a}\right)\right]; \end{aligned}$$

$$\begin{aligned} |h_2^y(t)| &\le \frac{1}{2}\|\nabla^2 F(0)\||y(t)|^2 + \varepsilon|y(t)|^2 \\ &\le \frac{R^2}{(t-a)^{\frac{3}{2}}}(\|\nabla^2 F(0)\| + 2\varepsilon)\left(\frac{1}{\sqrt{a}} + \frac{1}{a^{\frac{3}{2}}}\right). \end{aligned}$$

Thus, we can briefly write, for $t \leq 0$, that

$$|h_1^y(t)| \leq \frac{R^2}{(t-a)^2} M(a,\varepsilon), \quad \text{with } M(a,\varepsilon) \to 0, \text{ as } a \to +\infty, \varepsilon \to 0^+, \tag{5.11}$$

and

$$|h_2^y(t)| \leq \frac{R^2}{|t-a|^{\frac{3}{2}}} \tilde{M}(a,\varepsilon), \quad \text{with } \tilde{M}(a,\varepsilon) \to 0, \text{ as } a \to +\infty. \tag{5.12}$$

We consider the auxiliary problems

$$\begin{cases} \dot{x}_\omega(t) - x_\omega^2(t) = h_1^y(t), & t \in (-\infty, 0] \\ x_\omega(0) = \frac{1}{a}, \end{cases} \tag{5.13}$$

and

$$\begin{cases} \dot{x}_2(t) - \nabla F(0) x_2(t) = h_2^y(t), & t \in (-\infty, 0] \\ \lim_{t \to -\infty} x_2(t) = 0. \end{cases} \tag{5.14}$$

We are going to prove, in step 1 and 2, that problems (5.13) and (5.14) have unique solutions and that the solution of problem (5.13) tends to 0 as t tends to $-\infty$. Therefore, if $x = y$, such problems are equivalent to (5.4).
Step 1. If $y \in B_R$ and (5.10) holds, (5.11) implies that there exists $H_1^y \in L^\infty(-\infty, 0)$ such that

$$h_1^y(t) = \frac{H_1^y}{(t-a)^2}, \quad t \leq 0,$$

and $\|H_1^y\|_{\infty,(-\infty,0)} \leq R^2 M(a,\varepsilon)$. Now, by observing that the equation in (5.13) is a particular Riccati equation and by setting $x_\omega = \frac{u}{(t-a)}$, we have that problem (5.13) is equivalent to

$$\begin{cases} \dot{u}(t) = \frac{u^2(t) + u(t) + H_1^y(t)}{(t-a)}, & t \in (-\infty, 0] \\ u(0) = -1, \end{cases} \tag{5.15}$$

Let w be the solution of (5.15) with $-R^2 M(a,\varepsilon)$ in place of H_1^y and v the solution of (5.15) with $R^2 M(a,\varepsilon)$ in place of H_1^y. It is easy to check that, if

$$M(a,\varepsilon) < \frac{1}{4R^2}, \tag{5.16}$$

then $\frac{-1-\sqrt{1+4R^2M(a,\varepsilon)}}{2} < w \leq -1 \leq v < \frac{-1-\sqrt{1-4R^2M(a,\varepsilon)}}{2}$. Therefore, by differential inequalities (see, e. g., [8]), we obtain that for every $t \leq 0$

$$\frac{-1-\sqrt{1+4R^2M(a,\varepsilon)}}{2} < u(t) = (t-a)x_\omega(t) < \frac{-1-\sqrt{1-4R^2M(a,\varepsilon)}}{2}, \tag{5.17}$$

and in turn, from (5.16), that

$$\sup_{t \leq 0} |t-a||x_\omega(t)| < \frac{1+\sqrt{1+4R^2M(a,\varepsilon)}}{2} < \frac{1+\sqrt{2}}{2}.$$

Step 2. By the variation of constants formula, we can write a solution of the equation in (5.14) as

$$x_2(t) = e^{(t-t_0)\nabla F(0)} x_2(t_0) + \int_{t_0}^t e^{(t-\tau)\nabla F(0)} h_2^y(\tau) d\tau.$$

By using (5.3), we have that there exists a constant $C_\beta > 0$, depending on the spectral gap $\beta > 0$ of $\nabla F(0)$, such that

$$\lim_{t_0 \to -\infty} \left| e^{(t-t_0)\nabla F(0)} x_2(t_0) \right| \leq \lim_{t_0 \to -\infty} C_\beta e^{-\frac{\beta}{2}(t-t_0)} |x_2(t_0)| = 0.$$

Therefore, the solution of problem (5.14) is

$$x_2(t) = \int_{-\infty}^{t} e^{(t-\tau)\nabla F(0)} h_2^y(\tau) d\tau, \quad t \le 0.$$

Now, if $y \in B_R$ and (5.10) holds, it easy to check, by using (5.3) and (5.12), that

$$\|x_2\|_{Y_2^a} \le \frac{2}{\beta} C_\beta R^2 \tilde{M}(a, \varepsilon).$$

Observe that $\|x_2\|_{Y_2^a} \le \frac{R}{2}$ if

$$\tilde{M}(a, \varepsilon) \le \frac{\beta}{4 C_\beta R}. \tag{5.18}$$

From step 1 and 2 we have obtained that, if

$$R := 1 + \sqrt{2},$$

and a is large enough and ε small enough such that (5.10), (5.16) and (5.18) are satisfied, then the operator

$$\Gamma : B_R \to B_R$$

which associates to $y \in B_R$ the function $x = x_\omega \omega + x_2$, with x_ω and x_2 the solutions of (5.13) and (5.14) respectively, is well-defined.

To conclude the proof, it remains to show that Γ is a contraction. Given y, $y^* \in B_R$, set

$$\Gamma(y) = x = x_1 + x_2, \qquad \Gamma(y^*) = x^* = x_1^* + x_2^*.$$

Let us handle the first component and the second one separately, by proceeding in two steps. The following estimates can be obtained similarly to (5.11) and (5.12). They hold if $R / \min\{a, a^{\frac{3}{2}}\} \le \tilde{\delta}(\varepsilon)$ for some $\tilde{\delta}(\varepsilon) > 0$ which can be supposed to be equal to $\delta(\varepsilon)$ (see (5.10)).

$$|h_1^y(t) - h_1^{y^*}(t)| \le \frac{R}{(t-a)^2} N(a, \varepsilon) \|y - y^*\|_{Y^a}, \quad \text{for every } t \le 0, \tag{5.19}$$

where

$$N(a, \varepsilon) := \frac{1}{|\omega \cdot \nu|} \left[\|\nabla^2 F(0)\| \left(\frac{1}{\sqrt{a}} + \frac{1}{a} \right) + 4\varepsilon \left(1 + \frac{1}{a} \right) \right],$$

and

$$|h_2^y(t) - h_2^{y^*}(t)| \le \frac{R}{|t-a|^{\frac{3}{2}}} \tilde{N}(a, \varepsilon) \|y - y^*\|_{Y^a}, \quad \text{for every } t \le 0, \tag{5.20}$$

where

$$\tilde{N}(a, \varepsilon) := \frac{1}{\sqrt{a}} \left[\|\nabla^2 F(0)\| \left(1 + \frac{1}{a} + \frac{2}{a} \right) + 4\varepsilon \left(1 + \frac{1}{a} \right) \right].$$

Step 3. As already done in step 1, let us set $x_\omega = \frac{u}{(t-a)}$ and $x_\omega^* = \frac{u^*}{(t-a)}$. From (5.17) we deduce that, for a large enough and ε small enough,

$$\alpha := u + u^*, \quad \text{is such that} \quad 1 + \alpha(t) < -\frac{1}{2}, \quad \text{for every } t \le 0. \tag{5.21}$$

Observe that the function $z := u - u^*$ satisfies the equation

$$\dot{z}(t) = \frac{1}{(t-a)} \{[1 + \alpha(t)] z(t) + H_1^y(t) - H_1^{y^*}(t)\}, \quad t \le 0,$$

and the condition $z(0) = 0$. Therefore, by the variation of constants formula, z satisfies the following estimate:

$$|z(t)| \le \|H_1^y - H_1^{y^*}\|_{\infty,(-\infty,0)} \int_t^0 \frac{\exp\left(\int_\tau^t \frac{1+\alpha(s)}{s-a}) ds \right)}{a - \tau} d\tau. \tag{5.22}$$

From (5.21), it turns out that

$$\int_t^0 \frac{\exp\left(\int_\tau^t \frac{1+\alpha(s)}{s-a})ds\right)}{a-\tau}d\tau \le \int_t^0 \frac{\exp\left(-\frac{1}{2}\int_t^\tau \frac{1}{a-s})ds\right)}{a-\tau}d\tau$$
$$= |t-a|^{-\frac{1}{2}}\int_t^0 (a-\tau)^{-\frac{1}{2}}d\tau \le 2. \quad (5.23)$$

Thus, since $\|H_1^y - H_1^{y^*}\|_{\infty,(-\infty,0)} = \sup_{t\le 0}(t-a)^2|h_1^y(t) - h_1^{y^*}(t)|$, from (5.22), (5.23) and (5.19) we obtain that

$$\|x_1 - x_1^*\|_{Y_1^a} \le 2RN(a,\varepsilon)\|y-y^*\|_{Y^a}. \quad (5.24)$$

Step 4. Since

$$|x_2(t) - x_2^*(t)| \le C_\beta \int_{-\infty}^t e^{-\frac{\beta}{2}(t-\tau)}|h_2^y(\tau) - h_2^{y^*}(\tau)|d\tau,$$

from (5.20) we have that

$$\begin{aligned}\|x_2 - x_2^*\|_{Y_2^a} &\le C_\beta R\tilde{N}(a,\varepsilon)\|y-y^*\|_{Y^a}\int_{-\infty}^t \frac{e^{-\frac{\beta}{2}(t-\tau)}}{|t-a|^{\frac{3}{2}}}d\tau \\ &\le \frac{2}{\beta}C_\beta R\tilde{N}(a,\varepsilon)\|y-y^*\|_{Y^a}. \quad (5.25)\end{aligned}$$

Finally, if we choose $a>0$ sufficiently large and $\varepsilon>0$ sufficiently small such that, for $R = 1+\sqrt{2}$, (5.10), (5.16), (5.18) and (5.21) hold together with

$$N(a,\varepsilon) \le \frac{1}{8R}, \qquad \tilde{N}(a,\varepsilon) \le \frac{\beta}{8C_\beta R},$$

from (5.24) and (5.25) we obtain that

$$\|\Gamma(y) - \Gamma(y^*)\|_{Y^a} = \|x_1 - x_1^*\|_{Y_1^a} + \|x_2 - x_2^*\|_{Y_2^a} \le \frac{1}{2}\|y-y^*\|_{Y^a},$$

that is Γ is a contraction from B_R to B_R. □

Proof of Proposition 5.1. The existence of a solution of problem (5.1), different from the constant η, is proved by Lemma 5.2. It remains to show the uniqueness of such a solution, up to time-translations. Clearly, we can suppose $\eta = 0$ in (5.1). The idea is to show that for every solution x of (5.1), different from the constant solution, there exists a sequence $t_n \to -\infty$ such that $x_\omega(t_n) > 0$ (and $x_\omega(t_n) \to 0$). In this way, it is possible to prove that the projections of the trajectories on X_1 intersect, and conclude by using Lemma 5.2.

Let x be a solution of (5.1). As shown in Lemma 5.2, the system $\dot{x}(t) = F(x(t))$ is equivalent to

$$\begin{aligned}\dot{x}_\omega(t) &= x_\omega^2(t) \\ &\quad + \frac{\nu}{\omega\cdot\nu}\left\{\nabla^2 F(0)[x_1(t), x_2(t)] + \frac{1}{2}\nabla^2 F(0)[x_2(t)]^2 + o(|x(t)|^2)\right\}, \quad (5.26)\end{aligned}$$

and

$$\dot{x}_2(t) = \nabla F(0)x_2(t) + h_2^x(t), \quad (5.27)$$

where $h_2^x(t) := \pi_2\left\{\frac{1}{2}\nabla^2 F(0)[x(t)]^2 + o(|x(t)|^2\right\}$, $t\le 0$. Observe that for every $\delta>0$ small enough, if $\|x\|_{\infty,(-\infty,0]} \le \delta$, then

$$|h_2^x(t)| \le \frac{1}{2}(\|\nabla^2 F(0)\| + 1)|x(t)|^2, \quad \text{for every } t\le 0. \quad (5.28)$$

Since $x(t) \to 0$ as $t \to -\infty$, there exists $t_0 = t_0(\delta)$ such that $|x(t)| \le \delta$ for every $t \le t_0$. Therefore, up to change x with $y(t) := x(t+t_0)$, we can suppose $t_0 = 0$ and then $\|x\|_{\infty,(-\infty,0]} \le \delta$. This assumption, together with (5.28), gives that

$$|h_2^x(t)| \le (\|\nabla^2 F(0)\| + 1)(|x_\omega(t)|^2 + \delta|x_2(t)|), \quad t \le 0. \tag{5.29}$$

From equation (5.27) and estimates (5.3) and (5.29), we obtain the following inequalities for every $t \le \hat{t} \le 0$:

$$\begin{aligned} |x_2(t)| &= \left| \int_{-\infty}^t e^{(t-\tau)\nabla F(0)} h_2^x(\tau) d\tau \right| \\ &\le C_\beta \int_{-\infty}^t e^{-\frac{\beta}{2}(t-\tau)} |h_2^x(\tau)| d\tau \\ &\le \frac{2}{\beta} C_\beta (\|\nabla^2 F(0)\| + 1)(\sup_{\tau \le \hat{t}} |x_\omega(t)|^2 + \delta \sup_{\tau \le \hat{t}} |x_2(t)|). \end{aligned}$$

Finally, if we choose δ such that $\delta \frac{2}{\beta} C_\beta (\|\nabla^2 F(0)\| + 1) < 1$, we can state that there exists $K > 0$ such that

$$\sup_{\tau \le \hat{t}} |x_2(\tau)| \le K \sup_{\tau \le \hat{t}} x_\omega^2(\tau), \quad \text{for every } \hat{t} \le 0. \tag{5.30}$$

Now, notice that it is possible to construct a sequence $\{t_n\}$ such that $t_n \to -\infty$ as $n \to \infty$ and

$$|x_\omega(t_n)| = \max_{t \le t_n} |x_\omega(t)|. \tag{5.31}$$

Thus, from (5.30), we have that

$$|x_2(t_n)| \le K x_\omega^2(t_n), \quad \text{for every } n.$$

From the last inequality and from (5.26), up to a smaller δ depending on some $\varepsilon > 0$ such that $\frac{2\varepsilon}{|\omega \cdot \nu|} < 1$, it descends that

$$\begin{aligned} &\dot{x}_\omega(t_n) \ge \\ &\quad x_\omega^2(t_n) - \frac{1}{|\omega\cdot\nu|} \left[\|\nabla^2 F(0)\| \left(|x_\omega(t_n)||x_2(t_n)| + \frac{|x_2(t_n)|^2}{2} \right) + \varepsilon |x(t_n)|^2 \right] \\ &\ge \left(1 - \frac{2\varepsilon}{|\omega\cdot\nu|}\right) x_\omega^2(t_n) - K \frac{|x_\omega(t_n)|^3}{|\omega\cdot\nu|} \left[\|\nabla^2 F(0)\| \left(1 + \frac{K}{2}|x_\omega(t_n)|\right) + 2\varepsilon K |x_\omega(t_n)| \right]. \end{aligned} \tag{5.32}$$

Now, if $x_\omega(t_n) = 0$ for some n, then $x \equiv 0$, in view of (5.31) and (5.30). Otherwise, from (5.32) we have that $\dot{x}_\omega(t_n) > 0$ for every n, and this implies, from the definition of t_n, that

$$x_\omega(t_n) > 0, \quad \text{for every } n. \tag{5.33}$$

Let x and x^* be solutions of (5.1) (with $\eta = 0$). The above arguments allow us to affirm that (5.33) hold for x_ω and x_ω^* on some sequences $\{t_n\}$ and $\{t_n^*\}$, respectively. We conclude by considering two cases:

(i) if there exist n and m such that $x_\omega(t_n) = x_\omega^*(t_m^*)$, we define $y(t) := x(t+t_n)$ and $y^*(t) := x^*(t+t_m^*)$. y and y^* satisfy problem (5.4) (with $\eta = 0$) with $a = \frac{1}{x_\omega(t_n)}$ sufficiently large. Therefore, y and y^* coincide and, in turn, x and x^* coincide up to time-translations.

(ii) if $x_\omega(t_n) \ne x_\omega^*(t_m^*)$ for every n and m, there exist n and $k > m$ such that $x_\omega(t_k) < x_\omega^*(t_n^*) < x_\omega(t_m)$. Thus, there exists $\bar{t} \in (t_k, t_m)$ such that $x_\omega(\bar{t}) = x_\omega^*(t_n^*)$. By defining $y(t) := x(t+\bar{t})$ and $y^*(t) := x^*(t+t_n^*)$, we conclude as in (i).

□

Acknowledgments

I am very grateful to Professor Gianni Dal Maso for suggesting me this work and for his supervision during the preparation of the paper.

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