COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **12**, Number **4**, July **2013**

doi:10.3934/cpaa.2013.12.

pp. –

FRACTURE MODELS AS Γ-LIMITS OF DAMAGE MODELS

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Abstract. We analyze the asymptotic behavior of a variational model for damaged elastic materials. This model depends on two small parameters, which govern the width of the damaged regions and the minimum elasticity constant attained in the damaged regions. When these parameters tend to zero, we find that the corresponding functionals Γ-converge to a functional related to fracture mechanics. The corresponding problem is brittle or cohesive, depending on the asymptotic ratio of the two parameters.

1. Introduction.

In damage models for linearly elastic materials the elasticity tensor depends continuously and monotonically on an internal variable v. This is a function defined on the reference configuration $\Omega \subset \mathbb{R}^n$ with values in the interval $[0,1]$. We assume that $v = 1$ corresponds to the original elastic material, while $v = 0$ represents the totally damaged material, with vanishing elasticity tensor. In the regions where v is very small, the displacement gradient ∇u of the solution u of a stationary elastic problem can be very large, so we expect that u develops a jump discontinuity when v tends to zero.

To study this problem in a definite setting, we consider a damage model depending on a small parameter ϵ, and investigate the behavior of the displacement u_ϵ and of the internal variable v_ϵ as $\epsilon \to 0$. We assume that the model forces damage concentration, so that $v_\epsilon \to 1$ in $L^1(\Omega)$. We assume also that $\min v_\epsilon \to 0$. In other words, there are regions with smaller and smaller volume where the elasticity tensor tends to zero. It is possible that ∇u_ϵ becomes larger and larger in these regions and that u_ϵ converges in $L^1(\Omega)$ to some function u that exhibits jump discontinuities along sets of codimension one. We expect that u can be considered as the displacement obtained in some fracture model.

In this paper we consider this problem in the simplest situation: the antiplane case for a homogeneous and isotropic material. Then the displacement u is scalar, and the elasticity tensor reduces to a single constant. We also assume that the material remains isotropic during the damage process, so that the internal variable v can be chosen equal to the elasticity coefficient of the damaged material, up to a multiplicative constant.

The stored elastic energy corresponding to the displacement u and to the internal variable v is then given by

$$\int_\Omega v|\nabla u|^2 dx.$$

2000 *Mathematics Subject Classification.* Primary: 74A45, 49J45; Secondary: 35A35.

Key words and phrases. Damage problems, fracture mechanics, Γ-convergence, free discontinuity problems.

This paper is dedicated to Igor V. Skrypnik.

The energy dissipated by the damage process has the form

$$\int_\Omega \phi_\epsilon(v)dx,$$

where $\phi_\epsilon\colon [0,1]\to\mathbb{R}$ is strictly decreasing and satisfies $\phi_\epsilon(1)=0$. The total energy for the damage model is then

$$\int_\Omega v|\nabla u|^2dx+\int_\Omega \phi_\epsilon(v)dx.$$

To force damage concentration as $\epsilon\to 0$ we assume that $\phi_\epsilon=\frac{1}{\epsilon}\psi$ for some strictly decreasing function $\psi\colon[0,1]\to\mathbb{R}$ satisfying $\psi(1)=0$. In the limit as $\epsilon\to 0$ this leads to the condition $v=1$ $\mathcal{L}^n$-a.e. in Ω. Under this assumption the total energy is

$$\int_\Omega v|\nabla u|^2dx+\frac{1}{\epsilon}\int_\Omega \psi(v)dx. \tag{1}$$

It is reasonable to require that neighboring points are similarly damaged. For this reason we assume that the internal variable v satisfies an inequality like

$$|\nabla v|\le a_\epsilon \quad \mathcal{L}^n\text{-a.e. on } \Omega, \tag{2}$$

for a certain constant a_ϵ. In this paper we develop the case $a_\epsilon=1/\epsilon$. The arguments for our proofs show that, in this case, the damaged regions $\{v_\varepsilon<\frac{1}{2}\}$ tend to concentrate around sets of codimension one, on strips whose width is proportional to ϵ.

Finally, we suppose that the material is never totally damaged. This leads to the condition

$$\eta_\epsilon\le v\le 1, \tag{3}$$

where η_ϵ is a positive constant. We assume that $\eta_\epsilon/\epsilon\to\alpha$, with $\alpha\in[0,\infty]$.

In this model a solution of the stationary damage problem is a minimizer of (1), with suitable boundary conditions, under the constraints (2) and (3). Of course, lower order terms should be added to (1) if external volume forces are present. To study the asymptotic behavior of these minimizers as $\epsilon\to 0$ we fix two sequences $\epsilon_k>0$ and $\eta_k>0$, with $\epsilon_k\to 0$ and $\eta_k\to 0$, and we determine the Γ-limit in $L^1(\Omega)\times L^1(\Omega)$ of the sequence of functionals defined by

$$F_k(u,v):=\int_\Omega v|\nabla u|^2dx+\frac{1}{\epsilon_k}\int_\Omega \psi(v)dx$$

if $u\in H^1(\Omega)$ and v satisfies the the constraints

$$\eta_k\le v\le 1 \quad \text{and} \quad |\nabla v|\le\frac{1}{\epsilon_k}\,\mathcal{L}^n\text{-a.e. on } \Omega, \tag{4}$$

and by $F_k(u,v):=+\infty$ if these conditions are not satisfied. We assume that $\eta_k/\epsilon_k\to\alpha$, with $\alpha\in[0,\infty]$. Moreover we assume that Ω is a bounded open set with Lipschitz boundary and that ψ satisfies a very mild technical condition, which is fulfilled in the standard examples $\psi(z)=1-z^\beta$, with $\beta>0$.

The Γ-limit depends on α. Its domain is contained in the spaces $GSBV^2(\Omega)$, $SBV^2(\Omega)$, or $H^1(\Omega)$, depending on the value of α. For the definition of the first two spaces we refer to 7 in Section 2, which contains also a short account on the notation used for free discontinuity problems. To describe the Γ-limit when $0 < \alpha < \infty$ we introduce the functional $\Phi_\alpha\colon L^1(\Omega) \mapsto [0, +\infty]$ defined as follows:

$$\Phi_\alpha(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 dx + a\mathcal{H}^{n-1}(J_u) + b_\alpha \int_{J_u} |[u]| d\mathcal{H}^{n-1} & \text{if } u \in SBV^2(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$a := 2\int_0^1 \psi(s) ds \quad \text{and} \quad b_\alpha := 2(\alpha\psi(0))^{1/2}.$$

In the limiting cases $\alpha = 0$ and $\alpha = \infty$ we define

$$\Phi_0(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 dx + a\mathcal{H}^{n-1}(J_u) & \text{if } u \in GSBV^2(\Omega) \cap L^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

$$\Phi_\infty(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 dx & \text{if } u \in H^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

First we prove the following theorem (see Theorem 3.1).

Theorem 1.1. *The Γ-limit of (F_k) in $L^1(\Omega)\times L^1(\Omega)$ is given by*

$$F_\alpha(u, v) := \begin{cases} \Phi_\alpha(u) & \text{if } v = 1 \ \mathcal{L}^n\text{-a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

For $\alpha = 0$ the limit functional is used to determine stationary solutions in some brittle fracture models (see [7]). For $0 < \alpha < \infty$ the limit functional is related to fracture models with a cohesive zone. It is also used in the study of plastic slips (see [4]). For $\alpha = \infty$ the limit functional corresponds to an elasticity problem without cracks.

In the case $\alpha = 0$ our result is similar to the approximation of Φ_0 obtained in [6] (see also [5]) by means of the functionals

$$G_\epsilon(u, v) = \int_\Omega (v^2 + \eta_\epsilon)|\nabla u|^2 dx + \epsilon \int_\Omega |\nabla v|^2 dx + \frac{1}{4\epsilon}\int_\Omega (v-1)^2 dx. \tag{5}$$

In our result the integral term $\epsilon \int_\Omega |\nabla v|^2 dx$ is replaced by the constraint $|\nabla v| \leq 1/\epsilon$. To our knowledge, no result has been proved for (5) in the case $0 < \alpha < \infty$. In [4] the functional Φ_α, with $0 < \alpha < \infty$, has been obtained as Γ-limit of the sequence

$$G_\epsilon(u, v) = \int_\Omega (v^2 + \eta_\epsilon)|\nabla u|^2 dx + a\epsilon \int_\Omega |\nabla v|^2 dx + \frac{a}{4\epsilon}\int_\Omega (v-1)^2 dx + b_\alpha \int_\Omega (v-1)^2 |\nabla u| dx.$$

In a future work [16] the results of the present paper will be extended to other variants of the Ambrosio-Tortorelli approximation.

Theorem 1.1 enables us to prove the following result about the convergence of minimizers of some variational problems involving the functionals F_k and F_α (see Theorem 7.1).

Theorem 1.2. *Let $q > 1$. For every k, let (u_k, v_k) be a minimizer of the functional*

$$\int_\Omega v|\nabla u|^2 dx + \frac{1}{\epsilon_k}\int_\Omega \psi(v) dx + \int_\Omega |u - g|^q dx \tag{6}$$

with the constraints (4). Then $v_k \to 1$ in $L^1(\Omega)$ and a subsequence of (u_k) converges in $L^q(\Omega)$ to a minimizer u of the following limit problem:

$$\min_{u\in SBV^2(\Omega)} \Big(\int_\Omega |\nabla u|^2 dx + a\mathcal{H}^{n-1}(J_u) + b_\alpha \int_{J_u} |[u]| d\mathcal{H}^{n-1} + \int_\Omega |u-g|^q dx \Big),$$

if $0<\alpha<\infty$; whereas in the extreme cases $\alpha = 0$ and $\alpha=\infty$ the limit problems are

$$\min_{u\in GSBV^2(\Omega)} \Big(\int_\Omega |\nabla u|^2 dx + a\mathcal{H}^{n-1}(J_u) + \int_\Omega |u-g|^q dx \Big),$$
$$\min_{u\in H^1(\Omega)} \Big(\int_\Omega |\nabla u|^2 dx + \int_\Omega |u-g|^q dx \Big),$$

respectively. Moreover for $0 \le \alpha \le \infty$ the minimum values of (6) with the constraints 4 tend to the minimum value of the limit problem.

The paper is composed of seven sections. After a brief introduction, in Section 2 we fix the notation for functions with bounded variation. In Section 3 we state the Γ-convergence result, which is proved in Sections 4, 5, and 6. In particular in Section 4 we face the one-dimensional problem in the cases $\alpha = 0$, $0<\alpha<\infty$, and $\alpha=\infty$; in Section 5 we prove the Γ-$\liminf$ inequality in the n-dimensional case by a slicing argument, whereas in Section 6 we prove the Γ-$\limsup$ inequality through the construction of a recovery sequence. Finally we deal with the convergence of minimizers in Section 7.

2. **Notation and preliminaries.** Let $n \ge 1$ be a fixed integer. The Lebesgue measure and the k-dimensional Hausdorff measure in $\mathbb{R}^n$ are denoted by $\mathcal{L}^n$ and $\mathcal{H}^k$. For the general properties of the Hausdorff measure we refer to [13] and [12].

The open ball of $\mathbb{R}^n$ with centre x and radius r is indicated by $B(x,r)$ or $B_r(x)$; if $x=0$, we write also B_r in place of $B_r(0)$. The Lebesgue measure of the unit ball of $\mathbb{R}^n$ is denoted by ω_n. Moreover let $d(x,E)$ be the Euclidean distance of the point x from the set $E\subset\mathbb{R}^n$, let $\mathrm{diam}(E)$ be the diameter of E, and let $E\triangle F$ be the symmetric difference of E and F. The symbols $\vee$ and $\wedge$ denote the maximum and the minimum operators respectively.

For the general theory of BV-functions we refer to [3], [13], [14] and [12]; here we just recall the notation and some results we use in the sequel.

Let Ω be an open subset of $\mathbb{R}^n$. For every $u \in BV(\Omega)$ the distributional gradient Du, the one-sided approximate limits u^+ and u^-, the approximate differential ∇u and the jump set J_u are defined in [3, Sections 3.1, 3.6]. Moreover the jump function $u^+ - u^-$ is denoted by $[u]$.

The jump set J_u is countably $\mathcal{H}^{n-1}$-rectifiable according to [3, Definition 2.57]. Moreover there exists a Borel function $\nu : J_u \to S^{n-1}$ such that the vector ν is normal to J_u in the sense that, if M is a C^1-manifold of dimension $n-1$, then $\nu(x)$ is normal to M for $\mathcal{H}^{n-1}$-a.e. $x\in M\cap J_u$. In particular the triplet $(u^+(x), u^-(x), \nu(x))$ is uniquely determined up to a change of sign of $\nu(x)$ and a simultaneous interchange between $u^+(x)$ and $u^-(x)$.

If $u\in BV(\Omega)$ then

$$Du = D^a u + D^j u + D^c u,$$

where $D^a u$ is absolutely continuous and $D^j u + D^c u$ is singular with respect to the Lebesgue measure; in particular $D^j u$ denotes the jump derivative of u and

$$D^j u = (u^+ - u^-)\nu \mathcal{H}^{n-1} \lfloor J_u,$$

whereas $D^c u$ is the Cantor part of the derivative of u (see [3, Section 3.9]). In particular the approximate differential ∇u coincides with the density of $D^a u$.

The spaces $SBV(\Omega)$, $GBV(\Omega)$, $GSBV(\Omega)$ are defined as in [3]. We recall that a GBV-function is weakly approximately differentiable $\mathcal{L}^n$-a.e. in Ω (see [3, Definition 4.31, Theorem 4.34]). Since an approximately differentiable function u is also weakly approximately differentiable and the approximate differential coincides with the weak approximate differential $\mathcal{L}^n$-a.e. in Ω, we also denote the weak approximate differential by ∇u.

Let $p \in \,]1, +\infty[$; let us define

$$\begin{aligned} SBV^p(\Omega) &:= \big\{u \in SBV(\Omega) : \nabla u \in L^p(\Omega, \mathbb{R}^n) \text{ and } \mathcal{H}^{n-1}(J_u) < +\infty\big\}, \\ GSBV^p(\Omega) &:= \big\{u \in GSBV(\Omega) : \nabla u \in L^p(\Omega, \mathbb{R}^n) \text{ and } \mathcal{H}^{n-1}(J_u) < +\infty\big\}. \end{aligned} \tag{7}$$

In the case $\Omega \subset \mathbb{R}$, if $u \in SBV^2(\Omega)$, then $u \in H^1(\Omega \setminus J_u)$. Conversely, if $\Omega \subset \mathbb{R}$ and there exists a finite set F such that $u \in H^1(\Omega \setminus F)$, then $u \in SBV^2(\Omega)$ and $J_u \subset F$.

Finally for the Γ-convergence theory we refer to [11].

3. **The Γ-convergence result.** Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $\epsilon_k > 0$, $\epsilon_k \to 0$. We shall study the Γ-limit in $L^1(\Omega)\times L^1(\Omega)$ of the sequence of functionals $F_k\colon L^1(\Omega)\times L^1(\Omega) \to [0, +\infty]$ defined by

$$F_k(u, v) := \begin{cases} \displaystyle\int_\Omega v\,|\nabla u|^2\,dx + \frac{1}{\epsilon_k}\int_\Omega \psi(v)dx & \text{if } (u, v) \in H^1(\Omega)\times V_k, \\ +\infty & \text{otherwise,} \end{cases} \tag{8}$$

where

$$\psi \in C^1([0, 1]) \text{ is strictly decreasing with } \psi(1) = 0, \tag{9}$$

$$V_k := \left\{v \in W^{1,\infty}(\Omega) : \eta_k \le v \le 1,\ |\nabla v| \le \frac{1}{\epsilon_k}\ \mathcal{L}^n\text{-a.e. in } \Omega\right\}, \tag{10}$$

with $\eta_k \ge 0$, $\eta_k \to 0$. We assume that for every $c \ge 0$

$$\text{the equation} \quad s^2\psi'(s) = -c \quad \text{has a finite number of solutions.} \tag{11}$$

We assume that the limit

$$\alpha := \lim_{k\to\infty} \frac{\eta_k}{\epsilon_k} \tag{12}$$

exists. For $0 < \alpha < \infty$ let $\Phi_\alpha\colon L^1(\Omega) \mapsto [0, +\infty]$ is defined by

$$\Phi_\alpha(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 dx + a\mathcal{H}^{n-1}(J_u) + b_\alpha \int_{J_u} |[u]|d\mathcal{H}^{n-1} & \text{if } u \in SBV^2(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{13}$$

where

$$a := 2\int_0^1 \psi(s)ds \quad \text{and} \quad b_\alpha := 2(\alpha\psi(0))^{\frac{1}{2}}. \tag{14}$$

In the limiting cases $\alpha = 0$ and $\alpha = \infty$ we define

$$\Phi_0(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 dx + a\mathcal{H}^{n-1}(J_u) & \text{if } u \in GSBV^2(\Omega) \cap L^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{15}$$

$$\Phi_\infty(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 dx & \text{if } u \in H^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

We are now in a position to state the main result of the section.

Theorem 3.1. *Assume (8)-(12) and assume that Ω has Lipschitz boundary. The Γ-limit of (F_k) in $L^1(\Omega)\times L^1(\Omega)$ exists and is given by*

$$F_\alpha(u,v) := \begin{cases} \Phi_\alpha(u) & \text{if } v = 1 \ \mathcal{L}^n\text{-a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases} \tag{16}$$

Theorem 3.1 is an immediate consequence of the estimates for the functionals

$$F'_\alpha := \Gamma\text{-}\liminf_{k\to\infty} F_k \qquad \text{and} \qquad F''_\alpha := \Gamma\text{-}\limsup_{k\to\infty} F_k \tag{17}$$

contained in the following results.

Theorem 3.2. *Assume (8)-(12). Let $(u,v) \in L^1(\Omega)\times L^1(\Omega)$ be such that $F'_\alpha(u,v) < +\infty$. Then $v = 1$ $\mathcal{L}^n$-a.e. in Ω, $u \in GSBV^2(\Omega) \cap L^1(\Omega)$, and*

$$\Phi_\alpha(u) \le F'_\alpha(u,1). \tag{18}$$

Theorem 3.3. *Assume (8)-(12) and assume that Ω has Lipschitz boundary. Let $u \in GSBV^2(\Omega) \cap L^1(\Omega)$. Then the following estimate holds*

$$F''_\alpha(u,1) \le \Phi_\alpha(u). \tag{19}$$

Theorem 3.3 will be proved in Sections 4 and 6. Theorem 3.2 can be obtained as a consequence of the following proposition that will be proved in Sections 4 and 5.

Proposition 1. *Assume (8)-(12). Let (u_k, v_k) be a sequence in $L^1(\Omega)\times L^1(\Omega)$ such that*

$$(u_k, v_k) \to (u,v) \text{ in } L^1(\Omega)\times L^1(\Omega), \tag{20}$$

$$(F_k(u_k, v_k)) \text{ is bounded.} \tag{21}$$

Then $u \in GSBV^2(\Omega) \cap L^1(\Omega)$, $v = 1$ $\mathcal{L}^n$-a.e. in Ω, and

$$\int_\Omega |\nabla u|^2 dx \le \liminf_{k\to\infty} \int_\Omega v_k |\nabla u_k|^2 dx, \tag{22}$$

$$a\mathcal{H}^{n-1}(J_u) \le \liminf_{k\to\infty} \frac{1}{\epsilon_k} \int_\Omega \psi(v_k) dx, \tag{23}$$

$$\Phi_\alpha(u) \le \liminf_{k\to\infty} \int_\Omega \Big[v_k |\nabla u_k|^2 + \frac{1}{\epsilon_k}\psi(v_k) \Big] dx. \tag{24}$$

Remark 1. Estimates (22) and (23) cannot be deduced from (24), so that they require a direct proof.

Let us show that Proposition 1 implies Theorem 3.2.

Proof of Theorem 3.2. If $F'_\alpha(u,v) < +\infty$, there exists a sequence (u_k, v_k) such that $(u_k, v_k) \to (u,v)$ in $L^1(\Omega)\times L^1(\Omega)$ and

$$\liminf_{k\to\infty} F_k(u_k, v_k) = F'_\alpha(u,v).$$

Passing to a subsequence, not relabeled, we can assume that $F_k(u_k, v_k) \to F'_\alpha(u,v)$, so that (21) holds. By Proposition 1 we have that $u \in GSBV^2(\Omega) \cap L^1(\Omega)$, $v = 1$ $\mathcal{L}^n$-a.e. in Ω, and (18) follows from (24). □

Remark 2. The hypothesis of Lipschitz boundary for Ω is used only to state the estimate from above in the case $n > 1$. Indeed, in the proof of that estimate we shall use a local reflection argument and the approximation Theorem 6.1 which is proved under this hypothesis.

In the case $n = 1$ the functional Φ_α used to express the Γ-limit F_α is finite only on the space $SBV^2(\Omega)$. This is stated in the following proposition which will be proved at the end of Section 4. Note that the last statements of Section 2 imply $SBV^2(\Omega) \subset L^\infty(\Omega)$ in the one-dimensional case.

Proposition 2. *Let $\Omega \subset \mathbb{R}$ be a bounded open set. Then $\Phi_0(u) < +\infty$ if and only if $u \in SBV^2(\Omega)$.*

Remark 3. Let us note that, if $n > 1$, then the inequality $F_\alpha(u, 1) < +\infty$ does not imply $u \in BV(\Omega)$ nor $u \in L^2(\Omega)$. Indeed, let Ω be a bounded open set in $\mathbb{R}^n$ and consider a sequence of pairwise disjoint balls $B_{r_i}(x_i)$, contained in Ω, with centres x_i and radii $r_i := 2^{-i}$. Moreover assume that also the balls $B_{3r_i}(x_i)$ are contained in Ω and pairwise disjoint. Let $u \in L^1(\Omega)$ be defined by

$$u(x) := \begin{cases} a_i & \text{if } x \in B_{r_i}(x_i), \\ 0 & \text{otherwise,} \end{cases} \tag{25}$$

where $a_i := 2^{(n-1)i}$. Clearly $u \in L^1(\Omega) \setminus L^2(\Omega)$. Moreover u belongs to $GSBV(\Omega)$ but does not to $BV(\Omega)$ since

$$|D^j u|(\Omega) = \sum_{i=1}^{+\infty} a_i {r_i}^{n-1} = +\infty.$$

Let $\sigma \geq 2$, $\epsilon_k := 2^{-nk}$, and $\eta_k := \epsilon_k^\sigma$; this implies $\alpha = 0$. Let us show that $F'_\alpha(u, 1) < +\infty$. To this aim let us consider $\delta_k := 2^{nk(1-\sigma)}$ and let us define u_k as a_i in $B_{r_i-\delta_k}(x_i)$, 0 out of $B_{r_i+\delta_k}(x_i)$, and with constant slope in $B_{r_i+\delta_k}(x_i) \setminus B_{r_i-\delta_k}(x_i)$, for $i \leq k$; we set $u_k := 0$ otherwise. Let v_k be defined as η_k in $B_{r_i+\delta_k}(x_i) \setminus B_{r_i-\delta_k}(x_i)$, with constant slope in $(B_{r_i+\delta_k+\epsilon_k(1-\eta_k)}(x_i) \setminus B_{r_i+\delta_k}(x_i)) \cup (B_{r_i-\delta_k}(x_i) \setminus B_{r_i-\delta_k-\epsilon_k(1-\eta_k)}(x_i))$, for $i \leq k$, and as 1 otherwise. Note that $(u_k, v_k) \in H^1(\Omega) \times V_k$ and $(u_k, v_k) \to (u, 1)$ in $L^1(\Omega) \times L^1(\Omega)$. A direct computation shows that

$$\liminf_{k\to\infty} F_k(u_k, v_k) < +\infty,$$

so that $F_\alpha(u, 1) < +\infty$.

4. Proof of the Γ-convergence result in the case n = 1.

Proof of Proposition 1. It is sufficient to prove the statement when Ω is an interval, since the left-hand sides of (22), (23) and (24) are σ-additive with respect to Ω, whereas the right-hand sides are σ-superadditive. Therefore we can assume $\Omega =]0, 1[$.

Let (u_k, v_k) be a sequence satisfying (20) and (21) with bounding constant c. Note that $\psi(v_k) \to 0$ in $L^1(\Omega)$ by (8) and (21); as $\psi(v_k) \to \psi(v)$ in $L^1(\Omega)$ we deduce $v = 1$ $\mathcal{L}^1$-a.e. on Ω.

Proof of (22). It is not restrictive to assume that the lower limit in the right-hand side of (22) is actually a limit. Let us divide the proof into two steps.

(a) Since v_k is a Lipschitz function, the set

$$B_k = \{x \in \overline{\Omega} : v_k(x) > 1/2\}$$

is relatively open in $\overline{\Omega}$. By Chebyshev inequality we get

$$\psi(1/2)\mathcal{L}^1(B_k^c) \leq \int_0^1 \psi(v_k)dx,$$

so that (8) and (21) imply

$$\mathcal{L}^1(B_k^c) \to 0. \tag{26}$$

We write

$$B_k = \bigcup_{1\le j\le N_k} I_j^k \cup \bigcup_{j>N_k} J_j^k, \tag{27}$$

where $I_1^k, \dots, I_{N_k}^k$ are the connected components of B_k such that $\mathcal{L}^1(I_j^k) \ge \epsilon_k/4$, whereas J_j^k are the connected components satisfying the opposite inequality. Let a_j^k and b_j^k be the end points of the interval I_j^k. By changing the numeration, we may assume that $0 \le a_1^k \le b_1^k < a_2^k < b_2^k < \dots < a_{N_k}^k \le b_{N_k}^k \le 1$. Moreover we set $b_0^k := 0$ and $a_{N_k+1}^k := 1$.

By definition $v_k \le 1/2$ on B_k^c; moreover $v_k \le 3/4$ on each J_j^k, since at least one end point belongs to B_k^c, the length of J_j^k is less than $\epsilon_k/4$, and $|\nabla v_k| \le 1/\epsilon_k\ \mathcal{L}^1$-a.e. in Ω by (8), (9), and (21). Then $v_k \le 3/4$ in $[b_j^k, a_{j+1}^k]$ for $j = 0, \dots, N_k$. From this estimate and from (21) it follows that

$$\sum_{j>N_k} \mathcal{L}^1(J_j^k) \le \frac{\epsilon_k c}{C_1}, \tag{28}$$

where $C_1 := \psi(3/4)$.

Let us show that (N_k) is bounded. To this aim we choose a point r_j in each interval $[b_{j-1}^k, a_j^k]$. We have $v_k \le 7/8$ in $]r_j - \frac{\epsilon_k}{8}, r_j + \frac{\epsilon_k}{8}[$, since $v_k(r_j) \le 3/4$ and $|\nabla v_k| \le 1/\epsilon_k\ \mathcal{L}^1$-a.e. in Ω. Then

$$\frac{1}{\epsilon_k}\int_{r_j-\frac{\epsilon_k}{8}}^{r_j+\frac{\epsilon_k}{8}} \psi(v_k)dx \ge C_2,$$

where $C_2 := 1/4\psi(7/8)$. We note that the intervals $]r_j - \frac{\epsilon_k}{8}, r_j + \frac{\epsilon_k}{8}[$ are pairwise disjoint, since $\mathcal{L}^1(I_j^k) \ge \epsilon_k/4$. By summing on the index j we find

$$C_2(N_k + 1) \le c.$$

This shows that (N_k) is a bounded sequence of integers. Up to subsequences, we can assume $N_k = N$ for a certain N; by compactness we can also assume the existence of the limits

$$\lim_{k\to\infty} b_j^k =: b_j \quad \text{and} \quad \lim_{k\to\infty} a_j^k =: a_j, \tag{29}$$

with $0 = b_0 \le a_1 \le b_1 \le \dots \le a_N \le b_N \le a_{N+1} = 1$. Now, by (26) and (28) we have

$$\sum_{j=0}^{N} (a_{j+1}^k - b_j^k) = \mathcal{L}^1(B_k^c) + \sum_{j>N} \mathcal{L}^1(J_j^k) \to 0; \tag{30}$$

it follows that $b_j = a_{j+1}$, for $j = 0, \dots, N$. Let $0 = x_0 < x_1 < \dots < x_m = 1$ be an increasing enumeration of the set $\{b_0, \dots, b_N\}$.

Let $\sigma > 0$ be such that $x_{i-1} + \sigma < x_i - \sigma$ for $i = 1, \dots, m$. For large values of k we have $a_j^k, b_j^k \notin [x_{i-1} + \sigma, x_i - \sigma]$. Using (30) and (29), we can deduce that for every k and every i there exists j such that

$$[x_{i-1} + \sigma, x_i - \sigma] \subset]a_j^k, b_j^k[;$$

therefore $v_k > 1/2$ in $[x_{i-1} + \sigma, x_i - \sigma]$, for $i = 1, \dots, m$. By (8) and (21) we find

$$\int_{x_{i-1}+\sigma}^{x_i-\sigma} |\nabla u_k|^2 dx \le 2c, \tag{31}$$

i.e., (∇u_k) is bounded in $L^2(x_{i-1}+\sigma, x_i-\sigma)$, for $i=1,\dots,m$.

(b) Using the Poincaré-Wirtinger Inequality, we deduce from (20) and (31) that (u_k) is bounded in $H^1(]x_{i-1}+\sigma, x_i-\sigma[)$. This ensures that $u \in H^1(x_{i-1}+\sigma, x_i-\sigma)$ and that $u_k \rightharpoonup u$ weakly in $H^1(x_{i-1}+\sigma, x_i-\sigma)$.

By the Severini-Egorov Theorem for every $\mu > 0$ there exists a measurable set $A_\mu \subset [x_{i-1}+\sigma, x_i-\sigma]$, with $\mathcal{L}^1(A_\mu) < \mu$, such that, up to a subsequence, $v_k \to 1$ uniformly in $[x_{i-1}+\sigma, x_i-\sigma] \setminus A_\mu$. Then, fixed $\delta > 0$, we have $v_k > 1-\delta$ in $[x_{i-1}+\sigma, x_i-\sigma]\setminus A_\mu$ for large k. By the weak lower semicontinuity of the L^2-norm, we have

$$(1-\delta)\int_{[x_{i-1}+\sigma,x_i-\sigma]\setminus A_\mu} |\nabla u|^2 dx \le \liminf_{k\to\infty} \int_{x_{i-1}+\sigma}^{x_i-\sigma} v_k |\nabla u_k|^2 dx.$$

We pass to the limit first as $\delta \to 0$ and then as $\mu \to 0$; adding on the index i we find

$$\sum_{i=1}^m \int_{x_{i-1}+\sigma}^{x_i-\sigma} |\nabla u|^2 dx \le \liminf_{k\to\infty} \sum_{i=1}^m \int_{x_{i-1}+\sigma}^{x_i-\sigma} v_k |\nabla u_k|^2 dx. \tag{32}$$

As $\sigma \to 0$, from (21) we obtain $u \in H^1(x_{i-1}, x_i)$ for $i=1,\dots,m$. Inequality (22) follows.

Proof of (23). If u is continuous in a certain x_i, then $u \in H^1(x_{i-1}, x_{i+1})$ and we can remove x_i from the list. Therefore it is not restrictive to assume that every x_i is a jump point for u, for $i=1,\dots,m-1$, so that $\mathcal{H}^0(J_u) = m-1$. Fix $\sigma > 0$ such that $2\sigma < x_i - x_{i-1}$ for every i and let

$$\delta_k^i = \min\{v_k(x) : x \in [x_i - \tfrac{\sigma}{2}, x_i + \tfrac{\sigma}{2}]\}.$$

Let us prove that $\delta_k^i \to 0$ as $k \to \infty$; by contradiction, we suppose that there exists a subsequence of (δ_k^i), not relabeled, and a constant $K > 0$ such that $\delta_k^i > K$ for every k, i.e., $v_k > K > 0$ in $[x_i - \frac{\sigma}{2}, x_i + \frac{\sigma}{2}]$. By repeating the argument used in steps (a) and (b) we find that $u \in H^1(x_i - \frac{\sigma}{2}, x_i + \frac{\sigma}{2})$ and this contradicts the assumption that x_i is a jump point.

Now let t_k^i be a minimum point for v_k in $[x_i - \frac{\sigma}{2}, x_i + \frac{\sigma}{2}]$. For large value of k we have $[t_k^i - \epsilon_k(1-\delta_k^i), t_k^i + \epsilon_k(1-\delta_k^i)] \subset]x_i - \sigma, x_i + \sigma[$. Since $v_k(t_k^i) = \delta_k^i$ and $|\nabla v_k| \le 1/\epsilon_k$ $\mathcal{L}^1$-a.e. in Ω, it follows that $v_k \le \frac{1}{\epsilon_k}|x - t_k^i| + \delta_k^i$. Since ψ is decreasing we deduce

$$2\int_{\delta_k^i}^1 \psi(s)ds = \frac{1}{\epsilon_k}\int_{t_k^i-\epsilon_k(1-\delta_k^i)}^{t_k^i+\epsilon_k(1-\delta_k^i)} \psi\Big(\frac{|x-t_k^i|}{\epsilon_k} + \delta_k^i\Big)dx \le \frac{1}{\epsilon_k}\int_{x_i-\sigma}^{x_i+\sigma} \psi(v_k)dx;$$

adding with respect to i and passing to the lower limit we obtain (23).

Proof of (24). In the case $\alpha = 0$ inequality (24) is obtained by adding (22) and (23).

Let $\alpha > 0$. Up to subsequences, we have $u_k \to u$ $\mathcal{L}^1$-a.e. on Ω; we write $J_u = \{x_1 \dots x_{m-1}\}$, where $0 = x_0 < x_1 < \dots < x_{m-1} < x_m = 1$, and we choose $\sigma > 0$, with $2\sigma < x_i - x_{i-1}$, such that

$$u_k(x_i - \sigma) \to u(x_i - \sigma) \quad \text{and} \quad u_k(x_{i-1}+\sigma) \to u(x_{i-1}+\sigma) \quad \text{for } i = 1,\dots,m. \tag{33}$$

We want to estimate from below the integrals

$$I_k^i := \int_{x_i-\sigma}^{x_i+\sigma} v_k(\nabla u_k)^2 dx + \frac{1}{\epsilon_k}\int_{x_i-\sigma}^{x_i+\sigma} \psi(v_k)dx. \tag{34}$$

To this aim fix $1 \le i \le m-1$ and for k large we define

$$W_k := \{w \in H^1(x_i-\sigma, x_i+\sigma),\ w(x_i-\sigma) = u_k(x_i-\sigma),\ w(x_i+\sigma) = u_k(x_i+\sigma)\},$$

$$Z_k := \{z \in W^{1,\infty}(x_i-\sigma, x_i+\sigma),\ \eta_k \le z \le 1,\ |\nabla z| \le 1/\epsilon_k\ \mathcal{L}^1\text{-a.e. on }]x_i-\sigma, x_i+\sigma[\},$$

$$H_k(w,z) := \int_{x_i-\sigma}^{x_i+\sigma} z|\nabla w|^2 dx + \frac{1}{\epsilon_k}\int_{x_i-\sigma}^{x_i+\sigma} \psi(z)dx, \quad \text{for } (w,z) \in W_k \times Z_k,$$

$$h_k(z) := \min_{w \in W_k} H_k(w,z).$$

By elementary computation we find that this minimum is achieved and that

$$h_k(z) = \frac{{\beta_k}^2}{\displaystyle\int_{x_i-\sigma}^{x_i+\sigma} \frac{1}{z} dx} + \frac{1}{\epsilon_k}\int_{x_i-\sigma}^{x_i+\sigma} \psi(z)dx, \tag{35}$$

where

$$\beta_k := |u_k(x_i+\sigma) - u_k(x_i-\sigma)|. \tag{36}$$

Let z_k be a minimum point for h_k in Z_k. It follows from the definition of h_k and from (34) that

$$h_k(z_k) \le I_k^i. \tag{37}$$

We note that h_k is invariant with respect to symmetric rearrangements of z (see [2], [9], [13], [15], [17]), therefore we can assume that z_k is symmetric with respect to x_i and nondecreasing on $[x_i, x_i+\sigma[$. Now we want to prove that z_k is piecewise affine.

First of all, by monotonicity and continuity, the sets

$$A_k := \{z_k = \eta_k\} \cap [x_i, x_i+\sigma[\quad \text{and} \quad B_k := \{z_k = 1\} \cap [x_i, x_i+\sigma[$$

are closed intervals of $[x_i, x_i+\sigma[$. Let us define

$$C_k := \{\eta_k < z_k < 1,\ |\nabla z_k| < 1/\epsilon_k\} \cap [x_i, x_i+\sigma[,$$
$$U_{j,k} := \{\eta_k + \tfrac{1}{j} < z_k < 1 - \tfrac{1}{j}\} \cap [x_i, x_i+\sigma[, \quad E_{j,k} = \{|\nabla z_k| < \tfrac{1}{\epsilon_k} - \tfrac{1}{j}\} \cap U_{j,k},$$

so that C_k is the union of the sets $E_{j,k}$ for $j \in \mathbb{N}$. For every j, $U_{j,k}$ is open in $[x_i, x_i+\sigma[$ and $E_{j,k}$ is measurable. Suppose $\mathcal{L}^1(C_k) > 0$ and fix j such that $\mathcal{L}^1(E_{j,k}) > 0$; let φ be a Lipschitz function such that

$$\{\varphi \ne 0\} \subset U_{j,k} \quad \text{and} \quad |\nabla\varphi| \le 1_{E_{j,k}}\ \mathcal{L}^1\text{-a.e. on } \mathbb{R}; \tag{38}$$

then $z_k + t\varphi \in Z_k$ for t small enough. So 0 is a a minimizer for the function $t \mapsto h_k(z_k + t\varphi)$ and, imposing that 0 is a critical point, we find

$$\int_{U_{j,k}} \left[\frac{\lambda_k}{z_k^2} + \frac{\psi'(z_k)}{\epsilon_k}\right]\varphi dx = 0, \tag{39}$$

where $\lambda_k := \beta_k^2\big(2\int_{x_i}^{x_i+\sigma} \frac{1}{z_k} dx\big)^{-2}$.

Let us prove that

$$\frac{\lambda_k}{z_k^2} + \frac{\psi'(z_k)}{\epsilon_k} = 0 \quad \mathcal{L}^1\text{-a.e. on } E_{j,k}, \tag{40}$$

arguing by contradiction. Let

$$E_{j,k}^+ := E_{j,k} \cap \left\{\frac{\lambda_k}{z_k^2} + \frac{\psi'(z_k)}{\epsilon_k} > 0\right\}$$

and suppose $\mathcal{L}^1(E^+_{j,k}) > 0$. By the continuity of z_k and ψ' and by the Lebesgue Differentiation Theorem there exist $x_0 \in E^+_{j,k}$ and $\delta > 0$ such that

$$[x_0-\delta, x_0+\delta] \subset U_{j,k} \cap \left\{ \frac{\lambda_k}{z_k^2} + \frac{\psi'(z_k)}{\epsilon_k} > 0 \right\} \quad \text{and} \quad \mathcal{L}^1\big(E_{j,k} \cap [x_0-\delta, x_0+\delta]\big) > 0.$$

Now let y be such that

$$\mathcal{L}^1\big(E_{j,k} \cap [x_0-\delta, y]\big) = \mathcal{L}^1\big(E_{j,k} \cap [y, x_0+\delta]\big),$$

and let

$$\theta(x) := \mathcal{L}^1\big(E_{j,k} \cap [x_0-\delta, y] \cap [x_0-\delta, x]\big) - \mathcal{L}^1\big(E_{j,k} \cap [x_0-\delta, x] \cap [y, x_0+\delta]\big),$$

for $x \in [x_i, x_i+\sigma[$. In particular θ is a Lipschitz function satisfying (38), so that (39) implies

$$\int_{x_0-\delta}^{x_0+\delta} \left[\frac{\lambda_k}{z_k^2} + \frac{\psi'(z_k)}{\epsilon_k} \right] \theta dx = 0; \tag{41}$$

since $\theta \geq 0$, $\theta(y) > 0$, and $\frac{\lambda_k}{z_k^2} + \frac{\psi'(z_k)}{\epsilon_k} > 0$ in $[x_0-\delta, x_0+\delta]$ the integral in (41) is positive and we get a contradiction. This concludes the proof of (40).

From (40) it follows that z_k maps C_k into the set of solutions of the equation $s^2\psi'(s) = -\lambda_k\epsilon_k$, where $\lambda_k\epsilon_k$ is infinitesimal since (λ_k) is bounded. Then, assumption (11) implies that z_k takes only a finite number of different values on C_k and, by monotonicity and continuity, C_k is a finite union of intervals. It follows that $[x_i, x_i+\sigma[$ can be written as union of a finite number of intervals, where either z_k is constant or $\nabla z_k = 1/\epsilon_k$.

We now estimate from below $h_k(z_k)$. In order to simplify the computation, we suppose that z_k assumes a unique value ξ_k in C_k, $\eta_k < \xi_k < 1$, so that C_k is an interval. Let $\alpha_k := \mathcal{L}^1(A_k)$ and $\gamma_k := \mathcal{L}^1(C_k)$; since $\nabla z_k = 1/\epsilon_k$ in $[x_i, x_i+\sigma[\setminus(A_k \cup B_k \cup C_k)$, the measure of $[x_i, x_i+\sigma[\setminus(A_k \cup B_k \cup C_k)$ is $-\epsilon_k\eta_k + \epsilon_k$ so that $\mathcal{L}^1(B_k) = \sigma - \gamma_k - \alpha_k + \epsilon_k\eta_k - \epsilon_k$.

By (35) we get

$$\begin{aligned} h_k(z_k) &= \frac{{\beta_k}^2}{2\alpha_k\frac{1-\eta_k}{\eta_k} + 2\gamma_k\frac{1-\xi_k}{\xi_k} + \zeta_k} + 2\alpha_k\frac{\psi(\eta_k)}{\epsilon_k} + 2\gamma_k\frac{\psi(\xi_k)}{\epsilon_k} + \kappa_k \\ &\geq \frac{{\beta_k}^2}{2\frac{1-\eta_k}{\eta_k}(\alpha_k+\gamma_k) + \zeta_k} + 2(\alpha_k+\gamma_k)\frac{\psi(\xi_k)}{\epsilon_k} + \kappa_k, \end{aligned}$$

where $\zeta_k = 2\sigma + 2\epsilon_k\eta_k - 2\epsilon_k - 2\epsilon_k \log\eta_k$ and $\kappa_k = 2\int_{\eta_k}^1 \psi(s)ds$.

The map

$$t \mapsto \frac{{\beta_k}^2}{t+\zeta_k} + \frac{\eta_k}{\epsilon_k}\frac{\psi(\xi_k)}{1-\eta_k}t + \kappa_k$$

can be estimated differently in the cases $\alpha = \infty$ and $0 < \alpha < \infty$.

If $\alpha = \infty$, by (21), (34), and (37) we find

$$\frac{\beta_k^2}{\zeta_k} \leq h_k(z_k) \leq I_k^i \leq c.$$

By (33), this implies, as $k \to \infty$,

$$\frac{(u(x_i+\sigma) - u(x_i-\sigma))^2}{2\sigma} \leq c.$$

As $\sigma \to 0$, we obtain $|[u(x_i)]| = 0$; this contradicts our assumption that x_i is a jump point and proves that $\mathcal{H}^0(J_u) = 0$, so that $u \in H^1(\Omega)$ and (24) follows from (22).

If $0 < \alpha < \infty$ we have

$$2\beta_k \Big(\frac{\psi(\xi_k)}{1-\eta_k} \frac{\eta_k}{\epsilon_k} \Big)^{\frac{1}{2}} - \frac{\psi(\xi_k)}{1-\eta_k} \frac{\eta_k}{\epsilon_k} \zeta_k + \kappa_k \le h_k(z_k) \le I_k^i,$$

then taking $k \to \infty$ and summing on the index i we get

$$\sum_{i=1}^{m-1} 2\Big[(\alpha\psi(0))^{\frac{1}{2}} |u(x_i+\sigma) - u(x_i-\sigma)| - \alpha\psi(0)\sigma + \int_0^1 \psi(s)ds \Big]$$
$$\le \sum_{i=1}^{m-1} \liminf_{k\to\infty} \int_{x_i-\sigma}^{x_i+\sigma} \Big[v_k(\nabla u_k)^2 dx + \frac{\psi(v_k)}{\epsilon_k} \Big] dx. \tag{42}$$

By adding (32) and (42), as $\sigma \to 0$, we obtain (24). □

Proof of Theorem 3.3. By Proposition 2 it is sufficient to take $u \in SBV^2(\Omega)$. We are going to construct a recovery sequence converging to $(u,1)$ in $L^1(\Omega)\times L^1(\Omega)$.

The case $\alpha = \infty$ is trivial since the right-hand side of (19) is finite if and only if $u \in H^1(\Omega)$ and in this case it is sufficient to choose the recovery sequence identically equal to $(u,1)$.

Now we suppose $\alpha < \infty$. In order to simplify the discussion we assume u has only one jump point $\overline{x}$. Let (δ_k^α) be an infinitesimal sequence and let

$$A_k := [\overline{x} - \delta_k^\alpha, \overline{x} + \delta_k^\alpha] \quad \text{and} \quad B_k := [\overline{x} - \delta_k^\alpha - \epsilon_k(1-\eta_k), \overline{x} + \delta_k^\alpha + \epsilon_k(1-\eta_k)];$$

moreover let us define v_k by η_k in A_k, by 1 out of B_k, and connecting linearly in $B_k \setminus A_k$; finally let us define u_k by u out of A_k and linking linearly in A_k.

Then $(u_k, v_k) \in H^1(\Omega)\times V_k$ and $(u_k, v_k) \to (u,1)$ in $L^1(\Omega)\times L^1(\Omega)$. We have

$$\lim_k \int_{\Omega\setminus A_k} \Big(v_k|\nabla u_k|^2 + \frac{1}{\epsilon_k}\psi(v_k) \Big) dx = \int_\Omega |\nabla u|^2 dx + 2\int_0^1 \psi(s)ds,$$

$$\int_{A_k} \Big(v_k|\nabla u_k|^2 + \frac{1}{\epsilon_k}\psi(v_k) \Big) dx = \frac{\eta_k}{2\delta_k^\alpha}(u(\overline{x}+\delta_k^\alpha) - u(\overline{x}-\delta_k^\alpha))^2 + 2\psi(\eta_k)\frac{\delta_k^\alpha}{\epsilon_k}. \tag{43}$$

If $\alpha = 0$ we take δ_k^0 such that $\eta_k/\delta_k^0 \to 0$ and $\delta_k^0/\epsilon_k \to 0$; by this choice the integral in (43) converges to 0. Whereas if $0 < \alpha < \infty$ we define $\delta_k^\alpha := \frac{1}{2}(\frac{\alpha}{\psi(0)})^{\frac{1}{2}}|[u(\overline{x})]|\epsilon_k$ and the integral in (43) tends to $b_\alpha|[u(\overline{x})]|$. □

We conclude this section by proving Proposition 2 on the effective domain of the Γ-limit in dimension one.

Proof of Proposition 2. Let $u \in GSBV^2(\Omega) \cap L^1(\Omega)$. Since J_u is a finite set it is sufficient to prove that $u \in H^1(\Omega\setminus J_u)$. Let $M > 0$ and let u^M be the truncated function $u^M := (-M \vee u) \wedge M$. From $u^M \in GSBV^2(\Omega) \cap L^\infty(\Omega)$ we deduce $u^M \in SBV^2(\Omega)$. This fact implies $u^M \in H^1(\Omega\setminus J_u)$ since $J_{u^M} \subset J_u$. The sequence (u^M) is bounded in $H^1(\Omega\setminus J_u)$ and $u^M \to u$ in $L^1(\Omega)$, therefore we conclude $u \in H^1(\Omega\setminus J_u)$. □

5. **Proof of the estimate from below in the case n > 1.** In this section we use the slicing argument to prove the estimate from below (18) when $n > 1$. For every $\xi \in S^{n-1}$ we define

$$\Pi^\xi := \{y \in \mathbb{R}^n : y \cdot \xi = 0\} \quad \text{and} \quad \Omega^\xi_y := \{t \in \mathbb{R} : y + t\xi \in \Omega\},$$

for every $y \in \Pi^\xi$. For every $u : \Omega \mapsto \mathbb{R}$ we define $u^\xi_y : \Omega^\xi_y \mapsto \mathbb{R}$ by

$$u^\xi_y(t) := u(y + t\xi).$$

For the main properties of slicing we refer to [3, Section 3.11]. We collect here only few properties used in this section.

If (u_k) is a sequence in $L^1(\Omega)$ such that $u_k \to u$ in $L^1(\Omega)$ then for every $\xi \in S^{n-1}$ there exists a subsequence (u_{k_j}) such that, for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$,

$$(u_{k_j})^\xi_y \to u^\xi_y \text{ in } L^1(\Omega^\xi_y).$$

If $u \in BV(\Omega)$, then for every $\xi \in S^{n-1}$ the following properties hold:

$$\int_{J_u} |\nu_u \cdot \xi| d\mathcal{H}^{n-1}(y) = \int_{\Pi^\xi} \mathcal{H}^0((J_u)^\xi_y) d\mathcal{H}^{n-1}(y), \tag{44}$$

$$\int_{J_u} |\nu_u \cdot \xi||[u]| d\mathcal{H}^{n-1}(y) = \int_{\Pi^\xi} \left[\int_{(J_u)^\xi_y} |[u]^\xi_y| d\mathcal{H}^0(t) \right] d\mathcal{H}^{n-1}(y), \tag{45}$$

for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have $|\nabla(u^\xi_y)| = |(\nabla u)^\xi_y \cdot \xi| \leq |(\nabla u)^\xi_y|$ $\mathcal{L}^1$-a.e. on Ω^ξ_y. (46)

Moreover for every $\xi \in S^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have

$$(J_u)^\xi_y = J_{u^\xi_y} \quad \text{and} \quad |[u]^\xi_y| = |[u^\xi_y]| \quad \text{on } \Omega^\xi_y. \tag{47}$$

We also make use of the fine properties of GBV-functions collected in [3, Theorem 4.34].

As we have already seen in Section 4, in order to obtain the Γ-lim inf inequality it is sufficient to prove Proposition 1.

Proof of Proposition 1. Let (u_k, v_k) be a sequence satisfying (20) and (21) with bounding constant c; as in the one-dimensional case we can deduce that $v = 1$ $\mathcal{L}^n$-a.e. in Ω. In the first part of the proof we assume that (u_k) is bounded in $L^\infty(\Omega)$ and we want to prove $u \in SBV^2(\Omega)$ in this case.

Proof of (22) in the bounded case. Given $\xi \in S^{n-1}$, we extract a subsequence (u_r, v_r) of (u_k, v_k) such that

$$((u_r)^\xi_y, (v_r)^\xi_y) \to (u^\xi_y, 1) \text{ in } L^1(\Omega^\xi_y) \times L^1(\Omega^\xi_y) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi \tag{48}$$

and

$$\lim_{r\to\infty} \int_\Omega v_r |\nabla u_r \cdot \xi|^2 dx = \liminf_{k\to\infty} \int_\Omega v_k |\nabla u_k \cdot \xi|^2 dx. \tag{49}$$

Let $0 < \kappa < 1$; by the Fubini Theorem and (46) we can write

$$\int_{\Pi^\xi} \left[\int_{\Omega^\xi_y} \left((v_r)^\xi_y \left|\nabla((u_r)^\xi_y)\right|^2 + \frac{\kappa}{\epsilon_r} \psi(v_r)^\xi_y \right) dt \right] d\mathcal{H}^{n-1}(y)$$
$$\leq \int_\Omega \left(v_r |\nabla u_r|^2 + \frac{\kappa}{\epsilon_r} \psi(v_r) \right) dx \leq c,$$

where the last inequality follows from (21). From the Fatou Lemma it follows that

$$\int_{\Pi^\xi} \liminf_{r\to\infty} \left[\int_{\Omega^\xi_y} \left((v_r)^\xi_y \left|\nabla((u_r)^\xi_y)\right|^2 + \frac{\kappa}{\epsilon_r} \psi(v_r)^\xi_y \right) dt \right] d\mathcal{H}^{n-1}(y) \leq c;$$

then for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$

$$\liminf_{r\to\infty} \int_{\Omega_y^\xi} \left((v_r)_y^\xi \left|\nabla((u_r)_y^\xi)\right|^2 + \frac{\kappa}{\epsilon_r}\psi(v_r)_y^\xi \right) dt < +\infty. \tag{50}$$

Let $F_{y,r}$ be the one-dimensional functional on the set Ω_y^ξ, defined by

$$F_{y,r}(w,z) := \begin{cases} \displaystyle\int_{\Omega_y^\xi} z\left|\nabla w\right|^2 dt + \frac{1}{\epsilon_r}\int_{\Omega_y^\xi}\varphi(z)dt & \text{if } (w,z)\in H^1(\Omega_y^\xi)\times V_{y,r}, \\ +\infty & \text{otherwise,} \end{cases} \tag{51}$$

where $\varphi := \kappa\psi$ and

$$V_{y,r} := \left\{ z\in W^{1,\infty}(\Omega_y^\xi) : \eta_r \le z \le 1, |\nabla z| \le \frac{1}{\epsilon_r}\ \mathcal{L}^1\text{-a.e. in } \Omega_y^\xi \right\}. \tag{52}$$

The corresponding Γ-lim inf will be denoted by $F'_{y,\alpha}$.

For $0<\alpha<\infty$ let $\Phi_{y,\alpha}\colon L^1(\Omega_y^\xi)\mapsto[0,+\infty]$ be defined by

$$\Phi_{y,\alpha}(w) := \begin{cases} \displaystyle\int_{\Omega_y^\xi\setminus J_w} |\nabla w|^2 dx + a\mathcal{H}^0(J_w) + \beta_\alpha \int_{J_w} |[w]| d\mathcal{H}^0 & \text{if } w\in SBV^2(\Omega_y^\xi) \\ +\infty & \text{otherwise,} \end{cases}$$

where a is as in (14) and $\beta_\alpha := 2(\alpha\varphi(0))^{1/2}$. In the limiting cases $\alpha=0$ and $\alpha=\infty$ we define

$$\Phi_{y,0}(w) := \begin{cases} \displaystyle\int_{\Omega_y^\xi\setminus J_w} |\nabla w|^2 dx + a\mathcal{H}^0(J_w) & \text{if } w\in SBV^2(\Omega_y^\xi)\cap L^1(\Omega_y^\xi) \\ +\infty & \text{otherwise,} \end{cases}$$

$$\Phi_{y,\infty}(w) := \begin{cases} \displaystyle\int_{\Omega_y^\xi\setminus J_w} |\nabla w|^2 dx & \text{if } w\in H^1(\Omega_y^\xi) \\ +\infty & \text{otherwise.} \end{cases}$$

Thanks to (46) we have $|\nabla((v_r)_y^\xi)| \le 1/\epsilon_r\ \mathcal{L}^1$-a.e. in Ω_y^ξ and then

$$F_{y,r}((u_r)_y^\xi,(v_r)_y^\xi) = \int_{\Omega_y^\xi} (v_r)_y^\xi \left|\nabla((u_r)_y^\xi)\right|^2 dt + \frac{\kappa}{\epsilon_r}\int_{\Omega_y^\xi}\psi(v_r)_y^\xi dt.$$

By (50), for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$ we can find a subsequence (u_m,v_m) of (u_r,v_r) such that

$$\begin{aligned} &\lim_{m\to\infty}\int_{\Omega_y^\xi}\left((v_m)_y^\xi \left|\nabla((u_m)_y^\xi)\right|^2 + \frac{\kappa}{\epsilon_m}\psi(v_m)_y^\xi \right) dt \\ = \;&\liminf_{r\to\infty}\int_{\Omega_y^\xi}\left((v_r)_y^\xi \left|\nabla((u_r)_y^\xi)\right|^2 + \frac{\kappa}{\epsilon_r}\psi(v_r)_y^\xi \right) dt < +\infty, \end{aligned} \tag{53}$$

so that (48) and (53) in particular imply

$$F'_{y,\alpha}(u_y^\xi,1) \le \lim_{m\to\infty} F_{y,m}((u_m)_y^\xi,(v_m)_y^\xi) < +\infty,$$

for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$. Applying Theorem 3.2 in the case $n=1$ (and Proposition 2) we obtain that $u_y^\xi\in SBV^2(\Omega_y^\xi)$,

$$\Phi_{y,\alpha}(u_y^\xi) \le F'_{y,\alpha}(u_y^\xi,1), \tag{54}$$

and that (22) is true for $((u_m)_y^\xi,(v_m)_y^\xi)$.

Now let us prove that $u \in SBV(\Omega)$. Let $M < +\infty$ be such that $||u_m||_{L^\infty(\Omega)} \le M$ for every m. Then decomposing the derivative of u^ξ_y (see [3, Section 3.9]) we get

$$|D(u^\xi_y)|(\Omega^\xi_y) = \int_{\Omega^\xi_y \setminus J_{u^\xi_y}} |\nabla(u^\xi_y)|dt + \sum_{J_{u^\xi_y}} |[u^\xi_y]|$$
$$\le \mathcal{L}^1(\Omega^\xi_y) + \int_{\Omega^\xi_y \setminus J_{u^\xi_y}} |\nabla(u^\xi_y)|^2 dt + 2M\mathcal{H}^0(J_{u^\xi_y}) \le A[1 + F'_{y,\alpha}(u^\xi_y, 1)],$$

where in the last inequality $A := \text{diam}(\Omega) + 1 + \frac{2M}{a}$ and we have used (54). Since (u_r) does not depend on y, we can integrate on the projection $\Pi^\xi(\Omega)$ of Ω on Π^ξ and we obtain

$$\begin{aligned} &\int_{\Pi^\xi(\Omega)} |D(u^\xi_y)|(\Omega^\xi_y) d\mathcal{H}^{n-1}(y) \\ \le\ & A\,\mathcal{H}^{n-1}(\Pi^\xi(\Omega)) + A \int_{\Pi^\xi} \liminf_{r\to\infty} F_{y,r}((u_r)^\xi_y, (v_r)^\xi_y) d\mathcal{H}^{n-1}(y) \\ \le\ & A\mathcal{H}^{n-1}(\Pi^\xi(\Omega)) + Ac < +\infty. \end{aligned}$$

By taking $\xi = e_1, \dots, e_n$, the elements of the canonical basis of $\mathbb{R}^n$, we get $u \in BV(\Omega)$ by [3, Remark 3.104]; since $u^\xi_y \in SBV^2(\Omega^\xi_y)$, we obtain also $u \in SBV(\Omega)$ by [3, Theorem 3.108].

From (22) applied to $((u_m)^\xi_y, (v_m)^\xi_y)$ and from (53) it follows that

$$\int_{\Omega^\xi_y \setminus J_{u^\xi_y}} |\nabla(u^\xi_y)|^2 dt \le \liminf_{r\to\infty} \int_{\Omega^\xi_y} \left((v_r)^\xi_y \left|\nabla((u_r)^\xi_y)\right|^2 + \frac{\kappa}{\epsilon_r}(\psi(v_r)^\xi_y) \right) dt,$$

for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$. Integrating on Π^ξ and applying the Fatou Lemma we get

$$\begin{aligned} &\int_{\Pi^\xi} \left[\int_{\Omega^\xi_y \setminus J_{u^\xi_y}} |\nabla(u^\xi_y)|^2 dt \right] d\mathcal{H}^{n-1}(y) \\ \le\ & \liminf_{r\to\infty} \int_{\Pi^\xi} \left[\int_{\Omega^\xi_y} \left((v_r)^\xi_y \left|\nabla((u_r)^\xi_y)\right|^2 + \frac{\kappa}{\epsilon_r}(\psi(v_r)^\xi_y) \right) dt \right] d\mathcal{H}^{n-1}(y) \\ \le\ & \liminf_{k\to\infty} \int_{\Pi^\xi} \left[\int_{\Omega^\xi_y} (v_k)^\xi_y \left|\nabla((u_k)^\xi_y)\right|^2 dt \right] d\mathcal{H}^{n-1}(y) + \kappa\, c, \end{aligned}$$

where the last inequality follows from (21) and (49). We observe that (u_k, v_k) does not depend on κ; as $\kappa \to 0$ in the previous inequality we find

$$\int_\Omega |\nabla u \cdot \xi|^2 dx \le \liminf_{k\to\infty} \int_\Omega v_k \left|\nabla u_k \cdot \xi\right|^2 dx, \tag{55}$$

using (46) and the Fubini Theorem. By taking $\xi = e_1, \dots, e_n$ and summing the results we obtain (22).

Proof of (23) in the bounded case. Given $\xi \in S^{n-1}$, the first subsequence (u_r, v_r) of (u_k, v_k) is now chosen so that (48) holds and (49) is replaced by

$$\lim_{r\to\infty} \int_{\Pi^\xi} \left[\int_{\Omega^\xi_y} \frac{1}{\epsilon_r}(\psi(v_r)^\xi_y) dt \right] d\mathcal{H}^{n-1}(y) = \liminf_{k\to\infty} \int_{\Pi^\xi} \left[\int_{\Omega^\xi_y} \frac{1}{\epsilon_k}(\psi(v_k)^\xi_y) dt \right] d\mathcal{H}^{n-1}(y). \tag{56}$$

Let $0<\kappa<1$; by the Fubini Theorem and the Fatou Lemma we find

$$\int_{\Pi^\xi}\liminf_{r\to\infty}\Big[\int_{\Omega^\xi_y}\Big(\kappa\,(v_r)^\xi_y\left|\nabla((u_r)^\xi_y)\right|^2+\frac{1}{\epsilon_r}(\psi(v_r)^\xi_y)\Big)dt\Big]d\mathcal{H}^{n-1}(y)\le c$$

and this implies, for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$,

$$\liminf_{r\to\infty}\int_{\Omega^\xi_y}\Big(\kappa\,(v_r)^\xi_y\left|\nabla((u_r)^\xi_y)\right|^2+\frac{1}{\epsilon_r}(\psi(v_r)^\xi_y)\Big)dt<+\infty.$$

It follows that for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$ there exists a subsequence (u_m,v_m) of (u_r,v_r) such that

$$\begin{aligned}&\lim_{m\to\infty}\int_{\Omega^\xi_y}\Big(\kappa\,(v_m)^\xi_y\left|\nabla((u_m)^\xi_y)\right|^2+\frac{1}{\epsilon_m}(\psi(v_m)^\xi_y)\Big)dt\\ =\;&\liminf_{r\to\infty}\int_{\Omega^\xi_y}\Big(\kappa\,(v_r)^\xi_y\left|\nabla((u_r)^\xi_y)\right|^2+\frac{1}{\epsilon_r}(\psi(v_r)^\xi_y)\Big)dt<+\infty.\end{aligned}\tag{57}$$

Let us consider the one-dimensional functional $F_{y,r}$ defined in (51), where φ is now set equal to ψ and $V_{y,r}$ is as in (52).

By (48) and (57) the sequence $F_{y,m}((\kappa^{1/2}u_m)^\xi_y,(v_m)^\xi_y)$ is bounded, so that Theorem 3.2 in the case $n=1$ implies that inequality (23) holds for $((\kappa^{1/2}u_m)^\xi_y,(v_m)^\xi_y)$; using formula (57) we get

$$a\mathcal{H}^0(J_{u^\xi_y})\le\liminf_{r\to\infty}\int_{\Omega^\xi_y}\Big(\kappa\,(v_r)^\xi_y\left|\nabla((u_r)^\xi_y)\right|^2+\frac{1}{\epsilon_r}(\psi(v_r)^\xi_y)\Big)dt.$$

Let us observe that (u_r) does not depend on y. Then we can integrate on Π^ξ both sides of the previous inequality and apply the Fatou Lemma

$$\begin{aligned}&a\int_{\Pi^\xi}\mathcal{H}^0(J_{u^\xi_y})d\mathcal{H}^{n-1}(y)\\ \le\;&\liminf_{r\to\infty}\int_{\Pi^\xi}\Big[\int_{\Omega^\xi_y}\Big(\kappa\,(v_r)^\xi_y\left|\nabla((u_r)^\xi_y)\right|^2+\frac{1}{\epsilon_r}(\psi(v_r)^\xi_y)\Big)dt\Big]d\mathcal{H}^{n-1}(y)\\ \le\;&\liminf_{k\to\infty}\int_{\Pi^\xi}\Big[\int_{\Omega^\xi_y}\frac{1}{\epsilon_k}(\psi(v_k)^\xi_y)dt\Big]d\mathcal{H}^{n-1}(y)+\kappa\,c,\end{aligned}$$

by (21) and (56). As $\kappa\to0$, using (44) and (47) we find

$$a\int_{J_u}|\nu_u\cdot\xi|d\mathcal{H}^{n-1}\le\liminf_{k\to\infty}\frac{1}{\epsilon_k}\int_\Omega\psi(v_k)dx\le c.\tag{58}$$

Applying (58) with $\xi=e_1,\dots,e_n$ we get $\mathcal{H}^{n-1}(J_u)<+\infty$. Since we have already proved that $u\in SBV(\Omega)$, we deduce from (21) and (22) that $u\in SBV^2(\Omega)$.

In order to obtain (23) we use a particular case of the localization method developed in [8, Theorem 2.3.1]. First we note that (58) holds also for an open set $A\subset\Omega$, hence

$$a\int_{J_u\cap A}|\nu_u\cdot\xi|d\mathcal{H}^{n-1}\le\liminf_{k\to\infty}\frac{1}{\epsilon_k}\int_A\psi(v_k)dx.\tag{59}$$

Since ν_u is a Borel function with values in S^{n-1}, there exists a sequence (ω_j) of simple functions with values in S^{n-1} converging to ν_u pointwise $\mathcal{H}^{n-1}$-a.e. in J_u.

We can write $\omega_j = \xi_j^1 1_{B_j^1} + \cdots + \xi_j^{m_j} 1_{B_j^{m_j}}$, where ξ_j^i are unit vectors and $B_j^1, \dots, B_j^{m_j}$ is a Borel partition of J_u. By the Dominated Convergence Theorem we have

$$\lim_{j\to\infty} \sum_{i=1}^{m_j} \int_{B_j^i} |\nu_u \cdot \xi_j^i| d\mathcal{H}^{n-1} = \mathcal{H}^{n-1}(J_u). \tag{60}$$

For every j we can find $A_j^1, \dots, A_j^{m_j}$ a family of pairwise disjoint open subsets of Ω such that $\mathcal{H}^{n-1}((A_j^i \cap J_u) \triangle B_j^i) \leq 1/(jm_j)$. Then (60) holds with B_j^i replaced by $J_u \cap A_j^i$. Since by (59)

$$a \sum_{i=1}^{m_j} \int_{J_u \cap A_j^i} |\nu_u \cdot \xi_j^i| d\mathcal{H}^{n-1} \leq \liminf_{k\to\infty} \frac{1}{\epsilon_k} \int_\Omega \psi(v_k) dx,$$

we obtain (23) as $j \to \infty$.

Proof of (24) in the bounded case. If $\alpha = 0$ inequality (24) can be obtained by adding (22) and (23).

When $\alpha = \infty$ by Theorem 3.2 in the case $n = 1$ we get $u_y^\xi \in H^1(\Omega_y^\xi)$ for every ξ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$. Since $u \in SBV^2(\Omega)$, this implies $u \in H^1(\Omega)$ by [3, Theorem 3.108], therefore (24) follows from (22).

Let now $0 < \alpha < \infty$. Given $\xi \in S^{n-1}$, we choose a subsequence (u_r, v_r) of (u_k, v_k) such that (48) holds and

$$\begin{aligned}
&\lim_{r\to\infty} \int_{\Pi^\xi} \left[\int_{\Omega_y^\xi} \left((v_r)_y^\xi \left| \nabla (u_r)_y^\xi \right|^2 + \frac{1}{\epsilon_r} (\psi(v_r)_y^\xi) \right) dt \right] d\mathcal{H}^{n-1}(y) \\
&= \liminf_{k\to\infty} \int_{\Pi^\xi} \left[\int_{\Omega_y^\xi} \left((v_k)_y^\xi \left| \nabla (u_k)_y^\xi \right|^2 + \frac{1}{\epsilon_k} (\psi(v_k)_y^\xi) \right) dt \right] d\mathcal{H}^{n-1}(y).
\end{aligned}$$

By (46), using the Fubini Theorem and the Fatou Lemma we get

$$\int_{\Pi^\xi} \liminf_{r\to\infty} \left[\int_{\Omega_y^\xi} \left((v_r)_y^\xi \left| \nabla ((u_r)_y^\xi) \right|^2 + \frac{1}{\epsilon_r} (\psi(v_r)_y^\xi) \right) dt \right] d\mathcal{H}^{n-1}(y) \leq c$$

and then for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have

$$\liminf_{r\to\infty} \int_{\Omega_y^\xi} \left((v_r)_y^\xi \left| \nabla ((u_r)_y^\xi) \right|^2 + \frac{1}{\epsilon_r} (\psi(v_r)_y^\xi) \right) dt < +\infty.$$

Let $F_{y,r}$ be the one-dimensional functional defined in (51), where $\varphi := \psi$ and $V_{y,r}$ is as in (52). For $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we can find a subsequence (u_m, v_m) of (u_r, v_r) such that

$$\begin{aligned}
&\lim_{m\to\infty} \int_{\Omega_y^\xi} \left((v_m)_y^\xi \left| \nabla ((u_m)_y^\xi) \right|^2 + \frac{1}{\epsilon_m} (\psi(v_m)_y^\xi) \right) dt \\
&= \liminf_{r\to\infty} \int_{\Omega_y^\xi} \left({v_r}_y^\xi \left| \nabla ({u_r}_y^\xi) \right|^2 + \frac{1}{\epsilon_r} (\psi(v_r)_y^\xi) \right) dt < +\infty;
\end{aligned} \tag{61}$$

then Theorem 3.2 in the case $n = 1$ implies

$$\Phi_{y,\alpha}(u_y^\xi) \leq \liminf_{r\to\infty} F_{y,r}((u_r)_y^\xi, (v_r)_y^\xi).$$

Let us observe that (u_r) does not depend on y; integrating on Π^ξ both sides of the previous inequality and applying the Fatou Lemma we get

$$\int_{\Pi^\xi} \Phi_{y,\alpha}(u^\xi_y) d\mathcal{H}^{n-1}(y)$$
$$\leq \liminf_{k\to\infty} \int_{\Pi^\xi} \Big[\int_{\Omega^\xi_y} \Big((v_k)^\xi_y \left|\nabla((u_k)^\xi_y)\right|^2 + \frac{1}{\epsilon_k}(\psi(v_k)^\xi_y) \Big) dt \Big] d\mathcal{H}^{n-1}(y). \tag{62}$$

We now apply the localization method to the measure $\mu = \mathcal{L}^n\lfloor\Omega + \mathcal{H}^{n-1}\lfloor J_u$ instead of $\mathcal{H}^{n-1}\lfloor J_u$. Since (62) holds with an open set $A \subset \Omega$ in place of Ω, by (44)-(47) and by the Fubini Theorem we get

$$\int_A \Big[|\nabla u\cdot\xi|^2 1_{\Omega\setminus J_u} + |\nu_u\cdot\xi|(a + b_\alpha|[u]|)1_{J_u} \Big] d\mu$$
$$\leq \liminf_{k\to\infty} \int_A \Big[v_k|\nabla u_k\cdot\xi|^2 + \frac{1}{\epsilon_k}\psi(v_k) \Big] dx. \tag{63}$$

Let us define $\omega := \nu_u$ on J_u, $\omega := \nabla u/|\nabla u|$ on $\{\nabla u \neq 0\}\setminus J_u$, and $\omega := e_1$ elsewhere. Since ω is a μ-measurable function with values in S^{n-1}, there exists a sequence (ω_j) of simple functions with values in S^{n-1}, converging to ω μ-a.e. in Ω. We can write $\omega_j = \xi^1_j 1_{B^1_j} + \cdots + \xi^{m_j}_j 1_{B^{m_j}_j}$, where ξ^i_j are unit vectors and $B^1_j, \ldots, B^{m_j}_j$ is a Borel partition of Ω. By the Dominated Convergence Theorem we have

$$\lim_{j\to\infty} \sum_{i=1}^{m_j} \int_{B^i_j} \Big[|\nabla u\cdot\xi^i_j|^2 1_{\Omega\setminus J_u} + |\nu_u\cdot\xi^i_j|(a + b_\alpha|[u]|)1_{J_u} \Big] d\mu = \Phi_\alpha(u). \tag{64}$$

For every j we can find a family $A^1_j, \ldots, A^{m_j}_j$ of pairwise disjoint open subsets of Ω such that $\mu(A^i_j \triangle B^i_j) \leq 1/(jm_j)$. Then (64) holds with B^i_j replaced by A^i_j. By (63) we find

$$\sum_{i=1}^{m_j} \int_{A^i_j} \Big[|\nabla u\cdot\xi^i_j|^2 1_{\Omega\setminus J_u} + |\nu_u\cdot\xi^i_j|(a + b_\alpha|[u]|)1_{J_u} \Big] d\mu$$
$$\leq \liminf_{k\to\infty} \int_\Omega \Big[v_k|\nabla u_k|^2 + \frac{1}{\epsilon_k}\psi(v_k) \Big] dx$$

and we obtain (24) as $j \to \infty$.

The general case. We now remove the assumption that (u_k) is bounded in $L^\infty(\Omega)$. Let us fix $M > 0$ and let us consider the sequence of truncated functions $u^M_k = (-M \vee u) \wedge M$. We have that $u^M_k \to u^M$ in $L^1(\Omega)$, $v_k \to 1$ in $L^1(\Omega)$, and by (21)

$$F_k(u^M_k, v_k) \leq F_k(u_k, v_k) \leq c.$$

From the proof in the bounded case it follows that $u^M \in SBV^2(\Omega)$ and that

$$\int_\Omega |\nabla u^M|^2 dx \leq \liminf_{k\to\infty} \int_\Omega v_k|\nabla u_k|^2 dx, \tag{65}$$

$$a\mathcal{H}^{n-1}(J_{u^M}) \leq \liminf_{k\to\infty} \frac{1}{\epsilon_k} \int_\Omega \psi(v_k) dx. \tag{66}$$

This implies $u \in GSBV(\Omega)$. As $|\nabla u^M| = |\nabla u| 1_{\{|u|\leq M\}}$ by Theorem [3, Theorem 4.34], using the Monotone Convergence Theorem we obtain

$$\int_\Omega |\nabla u|^2 dx = \lim_{M\to\infty} \int_\Omega |\nabla u^M|^2 dx,$$

which together with (65) proves (22). Moreover, taking $M \to \infty$ in (66) we find (23). Therefore $u \in GSBV^2(\Omega)$.

Let us prove now (24). When $\alpha = 0$, this inequality can be obtained by adding (22) and (23).

Assume $\alpha = \infty$. In this case we prove that $u \in H^1(\Omega)$. The proof in the bounded case, applied to (u_k^M, v_k), gives $u^M \in H^1(\Omega)$ and by (65) we have that the sequence (∇u^M) is bounded in $L^2(\Omega, \mathbb{R}^n)$; using the Poincaré-Wirtinger Inequality and the fact that $u^M \to u$ in $L^1(\Omega)$ we obtain $u \in H^1(\Omega)$. Inequality (24) follows from (22).

Let $0 < \alpha < \infty$. The proof in the bounded case, applied to (u_k^M, v_k), gives

$$\Phi_\alpha(u^M) \leq \liminf_{k\to\infty} \int_\Omega \Big[v_k |\nabla u_k|^2 + \frac{1}{\epsilon_k}\psi(v_k) \Big] dx \leq c. \tag{67}$$

Since $u^M \in SBV^2(\Omega)$, inequality (67) gives $|Du^M|(\Omega) \leq \mathcal{L}^n(\Omega) + c\max(1, 1/b_\alpha)$ for every $M > 0$. From $u^M \to u$ in $L^1(\Omega)$, we conclude that $u \in BV(\Omega)$ and $u^M \rightharpoonup u$ weakly* in $BV(\Omega)$. Using the Closure Theorem for SBV [3, Theorem 4.7], we deduce from (67) that $u \in SBV^2(\Omega)$. Estimate (67), as $M \to \infty$, leads to (24). □

6. **Proof of the estimate from above in the case n > 1.** Now our purpose is to prove the Γ-lim sup inequality. In order to work with more regular functions and jump sets, we first introduce an approximation result. The following theorem is a small modification of a theorem due to Cortesani and Toader (see [10, Theorem 3.1]).

Theorem 6.1. *Let $Q \subset \mathbb{R}^n$ be an open cube, let $1 < p \leq 2$, and let $u \in SBV^p(Q) \cap L^\infty(Q)$. Then for every $\epsilon > 0$ there exist a function $v \in SBV^p(Q)$ and a set $S = \cup_{i=1}^m S_i$, with S_i closed and pairwise disjoint $(n-1)$-simplexes contained in Q, such that*

(a) $\mathcal{H}^{n-1}(S \setminus J_v) = 0$;
(b) $v \in W^{k,\infty}(Q \setminus S)$ *for every* k;
(c) $||v - u||_{L^1(Q)} < \epsilon$;
(d) $||\nabla v - \nabla u||_{L^p(Q,\mathbb{R}^n)} < \epsilon$;
(e) $\mathcal{H}^{n-1}(J_v) < \mathcal{H}^{n-1}(J_u) + \epsilon$;
(f) $\displaystyle\int_{J_v} |[v]| d\mathcal{H}^{n-1} < \int_{J_u} |[u]| d\mathcal{H}^{n-1} + \epsilon.$

Proof. Using [10, Theorem 3.1] and [10, Remark 3.5] we can find a function $w \in SBV^p(Q)$ and a set $T = \cup_{i=1}^m T_i$, not necessarily contained in Q, with T_i closed and pairwise disjoint $(n-1)$-simplexes, such that conditions (a)-(f) hold for w in place of v and $T \cap Q$ in place of S. Since $T \cap Q$ is a polyhedron, we can adapt the arguments in [10, Remark 3.5] to obtain a function v and a set $S \subset Q$ satisfying conditions (a)-(f). □

Now we can prove the estimate from above of the Γ-limit.

Proof of Theorem 3.3. Let $u \in GSBV^2(\Omega) \cap L^1(\Omega)$. We have to construct a recovery sequence (u_k, v_k) converging to $(u, 1)$ in $L^1(\Omega) \times L^1(\Omega)$.

The case $\alpha = \infty$ is trivial since the right-hand side in (19) is finite if and only if $u \in H^1(\Omega)$, and in this case it is sufficient to define $(u_k, v_k) := (u, 1)$.

Let $\alpha < \infty$. We consider first the case $u \in L^\infty(\Omega)$, so that u belongs to $SBV^2(\Omega) \cap L^\infty(\Omega)$. It is enough to prove (19) for a cube Q and for a function u satisfying properties (a) and (b) of Theorem 6.1. Indeed, if Ω is an arbitrary bounded open set Ω with Lipschitz boundary and $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$, then a local reflection argument provides an extension of u to a function $\tilde{u} \in SBV^2(Q) \cap L^\infty(Q)$ such that $\mathcal{H}^{n-1}(J_{\tilde{u}} \cap \partial\Omega) = 0$. Through this paragraph we shall write explicitly the domain of the integrals in the functionals (8), (13), (15), (16), and (17). By Theorem 6.1 for every k we can find a function $w_k \in SBV^2(Q)$ satisfying properties (a)-(f). Assuming that (19) holds for w_k, we have $F''_{\alpha,Q}(w_k, 1) \le \Phi_{\alpha,Q}(w_k)$. Then by the lower semicontinuity of $F''_{\alpha,Q}$ we obtain

$$\begin{aligned} F''_{\alpha,Q}(\tilde{u}, 1) &\le \limsup_{k\to\infty} \Phi_{\alpha,Q}(w_k) \\ &\le \limsup_{k\to\infty} \Big[\Phi_{\alpha,Q}(\tilde{u}) + \frac{1}{k^2} + \frac{2}{k}||\nabla \tilde{u}||_{L^2(Q,\mathbb{R}^n)} + \frac{a+b_\alpha}{k}\Big] \\ &= \Phi_{\alpha,Q}(\tilde{u}). \end{aligned} \tag{68}$$

Let us check that this implies $F''_{\alpha,\Omega}(u, 1) \le \Phi_{\alpha,\Omega}(u)$. By Theorem 3.2 and inequality (68) we have

$$\begin{gathered} F_{\alpha,Q}(\tilde{u}, 1) = \Phi_{\alpha,\Omega}(u) + \Phi_{\alpha,Q\setminus\overline{\Omega}}(\tilde{u}), \\ \Phi_{\alpha,\Omega}(u) \le F'_{\alpha,\Omega}(u, 1), \quad \Phi_{\alpha,Q\setminus\overline{\Omega}}(\tilde{u}) \le F'_{\alpha,Q\setminus\overline{\Omega}}(\tilde{u}), \end{gathered} \tag{69}$$

so that

$$F_{\alpha,Q}(\tilde{u}, 1) \le F'_{\alpha,\Omega}(u, 1) + F'_{\alpha,Q\setminus\overline{\Omega}}(\tilde{u}). \tag{70}$$

Moreover [11, Proposition 6.17] implies

$$F''_{\alpha,\Omega}(u, 1) + F'_{\alpha,Q\setminus\overline{\Omega}}(\tilde{u}) \le F_{\alpha,Q}(\tilde{u}, 1);$$

this estimate together with (69) and (70) gives $F''_{\alpha,\Omega}(u, 1) = \Phi_{\alpha,\Omega}(u)$.

Therefore, in the rest of the proof we assume that $\Omega = Q$, $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$, and that properties (a) and (b) of Theorem 6.1 hold for u. Finally, in order to simplify the computation, we suppose that S is a unique $(n-1)$-simplex and that $S \subset \{x_n = 0\}$. We write a point $x \in \mathbb{R}^n$ as $x = (\overline{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and we orient J_u so that $\nu_u = (0, 1)$. Let

$$\Omega^\pm := \big\{x \in \Omega : \pm x_n > 0\big\}$$

and let L be the maximum between the Lipschitz constants of u in Ω^+ and Ω^-.

If $0 < \alpha < \infty$ we define $\delta_k^\alpha(\overline{x}) := \frac{1}{2}\epsilon_k(\frac{\alpha}{\psi(0)})^{1/2}|[u(\overline{x}, 0)]|$ for $x = (\overline{x}, x_n) \in \Omega$; whereas for $\alpha = 0$ we define δ_k^0 as any sequence of constant functions such that $\eta_k/\delta_k^0 \to 0$ and $\delta_k^0/\epsilon_k \to 0$. Note that δ_k^α is a Lipschitz function since u^+ and u^- are; moreover in the case $0 < \alpha < \infty$, $\delta_k^\alpha(\overline{x}) = 0$ for $(\overline{x}, 0) \in \partial S$, where ∂S is the boundary of S in the relative topology of $\mathbb{R}^{n-1} \times \{0\}$.

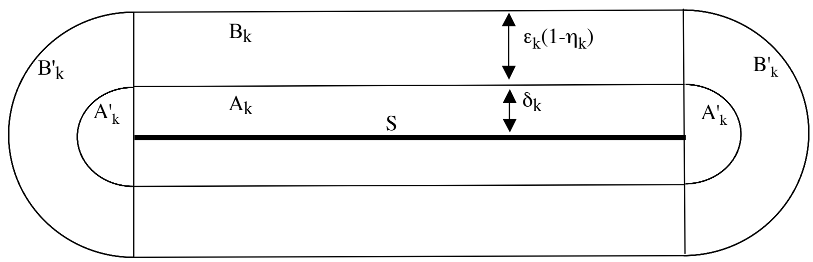

FIGURE 1. The geometry for $\alpha = 0$.

We define

$$A_k := \left\{ x \in \mathbb{R}^n : (\overline{x}, 0) \in S,\ |x_n| < \delta_k^\alpha(\overline{x}) \right\},$$

$$B_k := \left\{ x \in \mathbb{R}^n : (\overline{x}, 0) \in S,\ \delta_k^\alpha(\overline{x}) \leq |x_n| \leq \delta_k^\alpha(\overline{x}) + \frac{\epsilon_k(1-\eta_k)}{c_{k,\alpha}} \right\},$$

$$A'_k := \left\{ x \in \mathbb{R}^n : (\overline{x}, 0) \notin S,\ d(x, \partial S) < \delta_k^\alpha(\overline{x}) \right\},$$

$$B'_k := \left\{ x \in \mathbb{R}^n : (\overline{x}, 0) \notin S,\ \delta_k^\alpha(\overline{x}) \leq d(x, \partial S) \leq \delta_k^\alpha(\overline{x}) + \frac{\epsilon_k(1-\eta_k)}{c_{k,\alpha}} \right\},$$

where $d(x, S)$ is the distance from the point x to the set S and $c_{k,\alpha} := 1$ for $\alpha = 0$, whereas $c_{k,\alpha} := 1 - \epsilon_k(\frac{\alpha}{\psi(0)})^{1/2}L$ for $0 < \alpha < \infty$ (see Figures 1 and 2). For k large we have that the closure of $A_k \cup B_k \cup A'_k \cup B'_k$ is contained in Ω.

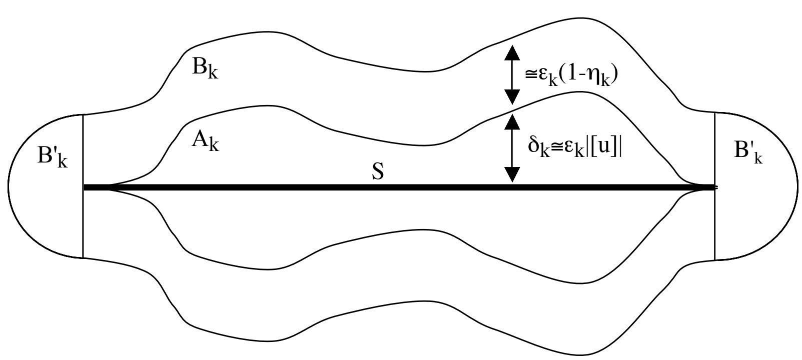

FIGURE 2. The geometry for $0 < \alpha < \infty$.

Let

$$u_k(\overline{x}, x_n) := \begin{cases} \dfrac{x_n + \delta_k^\alpha}{2\delta_k^\alpha}(u(\overline{x}, \delta_k^\alpha) - u(\overline{x}, -\delta_k^\alpha)) + u(\overline{x}, -\delta_k^\alpha) & \text{if } x \in A_k, \\ u(x) & \text{if } x \in \Omega \setminus (A_k \cup A'_k). \end{cases}$$

Here and henceforth δ_k^α denotes $\delta_k^\alpha(\overline{x})$. Let us verify that $u_k \in W^{1,\infty}(\Omega \setminus A'_k)$. If $x = (\overline{x}, x_n) \in A_k$, we have

$$\begin{aligned} & |D_n u_k(\overline{x}, x_n)| \\ = \; & \left| \frac{u(\overline{x}, \delta_k^\alpha) - u(\overline{x}, -\delta_k^\alpha)}{2\delta_k^\alpha} \right| \\ = \; & \left| \frac{u(\overline{x}, \delta_k^\alpha) - u^+(\overline{x}, 0)}{2\delta_k^\alpha} + \frac{u^+(\overline{x}, 0) - u^-(\overline{x}, 0)}{2\delta_k^\alpha} + \frac{u^-(\overline{x}, 0) - u(\overline{x}, -\delta_k^\alpha)}{2\delta_k^\alpha} \right| \\ \leq \; & L + \frac{|[u(\overline{x}, 0)]|}{2\delta_k^\alpha}, \end{aligned} \tag{71}$$

where the last inequality follows from the Lipschitz continuity of u on $\Omega^\pm$. Using the previous estimate we also obtain

$$\begin{aligned} & |D_j u_k(\overline{x}, x_n)| \\ \leq \; & \left| \frac{x_n}{\delta_k^\alpha} D_j \delta_k^\alpha \frac{u(\overline{x}, \delta_k^\alpha) - u(\overline{x}, -\delta_k^\alpha)}{2\delta_k^\alpha} \right| + \left| D_j u(\overline{x}, -\delta_k^\alpha) - D_n u(\overline{x}, -\delta_k^\alpha) D_j \delta_k^\alpha \right| \\ & + \left| D_j u(\overline{x}, \delta_k^\alpha) + D_n u(\overline{x}, \delta_k^\alpha) D_j \delta_k^\alpha - D_j u(\overline{x}, -\delta_k^\alpha) + D_n u(\overline{x}, -\delta_k^\alpha) D_j \delta_k^\alpha \right| \\ \leq \; & D_j \delta_k^\alpha \Big(\frac{|[u(\overline{x}, 0)]|}{2\delta_k^\alpha} + 4L \Big) + 3L, \end{aligned} \tag{72}$$

for $j = 1, \dots, n-1$ and for every $(\overline{x}, x_n) \in A_k$.

By the definition of δ_k^α and the boundedness of u, the quotient $|[u(\overline{x}, 0)]|/\delta_k^\alpha$ is bounded uniformly with respect to $\overline{x}$; since $D_j \delta_k^\alpha \leq (\frac{\alpha}{\psi(0)})^{1/2} L \epsilon_k$, we deduce from (71) and (72) that $u_k \in W^{1,\infty}(\Omega \setminus A'_k)$, so that in the case $0 < \alpha < \infty$ we obtain $u_k \in W^{1,\infty}(\Omega)$. In the case $\alpha = 0$ inequalities (71) and (72) imply that u_k is Lipschitz continuous in $\{x \in \Omega : (\overline{x}, 0) \in S\}$, with Lipschitz constant $(M/\delta_k^0) + 3nL$, where $M := ||u||_{L^\infty(\Omega)}$.

To prove that u_k is Lipschitz continuous in $\Omega \setminus A'_k$ we will show that

$$|u_k(x) - u_k(y)| \leq \Big(\frac{4M}{\delta_k^0} + 12nL \Big) (|\overline{x} - \overline{y}| + |x_n - y_n|) \quad \text{for } x, y \in \Omega \setminus A'_k. \tag{73}$$

Let $x, y \in A_k \cup B_k \cup B'_k$. It is enough to prove (73) when x_n and y_n have the same sign. Indeed, if $(\overline{x}, 0) \in S$ we can write

$$|u_k(x) - u_k(y)| \leq |u_k(\overline{x}, x_n) - u_k(\overline{x}, y_n)| + |u_k(\overline{x}, y_n) - u_k(\overline{y}, y_n)| \tag{74}$$

and the estimate for the first term in the right-hand side comes from the Lipschitz continuity of u_k in $\{x \in \Omega : (\overline{x}, 0) \in S\}$. If $(\overline{x}, 0) \notin S$ and $(\overline{y}, 0) \notin S$, then

$$|u_k(x) - u_k(y)| = |u(x) - u(y)| \leq |u(\overline{x}, x_n) - u(\overline{x}, y_n)| + |u(\overline{x}, y_n) - u(\overline{y}, y_n)|.$$

Since the segment with end points $(\overline{x}, x_n)$ and $(\overline{x}, y_n)$ is contained in $\Omega \setminus S$, the first term in the right-hand side is estimated by $L|x_n - y_n|$, whereas the second term is estimated by $L|\overline{x} - \overline{y}|$ due to the Lipschitz continuity of u in $\Omega^\pm$.

Therefore, it is enough to prove (73) when $x_n > 0$ and $y_n > 0$. If $y_n > \delta_k^0$, then we can write (74) and the right-hand side reduces to $|u_k(\overline{x}, x_n) - u_k(\overline{x}, y_n)| + |u(\overline{x}, y_n) - u(\overline{y}, y_n)|$. The second term is estimated by L as before. If $(\overline{x}, 0) \in S$ the first term is estimated using the Lipschitz continuity of u_k in $\{x \in \Omega : (\overline{x}, 0) \in S\}$. If $(\overline{x}, 0) \notin S$, the first term can be written as $|u(\overline{x}, x_n) - u(\overline{x}, y_n)|$, which is estimated by $L|x_n - y_n|$, since $x, y \in \Omega^+$.

It remains to consider the case $0 < x_n < \delta_k^0$ and $0 < y_n < \delta_k^0$. If $(\overline{x}, 0), (\overline{y}, 0) \in S$ then $x, y \in A_k$ and the estimate has already been proved. If $(\overline{x}, 0), (\overline{y}, 0) \notin S$ then $|u_k(x) - u_k(y)| = |u(x) - u(y)|$, which can be estimated by the Lipschitz continuity of u in Ω^+. Assume now $(\overline{x}, 0) \notin S$ and $(\overline{y}, 0) \in S$. Let $(\overline{z}, 0)$ be an element of ∂S in the segment of end points $\overline{x}$ and $\overline{y}$, and let $z := (\overline{z}, \delta_k^0)$. Then

$$|u_k(x)-u_k(y)| \leq |u(x)-u(z)|+|u_k(z)-u_k(y)| \leq \Big(\frac{M}{\delta_k^0}+3nL\Big)(|x-z|+|z-y|). \tag{75}$$

We have

$$\begin{aligned} |x - z| + |z - y| &\leq |\overline{x} - \overline{z}| + |x_n - \delta_k^0| + |\overline{z} - \overline{y}| + |y_n - \delta_k^0| \\ &= |\overline{x} - \overline{z}| + |\overline{z} - \overline{y}| + 2|x_n - \delta_k^0| + |x_n - y_n|; \end{aligned} \tag{76}$$

since $x \notin A'_k$ we obtain

$$(\delta_k^0)^2 \leq |(\overline{x}, x_n) - (\overline{z}, 0)|^2 \leq |\overline{x} - \overline{z}|^2 + x_n^2,$$

so that we can estimate $(\delta_k^0 - x_n)^2$ as follows

$$(\delta_k^0 - x_n)^2 \leq (\delta_k^0)^2 - x_n^2 \leq |\overline{x} - \overline{z}|^2. \tag{77}$$

Inequality (73) follows from (75), (76), (77), and from $|\overline{x} - \overline{z}| + |\overline{z} - \overline{y}| = |\overline{x} - \overline{y}|$. This concludes the proof of the Lipschitz continuity of u_k in $\Omega \setminus A'_k$. We are now in a position to apply the McShane Theorem, so that there exists a function, still denoted u_k, that extends u_k to A'_k and has the same Lipschitz constant as u_k, i.e.,

$$|u_k(x) - u_k(y)| \leq \Big(\frac{4M}{\delta_k^0} + 12nL\Big)(|\overline{x} - \overline{y}| + |x_n - y_n|) \quad \text{for } x, y \in \Omega. \tag{78}$$

Let us define

$$v_k(x) := \begin{cases} \eta_k & \text{if } x \in A_k \cup A'_k, \\ \eta_k + \dfrac{c_{k,\alpha}}{\epsilon_k}(|x_n| - \delta_k^\alpha) & \text{if } x \in B_k, \\ \eta_k + \dfrac{c_{k,\alpha}}{\epsilon_k}(d(x, \partial S) - \delta_k^\alpha) & \text{if } x \in B'_k, \\ 1 & \text{otherwise.} \end{cases}$$

Then $u_k \to u$ in $L^1(\Omega)$, $\eta_k \leq v_k \leq 1$ $\mathcal{L}^n$-a.e. in Ω and $v_k \to 1$ in $L^1(\Omega)$; moreover $v_k \in W^{1,\infty}(\Omega)$ and $|\nabla v_k| \leq 1/\epsilon_k$ $\mathcal{L}^n$-a.e. in Ω by the choice of the constant $c_{k,\alpha}$. The sequence $F_k(u_k, v_k)$ can be written now as

$$\begin{aligned} F_k(u_k, v_k) &= \int_{A_k} \eta_k |\nabla u_k|^2 dx + \int_{A'_k} \eta_k |\nabla u_k|^2 dx + \int_{\Omega \setminus (A_k \cup A'_k)} v_k |\nabla u|^2 dx \\ &\quad + \frac{1}{\epsilon_k} \int_{A_k} \psi(\eta_k) dx + \frac{1}{\epsilon_k} \int_{B_k} \psi(v_k) dx + \frac{1}{\epsilon_k} \int_{A'_k \cup B'_K} \psi(v_k) dx. \end{aligned} \tag{79}$$

Let us study each term in the previous expression. Let us start with the first one. Using (71) and (72) we obtain that there exists a constant K_1 such that

$$\begin{aligned} \int_{A_k} \eta_k |\nabla u_k|^2 dx &= \int_{A_k} \eta_k (D_n u_k)^2 dx + \sum_{j=1}^{n-1} \int_{A_k} \eta_k (D_j u_k)^2 dx \\ &\leq \int_{J_u} \eta_k \frac{(u(\overline{x}, \delta_k^\alpha) - u(\overline{x}, -\delta_k^\alpha))^2}{2\delta_k^\alpha} d\mathcal{H}^{n-1} + K_1 \eta_k; \end{aligned}$$

if $\alpha = 0$ the right-hand side of the previous inequality tends to 0, since $u \in L^\infty(\Omega)$ and $\eta_k/\delta_k^0 \to 0$; if $0 < \alpha < \infty$, by the Dominated Convergence Theorem it tends to

$$\frac{b_\alpha}{2}\int_{J_u} |[u]| d\mathcal{H}^{n-1}.$$

Let us consider the second term in (79) in the case $\alpha = 0$, when $A_k' \neq \emptyset$. By (78) we get

$$\int_{A_k'} \eta_k |\nabla u_k|^2 dx \leq K_2 \frac{\eta_k}{(\delta_k^0)^2} \mathcal{L}^n(A_k') + K_3 \eta_k,$$

where K_2 and K_3 are constants. First we note that $A_k' \subset (\partial S)_{\delta_k^0}$, where $(\partial S)_{\delta_k^0} := \{x \in \mathbb{R}^n : d(x, \partial S) < \delta_k^0\}$. From a well-known result about the Minkowski content, (see, for instance, [3, Theorem 2.106]), we can write

$$\mathcal{L}^n(A_k') \leq O((\delta_k^0)^2),$$

so that the second term in (79) tends to 0.

The third term in (79) is estimated by

$$\int_{\Omega \setminus (A_k \cup A_k')} |\nabla u|^2 v_k dx \leq \int_\Omega |\nabla u|^2 dx.$$

The fourth term in (79) is given by

$$\frac{1}{\epsilon_k}\int_{A_k} \psi(\eta_k) dx = \frac{\psi(\eta_k)}{\epsilon_k}\int_{J_u} 2\delta_k^\alpha d\mathcal{H}^{n-1}.$$

It tends to 0 if $\alpha = 0$, whereas in the case $0 < \alpha < \infty$ it tends to

$$\frac{b_\alpha}{2}\int_{J_u} |[u]| d\mathcal{H}^{n-1}.$$

As for the fifth term in (79), we get

$$\begin{aligned} \frac{1}{\epsilon_k}\int_{B_k} \psi(v_k) dx &= \frac{2}{\epsilon_k}\int_{J_u} \Big[\int_0^{\epsilon_k \frac{1-\eta_k}{c_{k,\alpha}}} \psi(\eta_k + \frac{c_{k,\alpha}}{\epsilon_k} x_n) dx_n \Big] d\mathcal{H}^{n-1} \\ &= \frac{2}{c_{k,\alpha}} \Big(\int_0^{1-\eta_k} \psi(x_n) dx_n \Big) \mathcal{H}^{n-1}(J_u) \end{aligned}$$

and this term tends to $a\mathcal{H}^{n-1}(J_u)$.

Finally, the last term in (79) can be estimated by

$$\frac{1}{\epsilon_k}\int_{A_k' \cup B_k'} \psi(v_k) dx \leq \frac{1}{\epsilon_k} \psi(\eta_k) \mathcal{L}^n(A_k' \cup B_k');$$

arguing as above we obtain

$$\mathcal{L}^n(A_k' \cup B_k') \leq O((\delta_k^\alpha + \epsilon_k \frac{1-\eta_k}{c_{k,\alpha}})^2),$$

so that the last term in (79) tends to 0. Estimate (19) follows.

In the general case when $u \notin L^\infty(\Omega)$, estimate (19) follows from the previous step applied to the truncated function u^M, from the lower semicontinuity of F''_α and from the fact that $\Phi_\alpha(u^M) \leq \Phi_\alpha(u)$. □

7. **Convergence of minimizers.** The most important result of the paper is the following theorem on the convergence of minimizers of some variational problems involving the functionals F_k and F_α.

Theorem 7.1. *Let $q > 1$; let (ϵ_k) and (η_k) be infinitesimal sequences of positive numbers with $\alpha := \lim_k \eta_k/\epsilon_k$. Let V_k be as in (9), let $\psi \in C^1([0,1])$ be a strictly decreasing function satisfying (11), and let $g \in L^q(\Omega)$. For every k, let $(u_k, v_k) \in H^1(\Omega)\times V_k$ be a minimizer of the problem*

$$\min_{(u,v)\in H^1(\Omega)\times V_k} \Big(\int_\Omega v|\nabla u|^2 dx + \frac{1}{\epsilon_k} \int_\Omega \psi(v) dx + \int_\Omega |u-g|^q dx \Big). \tag{80}$$

Then $v_k \to 1$ in $L^1(\Omega)$ and a subsequence of (u_k) converges in $L^q(\Omega)$ to a minimizer u of the following limit problem:

$$\min_{u\in SBV^2(\Omega)} \Big(\int_\Omega |\nabla u|^2 dx + a\mathcal{H}^{n-1}(J_u) + b_\alpha \int_{J_u} |[u]| d\mathcal{H}^{n-1} + \int_\Omega |u-g|^q dx \Big),$$

if $0 < \alpha < \infty$; whereas in the extreme cases $\alpha = 0$ and $\alpha = \infty$ the limit problems are

$$\min_{u\in GSBV^2(\Omega)} \Big(\int_\Omega |\nabla u|^2 dx + a\mathcal{H}^{n-1}(J_u) + \int_\Omega |u-g|^q dx \Big),$$

$$\min_{u\in H^1(\Omega)} \Big(\int_\Omega |\nabla u|^2 dx + \int_\Omega |u-g|^q dx \Big),$$

respectively. Moreover for $0 \le \alpha \le \infty$ the minimum values of (80) tend to the minimum value of the limit problem.

Remark 4. If $g \in L^\infty(\Omega)$, the limit problem for $\alpha = 0$ can be formulated in $SBV^2(\Omega) \cap L^\infty(\Omega)$, since the functionals considered in these problems decrease by truncation with constants larger than $||g||_{L^\infty(\Omega)}$. As a consequence of Proposition 2, when $n = 1$ the limit problem in the case $\alpha = 0$ can be formulated in $SBV^2(\Omega)$ even if $g \notin L^\infty(\Omega)$.

Remark 5. In Theorem 7.1 we assume $\eta_k > 0$ only to guarantee the existence of a minimum point for G_k. In the case $\eta_k \ge 0$, the thesis of Theorem 7.1 continues to hold if (u_k, v_k) is a sequence which satisfies

$$\lim_{k\to\infty} G_k(u_k, v_k) - \inf_{L^q(\Omega)\times L^1(\Omega)} G_k = 0.$$

The proof is essentially the same.

To prove Theorem 7.1 we shall consider the functionals $F_{q,k} \colon L^q(\Omega)\times L^1(\Omega) \to [0, +\infty]$ defined by

$$F_{q,k}(u, v) := F_k|_{L^q(\Omega)\times L^1(\Omega)},$$

where the functionals (F_k) are defined in (8).

The first step in the proof of Theorem 7.1 is the following lemma.

Lemma 7.2. *Under the hypotheses of Theorem 7.1, the functionals $F_{q,k}$ Γ-converge in $L^q(\Omega)\times L^1(\Omega)$ to the functional $F_{q,\alpha} := F_\alpha|_{L^q(\Omega)\times L^1(\Omega)}$, where F_α is defined in (16).*

Proof. Let $F'_{q,\alpha}$ and $F''_{q,\alpha}$ be the Γ-lim inf and the Γ-lim sup of $F_{q,k}$ in $L^q(\Omega)\times L^1(\Omega)$ and let $(u, v) \in L^q(\Omega)\times L^1(\Omega)$.

Proof of the estimate from below.
The Γ-$\liminf$ inequality follows from $F'_{q,\alpha} \geq F'_\alpha$ (see, for instance, [11, Proposition 6.3]) and from Theorem 3.1.

Proof of the estimate from above.
Let $u \in GSBV^2(\Omega) \cap L^q(\Omega)$ with $F_\alpha(u, 1) < +\infty$.

First we suppose $u \in L^\infty(\Omega)$. Theorem 3.3 ensures the existence of a sequence $(u_k, v_k) \in H^1(\Omega)\times V_k(\Omega)$ such that $(u_k, v_k) \to (u, 1)$ in $L^1(\Omega)\times L^1(\Omega)$ and

$$\lim_{k\to\infty} F_k(u_k, v_k) = F_\alpha(u, 1).$$

The Γ-$\limsup$ inequality follows from this equality, from the convergence of the truncated functions $u_k^M \to u$ in $L^q(\Omega)$ with $M := ||u||_{L^\infty(\Omega)}$, and from the fact that $F_{q,k}(u_k^M, v_k) \leq F_k(u_k, v_k)$.

In the general case when $u \notin L^\infty(\Omega)$ the Γ-$\limsup$ inequality follows from the previous step applied to the truncated function u^M, from the lower semicontinuity of $F''_{q,\alpha}$ and from the fact that $F_\alpha(u^M, 1) \leq F_\alpha(u, 1)$. □

In order to obtain Theorem 7.1 we also need a compactness result, whose proof makes use of the following theorem, due to Alberti, Bouchitté, and Seppecher (see [1]). For every set $\mathcal{F} \subset L^1(\Omega)$ we define $\mathcal{F}^\xi_y := \{u^\xi_y : u \in \mathcal{F}\}$ for every $\xi \in S^{n-1}$ and for every $y \in \Pi^\xi$.

Theorem 7.3. *Let $\mathcal{F}$ be an equibounded subset of $L^\infty(\Omega)$. Assume that there exist n linearly independent unit vectors ξ which satisfy the following property: for every $\delta > 0$ there exists an equibounded subset $\mathcal{F}_\delta$ of $L^\infty(\Omega)$ such that $\mathcal{F}$ lies in a δ-neighborhood of $\mathcal{F}_\delta$ with respect to the $L^1(\Omega)$ distance and $(\mathcal{F}_\delta)^\xi_y$ is pre-compact in $L^1(\Omega^\xi_y)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega$. Then $\mathcal{F}$ is pre-compact in $L^1(\Omega)$.*

The compactness result is given by the following theorem.

Theorem 7.4. *Let (u_k, v_k) be a sequence in $L^1(\Omega)\times L^1(\Omega)$ such that (u_k) is bounded in $L^1(\Omega)$ and*

$$\liminf_{k\to\infty} F_k(u_k, v_k) < +\infty.$$

Then there exists a subsequence (u_j, v_j) of (u_k, v_k) and a function $u \in GSBV(\Omega) \cap L^1(\Omega)$ such that $u_j \to u$ $\mathcal{L}^n$-a.e. on Ω and $v_j \to 1$ in $L^1(\Omega)$.

Proof. We can suppose, up to subsequences, that there exists a constant $M < +\infty$ such that

$$F_k(u_k, v_k) \leq M.$$

This implies in particular that $v_k \to 1$ in $L^1(\Omega)$. We divide the proof into three steps.

The bounded case for $n = 1$. Let $n = 1$ and let (u_k) be bounded in $L^\infty(\Omega)$. It is not restrictive to assume $\Omega =]0, 1[$; if this is not the case we prove the statement for each connected component and then we use a diagonal argument.

Repeating step (a) of the proof of Theorem 3.2 in the case $n = 1$, we can find $m+1$ points $0 = x_0 < \cdots < x_m = 1$ such that ∇u_k is bounded in $L^2(x_i+\mu, x_{i+1}-\mu)$ uniformly with respect to k, $\mu > 0$, and $i = 0, \ldots, m-1$. This implies by assumption that u_k is bounded in $H^1(x_i + \mu, x_{i+1} - \mu)$ uniformly with respect to k, μ, and i. For every $\mu > 0$, we can find a subsequence of (u_k), not relabeled, that converges in $L^2(x_i + \mu, x_{i+1} - \mu)$, for $i = 0, \ldots, m - 1$. Then by a diagonal argument we extract a further subsequence (u_j) of (u_k) that converges in $L^1(\Omega)$ to some $u \in L^\infty(\Omega)$. From this convergence and from Proposition 1 we also deduce $u \in SBV^2(\Omega)$.

The bounded case for $n > 1$. Let $n > 1$ and let (u_k) be bounded in $L^\infty(\Omega)$.

Let $\xi \in \mathbb{R}^n$ be a unit vector and let $V_{y,k}$, $F_{y,k}$ be defined as in (52), (51). Moreover we set

$$A_k := \{y \in \Pi^\xi : F_{y,k}((u_k)^\xi_y, (v_k)^\xi_y) \leq L\},$$

where L is a fixed constant, so that by the Chebyshev inequality we obtain

$$\mathcal{H}^{n-1}((A_k)^c) \leq \frac{M}{L}.$$

Let $\delta > 0$; we can choose L so that $\operatorname{diam}(\Omega) cM/L < \delta$, with $c := \sup_k ||u_k||_{L^\infty}$. Let us define

$$(w_k)^\xi_y(t) := \begin{cases} (u_k)^\xi_y & \text{if } y \in A_k, \\ 0 & \text{otherwise} \end{cases}$$

and let $w_k(y + t\xi) := (w_k)^\xi_y(t)$, for $y \in \Pi^\xi$ and $t \in \Omega^\xi_y$. Then

$$||w_k - u_k||_{L^1(\Omega)} \leq c \operatorname{diam}(\Omega)\mathcal{H}^{n-1}((A_k)^c) < \delta.$$

Let $\mathcal{F} := (u_k)$ and $\mathcal{F}_\delta := (w_k)$, then $\mathcal{F}$ lies in a δ-neighborhood of $\mathcal{F}_\delta$ with respect to the $L^1(\Omega)$ distance; moreover $\mathcal{F}_\delta$ is pre-compact by the first part of the proof. From Theorem 7.3, we deduce the existence of a function $u \in L^\infty(\Omega)$ and of a subsequence (u_j, v_j) of (u_k, v_k) such that $(u_j, v_j) \to (u, 1)$ in $L^1(\Omega) \times L^1(\Omega)$ and $||u||_{L^\infty(\Omega)} \leq c$. Since

$$F'_\alpha(u, 1) \leq \lim_{j\to\infty} F_j(u_j, v_j) \leq M,$$

by Theorem 3.1 we conclude $u \in GSBV^2(\Omega) \cap L^\infty(\Omega)$, i.e., $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$.

The general case. For every $\mu \in \mathbb{N}$ we can consider $u^\mu_k := (-\mu \vee u_k) \wedge \mu$, then

$$F_k(u^\mu_k, v_k) \leq F_k(u_k, v_k)$$

and by the first part of the proof there exists a subsequence (u^μ_j) of (u^μ_k) and a function $u_\mu \in SBV^2(\Omega) \cap L^\infty(\Omega)$, with $||u_\mu||_{L^\infty(\Omega)} \leq \mu$, such that $u^\mu_j \to u_\mu$ in $L^1(\Omega)$ and $\mathcal{L}^n$-a.e. in Ω. This implies that the complement of the set

$$A := \big\{x \in \Omega : (u^\mu_j(x)) \text{ converges for every } \mu \in \mathbb{N}\big\}$$

is negligible. Let us observe that

$$\big(u_\mu(x)\big)^\lambda = \lim_{j\to\infty} \big(u^\mu_j(x)\big)^\lambda = \lim_{j\to\infty} u^\lambda_j(x) = u_\lambda(x) \quad \text{for every } \mu > \lambda. \tag{81}$$

We claim that the subset of A

$$E := \big\{x \in A : |u_\lambda(x)| = \lambda \text{ for every } \lambda \in \mathbb{N}\big\}$$

has measure zero. Indeed, for every $\lambda \in \mathbb{N}$ and $\epsilon > 0$ we have

$$\mathcal{L}^n(E) \leq \mathcal{L}^n\big(\big\{|u^\lambda_j| > \lambda - \epsilon\big\}\big) \leq \frac{1}{\lambda - \epsilon}\int_\Omega |u_j| dx \leq \frac{c}{\lambda - \epsilon}$$

for j large enough, where c is the bounding constant of (u_j) in $L^1(\Omega)$; as $\epsilon \to 0$ and $\lambda \to \infty$ we obtain $\mathcal{L}^n(E) = 0$. Let now $x \in A \setminus E$, so that there exists $\lambda \in \mathbb{N}$ with $|u_\lambda(x)| < \lambda$; this condition, together with equalities (81) gives $u_\mu(x) = u_\lambda(x)$ for every $\mu > \lambda$.

Let us define for $\mathcal{L}^n$-a.e. $x \in \Omega$

$$u(x) := \lim_{\lambda\to\infty} u_\lambda(x),$$

then by (81) u_λ coincides with the truncated function u^λ $\mathcal{L}^n$-a.e. in Ω. This implies that $u_j \to u$ $\mathcal{L}^n$-a.e. in Ω; since (u_λ) is contained in $SBV(\Omega)$ we deduce that $u \in GSBV(\Omega)$. Finally, since u_j^λ is uniformly bounded in $L^1(\Omega)$ with respect to λ and j, we also conclude that $u \in L^1(\Omega)$. □

In the following lemma we compute the Γ-limit of the functionals introduced in (80).

Lemma 7.5. *Let $1 \le q < +\infty$ and let $g \in L^q(\Omega)$. Let us consider the sequence of functionals (G_k) defined by*

$$G_k(u, v) := F_k(u, v) + \int_\Omega |u - g|^q dx, \tag{82}$$

where $u, v \in L^1(\Omega)$ and F_k is as in (8). Then (G_k) Γ-converges in $L^1(\Omega){\times}L^1(\Omega)$ to the functional $G_\alpha\colon L^1(\Omega){\times}L^1(\Omega) \to [0, +\infty]$ defined by

$$G_\alpha(u, v) := F_\alpha(u, v) + \int_\Omega |u - g|^q dx.$$

Proof. Let G'_α and G''_α be the Γ-lim inf and the Γ-lim sup of G_k in $L^1(\Omega){\times}L^1(\Omega)$. First we observe that the functional $H : L^1(\Omega){\times}L^1(\Omega) \to [0, +\infty]$ defined by

$$H(u, v) := \int_\Omega |u - g|^q dx$$

is lower semicontinuous.

In the case $q = 1$ the functional H is continuous; since (F_k) Γ-converges to F_α by Theorem 3.1, we can apply [11, Proposition 6.21] about the sum of Γ-limits to conclude that G_k Γ-converges to $F_\alpha + H$.

Let $q > 1$. Since H is not continuous, we need a different argument. To this aim we introduce $G''_{q,\alpha}$, the Γ-lim sup of G_k in $L^q(\Omega){\times}L^1(\Omega)$.

If $(u, v) \in (L^1(\Omega) \setminus L^q(\Omega)){\times}L^1(\Omega)$ we obtain by [11, Proposition 6.17]

$$+\infty = F_\alpha(u, v) + H(u, v) \le G'_\alpha(u, v);$$

let now $(u, v) \in L^q(\Omega){\times}L^1(\Omega)$. By [11, Proposition 6.3, 6.17, and 6.21], by Theorem 3.1, and by Lemma 7.2 we can deduce that

$$\begin{aligned} F_\alpha(u, v) + H(u, v) &\le G'_\alpha(u, v) \le G''_\alpha(u, v) \le G''_{q,\alpha}(u, v) = F_{q,\alpha}(u, v) + H(u, v) \\ &= F_\alpha(u, v) + H(u, v), \end{aligned}$$

so that the functionals G_k Γ-converge to the functional G_α. □

We are now in a position to prove Theorem 7.1.

Proof of Theorem 7.1. We fix k and prove that each functional G_k, defined in (82), attains its minimum. Let (u_j, v_j) be a sequence such that

$$\lim_{j\to\infty} G_k(u_j, v_j) = \inf_{L^q(\Omega)\times L^1(\Omega)} G_k.$$

Since $(G_k(u_j, v_j))$ is bounded, from the definition of G_k we deduce $(u_j, v_j) \in H^1(\Omega){\times}V_k$. In particular (u_j) is bounded in $L^q(\Omega)$ and (∇u_j) is bounded in $L^2(\Omega, \mathbb{R}^n)$; this implies that (u_j) is bounded in $H^1(\Omega)$. Then we can find a function $u \in H^1(\Omega) \cap L^q(\Omega)$ and a subsequence of (u_j), not relabeled, such that $u_j \rightharpoonup u$

weakly in $H^1(\Omega)$ and $\mathcal{L}^n$-a.e. in Ω. From the boundedness of (v_j) in $W^{1,\infty}(\Omega)$ we can deduce the existence of a function $v \in W^{1,\infty}(\Omega)$, with $\eta_k \le v \le 1$ and $|\nabla v| \le 1/\epsilon_k$ $\mathcal{L}^n$-a.e. on Ω, and of a subsequence of (v_j), not relabeled, such that $v_j \to v$ in $L^1(\Omega)$ and $\mathcal{L}^n$-a.e. in Ω. By [8, Theorem 2.3.1] and by the Fatou lemma, this implies that the estimates

$$\int_\Omega |\nabla u|^2 v dx \le \liminf_{j\to\infty} \int_\Omega |\nabla u_j|^2 v_j dx, \qquad \int_\Omega |u-g|^q dx \le \liminf_{j\to\infty} \int_\Omega |u_j - g|^q dx \quad (83)$$

hold, so that we obtain

$$G_k(u,v) \le \lim_{j\to\infty} G_k(u_j,v_j) = \inf_{L^q(\Omega)\times L^1(\Omega)} G_k.$$

This shows that the infimum of G_k is achieved.

Let now (u_k,v_k) be a minimizer of G_k, which obviously belongs to $H^1(\Omega)\times V_k$. Since the sequence $(F_k(u_k,v_k))$ is bounded, by the Compactness Theorem 7.4 there exists a function $u \in GSBV(\Omega)\cap L^q(\Omega)$ and a subsequence of (u_k,v_k), not relabeled, such that $u_k \to u$ $\mathcal{L}^n$-a.e. in Ω and $v_k \to 1$ in $L^1(\Omega)$. Let us prove that $u_k \to u$ in $L^1(\Omega)$. By the the Dominated Convergence Theorem we get $\int_\Omega |u_k - u| 1_{B_k^c} dx \to 0$, where $B_k := \big\{|u_k - u| > 1\big\}$; moreover using the Hölder inequality we obtain

$$\int_{B_k} |u_k - u| dx \le \Big(||u_k - g||_{L^q(\Omega)} + ||u-g||_{L^q(\Omega)} \Big) \mathcal{L}^n(B_k)^{1-\frac{1}{q}} \le 2||g||_{L^q(\Omega)} \mathcal{L}^n(B_k)^{1-\frac{1}{q}},$$

where the last inequality follows from the estimate $G_k(u_k,v_k) \le G_k(0,1) = ||g||^q_{L^q(\Omega)}$ and from (83). Since $u_k \to u$ in measure we conclude that $\mathcal{L}^n(B_k) \to 0$ and the convergence $u_k \to u$ in $L^1(\Omega)$ follows.

By the Γ-convergence of G_k to G_α (Lemma 7.5) and by a general property of Γ-convergence (see [11, Corollary 7.20]), we find that $(u,1)$ is a minimizer for G_α, so that $u \in GSBV^2(\Omega) \cap L^q(\Omega)$. Moreover we have the convergence of minimum values and the convergence of minimizer in $L^1(\Omega)\times L^1(\Omega)$.

Let us prove now that $u_k \to u$ in $L^q(\Omega)$, up to subsequences. Since

$$F_\alpha(u,1) + \int_\Omega |u-g|^q dx = \lim_{k\to\infty} \Big(F_k(u_k,v_k) + \int_\Omega |u_k - g|^q dx \Big),$$

$$F_\alpha(u,1) \le \liminf_{k\to\infty} F_k(u_k,v_k), \quad \text{and} \quad \int_\Omega |u-g|^q dx \le \liminf_{k\to\infty} \int_\Omega |u_k - g|^q dx,$$

we obtain

$$\int_\Omega |u-g|^q dx = \lim_{k\to\infty} \int_\Omega |u_k - g|^q dx. \quad (84)$$

This fact, together with the $\mathcal{L}^n$-a.e. convergence in Ω of $u_k - g$ to $u - g$, implies that $u_k \to u$ in $L^q(\Omega)$ by the Generalized Dominated Convergence Theorem. □

Acknowledgments. This paper was partially supported by the Project "Variational Problems with Multiple Scales" 2008 of the Italian Ministry of Education, University, and Research.

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Received June 2011; revised June 2012.

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