Γ-CONVERGENCE AND H-CONVERGENCE OF LINEAR ELLIPTIC OPERATORS

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Abstract. We consider a sequence of linear Dirichlet problems as follows

$$\begin{cases} -\operatorname{div}(\sigma_\varepsilon \nabla u_\varepsilon) = f \text{ in } \Omega, \\ u_\varepsilon \in H^1_0(\Omega), \end{cases}$$

with (σ_ε) uniformly elliptic and possibly non-symmetric. Using *purely variational arguments* we give an alternative proof of the compactness of H-convergence, originally proved by Murat and Tartar.

Keywords: linear elliptic operators, Γ-convergence, H-convergence.

2000 Mathematics Subject Classification: 35J15, 35J20, 49J45.

1. Introduction

The notion of H-convergence was introduced by Murat and Tartar in [10, 12] to study a wide class of homogenization problems for possibly non-symmetric elliptic equations. Let $\sigma_\varepsilon \in L^\infty(\Omega;\mathbb{R}^{n\times n})$ be a sequence of matrices satisfying uniform ellipticity and boundedness conditions on a bounded open set $\Omega \subset \mathbb{R}^n$. We say that σ_ε H-converges to matrix $\sigma_0 \in L^\infty(\Omega;\mathbb{R}^{n\times n})$ satisfying the same ellipticity and boundedness conditions if for every $f \in H^{-1}(\Omega)$ the sequence u_ε of the solutions to the problems

$$\begin{cases} -\operatorname{div}(\sigma_\varepsilon \nabla u_\varepsilon) = f \text{ in } \Omega, \\ u_\varepsilon \in H^1_0(\Omega), \end{cases} \tag{1.1}$$

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satisfy

$$u_\varepsilon \rightharpoonup u_0 \text{ weakly in } H^1_0(\Omega) \quad \text{and} \quad \sigma_\varepsilon \nabla u_\varepsilon \rightharpoonup \sigma_0 \nabla u_0 \text{ weakly in } L^2(\Omega;\mathbb{R}^n),$$

where u_0 is the solution to

$$\begin{cases} -\operatorname{div}(\sigma_0 \nabla u_0) = f \text{ in } \Omega, \\ u_0 \in H^1_0(\Omega). \end{cases}$$

The notion of Γ-convergence was introduced by De Giorgi and Franzoni in [5, 6] to study the asymptotic behavior of the solutions of a wide class of minimization problems depending on a parameter $\varepsilon > 0$, which varies in a sequence converging to 0. Let (X, d) be a metric space and let $F_\varepsilon : X \to \overline{\mathbb{R}}$ be a sequence of functionals, we say that F_ε $\Gamma(d)$-converges to a functional $F_0 : X \to \overline{\mathbb{R}}$ if for all $x \in X$ we have

(i) (liminf inequality) for every sequence $x_\varepsilon \xrightarrow{d} x$ in X

$$F_0(x) \le \liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon);$$

(ii) (limsup inequality) there exists a sequence $\bar{x}_\varepsilon \xrightarrow{d} x$ in X such that

$$F_0(x) \ge \limsup_{\varepsilon\to 0} F_\varepsilon(\bar{x}_\varepsilon).$$

Preprint SISSA 04/2012/M (March 2012).

It has been proved that when σ_ε is symmetric, the equation (1.1) has a variational structure since it can be seen as the Euler-Lagrange equation associated with

$$\mathcal{F}_\varepsilon(u) = \frac{1}{2}\int_\Omega \sigma_\varepsilon(x)\nabla u \cdot \nabla u \, dx - \int_\Omega f\, u \, dx \,,$$

or, equivalently, as the solution to the minimization problem

$$\min\{\mathcal{F}_\varepsilon(u) : \; u \in H^1_0(\Omega)\}. \tag{1.2}$$

i:min

Therefore, in this case (1.2) provides a variational principle for the Dirichlet problem (1.1) and the convergence of the solutions of (1.1) can be equivalently studied by means of the Γ-convergence, with respect to the weak topology of $H^1_0(\Omega)$, of the associated functionals $\mathcal{F}_\varepsilon$ or in terms of the G-convergence of the uniformly elliptic, symmetric matrices (σ_ε) (see De Giorgi and Spagnolo [7]).

In this paper we consider the equivalence between H-convergence and Γ-convergence in the possibly non-symmetric case. To every elliptic matrix $\sigma \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ we associate a suitable quadratic integral functional $F : L^2(\Omega; \mathbb{R}^n) \times H^1_0(\Omega) \to [0, +\infty)$ (see (2.12)) and we consider the Γ-convergence with respect to the distance d defined by

$$d((\alpha, \varphi), (\beta, \psi)) = \|\alpha - \beta\|_{H^{-1}(\Omega;\mathbb{R}^n)} + \|\mathrm{div}(\alpha - \beta)\|_{H^{-1}(\Omega)} + \|\varphi - \psi\|_{L^2(\Omega)} \,.$$

We prove (Theorem 3.2) that the H-convergence of σ_ε to σ_0 is equivalent to the $\Gamma(d)$-convergence of the functionals F_ε corresponding to σ_ε to the functional F_0 corresponding to σ_0. In [2] this result was proved using compactness properties of H-convergence [10, 12], while in the present paper the equivalence is obtained as a consequence of a general compactness theorem for integral functionals with respect to $\Gamma(d)$-convergence [1]. Moreover, as a consequence of the results proved in [1], we also give an independent proof (Theorem 3.1) of the compactness of H-convergence based only on Γ-convergence arguments.

2. Notation and preliminaries

In this section we introduce a few notation and we recall some preliminary results we employ in the sequel. For any $A \in \mathbb{R}^{n\times n}$ we denote by A^s and A^a the symmetric and the anti-symmetric part of A, respectively; *i.e.,*

$$A^s := \frac{A + A^T}{2}, \quad A^a := \frac{A - A^T}{2},$$

where A^T is the transpose matrix of A. We use bold capital letters to denote matrices in $\mathbb{R}^{2n\times 2n}$. The scalar product of two vectors ξ and η is denoted by $\xi \cdot \eta$.

Let Ω be an open bounded subset of $\mathbb{R}^n$. For $0 < c_0 \le c_1 < +\infty$, $\mathcal{M}(c_0, c_1, \Omega)$ denotes the set of matrix-valued functions $\sigma \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ satisfying

$$\sigma(x)\xi \cdot \xi \ge c_0|\xi|^2, \quad \sigma^{-1}(x)\xi \cdot \xi \ge c_1^{-1}|\xi|^2, \quad \text{for every } \xi \in \mathbb{R}^n, \text{ for a.e. } x \in \Omega, \tag{2.1}$$

ellipticity

or, equivalently, satisfying

$$\sigma(x)\xi \cdot \xi \ge c_0|\xi|^2, \quad \sigma(x)\xi \cdot \xi \ge c_1^{-1}|\sigma(x)\xi|^2, \quad \text{for every } \xi \in \mathbb{R}^n, \text{ for a.e. } x \in \Omega. \tag{2.2}$$

ellipticity-equi

Note that (2.1) (or (2.2)) implies that

$$|\sigma(x)| \le c_1 \quad \text{for a.e. } x \in \Omega,$$

and that necessarily $c_0 \le c_1$. To not overburden notation, in all that follows we always write σ in place of $\sigma(x)$.

Given $\sigma \in \mathcal{M}(c_0, c_1, \Omega)$ we consider the $(2n \times 2n)$-matrix-valued function $\mathbf{\Sigma} \in L^\infty(\Omega; \mathbb{R}^{2n\times 2n})$ having the following block structure

$$\mathbf{\Sigma} := \begin{pmatrix} (\sigma^s)^{-1} & -(\sigma^s)^{-1}\sigma^a \\ \sigma^a(\sigma^s)^{-1} & \sigma^s - \sigma^a(\sigma^s)^{-1}\sigma^a \end{pmatrix}. \tag{2.3}$$

Sigma

Notice that $\mathbf{\Sigma}$ is symmetric. Moreover, the assumption $\sigma \in \mathcal{M}(c_0, c_1, \Omega)$ easily implies that $\mathbf{\Sigma}$ is uniformly coercive (see [2, Section 3.1.1] for the details); specifically, there exists a constant $C(c_0, c_1) > 0$, depending only on c_0 and c_1, such that

$$\mathbf{\Sigma}\mathrm{w} \cdot \mathrm{w} \geq C(c_0, c_1)|\mathrm{w}|^2, \tag{2.4}$$ $\boxed{\texttt{unif-coer}}$

for every $\mathrm{w} \in \mathbb{R}^{2n}$, and a.e. in Ω.

If we consider the matrix-valued functions $A, B, C \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ defined as

$$A = (\sigma^s)^{-1}, \quad B = -(\sigma^s)^{-1}\sigma^a, \quad C = \sigma^s - \sigma^a(\sigma^s)^{-1}\sigma^a, \tag{2.5}$$ $\boxed{\texttt{notaz-comp}}$

the matrix $\mathbf{\Sigma}$ can be rewritten as

$$\mathbf{\Sigma} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}. \tag{2.6}$$ $\boxed{\texttt{Sigma-comp}}$

We notice that, for a.e. $x \in \Omega$, the matrix $\mathbf{\Sigma}$ belongs to the indefinite special orthogonal group $SO(n, n)$; *i.e.,*

$$\mathbf{\Sigma}\mathbf{J}\mathbf{\Sigma} = \mathbf{J} \quad \text{a.e. in } \Omega, \quad \text{with} \quad \mathbf{J} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{2.7}$$ $\boxed{\texttt{prop\_sigma}}$

where $I \in \mathbb{R}^{n\times n}$ is the identity matrix (see [2, Section 3.1.1]). Moreover, taking into account the symmetry of $\mathbf{\Sigma}$, it is immediate to show that (2.7) is equivalent to the following system of identities for the block decomposition (2.6):

$$\begin{cases} AB^T + BA = 0 \\ AC + B^2 = I, \\ CB + B^TC = 0 \end{cases} \qquad \text{a.e. in } \Omega. \tag{2.8}$$ $\boxed{\texttt{conditions}}$

Conversely, one can prove that, if $\mathbf{M} \in L^\infty(\Omega; \mathbb{R}^{2n\times 2n})$ is symmetric and has the block decomposition

$$\mathbf{M} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}, \tag{2.9}$$ $\boxed{\texttt{blockM}}$

with $A, B, C \in L^\infty(\Omega; \mathbb{R}^{n\times n})$, A and C symmetric, and $\det A \neq 0$, then the first two equations in (2.8) imply the third one, and (2.8) implies that $\mathbf{M}$ is equal to the matrix $\mathbf{\Sigma}$ defined in (2.3) with $\sigma = A^{-1} - A^{-1}B$ (see [2, Proposition 3.1]).

Throughout the paper the parameter ε varies in a strictly decreasing sequence of positive real numbers converging to zero. Let (σ_ε) be a sequence in $\mathcal{M}(c_0, c_1, \Omega)$ and consider the sequence $(\mathbf{\Sigma}_\varepsilon) \subset L^\infty(\Omega; \mathbb{R}^{2n\times 2n})$ defined by (2.3) with $\sigma = \sigma_\varepsilon$. Let $Q_\varepsilon \colon L^2(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{R}^n) \to [0, +\infty)$ be the quadratic forms associated with $\mathbf{\Sigma}_\varepsilon$; *i.e.,*

$$Q_\varepsilon(a, b) := \int_\Omega \mathbf{\Sigma}_\varepsilon \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} dx. \tag{2.10}$$ $\boxed{\texttt{general-Qe}}$

Their gradients $\operatorname{grad} Q_\varepsilon \colon L^2(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{R}^n) \to L^2(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{R}^n)$ are given by

$$\operatorname{grad} Q_\varepsilon(a, b) = (A_\varepsilon a + B_\varepsilon b, B_\varepsilon^T a + C_\varepsilon b), \tag{2.11}$$ $\boxed{\texttt{differential}}$

where $A_\varepsilon, B_\varepsilon$, and C_ε are as in (2.5) with $\sigma = \sigma_\varepsilon$. We also consider the quadratic forms $F_\varepsilon : L^2(\Omega; \mathbb{R}^n) \times H_0^1(\Omega) \to [0, +\infty)$ defined by

$$F_\varepsilon(\alpha, \psi) := Q_\varepsilon(\alpha, \nabla\psi) \tag{2.12}$$ $\boxed{\texttt{Feps}}$

For every $\lambda, \mu \in H^{-1}(\Omega)$; we consider the sequence of constrained functionals $F_\varepsilon^{\lambda,\mu} \colon L^2(\Omega; \mathbb{R}^n) \times H_0^1(\Omega) \to [0, +\infty]$ defined as follows

$$F_\varepsilon^{\lambda,\mu}(\alpha, \psi) := \begin{cases} F_\varepsilon(\alpha, \psi) - \langle \mu, \psi \rangle & \text{if } \quad -\operatorname{div}\alpha = \lambda, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.13}$$ $\boxed{\texttt{constr-Fe}}$

where $\langle \cdot, \cdot \rangle$ denotes the dual paring between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Given a symmetric matrix $\mathbf{M} \in L^\infty(\Omega;\mathbb{R}^{2n\times 2n})$, we consider the quadratic functionals $Q_{\mathbf{M}} : L^2(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{R}^n) \to [0,+\infty)$ and $F_{\mathbf{M}} : L^2(\Omega;\mathbb{R}^n)\times H^1_0(\Omega) \to [0,+\infty)$ defined by

$$Q_{\mathbf{M}}(a,b) := \int_\Omega \mathbf{M}\begin{pmatrix} a \\ b \end{pmatrix}\cdot\begin{pmatrix} a \\ b \end{pmatrix} dx \qquad \text{and} \qquad F_{\mathbf{M}}(\alpha,\psi) := Q_{\mathbf{M}}(\alpha,\nabla\psi)\,. \tag{2.14}$$

Considering the block decomposition (2.9), the gradient of $Q_{\mathbf{M}}$ is given by

$$\operatorname{grad} Q_{\mathbf{M}}(a,b) = (Aa + Bb, B^T a + Cb)\,. \tag{2.15}$$

Finally, for every $\lambda,\mu \in H^{-1}(\Omega)$; we consider the constrained functional $F^{\lambda,\mu}_{\mathbf{M}}\colon L^2(\Omega;\mathbb{R}^n)\times H^1_0(\Omega) \to [0,+\infty]$ defined as follows

$$F^{\lambda,\mu}_{\mathbf{M}}(\alpha,\psi) := \begin{cases} F_{\mathbf{M}}(\alpha,\psi) - \langle \mu,\psi\rangle & \text{if} \quad -\operatorname{div}\alpha = \lambda, \\ +\infty & \text{otherwise.} \end{cases}$$

Let w be the weak topology of $L^2(\Omega;\mathbb{R}^n)\times H^1_0(\Omega)$ and let d be the distance in $L^2(\Omega;\mathbb{R}^n)\times H^1_0(\Omega)$ defined by

$$d((\alpha,\varphi),(\beta,\psi)) = \|\alpha-\beta\|_{H^{-1}(\Omega;\mathbb{R}^n)} + \|\operatorname{div}(\alpha-\beta)\|_{H^{-1}(\Omega)} + \|\varphi-\psi\|_{L^2(\Omega)}\,.$$

The following result is proved in [1, Corollary 2.9].

Theorem 2.1. *Let (σ_ε) be a sequence in $\mathcal{M}(c_0,c_1,\Omega)$. There exist a subsequences of ε, not relabeled, and a symmetric matrix $\mathbf{M} \in L^\infty(\Omega;\mathbb{R}^{2n\times 2n})$, such that the functionals F_ε defined by (2.12) $\Gamma(d)$-converge to the functional $F_{\mathbf{M}}$ defined in (2.14). Moreover, $\mathbf{M}$ is positive definite and satisfies the coercivity condition* (2.4).

The following result is a consequence of [1, Theorem 3.3] and of the stability of Γ-convergence under continuous perturbations.

Theorem 2.2. *Let (σ_ε) be a sequence in $\mathcal{M}(c_0,c_1,\Omega)$ and let $\mathbf{M} \in L^\infty(\Omega;\mathbb{R}^{2n\times 2n})$ be a symmetric, positive define matrix satisfying* (2.4). *Assume that the functionals F_ε defined by (2.12) $\Gamma(d)$-converge to the functional $F_{\mathbf{M}}$ defined in (2.14). Then, for every $\lambda,\mu \in H^{-1}(\Omega)$, the functionals $(F^{\lambda,\mu}_\varepsilon)$ defined by (2.13) $\Gamma(w)$-converges to the functional $F^{\lambda,\mu}$ defined by*

$$F^{\lambda,\mu}(\alpha,\psi) := \begin{cases} F_{\mathbf{M}}(\alpha,\psi) - \langle \mu,\psi\rangle & \textit{if} \quad -\operatorname{div}\alpha = \lambda, \\ +\infty & \textit{otherwise.} \end{cases}$$

For the reader's sake, here we briefly recall a fundamental tool we employ in what follows, the Cherkaev-Gibiansky variational principle [3] (see also Fannjiang-Papanicolaou [8] and Milton [9]) , which will be presented in the notational setting which is suitable for our purposes. Loosely speaking, this variational principle amounts to associate to the two following Dirichlet problems

$$\begin{cases} -\operatorname{div}(\sigma_\varepsilon\nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon \in H^1_0(\Omega), \end{cases} \qquad \begin{cases} -\operatorname{div}(\sigma_\varepsilon^T\nabla v_\varepsilon) = g & \text{in } \Omega, \\ v_\varepsilon \in H^1_0(\Omega). \end{cases} \tag{2.16}$$

with $f,g \in H^{-1}(\Omega)$, a quadratic functional whose Euler-Lagrange equation is solved by a suitable combination of solutions to (2.16) and of their momenta. We set

$$a_\varepsilon := \sigma_\varepsilon\nabla u_\varepsilon \qquad \text{and} \qquad b_\varepsilon := \sigma_\varepsilon^T\nabla v_\varepsilon\,. \tag{2.17}$$

For every $\varepsilon > 0$, $\lambda, \mu \in H^{-1}(\Omega)$ the unique minimizer $(\alpha_\varepsilon, \psi_\varepsilon)$ of $F_\varepsilon^{\lambda,\mu}$ satisfies the constraint $-\mathrm{div}\alpha_\varepsilon = \lambda$ and the following system of Euler-Lagrange equations:

$$\begin{cases} \displaystyle\int_\Omega (A_\varepsilon \alpha_\varepsilon + B_\varepsilon \nabla \psi_\varepsilon) \cdot \beta \, dx = 0, \\ \displaystyle\int_\Omega (B_\varepsilon^T \alpha_\varepsilon + C_\varepsilon \nabla \psi_\varepsilon) \cdot \nabla \varphi \, dx = \langle \mu, \varphi \rangle, \end{cases} \tag{2.18}$$

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for every $\beta \in L^2(\Omega; \mathbb{R}^n)$ with $\mathrm{div}\beta = 0$ and for every $\varphi \in H_0^1(\Omega)$.

If $u_\varepsilon, v_\varepsilon$ satisfy (2.16) then we can prove (see [2, Section 3.2] for details) that the pair

$$(a_\varepsilon + b_\varepsilon, u_\varepsilon - v_\varepsilon) \tag{2.19}$$

pair1

solves (2.18), with $\lambda = f + g$, $\mu = f - g$, and thus minimizes $F_\varepsilon^{f+g,f-g}$.

In the same way, it can be seen that the pair

$$(a_\varepsilon - b_\varepsilon, u_\varepsilon + v_\varepsilon) \tag{2.20}$$

pair2

minimizes $F_\varepsilon^{f-g,f+g}$.

3. The main result

In this section we state and prove the main result of this paper: an alternative and purely variational proof of the sequential compactness of $\mathcal{M}(c_0, c_1, \Omega)$ with respect to H-convergence, originally proved by Murat and Tartar [10, 12].

main-theo

Theorem 3.1 (Compactness of H-convergence). *Let (σ_ε) be a sequence in $\mathcal{M}(c_0, c_1, \Omega)$. Then there exist a subsequence (not relabeled) and a matrix $\sigma_0 \in \mathcal{M}(c_0, c_1, \Omega)$ such that (σ_ε) H-converges to σ_0 and (σ_ε^T) H-converges to σ_0^T.*

Proof. By Theorem 2.1 there exist a subsequence of F_ε, not relabeled, and a symmetric, positive definite matrix $\mathbf{M} \in L^\infty(\Omega; \mathbb{R}^{2n \times 2n})$, with the block decomposition (2.9), such that F_ε $\Gamma(d)$-converges to $F_{\mathbf{M}}$. In the rest of this proof we show that (σ_ε) H-converges to σ_0 and (σ_ε^T) H-converges to σ_0^T, where $\sigma_0 := A^{-1} - A^{-1}B$.

Let $f, g \in H^{-1}(\Omega)$, let $u_\varepsilon, v_\varepsilon$ be as in (2.16), and let $a_\varepsilon, b_\varepsilon$ be as in (2.17). By standard variational estimates we have that (u_ε) and (v_ε) are bounded in $H_0^1(\Omega)$ while (a_ε) and (b_ε) are bounded in $L^2(\Omega; \mathbb{R}^n)$. Therefore, up to subsequences (not relabeled),

$$u_\varepsilon \rightharpoonup u_0, \quad v_\varepsilon \rightharpoonup v_0 \quad \text{weakly in } H_0^1(\Omega) \quad \text{and} \quad a_\varepsilon \rightharpoonup a_0, \quad b_\varepsilon \rightharpoonup b_0 \quad \text{weakly in } L^2(\Omega; \mathbb{R}^n), \tag{3.1}$$

convergence-1

for some $u_0, v_0 \in H_0^1(\Omega)$ and $a_0, b_0 \in L^2(\Omega; \mathbb{R}^n)$.

Since $(a_\varepsilon + b_\varepsilon, u_\varepsilon - v_\varepsilon)$ are minimizers of $F_\varepsilon^{f+g,f-g}$ and these functionals Γ-converge to $F_{\mathbf{M}}^{f+g,f-g}$ by Theorem 2.2, appealing to the fundamental property of Γ-convergence we find that

$$\lim_{\varepsilon \to 0} F_\varepsilon^{f+g,f-g}(a_\varepsilon + b_\varepsilon, u_\varepsilon - v_\varepsilon) = F_{\mathbf{M}}^{f+g,f-g}(a_0 + b_0, u_0 - v_0) = \min F_{\mathbf{M}}^{f+g,f-g}. \tag{3.2}$$

conv-min-1

Similarly, since $(a_\varepsilon - b_\varepsilon, u_\varepsilon + v_\varepsilon)$ minimizes $F_\varepsilon^{f-g,f+g}$, we have also

$$\lim_{\varepsilon \to 0} F_\varepsilon^{f-g,f+g}(a_\varepsilon - b_\varepsilon, u_\varepsilon + v_\varepsilon) = F_{\mathbf{M}}^{f-g,f+g}(a_0 - b_0, u_0 + v_0) = \min F_{\mathbf{M}}^{f-g,f+g}. \tag{3.3}$$

conv-min-2

Thanks to Theorem 2.2, (3.2), (3.3), and in view of [1, Proposition 2.10] we are now in a position to invoke the result about the convergence of momenta proved in [1, Corollary 4.6], hence we obtain

$$\operatorname{grad} Q_\varepsilon(a_\varepsilon + b_\varepsilon, \nabla u_\varepsilon - \nabla v_\varepsilon) \rightharpoonup \operatorname{grad} Q_{\mathbf{M}}(a + b, \nabla u - \nabla v)\,, \tag{3.4}$$

$$\operatorname{grad} Q_\varepsilon(a_\varepsilon - b_\varepsilon, \nabla u_\varepsilon + \nabla v_\varepsilon) \rightharpoonup \operatorname{grad} Q_{\mathbf{M}}(a_0 - b_0, \nabla u_0 + \nabla v_0) \tag{3.5}$$

weakly in $L^2(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{R}^n)$. By (2.11) and (2.15), considering only the first component, we get

$$A_\varepsilon(a_\varepsilon+b_\varepsilon)+B_\varepsilon(\nabla u_\varepsilon-\nabla v_\varepsilon)\rightharpoonup A(a_0+b_0)+B(\nabla u_0-\nabla v_0), \tag{3.6}$$
$$A_\varepsilon(a_\varepsilon-b_\varepsilon)+B_\varepsilon(\nabla u_\varepsilon+\nabla v_\varepsilon)\rightharpoonup A(a_0-b_0)+B(\nabla u_0+\nabla v_0) \tag{3.7}$$

weakly in $L^2(\Omega;\mathbb{R}^n)$. Since by (2.5) $A_\varepsilon(a_\varepsilon+b_\varepsilon)+B_\varepsilon(\nabla u_\varepsilon-\nabla v_\varepsilon)=\nabla u_\varepsilon+\nabla v_\varepsilon$ and $A_\varepsilon(a_\varepsilon-b_\varepsilon)+B_\varepsilon(\nabla u_\varepsilon+\nabla v_\varepsilon)=\nabla u_\varepsilon-\nabla v_\varepsilon$, from (3.6) and (3.7) we deduce that

$$\nabla u_\varepsilon+\nabla v_\varepsilon\rightharpoonup A(a_0+b_0)+B(\nabla u_0-\nabla v_0), \tag{3.8}$$
$$\nabla u_\varepsilon-\nabla v_\varepsilon\rightharpoonup A(a_0-b_0)+B(\nabla u_0+\nabla v_0) \tag{3.9}$$

weakly in $L^2(\Omega;\mathbb{R}^n)$. Hence, adding up (3.8) and (3.9) entails $\nabla u_\varepsilon\rightharpoonup Aa_0+B\nabla u_0$ in $L^2(\Omega;\mathbb{R}^n)$, which gives $\nabla u_0=Aa_0+B\nabla u_0$ by (3.1). This implies

$$a_0=\sigma_0\,\nabla u_0, \tag{3.10}$$

a*equal

with

$$\sigma_0:=A^{-1}-A^{-1}B. \tag{3.11}$$

sigma

Since $-\mathrm{div}a_\varepsilon=f$, by (2.16) and (2.17) we get that $-\mathrm{div}a_0=f$. Hence, (3.10) implies that u_0 is the solution to

$$\begin{cases}-\mathrm{div}(\sigma_0\nabla u_0)=f & \text{in } \Omega,\\ u_0\in H^1_0(\Omega).\end{cases} \tag{3.12}$$

Dirichletu

So far we have proved that for every $f\in H^{-1}(\Omega)$ the solutions u_ε of (2.16) converge weakly in $H^1_0(\Omega)$ to the solution u_0 of (3.12) and their momenta $\sigma_\varepsilon\nabla u_\varepsilon$ converge weakly in $L^2(\Omega;\mathbb{R}^n)$ to $\sigma_0\nabla u_0$. Thus, to conclude the proof of the H-convergence of (σ_ε) to σ_0 it remains to show that σ_0 belongs to $\mathcal{M}(c_0,c_1,\Omega)$. To this end, let $u\in H^1_0(\Omega)$ and choose

$$f:=-\mathrm{div}(\sigma_0\nabla u); \tag{3.13}$$

fw

in this way the solution u_0 of the equation (3.12) coincides with u.

Let $\varphi\in C^\infty_c(\Omega)$. Using φu_ε as a test function in the equation $-\mathrm{div}(\sigma_\varepsilon\nabla u_\varepsilon)=f$ and then passing to the limit on ε we get

$$\int_\Omega f\varphi u_0\,dx=\lim_{\varepsilon\to0}\int_\Omega f\varphi u_\varepsilon\,dx=\lim_{\varepsilon\to0}\Big(\int_\Omega(\sigma_\varepsilon\nabla u_\varepsilon\cdot\nabla u_\varepsilon)\varphi\,dx\Big)+\int_\Omega\sigma_0\nabla u_0\cdot u_0\nabla\varphi\,dx, \tag{3.14}$$

test1

where to compute the limit of the last term in (3.14) we appealed to the strong $L^2(\Omega)$ convergence of u_ε to u_0. On the other hand, since by (3.12)

$$\int_\Omega f\varphi u_0\,dx=\int_\Omega(\sigma_0\nabla u_0\cdot\nabla u_0)\varphi\,dx+\int_\Omega\sigma_0\nabla u_0\cdot u_0\nabla\varphi\,dx$$

from (3.14) we deduce that

$$\lim_{\varepsilon\to0}\int_\Omega(\sigma_\varepsilon\nabla u_\varepsilon\cdot\nabla u_\varepsilon)\varphi\,dx=\int_\Omega(\sigma_0\nabla u_0\cdot\nabla u_0)\varphi\,dx, \tag{3.15}$$

div-curl

for every $\varphi\in C^\infty_c(\Omega)$. Hence, choosing $\varphi\ge0$, combining (3.15), the first condition in (2.1), and the equality $u=u_0$, we have

$$\int_\Omega(\sigma_0\nabla u\cdot\nabla u)\varphi\,dx\ge c_0\liminf_{\varepsilon\to0}\int_\Omega|\nabla u_\varepsilon|^2\varphi\,dx\ge c_0\int_\Omega|\nabla u|^2\varphi\,dx,$$

the second inequality following from $\nabla u_\varepsilon\rightharpoonup\nabla u_0=\nabla u$ in $L^2(\Omega;\mathbb{R}^n)$. Since this inequality holds true for every $\varphi\in C^\infty_c(\Omega)$, $\varphi\ge0$, we get that

$$\sigma_0\nabla u\cdot\nabla u\ge c_0|\nabla u|^2\quad\text{a.e. in }\Omega, \tag{3.16}$$

ellip-1

for every $u \in H^1_0(\Omega)$. Using the second condition in (2.2), we find

$$\int_\Omega (\sigma_0 \nabla u \cdot \nabla u)\varphi\,dx \geq c_1^{-1} \liminf_{\varepsilon\to 0} \int_\Omega |\sigma_\varepsilon \nabla u_\varepsilon|^2 \varphi\,dx \geq c_1^{-1} \int_\Omega |\sigma_0 \nabla u|^2 \varphi\,dx,$$

since $\sigma_\varepsilon \nabla u_\varepsilon \rightharpoonup \sigma_0 \nabla u_0 = \sigma_0 \nabla u$ in $L^2(\Omega;\mathbb{R}^n)$. From the previous inequality we deduce

$$\sigma_0 \nabla u \cdot \nabla u \geq c_1^{-1} |\sigma_0 \nabla u|^2 \quad \text{a.e. in } \Omega, \tag{3.17}$$

ellip-2

for every $u \in H^1_0(\Omega)$. Finally, (2.2) follows from (3.16) and (3.17) by taking u to be affine in an open set $\omega \subset\subset \Omega$.

We now prove that σ_ε^T H-converges to σ_0^T. Subtracting (3.9) from (3.8) gives $\nabla v_\varepsilon \rightharpoonup Ab_0 - B\nabla v_0$ weakly in $L^2(\Omega;\mathbb{R}^n)$; the latter combined with (3.1) imply that $\nabla v_0 = Ab - B\nabla v_0$. We deduce then

$$b_0 = \tilde\sigma \nabla v_0, \tag{3.18}$$

conv-trasp

where

$$\tilde\sigma := A^{-1} + A^{-1}B. \tag{3.19}$$

sigmatilde

Since $-\mathrm{div} b_\varepsilon = g$ by (2.16) and (2.17), we get $-\mathrm{div} b_0 = g$, so that (3.18) implies that v_0 is the solution to

$$\begin{cases} -\mathrm{div}(\tilde\sigma \nabla v_0) = g & \text{in } \Omega, \\ v_0 \in H^1_0(\Omega). \end{cases} \tag{3.20}$$

Dirichletv

As in the previous part of the proof, this implies that σ_ε^T H-converges to $\tilde\sigma$. We want to prove that $\tilde\sigma = \sigma_0^T$.

To this end, we argue as in the previous step. Let $u, v \in H^1_0(\Omega)$. We choose $f := -\mathrm{div}(\sigma_0 \nabla u)$ and $g := -\mathrm{div}(\tilde\sigma \nabla v)$ and we consider the corresponding solutions u_ε and v_ε of (2.16). Since u coincides with the solution u_0 of (3.12) and v coincides with the solution v_0 of (3.20), the H-convergence of σ_ε entails

$$u_\varepsilon \rightharpoonup u_0 = u \quad \text{weakly in } H^1_0(\Omega) \quad \text{and} \quad \sigma_\varepsilon \nabla u_\varepsilon \rightharpoonup \sigma_0 \nabla u_0 = \sigma_0 \nabla u \quad \text{weakly in } L^2(\Omega;\mathbb{R}^n),$$

while the H-convergence of (σ_ε^T) yields

$$v_\varepsilon \rightharpoonup v_0 = v \quad \text{weakly in } H^1_0(\Omega) \quad \text{and} \quad \sigma_\varepsilon^T \nabla v_\varepsilon \rightharpoonup \tilde\sigma \nabla v_0 = \tilde\sigma \nabla v \quad \text{weakly in } L^2(\Omega;\mathbb{R}^n).$$

Let $\varphi \in C^\infty_c(\Omega)$; using φv_ε as test function in the equation for u_ε, we get

$$\int_\Omega f(\varphi v_\varepsilon)\,dx = \int_\Omega (\sigma_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon)\varphi\,dx + \int_\Omega \sigma_\varepsilon \nabla u_\varepsilon \cdot v_\varepsilon \nabla\varphi\,dx.$$

Therefore, appealing to the strong $L^2(\Omega)$ convergence of v_ε to v and using φv as a test function in (3.12), we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_\Omega (\sigma_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon)\varphi\,dx &= \int_\Omega f(\varphi v)\,dx - \int_\Omega \sigma_0 \nabla u \cdot v \nabla\varphi\,dx \\ &= \int_\Omega \sigma_0 \nabla u \cdot \nabla(\varphi v)\,dx - \int_\Omega \sigma_0 \nabla u \cdot v\nabla\varphi\,dx = \int_\Omega (\sigma_0 \nabla u \cdot \nabla v)\varphi\,dx. \end{aligned} \tag{3.21}$$

Moreover, arguing in a similar way, using now φu_ε as test function in the equation for v_ε, it is easy to show that

$$\lim_{\varepsilon\to 0} \int_\Omega (\sigma_\varepsilon^T \nabla v_\varepsilon \cdot \nabla u_\varepsilon)\varphi\,dx = \int_\Omega (\tilde\sigma \nabla v \cdot \nabla u)\varphi\,dx. \tag{3.22}$$

lim2

Then (3.21) and (3.22) yield

$$\int_\Omega (\sigma_0 \nabla u \cdot \nabla v)\varphi\,dx = \int_\Omega (\tilde\sigma \nabla u \cdot \nabla v)\varphi\,dx \quad \text{for every } \varphi \in C^\infty_c(\Omega).$$

Arguing as in the previous proof of (2.2) we deduce from this equality that

$$\sigma_0\xi\cdot\eta = \tilde\sigma\eta\cdot\xi \quad \text{a.e. in } \Omega,$$

for every $\xi,\eta\in\mathbb{R}^n$. This implies that $\tilde\sigma=\sigma_0^T$ a.e. in Ω which concludes the proof of the theorem. □

Given $\sigma_0\in\mathcal{M}(c_0,c_1,\Omega)$, the matrix Σ_0 and the functionals Q_0, F_0, and $F_0^{\lambda,\mu}$ are defined as in (2.6), (2.10), (2.12), and (2.13) with $\sigma=\sigma_0$.

Theorem 3.2. *Let (σ_ε) be a sequence in $\mathcal{M}(c_0,c_1,\Omega)$ and let $\sigma_0\in\mathcal{M}(c_0,c_1,\Omega)$. The following conditions are equivalent:*

(a) *σ_ε H-converges to σ_0;*
(b) *σ_ε^T H-converges to σ_0^T;*
(c) *F_ε $\Gamma(d)$-converges to F_0;*
(d) *$F_\varepsilon^{\lambda,\mu}$ $\Gamma(w)$ to $F_0^{\lambda,\mu}$ for every $\lambda,\mu\in H^{-1}(\Omega)$.*

Proof. The equivalence between (a) and (b) follows immediately from Theorem 3.1. The implication (c) $\Rightarrow$ (d) is given by Theorem 2.2. The implication (d) $\Rightarrow$ (a) is obtained in the proof of Theorem 3.1. It remains to prove that (a) and (b) imply (c). By Theorem 2.1 we may assume that F_ε $\Gamma(d)$-converges to $F_{\mathbf{M}}$ where $\mathbf{M}\in L^\infty(\Omega;\mathbb{R}^{2n\times 2n})$ is a positive definite, symmetric matrix satisfying the coercivity condition (2.4).

To prove that $\mathbf{M}\in SO(n,n)$ we consider the block decomposition (2.9). In Theorem 3.1 we proved that $\sigma_0=A^{-1}-A^{-1}B$ and $\sigma_0^T=\tilde\sigma=A^{-1}+A^{-1}B$; hence, we immediately deduce that

$$AB^T+BA=0 \quad \text{a.e. in } \Omega. \tag{3.23}$$

It remains to prove the second condition in (2.8). Let us fix $f,g\in H^{-1}(\Omega)$ and let $u_\varepsilon, v_\varepsilon, a_\varepsilon, b_\varepsilon$ be as in (2.16) and (2.17). By (2.11), (2.15), (3.4), and (3.5) using only the second component we get

$$B_\varepsilon^T(a_\varepsilon+b_\varepsilon)+C_\varepsilon(u_\varepsilon-v_\varepsilon)=\sigma_\varepsilon\nabla u_\varepsilon-\sigma_\varepsilon^T\nabla v_\varepsilon \rightharpoonup B^T(a_0+b_0)+C(\nabla u_0-\nabla v_0) \tag{3.24}$$

$$B_\varepsilon^T(a_\varepsilon-b_\varepsilon)+C_\varepsilon(u_\varepsilon+v_\varepsilon)=\sigma_\varepsilon\nabla u_\varepsilon+\sigma_\varepsilon^T\nabla v_\varepsilon \rightharpoonup B^T(a_0-b_0)+C(\nabla u_0-\nabla v_0) \tag{3.25}$$

weakly in $L^2(\Omega;\mathbb{R}^n)$. Then, adding up (3.24) and (3.25) we get

$$\sigma_\varepsilon\nabla u_\varepsilon \rightharpoonup B^Ta_0+C\nabla u_0$$

weakly in $L^2(\Omega;\mathbb{R}^n)$; on the other hand, since $\sigma_\varepsilon\nabla u_\varepsilon=a_\varepsilon\rightharpoonup a_0$ weakly in $L^2(\Omega;\mathbb{R}^n)$, we obtain

$$a_0=B^Ta_0+C\nabla u_0.$$

Since in the proof of Theorem 3.1 we already showed that $a_0=(A^{-1}-A^{-1}B)\nabla u_0$, we finally obtain

$$(I-B^T)(A^{-1}-A^{-1}B)\,\nabla u_0=C\,\nabla u_0 \quad \text{a.e. in } \Omega.$$

Therefore, suitably choosing f as in (3.13) and arguing as in the proof of Theorem 3.1 we can easily deduce that

$$(I-B^T)(A^{-1}-A^{-1}B)\,\xi=C\,\xi \quad \text{a.e. in } \Omega, \text{ for every } \xi\in\mathbb{R}^n,$$

thus, by the arbitrariness of $\xi\in\mathbb{R}^n$, we get

$$(I-B^T)(A^{-1}-A^{-1}B)=C \quad \text{a.e. in } \Omega.$$

The latter combined with (3.23) leads to

$$AC+B^2=I \quad \text{a.e. in } \Omega. \tag{3.26}$$

Eventually, by (3.23) and (3.26) we can apply [2, Proposition 3.1] and we deduce that $\mathbf{M}\in SO(n,n)$ a.e. in Ω and that $\mathbf{M}$ is equal to the matrix $\mathbf{\Sigma}$ defined in (2.3) with $\sigma=A^{-1}-A^{-1}B$. Since we have also $\sigma_0=A^{-1}-A^{-1}B$, we conclude that $\mathbf{M}=\mathbf{\Sigma}_0$. □

Acknowledgments. This material is based on work supported by the Italian Ministry of Education, University, and Research under the Project "Variational Problems with Multiple Scales" 2008 and by the European Research Council under Grant No. 290888 "Quasistatic and Dynamic Evolution Problems in Plasticity and Fracture".

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