On Hyperbolic Multi-Monopoles

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1931 Monopoles proposed by Dirac seeking singular solution to Maxwell's equations

Maxwell's Equations: $d=3$, $G=U(1)$

$d{*}F=0$, $dF=0$, $\Delta\varphi=0$

$F=dA$, $A=$ magnetic potential

$\varphi=$ electric potential

$\Rightarrow 0=\text{flux}=\int_S F=0$, $S=$ any closed surface

Maxwell has no solutions which vanish at ∞ in $\mathbb{R}^3$ except $\varphi=0$, $F=0$

<u>Dirac</u> <u>Equations</u>: add magnetic charge
$q_m = \delta$ (point charge at origin)

$$\boxed{d{*}F = 0,\quad dF = (4\pi){*}q_m,\quad \Delta\varphi = 0}$$

<u>Note</u>: $dF \neq 0$ forbids existence of global A

Solution of Dirac Equations:

$F = \sin\theta\, d\theta\, d\chi$, $A = (1-\cos\theta)\, d\chi$

${*}F = \frac{1}{r^2}\, dr$, $|F|^2 = \frac{1}{r^4}$

$\Rightarrow$ $F =$ <u>closed</u> 2-form (not <u>exact</u>)

$\int_S F = 2\pi$ $\qquad \int_{R^3} |F|^2\, dV = \infty$

$\boxed{\text{Energy not finite}}$

Replace $U(1)$ by $G = SU(2)$

non-Abelian gauge theory, non-singular

THE R^3- MODEL

A = connection 1-form
φ = Higgs field

$G = SU(2)$
A & φ are $su(2)$ value

$$D_A = d + [A, \cdot\,]$$

$$F = F_A = dA + \tfrac{1}{2}[A, A]$$

Energy $\mathcal{A}(A, \varphi) = \frac{1}{2}\|F\|_2^2 + \frac{1}{2}\|D_A \varphi\|_2^2$

Critical points satisfy 2nd order equations

Monopole Number = N (top. inv.)

$$4\pi N = \int_{\mathbb{R}^3} \mathrm{tr}(F \wedge D_A\varphi)$$

Bogomolnyi $\Rightarrow$ $\mathcal{a}(A,\varphi) = \frac{1}{2}\|F \mp D_A\varphi\|^2 + 4\pi N$

$$\boxed{(1) \quad *F = \pm D_A\varphi}$$

satisfied by minima

Theorem: (Taubes) $\mathbb{R}^3$, $G = SU(2)$, $\exists$ smooth solution of (1) with monopole no. N and

$$\lim_{|x| \to \infty} |\varphi(x)| = 1.$$

Question: M = Riemannian 3-manifold

When does (1) have interesting solutions?

Non-zero magnetic charge?

(Case 1) M compact, $F_A = * D_A\varphi$, $D_A F = 0$

$\Rightarrow D_A^* D_A \varphi = \nabla_A^2 \varphi = 0$

$\Rightarrow \Delta |\varphi|^2 \geqslant 0$

M compact $\Rightarrow$ $|\varphi| = \text{constant}$

Basic Identities

(a) $d(\sigma, \eta) = (d_A\sigma, \eta) + (\sigma, d_A\eta)$

(b) $*(\omega, \eta) = (\omega, *\eta)$

$\Rightarrow \quad 0 = \Delta|\varphi|^2 = 2(D_A\varphi, D_A\varphi) + 2(\varphi, \underbrace{\cancel{D_A^* D_A\varphi}}_{0})$

$\Rightarrow |D_A\varphi| = 0 \Rightarrow F_A = 0$

M cannot be compact

Possibilities of Interest:

$\mathbb{R}^3$, $\mathbb{H}^3$ or

M has Euclidean Ends (Floer, Ernst, Duienard)

or Hyperbolic Ends

Monopoles on $\mathbb{H}^3$

$c = (\varphi, A)$ smooth

$m = \lim_{|x| \to \infty} |\varphi(x)|$ = mass of monopole

magnetic Charge $= k = \frac{1}{4\pi m} \int_{\mathbb{H}^3} tr(F_A \wedge d_A \varphi)$

Minima $\boxed{d_A \varphi = * F_A}$ (1)

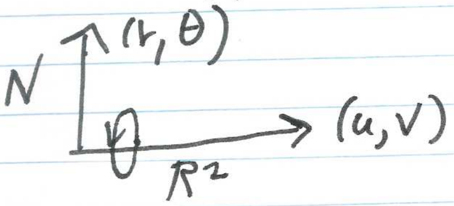
k = integer, $m > 0$ arbitrary

Atiyah - Manton Construction

Monopoles on $\mathbb{H}^3$ = Instantons on $\mathbb{R}^4 \setminus \mathbb{R}^2$

Conformal Equivalence:

$$r^2\Big\{\underbrace{\frac{du^2+dv^2+dr^2}{r^2}}_{\mathbb{H}^3} + \underbrace{d\theta^2}_{S^1}\Big\} \sim \underbrace{ds^2}_{\mathbb{R}^4\setminus\mathbb{R}^2}$$



Instanton on $\mathbb{R}^4$, $F = *F$ ind of θ

$$\Downarrow$$

Monopole on $\mathbb{H}^3$, F', $\varphi = A_\theta$, $D\varphi = *F'$

Finite Energy Preserved

$$c_2 = \frac{1}{8\pi^2} \int_{R^4} |F|^2 dV^4$$

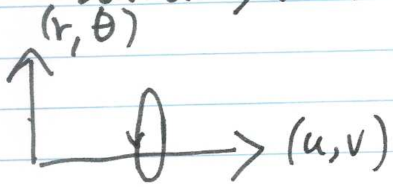
$$= \frac{1}{4\pi} \int_{H^3} (|F'|^2 + |D\varphi|^2) dV^3$$

$$= \frac{2}{4\pi} \int_{H^3} |F'|^2 dV^3$$

k = magnetic charge = winding no of φ
= no. zeros of φ
= integer

m = lim $|\varphi|$ at ∞ = holonomy on $R^4 \setminus R^2$

What is holonomy ?



bw $R^4 \setminus R^2$,

$$D = d + A$$

$$A = A_u\, du + \cdots + A_\theta\, d\theta$$

To gauge A_θ to zero, $g^{-1} A g = A + g^{-1} dg$

$\Rightarrow$ <u>IVP</u>: $\dfrac{dg}{d\theta} + A_\theta g = 0 \quad g(0) = I$

$$g(2\pi) = I$$

$$g : S^1 \longrightarrow SU(2)$$

has solution ??

IN GENERAL,

$$g(2\pi) = J \in SU(2)$$

<u>Not necessarily</u> I

DEFINITION: holonomy of connection A

$= [J] =$ conjugacy class of J in $SU(2)$

Properties

(1) $[J]$ gauge invariant; if A_θ^1 gauge equivalent to A_θ^2 then $[J^1] = [J^2]$

(2) $A \in L^2_{1,loc}(R^4 \setminus R^2)$, $F \in L^2(R^4)$

$\lim_{r \to 0} [J(r)] = [J]$ exists

& independent of point on R^2

Bundle FLAT if $F \equiv 0$

(3) Flat $SU(2)$ bundles $\overset{1-1}{=} [J] \in SU(2)$ over Y, $\pi_1(Y) = Z$

(4) $[J] \sim \begin{pmatrix} e^{-2\pi i m} & 0 \\ 0 & e^{2\pi i m} \end{pmatrix}$

$\operatorname{tr} J = \cos 2\pi m, \quad 0 \le m < 1$

(5) Prototype of flat connection

$$A^b = \begin{pmatrix} im & 0 \\ 0 & -im \end{pmatrix} d\theta = m\hat{i}\, d\theta$$

Solution of IVP with $A = m\hat{i}\, d\theta$

$$g(\theta) = \begin{pmatrix} e^{-im\theta} & 0 \\ 0 & e^{im\theta} \end{pmatrix} \in SU(2)$$

(6) $m_1 = m_2 + n$, n integer, then

A_1^b, A_2^b gauge equiv.

(7) $D = d + A$, A flat, holonomy m

$\Longrightarrow$ $\exists$ gauge in which $D = d + A^b$

(8) If m integer, D gauge equiv to d

<u>Classification Theorem:</u> (LMS, RJS)

If $\|F\|_2 < \infty$, $\exists m$, $A^b = m i\, d\theta$

$D = d + A$, with $A - A^b \in L_1^2(\mathbb{R}^4)$

$$\|A - A^b\|_{L_1^2} \le C \|F\|_{L^2}$$

(9) $\| |\varphi| - m \|_{L^4(\mathbb{H}^3)} \le C' \|A - A^b\|_{L_1^2(S^4)} \le C'' \|F\|_{L^2(\mathbb{H}^3)}$

$\Longrightarrow \lim_{|x| \to \infty} |\varphi| = m$

(10) m = integer $\Longrightarrow$ A extends from $\mathbb{R}^4 \setminus \mathbb{R}^2$ to all of $\mathbb{R}^4$

(11) Instanton over $\mathbb{R}^4$ independent of θ

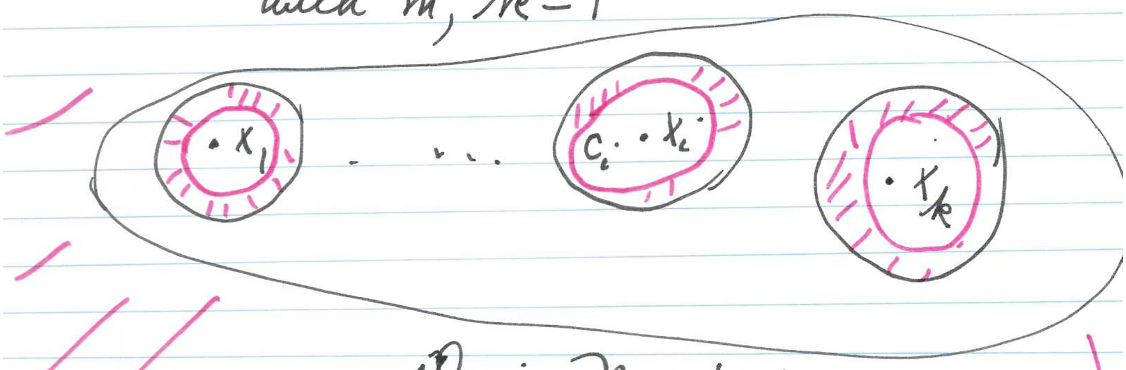
$\longleftrightarrow$ monopole over $\mathbb{H}^3$

$m =$ integer

<u>MAIN RESULT</u> (LMS, RJS) prescribe $k =$ integer, and $m > 0$. $\exists$ smooth $c = (\varphi, A)$ satisfying (1) with charge k and mass $m > 0$.

<u>SKETCH OF PROOF</u> (following Taubes on $\mathbb{R}^3$)

Explicit Chakrabarty monopole c with m, $k = 1$



Dirac Monopole

Glue together $c_0 = (\varphi_0, A_0)$

C_0 is an approximate monopole

Look for solutions of form $C = C_0 + \eta$

$$Lc = d_A\varphi - * F_A = Lc_0 + \cdots$$

$Lc = 0 \iff \mathcal{L}\eta = \Delta\eta +$ lower order terms $= G_0$

Need $\mathcal{L}$ invertible $\iff \Delta$ invertible on 1-forms

but on 1-forms on L^2, $\sigma(\Delta) = (0, \infty)$.

Weighted Spaces

$$\|\omega\|_{2,\beta} = \left(\int_{H^3} |\omega|^2 \cosh^2 \beta r \, dV \right)^{1/2}$$

$$a(c) = \frac{1}{2} \int_{H^3} (|F_A|^2 + |d_A\varphi|^2) \cosh^2 \beta r \, dV$$

Then, $\sigma_{ess}(\Delta) = [\beta^2, \infty)$

Are there eigenvalues ???

Theorem (Mazzeo): Under appropriate conditions

$$t^3 \int e^{2tr} |u|^2 dV + t^2 \int e^{2tr} |\nabla u|^2 dV \leq C \int e^{2tr} |\nabla^* \nabla u|^2$$

Using the theorem

$$\mathcal{L}\eta = \lambda \eta \Rightarrow \nabla^* \nabla \eta + \Phi \eta = \lambda \eta$$

$$\text{or } |\nabla^* \nabla \eta| \leq C(|\eta| + |\nabla \eta|)$$

$$\Rightarrow \eta \equiv 0$$

$\Rightarrow$ NO EIGENVALUES

$$\Rightarrow \sigma(\mathcal{L}) = \sigma_{ess}(\mathcal{L})$$

$\sigma_{ess}(\mathcal{L}) \subseteq [\beta^2, \infty)$

By Weyl's lemma,

$$\sigma_{ess}(\mathcal{L}) = \sigma_{ess}(\Delta) \subseteq [\beta^2, \infty)$$

Last result obtained by

"separation of variables"

By Raleigh Quotient Theorem:

$\mathcal{L}$ is invertible and satisfies

$$(\mathcal{L}f, f) \geq c\|f\|^2 \quad , c > 0$$

on some Hilbert space $\mathcal{H}$