Bilinear Control Systems: Theory and Applications

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Aim: Bilinear Systems are an important class of nonlinear control systems. The course aims at giving an overview of the main control problems and of some of the mathematical tools (notably differential geometric and Lie algebraic methods) required in the study of bilinear control systems.

Topics:

1. Introductory material
   - manifolds, vector fields, tangent spaces;
   - orbits of vector fields and Frobenius Theorem;
   - controllablity and Chow Theorem;
   - drift versus driftless systems, accessibility versus controllability;
2. Bilinear control systems
   - bilinear systems and matrix transition Lie groups;
   - structure of matrix Lie groups (homogeneous spaces, transitivity, exponential map and canonical coordinates);
   - Lie algebras (Levi decomposition, semisimpicity, solvability, nilpotency, Cartan criteria);
   - controllability properties for bilinear control systems on matrix Lie groups;
3. Control methods
   - feedback linearization;
   - system inversion and differential flatness;
   - feedback stabilization;
4. Applications
   - rigid body motion (rigid bodies on SO(3) and SE(3); system on a sphere);
   - nonholonomic systems (trailer systems, chained form);
   - switching systems (simultaneous stability);
   - quantum control systems (Shrodinger equation, Liouville equation).

References:

There is no book specific for the geometric aspect of bilinear control systems that will be treated in the course. Some parts can be found in

- Murray-Li-Sastry. *A mathematical Introduction to Robotic Manipulation* CRC press 1994. (Controllability; nonholonomic systems; rigid body motion)

Control system:

$$\dot{x} = f(x, u)$$

- autonomous
- $x \in M$ = smooth (real analytic) manifold
- $u \in \mathcal{U}$ = class of admissible control functions
  - * bounded measurable
  - * piecewise constant
  - * smooth

$f : M \times \mathcal{U} \to TM$ is called a (input-parametrized) vector field $f \in \mathcal{X}(M)$

- examples of manifolds

1) $M = \mathbb{R}^n$

2) sphere in $\mathbb{R}^n$ or $\mathbb{C}^n$

$$S^{n-1} = \{ x \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 = 1 \}$$ unit sphere centered at $x = 0$

3) ball in $\mathbb{R}^n$: $\mathbb{B}^n = \{ x \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 \leq 1 \}$

$\longrightarrow$ manifold with boundary

- * $\dim(\mathrm{int}(\mathbb{B}^n)) = n$
- * $\dim(\partial \mathbb{B}^n) = n-1$

4) group $GL(n) = \{ X \in M^{n \times n} \mid \det(X) \neq 0 \}$

group = set of elements with an operation (multiplication)

$$G \times G \to G \qquad X_1 \; X_2 \mapsto X_1 X_2$$ endowed with

- identity element I
- inverse element $X \in G \Rightarrow X^{-1} \in G$

manifolds: set with an atlas of charts of local coordinates

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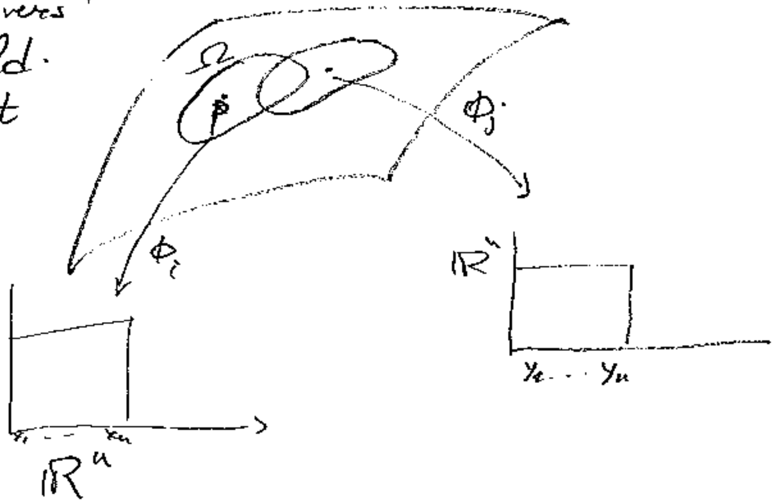
- Atlas of charts "covers" the entire manifold and locally maps it to $\mathbb{R}^n$
  → local coordinates (local bijection)



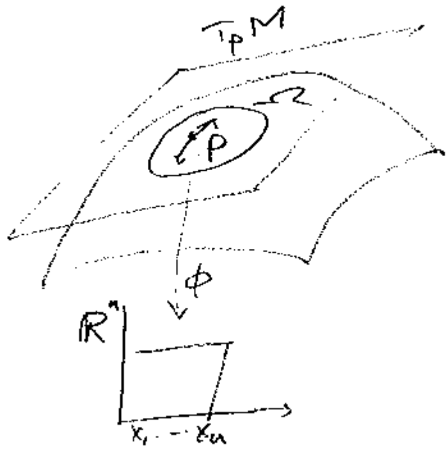
- charts are "compatible" in the overlaps

- charts are smooth (C^∞) or real analytic
- charts are countably many
- charts can be used to define a differentiable structure on the manifold: $f: M \to \mathbb{R}$ is differentiable if $\forall$ charts (Ω, ϕ) $f \circ \phi^{-1}: \mathbb{R}^n \to \mathbb{R}$ is differentiable

⇒ <u>differentiable manifold</u>

⇒ <u>tangent space</u> at $p \in M$ = collection of all tangent vector
= collection of all derivations at p

⇒ tang. sp. T_pM is a vector space



$(x_1 \dots x_n)$ local coord. at $p \in M$

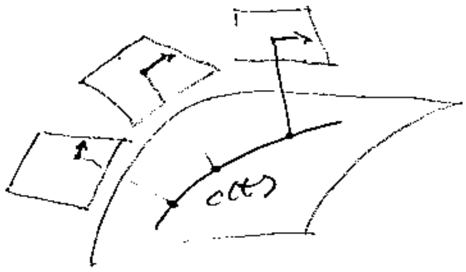
⇒ $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ local coord. induced on T_pM

⇒ $X_p \in T_pM$ s.t. $X_p = X_1 \frac{\partial}{\partial x_1} + \dots + X_n \frac{\partial}{\partial x_n}$

<u>tangent bundle</u> $TM = \bigcup_{p \in M} T_pM$ is a collection of tang. spaces of dim $2n$ and elements (p, T_pM)

$\Rightarrow$ <u>vector field</u> = smooth map $X: M \to TM$

that we use to represent ODEs on manifolds



$$C: I(\subseteq \mathbb{R}) \to M$$
$$t \mapsto c(t)$$

$c: I: M$ is an <u>integral curve</u> of the v.f. X

if $\frac{dc(t)}{dt} = X(c(t)) \quad (= X \circ c(t))$

A v.f. is <u>complete</u> if. $I = \mathbb{R}$ (i.e. $t \in \mathbb{R}$)

$\Rightarrow$ flow of X $\quad \Phi^X: \mathbb{R} \times M \to M$ $\quad$ form a group w.r.t. composition of mappings
$\quad t \times p \mapsto \Phi^X(t, p)$

$\Rightarrow$ one-parameter subgroup

If v.f. X is not complete (e.g. $t \in [0, +\infty)$)

$\Rightarrow \Phi^X: \mathbb{R}^+ \times M \to M \Rightarrow$ one-param. <u>semigroup</u>

example: linear vector field : $X = Ax$ $\quad A \in M^{n \times n}$, $x \in \mathbb{R}^n$

$\Rightarrow$ a single chart suffice (there might be constraints to represent M)

$$X = \sum X_i \frac{\partial}{\partial x_i} \quad \text{with} \quad \frac{\partial}{\partial x_i} \to \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{i-th component.}$$

$$\overset{!}{=} X_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + X_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + X_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum a_{1j} x_j \\ \sum a_{2j} x_j \\ \vdots \\ \sum a_{nj} x_j \end{bmatrix} = \sum a_{1j} x_j \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \sum a_{2j} x_j \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + \sum a_{nj} x_j \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \underline{X} = Ax \iff X_i = \sum a_{ij} x_j$$

Operations with vector fields

$X \in \mathcal{X}(M) \qquad X : M \longrightarrow TM$

$$p \longmapsto (p, X_p) = \Big(p, \underbrace{\sum_{i=1}^{n} X_i(p) \frac{\partial}{\partial x_i}}_{\text{tangent vector at } p}\Big)$$

$\varphi \in C^\infty(M) \qquad \varphi : M \to \mathbb{R} \qquad$ function

1) Lie derivative of a function w.r.t. a v.f.

= directional derivative of a function (in a point)

$$X(\varphi)(p) \triangleq L_X \varphi(p) : M \longrightarrow \mathbb{R}$$
$$p \longmapsto X_p(\varphi)$$

in local coord. $\quad x = (x_1, \dots x_n) = \phi(p)$

$$X(\varphi)(p) = X_p(\varphi) = \sum_{i=1}^{n} X_i(p) \frac{\partial \varphi}{\partial x_i}$$

2) Lie bracket $\qquad$ given $X, Y \in \mathcal{X}(M)$ and a test function $\varphi \in C^\infty(M)$

Lie bracket is the bilinear map

$$[X, Y] : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$

defined as

$$[X, Y](\varphi) \triangleq X(Y(\varphi)) - Y(X(\varphi))$$

in local coord. $\quad X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i} \qquad Y = \sum_{i=1}^{n} Y_i \frac{\partial}{\partial x_i}$

$$[X, Y] = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \left(X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i} \right) \frac{\partial}{\partial x_j} \right)$$

Proof in the local coordinates $X = \sum_{i=1}^{n} X_i(p) \frac{\partial}{\partial x_i}$

$$Y = \sum_{i=1}^{n} Y_i(p) \frac{\partial}{\partial x_i}$$

$$[X,Y](\varphi) = \sum_{i=1}^{n} \left(X_i \frac{\partial}{\partial x_i} \sum_{j=1}^{n} Y_j \frac{\partial \varphi}{\partial x_j} \right) - \sum_{i=1}^{n} Y_i \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} X_j \frac{\partial \varphi}{\partial x_j} \right)$$

$$= \sum_{i=1}^{n} \left(X_i \sum_{j=1}^{n} \left(\frac{\partial Y_j}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + \cancel{Y_j \frac{\partial^2 \varphi}{\partial x_i \partial x_j}} \right) - \right.$$

$$\left. - Y_i \sum_{j=1}^{n} \left(\frac{\partial X_j}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + \cancel{X_j \frac{\partial^2 \varphi}{\partial x_i \partial x_j}} \right) \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\left(X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i} \right) \frac{\partial \varphi}{\partial x_j} \right] \quad //$$

two v.f. X, Y <u>commute</u> if $[X,Y] = 0$

<u>example</u>: $X = Ax$ $\quad Y = Bx$ linear v.f.

$$[Ax, Bx] = ?$$

recall $\quad X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i} = X_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + X_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$

$$Ax = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj} x_j \end{bmatrix}$$

$\Rightarrow X_i = \sum_{j=1}^{n} a_{ij} x_j$ components of v.f. in local coordinates

$$\Rightarrow \frac{\partial X_j}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\sum_{k=1}^{n} a_{jk} x_k \right) = a_{ji}$$

$$\Rightarrow [Ax, Bx] = [A, B]x = \underbrace{(AB - BA)}_{\text{matrix commutator}} x$$

Lie algebra

<u>def</u> A vector space V (over the field $\mathbb{R}$) is a <u>Lie algebra</u> if it is endowed with a bilinear operation

$$[\cdot,\cdot] : V \times V \to V \quad \text{s.t.}$$

1) $[\alpha_1 v_1 + \alpha_2 v_2, w] = \alpha_1 [v_1, w] + \alpha_2 [v_2, w] \quad \forall v_1, v_2, w \in V, \ \forall \alpha_1, \alpha_2 \in \mathbb{R}$ (bilinearity)

2) $[v, w] = -[w, v]$ (skew-symmetry)

3) $[v,[w,z]] + [z,[v,w]] + [w,[z,v]] = 0 \quad \forall v, w, z \in V$ (Jacobi identity)

<u>def</u> A <u>subalgebra</u> of the Lie algebra V is a linear subspace $W \subseteq V$ s.t. $[v,w] \in W \ \forall v, w \in W$

<u>example</u> Lie algebra of $n \times n$ matrices $\Rightarrow$ operation is the matrix commutator $[A,B] = A\cdot B - B\cdot A$
$\forall A, B \in M_n$

<u>example</u>: $\mathcal{X}(M)$ v. sp. of vector fields over M is an ∞-dimens. Lie algebra.

Meaning of the Lie bracket

- consider two complete v.f. $X, Y \in \mathfrak{X}(M)$
- consider the associated flow

$$\Phi^X : \mathbb{R} \times M \longrightarrow M$$
$$(t, p) \longmapsto \Phi^X(t,p) = \Phi^X_t(p)$$

$$\Rightarrow X = \frac{d}{dt} \Phi^X_t(p)\Big|_{p=0} \quad = \text{infinitesimal generator of the flow (Lie derivative)}$$

then

$$[X, Y] = \frac{\partial^2}{\partial t \partial s}\left(\Phi^{-Y}_s \circ \Phi^{-X}_t \circ \Phi^Y_s \circ \Phi^X_t\right)\Big|_{t=s=0}$$

2nd order term of the expansion of the flow composition

proof: 1) flow along X for some time ϵ (small)

$\Rightarrow$ integral curve $\dot{c}(t) = X(c(t))$

$$\Rightarrow \ddot{c}(t) = \frac{\partial X}{\partial c}\dot{c} = \frac{\partial X}{\partial c} X(c(t))$$

$\Rightarrow$ local solution for a small time ϵ from $c(0) = p_0 \in M$

$$c(\epsilon) = \Phi^X_\epsilon(c(0)) \simeq \underbrace{c(0)}_{p_0} + \epsilon \dot{c}(0) + \frac{\epsilon^2}{2}\ddot{c}(0) + \mathcal{O}(\epsilon^3)$$
$$= p_0 + \epsilon X(p_0) + \frac{\epsilon^2}{2}\frac{\partial X}{\partial c} X + \mathcal{O}(\epsilon^3)$$

2) then flow along Y for another time ϵ

$$\Rightarrow c(2\epsilon) = \Phi^Y_\epsilon \circ \Phi^X_\epsilon(p_0) = \Phi^Y_\epsilon\left(p_0 + \epsilon X(p_0) + \frac{\epsilon^2}{2}\frac{\partial X}{\partial c}X + \mathcal{O}\right)$$
$$= p_0 + \epsilon X(p_0) + \frac{\epsilon^2}{2}\frac{\partial X}{\partial c}X + \epsilon Y(p_0 + \epsilon X(p_0)) + \frac{\epsilon^2}{2}\frac{\partial Y}{\partial c}Y(p_0) + \mathcal{O}$$

3) flow again along X in the opposite direction $\Rightarrow -X$

(can do because v.f. are complete)

$$c(3\epsilon) = \Phi^{-X}_{\epsilon} \circ \Phi^{Y}_{\epsilon} \circ \Phi^{X}_{\epsilon} = \Phi^{-X}_{\epsilon}\big(c(2\epsilon)\big) = c(2\epsilon) - \epsilon X\big(c(2\epsilon)\big) - \frac{\epsilon^2}{2}\frac{\partial X}{\partial c}\, c^{\circ}$$

$$-X$$

$$= p_0 + \epsilon\big(X(p_0) + Y(p_0)\big) + \epsilon^2\Big(\frac{1}{2}\frac{\partial X}{\partial c}X(p_0) + \underbrace{Y\big(X(p_0)\big)}_{=\frac{\partial Y}{\partial c}X(p_0)} + \frac{1}{2}\frac{\partial Y}{\partial c}Y(p_0)\Big)$$

$$-\epsilon X(p_0) - \epsilon^2\Big(X\big(X(p_0)\big) + \underbrace{X\big(Y(p_0)\big)}_{\frac{\partial X}{\partial c}Y} + \frac{\epsilon^2}{2}\frac{\partial X}{\partial c}X\Big)$$

$$= p_0 + \epsilon\big(\cancel{X(p_0)} + Y(p_0) - \cancel{X(p_0)}\big) + \epsilon^2\Big[\cancel{\frac{1}{2}\frac{\partial X}{\partial c}X} + \frac{\partial Y}{\partial c}X + \frac{1}{2}\frac{\partial Y}{\partial c}Y$$

$$-\cancel{\frac{\partial X}{\partial c}X} - \frac{\partial X}{\partial c}Y + \cancel{\frac{1}{2}\frac{\partial X}{\partial c}X}\Big]$$

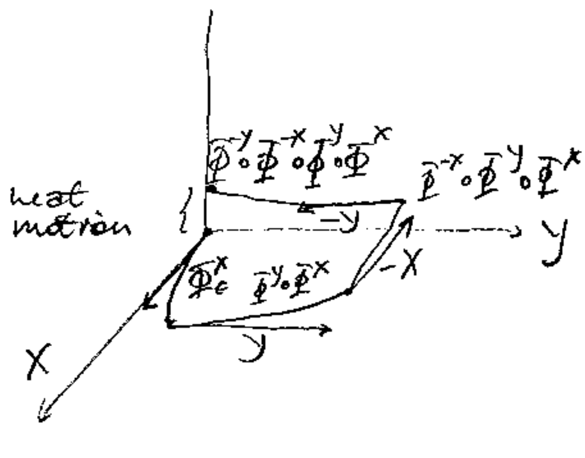
$$c(4\epsilon) = \Phi^{-Y}_{\epsilon} \circ \Phi^{-X}_{\epsilon} \circ \Phi^{Y}_{\epsilon} \circ \Phi^{X}_{\epsilon} = \Phi^{-Y}_{\epsilon}\big(c(3\epsilon)\big)$$

$$= p_0 + \epsilon Y(p_0) + \epsilon^2\Big[\frac{\partial Y}{\partial c}X + \frac{1}{2}\frac{\partial Y}{\partial c}Y - \frac{\partial X}{\partial c}Y\Big]$$

$$-\epsilon Y\big(p_0 + \epsilon Y(p_0)\big) + \frac{\epsilon^2}{2}\frac{\partial Y}{\partial c}Y$$

$$= p_0 + \epsilon\big(\cancel{Y(p_0)} - \cancel{Y(p_0)}\big) + \epsilon^2\Big[\frac{\partial Y}{\partial c}X + \cancel{\frac{1}{2}\frac{\partial Y}{\partial c}Y} - \frac{\partial X}{\partial c}Y - \cancel{\frac{\partial Y}{\partial c}Y} + \cancel{\frac{1}{2}\frac{\partial Y}{\partial c}Y}\Big]$$

$$= p_0 + \epsilon^2\Big(\frac{\partial Y}{\partial c}X - \frac{\partial X}{\partial c}Y\Big) + o(\epsilon^3) = p_0 + \epsilon^2[X, Y] + o(\epsilon^3)$$



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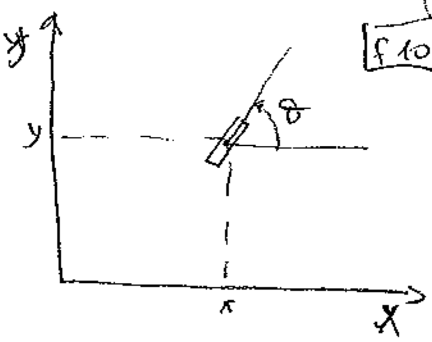
Example rolling wheel



$M = \mathbb{R}^2 \times S^1 \ni (x\ y\ \theta) = z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

$$\begin{cases} \dot{x} = v \cos\theta \\ \dot{y} = v \sin\theta \\ \dot{\theta} = \omega \end{cases}$$

$T_p M \ni \left(\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial \theta}\right)$

$\omega, v, \in \mathbb{R}$ controls

$$\dot{z} = v \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} + \omega \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = X \cdot v + Y\omega$$

$$X = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$$

$$Y = \frac{\partial}{\partial \theta}$$

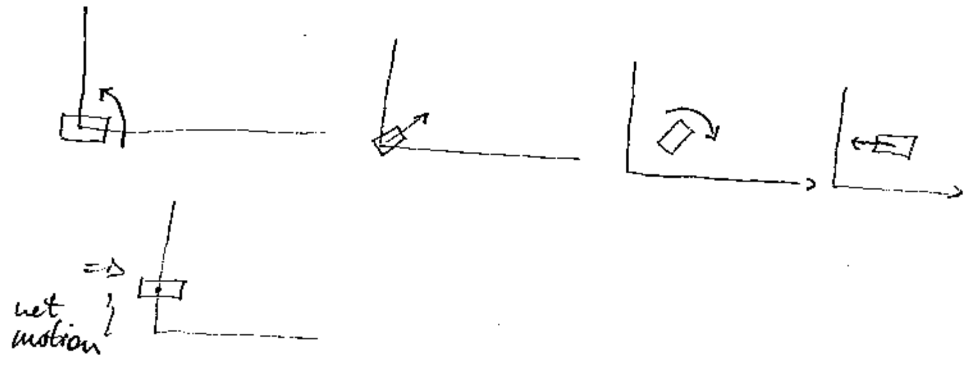
rank $\{X, Y\} = 2$

$$[X, Y] = \sum_{j=1}^{3}\left(\sum_{i=1}^{3}\left(X_i \frac{\partial Y_j}{\partial z_i} - Y_i \frac{\partial X_j}{\partial z_i}\right)\frac{\partial}{\partial x_j}\right) =$$

$$= \cancelto{0}{\frac{\partial Y}{\partial z}} X - \frac{\partial X}{\partial z} Y = -\begin{bmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} +\sin\theta \\ -\cos\theta \\ 0 \end{bmatrix} \triangleq Z$$

Z lin indep. from X and Y

flow composition:



DISTRIBUTIONS

def A distribution Δ on M is a map assigning to $p \in M$ a linear subspace of T_pM : $\Delta_p \in T_pM$

def A distribution is smooth if it is spanned at each point p by a set of smooth v.f. $X_1, \dots, X_m \in \mathfrak{X}(M)$

$$\Delta_p = \text{span}\{X_1(p), \dots, X_m(p)\}$$

def A distribution is regular if $\dim \Delta_p = m \quad \forall p \in M$

def A distrib. is involutive if $\forall X_1, X_2 \in \Delta \quad [X_1, X_2] \in \Delta$

def We denote the involutive closure $\bar{\Delta}$ of Δ the smallest involutive distribution containing Δ and closed w.r.t. the Lie bracket

$\Rightarrow \bar{\Delta}$ is a Lie algebra

def A submanifold (embedded, immersed manifold) N of M is an integral manifold of an involutive distribution Δ if

$$\Delta_p = T_pN \quad \forall p \in N$$

def Δ is integrable if $\exists$ an integral manifold thm (Frobenius)

A regular distribution Δ is integrable $\Leftrightarrow$ Δ is involutive

Practical meaning: Δ integrable if $\exists$ $n-m$ smooth functions $h_i : M \to \mathbb{R}$ s.t.

$$\begin{cases} \dfrac{\partial h_i}{\partial p} \text{ lin. indep.} \\ L_X h_i = X(h_i) = \left(\sum_{j=1}^{n} X_j \dfrac{\partial h_i}{\partial x_j} \right) = \dfrac{\partial h_i}{\partial p} X = 0 \quad \forall X \in \Delta \end{cases}$$

$\Rightarrow$ hypersurface given by the level sets

$$H(p) = \{ p \in M \text{ s.t. } h_1(p) = c_1, \dots, h_{n-m}(p) = c_{n-m} \}$$

is an integral manifold and $\Delta_p = T_p H$

- level sets $H(p)$ form a foliation
- each leaf of the foliation is determined by a choice of $\{c_1, \dots, c_{n-m}\}$.

example rolling wheel

$$\left.\begin{cases} X = \cos\theta \dfrac{\partial}{\partial x} + \sin\theta \dfrac{\partial}{\partial y} \\ Y = \dfrac{\partial}{\partial \theta} \\ Z = [X, Y] \end{cases}\right\} \Rightarrow \Delta = \operatorname{span}\{X, Y, Z\} = TM$$

$\Rightarrow$ smallest integrable distribution is the entire tang. bundle.

f.12 bis

example: bilinear driftless system

$$\begin{cases} \dot{x} = Ax u_1 + Bx u_2 \\ x \in \mathbb{R}^3 \end{cases} \quad \text{where } A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ +1 & 0 & 0 \end{bmatrix}$$

$$[A,B] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & +1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = C$$

$$\bar{\Delta}(x) = \mathrm{span}\{Ax, Bx, [A,B]x\} = \mathrm{span}\left\{\begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix}, \begin{bmatrix} -x_3 \\ 0 \\ x_1 \end{bmatrix}, \begin{bmatrix} 0 \\ x_3 \\ -x_2 \end{bmatrix}\right\}$$

$$\dim \bar{\Delta}(x) = \begin{cases} 0 & \text{if } x = 0 \\ 2 & \text{if } x \neq 0 \end{cases}$$

However A, B, C are linearly independent

$\Rightarrow$ matrix Lie algebra $\mathfrak{g}$ becomes involutive when dim $\mathfrak{g}$ is at least 3!

Is Frobenius theorem valid?

Compute how the norm of x changes along the trajectories of the system:

$$\frac{d}{dt}\|x\|^2 = \frac{d}{dt}\langle x, x\rangle = \langle \dot{x}, x\rangle + \langle x, \dot{x}\rangle = 2\langle x, \dot{x}\rangle = 2x^T\dot{x}$$

$$= 2x^T(Axu_1 + Bxu_2) = 2x^T\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}x u_1 + 2x^T\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}x u_2$$

$$= 2x^T\begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix}u_1 + 2x^T\begin{bmatrix} -x_3 \\ 0 \\ x_1 \end{bmatrix}u_2 = 2(x_1x_2 - x_2x_1)u_1 + 2(-x_1x_3 + x_3x_1)u_2 \equiv 0$$

$\Rightarrow$ we have found an integral manifold in $\mathbb{R}^3$: $\frac{d}{dt}\|x\|^2 = 0 \Leftrightarrow \|x\|^2 = \text{const}$

$$S^2 = \{x \in \mathbb{R}^3 \text{ s.t. } \|x\|^2 = \text{const} \geq 0\} = M$$

$\Rightarrow$ system lives on a 2-dimensional manifold in $\mathbb{R}^3$

$\Rightarrow \bar{\Delta}(x) = T_x S$ at each $x \neq 0 \Rightarrow$ Frobenius theorem holds

control affine vector fields

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \qquad f, g_1, \dots, g_m \in \mathcal{X}(M)$$

f = drift

$g_1, \dots, g_m$ = control vector fields

bilinear control systems

$$\dot{x} = Ax + \sum_{i=1}^{m} B_i u_i x \qquad Ax, B_1 x, \dots, B_m x \text{ linear v.f. : } \mathbb{R}^n \to \mathbb{R}^n$$

$A, B_1, \dots, B_m \in M_n(\mathbb{R})$ or $M_n(\mathbb{C})$

linear control systems

$$\dot{x} = Ax + \sum_{i=1}^{m} B_i u_i \qquad A, B_1, \dots, B_m \in M_n(\mathbb{R})$$

CONTROLLABILITY

(*) $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$ $\quad u_i : \mathbb{R} \to U_i \subseteq \mathbb{R}$ $\quad u = (u_1 \dots u_m)$

$u_i \in \mathcal{U}$ ex piecewise constant

- $f, g_1, \dots, g_m \in \mathcal{X}(M)$
- $(x_1 \dots x_n) = x = \phi(p)$
- f = drift
- $g_1 \dots g_m$ input v.f.

underactuated
autonomous
smooth v.f.

def (*) is controllable if $\forall\, x_1, x_2 \in M$ $\exists\, T$ finite and a control function $u : [0,T] \to U$ $u \in \mathcal{U}$

s.t. $\begin{cases} x(0) = x_1 \\ x(T) = x_2 \end{cases}$

Jp autonomous systems i.e. $f(x)$ and $g_1(x) \dots g_n(x)$ do not depend from time.

example for linear systems $\dot{x} = Ax + Bu$ $\quad x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$

controllability $\Rightarrow$ stabilizability
$\Rightarrow$ realization theory
$\Rightarrow$ LQ optimal control

controllable $\Leftrightarrow$ rank $[B \,\vdots\, AB \,\vdots\, \dots \,\vdots\, A^{n-1}B] = n$

i.e. $\mathrm{Im}[B \,\vdots\, AB \,\vdots\, \dots \,\vdots\, A^{n-1}B] = \mathcal{R}$ = reach space has dim n

controllability by linearization

Jacobian linearization of $\dot{x}=f(x)+\sum g_i(x)u_i$ at equil point x_0 (i.e. $f(x_0)=0$) is $\dot{x}=\frac{\partial f}{\partial x}(x_0)x+\sum g_i(x_0)u_i = \bar{A}x+\bar{B}u$

rank $[\bar{B}\ \bar{A}\bar{B} \ldots \bar{A}^{n-1}\bar{B}]=n \Rightarrow (*)$ is controllable locally around x_0 i.e. $\exists$ open neigh. V of x_0 that can be reached in time T by $u\in\mathcal{U}$ —

- controllability via linearization is often unsatisfactory

example rolling wheel

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix}=\begin{bmatrix}\cos\theta\\ \sin\theta\\ 0\end{bmatrix}v+\begin{bmatrix}0\\0\\1\end{bmatrix}\omega = g_1(z)v+g_2(z)\omega$$

- driftless $\Rightarrow f=0$
- equilibrium $z=0$

$$\dot{z}=\begin{bmatrix}1\\0\\0\end{bmatrix}v+\begin{bmatrix}0\\0\\1\end{bmatrix}\omega \qquad \bar{B}=\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}\begin{bmatrix}v\\ \omega\end{bmatrix} \qquad \bar{A}\equiv 0$$

rank $[\bar{B}, \bar{A}\bar{B} \ldots \bar{A}^{n-1}\bar{B}]=2 \Rightarrow$ lineariz. is not controllable

however, from experience, we know that (most) cars are controllable!

$\Rightarrow$ nonlinear controllability

REACHABLE SETS

given $x_0 \in M$ $\qquad \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \qquad (*)$

<u>def</u> • $\mathcal{R}^V(x_0, T)$ = reachable set at time $T > 0$ for $(*)$

$$= \{ x(t) \in M \mid \exists \; u : [0,T] \to U, \; u \in \mathcal{U} \text{ s.t. } \Phi_t(x_0, u(t)) = x(t) \in V \text{ and } \Phi_T(x_0, u(T)) = x(T) \}$$

• $\mathcal{R}^V(x_0, \leq T) = \bigcup_{0 < \tau \leq T} \mathcal{R}^V(x_0, \tau)$ = reachable set in time not greater than T.

STLC

<u>def</u> $(*)$ is <u>small-time locally controllable</u> at x_0 if we can reach nearby points in arbitrary small time while staying near x_0 for all times.

• $(*)$ STLC if $\mathcal{R}^V(x_0, \leq T)$ contains an open neighborhood of x_0 $\forall T$ and $\forall$ neighb.

i.e. if $\begin{cases} \mathcal{R}^V(x_0, \leq T) \text{ contains open neighb. of } M \\ x_0 \in \text{int}(\mathcal{R}^V(x_0, \leq T)) \quad \forall T \end{cases}$

<u>def</u> $(*)$ is <u>locally accessible</u> from x_0 if $\mathcal{R}^V(x_0, \leq T)$ contains nonempty open set of M $\forall V$ and $\forall T > 0$

ACCESSIBILITY

Accessibility algebra ($\triangleq$ smallest subalgebra containing $f, g_1, \ldots, g_m$)

$$C(x) \triangleq Lie(f, g_1, \ldots, g_m) = \text{Lie algebra} \subseteq \mathcal{X}(M)\big|_x$$

as a distribution, $C(x)$ is involutive

<u>thm</u> (<u>LARC</u> : Lie algebraic rank condition)

If $\dim C(x_0) = n \Rightarrow \check{\mathcal{R}}(x_0, \leq T)$ contains a nonempty open set of M (i.e. accessibility holds)

<u>proof</u> (sketch)

$\dim C(x_0) = n \Rightarrow$ by continuity $\forall x \in B(x_0, \epsilon)$ $\dim C(x) = n$

$\Rightarrow \exists$ n independent v.f. $X_1, \ldots, X_n \in \mathcal{X}(M)$ s.t.

$$(t_1, \ldots, t_n) \longmapsto X_n^{t_n} \circ X_{n-1}^{t_{n-1}} \circ \ldots \circ X_1^{t_1} \quad \text{has rank } n$$

$\Rightarrow$ the flow composition is s.t.

$$Im\left(\Phi_{t_n}^{X_n} \circ \Phi_{t_{n-1}}^{X_{n-1}} \circ \ldots \circ \Phi_{t_1}^{X_1}\right) \text{ has dim } n$$

$\Rightarrow$ accessibility holds //

- LARC $\Leftrightarrow$ local accessibility
- accessibility admits an infinitesimal (i.e. Lie-algebraic) characterization ($\Rightarrow$ easy to test!)
- $\dim C(x_0) = n \Rightarrow C(x_0) = T_{x_0} M$

ACCESSIBILITY ≠ CONTROLLABILITY

<u>example</u>

$$\begin{cases} \dot{x}_1 = x_2^2 \\ \dot{x}_2 = u \end{cases} \qquad f(x) = \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix} = x_2^2 \frac{\partial}{\partial x_1} \qquad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\partial}{\partial x_2}$$

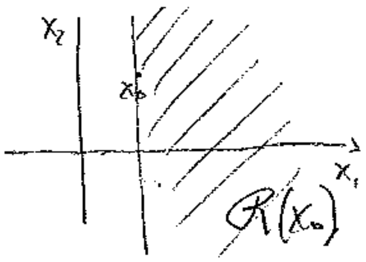
$$[f,g] = \cancel{\frac{\partial g}{\partial x}}^{\,0} f - \frac{\partial f}{\partial x} g = -\begin{bmatrix} 0 & 2x_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ 0 \end{bmatrix} = -2x_2 \frac{\partial}{\partial x_1}$$

$$\text{rank}\{f, g, [f,g]\} = 2 \text{ except for } x_2 = 0$$

$$[[f,g],g] = \overset{\nearrow 0}{\frac{\partial g}{\partial x}}[f,g] - \frac{\partial [f,g]}{\partial x} g = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \frac{\partial}{\partial x_1}$$

$$\text{rank}\{f, g, [f,g], [[f,g],g]\} \equiv 2 \Rightarrow \text{LARC}$$

$$C = \text{span}\{f, g, [f,g], [[f,g],g]\} \quad \text{has rank } 2$$



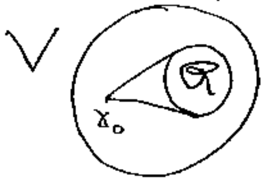
$$\dot{x}_1 \geq 0$$

$\Rightarrow$ system is not controllable.

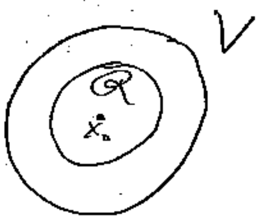
DIFFERENCE BETWEEN ACCESSIBILITY AND $\widetilde{\text{STLC}}$ $\boxed{f19}$

- accessibility: $\mathcal{R}^{\Omega}(x_0, \leq T)$ is an <u>open</u> set in M

$$\not\Rightarrow \quad x_0 \in \mathrm{int}\, \mathcal{R}^{\Omega}(x_0, \leq T)$$



STLC $\begin{cases} \mathcal{R}^{\Omega}(x_0, \leq T) \text{ open} \quad \text{and} \\ x_0 \in \mathrm{int}\, \mathcal{R}^{\Omega}(x_0, \leq T) \end{cases}$



- accessibility property admits an <u>infinitesimal</u> characterization in terms of rank of the Lie algebra of v.f. at a point x_0 (LARC) example: linear system

$$\dot{x} = Ax + \sum_{i=1}^{m} b_i u_i \qquad x \in \mathbb{R}^n$$

compute the accessibility algebra:

$$[b_i, b_j] = 0$$

$$[Ax, b_i] = -Ab_i \quad , \quad [Ax, -Ab_i] = A^2 b_i \quad \text{and so on}$$

$$\Rightarrow C(x) = \text{span}\{Ax,\ b_i,\ Ab_i,\ A^2 b_i, \dots, A^{n-1} b_i \quad i = 1,\dots,m\}$$

calling $B = [b_1, \dots, b_m]$

$$C(x) = \text{Im}\,[B\ AB \dots A^{n-1}B] + \text{span}\{Ax\}$$

if $x_0 = 0 \Rightarrow$ LARC $\equiv$ kalman rank cond.

however, reasoning in terms of nonlinear control this yields only accessibility, not controllability! (to get controllability: $M = \mathbb{R}^n$)

special case For the nonlin syst. $\dot{x} = f(x) + g(x)u$

If LARC is achieved only through so-called "ad-commutators"

$$\underbrace{[f, [f, \dots, [f, g] \dots]}_{k \text{ times}} \triangleq ad_f^k g$$

i.e. $\dim(\text{span}\{f, g, ad_f^k g \quad k = 1, \dots\}) = n$. If $f(x_0) = 0$

then $R^V(x_0, T)$ contains a full neighborhood of x_0

$\Rightarrow$ strong accessibility cond.

$\Rightarrow$ system is STLC

SUFFICIENT CONDITION FOR SMALL TIME LOCAL CONTROLLABILITY

$\dot{x} = f(x) + g(x)u$

$\mathcal{S}^1 = sp\{f, g, [f,g]\}$

$\mathcal{S}^k = sp\{f, g, [f,g], \dots \underbrace{[g[f,[g, \dots]]}_{\text{at most } k \text{ times } g}\}$

<u>thm</u> Assume x_0 equil. point $f(x_0) = 0$
" $\mathcal{S}^k(x_0) = T_{x_0}M$ for some k
" $\mathcal{S}^k(x_0) = \mathcal{S}^{k+1}(x_0)$ $\forall$ k odd

$\Rightarrow$ syst. is STLC

obstruction to STLC comes from $\mathcal{S}^k$ k <u>even</u> ("<u>bad" brackets</u> i.e. with even numb. of g)

$\Rightarrow$

1.) need to fill up all directions in $T_{x_0}M$ with only "good" brackets (i.e. containing odd n° of g)

2) need to "neutralize" the effect of the "bad" brackets
i.e. write them as linear combinations of brackets with a lower number of g.

Chow theorem — DRIFTLESS SYST..

for regular distributions of complete vector fields $\Delta = \text{span}\{g_1 \dots g_n\}$

$\dim \bar{\Delta}_p = T_p M \Rightarrow$ maximally nonintegrable distribution
$\Rightarrow$ locally controllable
$\rightarrow$ Chow theorem

. this is equivalent to LARC for driftless systems.
$\Rightarrow$ STLC $\Leftrightarrow$ LARC

in the example of the rolling wheel

$$\dot{z} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega = g_1(z)\, v + g_2(z)\, \omega$$

$$[g_1, g_2] = \cancel{\frac{\partial g_2}{\partial z} g_1} - \frac{\partial g_1}{\partial z} g_2 = - \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} = g_3(z)$$

$C(z) = \text{span}\{g_1 \;\; g_2 \;\; g_3\}$ has rank 3

$\Rightarrow$ LARC holds everywhere

$\Rightarrow$ syst is controll.

what is that gives nonintegrability i.e. controllability

$\rightarrow$ nonholonomic constraint

NONHOLONOMIC SYSTEMS

nonholonomic constraints = non integrable constraints between state variables of the system

→ express the impossibility of finding functions of the configuration variables which are "integrals" of the constr.

2 kinds :

- kinematic constraints (forbidden directions of the velocity) → bodies in contact which roll without slipping
  (1st order constr)
  ex : rolling wheel, wheeled vehicles, object manipulation with fingers
- dynamic constraints : conservation of angular momentum
  ex falling cats, multibody systems in space

ex rolling wheel

$\omega(z) = \sin\theta \, dx - \cos\theta \, dy = 0$

$\omega(z) \dot{z} = 0$ gives a constraints on the possible vector fields

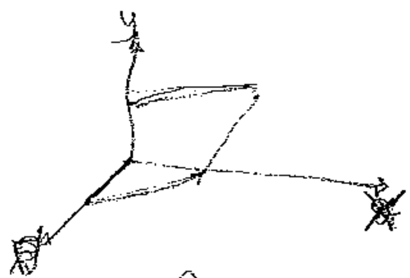
$$\begin{bmatrix} \sin\theta & -\cos\theta & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = 0$$

⇒ compute $g(z)$ s.t. $\omega(z) g(z) = 0$

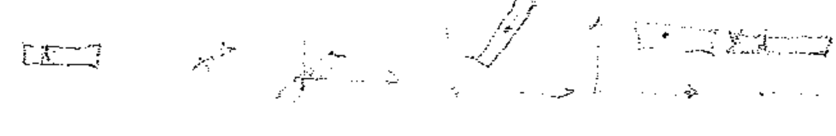
⇒ $\Delta = \mathrm{span}\{g_1 \; g_2\}$ distribution

$\Omega = \mathrm{span}\{\omega\}$ codistribution

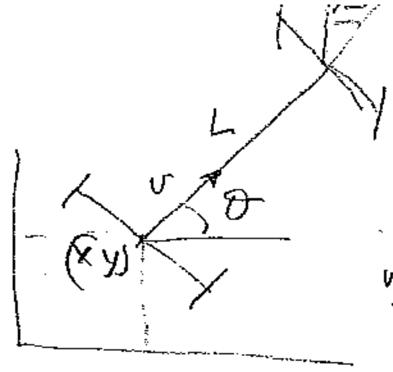
$\Delta = \Omega^{\perp}$

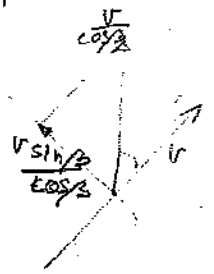


• interpretation of the Lie bracket



example kinematic car





$$\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \\ \dot{\theta} = \frac{v}{L}\tan\beta \\ \dot{\beta} = \omega \end{cases}$$

$x \in \mathbb{R}^2 \times S \times S$

$$\dot{z} = \begin{bmatrix} \cos z_3 \\ \sin z_3 \\ \frac{\tan z_4}{L} \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega = g_1(z)v + g_2(z)\omega$$

$\Delta = \text{span}\{g_1\ g_2\}$ rank $\Delta = 2$

$\Omega = \text{span}\{\omega_1(z)\ \omega_2(z)\}$

nonintegrable constr : on the midpoints

$$g_3 = [g_1\ g_2] = \frac{\partial g_1}{\partial z} g_2 - \frac{\partial g_2}{\partial z} g_1 = \begin{bmatrix} 0 & 0 & -\sin z_3 & 0 \\ 0 & 0 & \cos z_3 & \\ 0 & 0 & 0 & \frac{1}{L\cos^2 z_4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L\cos^2 z_4} \\ 0 \end{bmatrix}$$

$$g_4 = [g_3, g_1] = \frac{\partial g_1}{\partial z} g_3 - \frac{\partial g_3}{\partial z} g_1 = \begin{bmatrix} \frac{\sin z_3}{L\cos^2 z_4} \\ \frac{\cos z_3}{L\cos^2 z_4} \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & * \\ 0 & 0 \end{bmatrix} g_1$$

$$g_5 = [g_3, g_2] = -\frac{\partial g_3}{\partial z} g_2 \propto g_3$$

$T_p M = \text{span}\{g_1, \dots, g_4\} \Rightarrow$ controllability by Chow thm if $\bar{\Delta}_p = T_pM$ the the system is said completely nonholonomic

Filtration

$\Delta = \mathrm{span}\{g_1 \dots g_m\}$ $m<n$ Δ regular distribution

- construct the series of distrib.

$\Delta_1 = \Delta$

$\Delta_2 = \Delta_1 + [\Delta_1, \Delta_1]$ v.f. + 1st level bracket

$\vdots$

$\Delta_i = \Delta_{i-1} + [\Delta_1, \Delta_{i-1}]$ v.f. + i-th level bracket

def
The chain $\Delta_1, \Delta_2, \dots, \Delta_i$ is said the filtration of Δ

def the filtration is said regular in p_0 if .....

$\mathrm{rank}\, \Delta_i(p) = \mathrm{rank}\, \Delta_i(p_0)$ $\forall p \in B(p_0)$ $\forall i$

If a filtration is regular $\Rightarrow$ $\exists\, k \in \mathbb{N}$ s.t. $\Delta_{k+1} = \Delta_k$

$\Rightarrow$ filtration terminates and Δ_K is involutive

$(\Delta_{k+j} = \Delta_k \quad \forall j)$

$k \triangleq$ degree of nonholonomy of Δ

If $\mathrm{rank}\, \Delta_K = n$ $\Rightarrow$ LARC

if $\mathrm{rank}\, \Delta_K < n$ $\Rightarrow$ Frobenius thm. $\Rightarrow$ control system lives on a leaf of dim $= \mathrm{rank}\, \Delta_K$

Growth vector

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If regular filtration

$r_i \triangleq \operatorname{rank} \Delta_i \quad i=1,\dots,k$

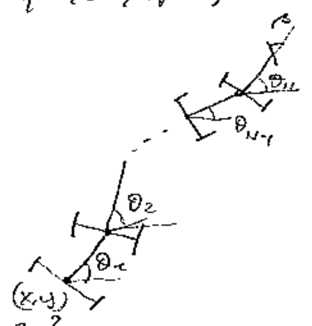
$r \triangleq \{r_1, \dots, r_k\}$ = <u>growth vector</u>

- r is an invariant that characterizes the control system
- r is an <u>invariant to feedback equivalence</u> (i.e. only systems having the same r can be mapped one into the other by means of an invertible change of coordinates and input transformation)..

<u>ex</u>: rolling wheel $\quad r = \{2, 3\}$

<u>ex</u>: kinematic car $\quad r = \{2, 3, 4\}$

<u>ex</u>: N-trailer system



$r = \{2, 3, 4, 5, \dots, N-2\}$

f27 (c 54)

CHAINED FORM

rank-2 distribution in n-dim space

$$\dot{x}_1 = u_1$$
$$\dot{x}_2 = u_2$$
$$\dot{x}_3 = x_2 u_1$$
$$\dot{x}_4 = x_3 u_1$$
$$\vdots$$
$$\dot{x}_n = x_{n-1} u_1$$

$$\dot{X} = \begin{bmatrix} 1 \\ 0 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_2$$

- $r = \{2, 3, \dots, n\}$
- degree of nonholonomy $n-2$
- N-trailer if feedback equivalent to chained form (in regular points)

$\dot{x} = \sum_{i=1}^{m} g_i(x) v_i$ $\exists$ $\begin{cases} z = \varphi(x) & \varphi \text{ diffeo} \\ v_i = \psi_i(x, u_i) & \psi_i \text{ invert.} \end{cases}$ s.t. $\dot{z} = \sum_{i=1}^{m} g_i(\varphi^{-1}(z)) \psi_i(x, u_i)$

- conversion of the kinematic car into chained form

$$\begin{cases} \dot{x} = v \cos\theta \\ \dot{y} = v \sin\theta \\ \dot{\theta} = \dfrac{v \tan\beta}{L} \\ \dot{\beta} = \omega \end{cases}$$

$u_1 \triangleq v \cos\theta$

$x_1 \triangleq x$

$x_4 \triangleq y$

$\dot{y} = \dot{x}_4 = u_1 \tan\theta \triangleq u_1 x_3$

$$\Rightarrow \begin{cases} \dot{x} = u_1 \\ \dot{y} = u_1 \tan\theta \\ \dot{\theta} = \dfrac{u_1 \tan\beta}{L \cos\theta} \\ \dot{\beta} = \omega \end{cases}$$

$$\dot{x}_3 = \frac{d}{dt} \tan\theta = \frac{\dot{\theta}}{\cos^2\theta} = u_1 \frac{\tan\beta}{L \cos^3\theta} \triangleq u_1 x_2$$

$$\dot{x}_2 = \frac{d}{dt}\left(\frac{\tan\beta}{L \cos^3\theta}\right) = \frac{\frac{L \cos^3\theta}{\cos^2\beta}\dot{\beta} + 3L \cos^2\theta \sin\theta \tan\beta\, \dot{\theta}}{L^2 \cos^6\theta} =$$

$$\dot{x}_2 = \frac{\cos^2\theta\, \omega + 3\sin\theta \tan^2\beta\, u_1}{L\cos^5\theta} \triangleq u_2$$

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \triangleq \begin{bmatrix} x \\ \frac{\tan\beta}{L\cos^3\theta} \\ \tan\theta \\ y \end{bmatrix} \qquad \begin{cases} u_1 = v\cos\theta \\ u_2 = \frac{\omega}{L\cos^3\theta} + 3\frac{\tan\theta}{\cos^3\theta}\tan^2 v \end{cases}$$

Chained form

- Heisenberg system rolling wheel is feedback equival

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2 = g_1(x)u_1 + g_2(x)u_2$$

$$[g_1, g_2] = g_3$$

- Kinematic car is feedback equivalent to Engel system

rank 2 Δ in 4-dim sp.

$$[g_1, g_2] = g_3$$

$$[g_1, [g_1, g_2]] = g_4$$

- the brackets of the chained form

$$\dot{X} = \begin{bmatrix} 1 \\ 0 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_2 = g_1(x) u_1 + g_2(x) u_2$$

1st level: Δ_2

$$[g_1 \; g_2] = -\frac{\partial g_1}{\partial x} g_2 = -\begin{bmatrix} 0 & & & & 0 \\ 0 & & & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & & \\ 0 & & & 0\,1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = -\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \triangleq g_3$$

$\Delta_2 = \text{span}\{g_1 \; g_2 \; g_3\}$ $\qquad \Delta_2 = \Delta_1 + [\Delta_1, \Delta_1]$

2nd level

$$[g_1, g_3] = g_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \Delta_3 = \Delta_2 + [\Delta_1, \Delta_2]$$

$\Delta_3 = \{g_1 \dots, g_4\}$

$r = \{2, 3, 4, 5 \dots n\}$ like in the N-trailer

<u>plus</u> only $n-2$ brackets are $\neq 0$

$\Rightarrow$ only the minimal number of brackets

$$\underbrace{[g_1, [g_1, \dots, [g_1, g_2] \dots]}_{n-2}$$

plus

$$\underbrace{[g_1, [g_1, \dots, [g_1, g_2] \dots]}_{n-1} = 0$$ all level $n-1$ brackets are 0

$\longrightarrow$ NILPOTENT SYSTEM (not approximation

MOTION PLANNING by DIFFERENTIAL FLATNESS

$\dot{x} = f(x,u)$ underactuated $u \in \mathbb{R}^m$

<u>def</u> A control system is <u>differentially flat</u> if $\exists$ a number of output functions $y_1(t) \dots y_m(t) \triangleq y(t)$ with m = number of inputs s.t. both x and u can be expressed as algebraic functions of y and its derivatives i.e

$$x = f_1(y, y^{(1)}, y^{(2)}, \dots)$$
$$u = f_2(y, y^{(1)}, y^{(2)}, \dots)$$

- system inversion y = <u>flat output</u> $\Rightarrow$ the whole system becomes a trivial algebraic expression of y

$\Rightarrow$ motion planning can be solved <u>exactly</u> by specifying $y(t)$

<u>example</u> chained form : 2 inputs $\Rightarrow$ 2 flat output

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_n \\ x_1 \end{bmatrix} \qquad \dot{y} = \begin{bmatrix} x_{n-1} u_1 \\ u_1 \end{bmatrix} \Rightarrow \begin{array}{l} u_1 = \dot{y}_2 \\ x_{n-1} = \dfrac{\dot{y}_1}{u_1} = \dfrac{\dot{y}_1}{\dot{y}_2} \end{array}$$

$$\ddot{y} = \begin{bmatrix} \dot{x}_{n-1} u_1 + x_{n-1} \dot{u}_1 \\ \dot{u}_1 \end{bmatrix} = \begin{bmatrix} x_{n-2} u_1^2 + x_{n-1} \dot{u}_1 \\ \dot{u}_1 \end{bmatrix} = \begin{bmatrix} x_{n-2} \dot{y}_2^2 + \dfrac{\dot{y}_1}{\dot{y}_2} \ddot{y}_2 \\ \end{bmatrix}$$

$$\Rightarrow x_{n-2} = \left(\ddot{y}_1 - \frac{\dot{y}_1 \ddot{y}_2}{\dot{y}_2} \right) \frac{1}{\dot{y}_2^2} = f_{n-2}(y, \dot{y}, \ddot{y})$$

$\vdots$

$$x_1 = f_1(y_1, \dot{y}, \dots y^{(n-2)})$$
$$u_1 = f_0(y, \dot{y}, \dots, y^{(n-1)})$$

singularity of the procedure
terms at the denominator $\Rightarrow u_1 \neq 0 \quad \dot{y}_2 \neq 0$

in the car example : $u_1 \simeq v$ = longitudinal velocity

$v \simeq u_1 = 0 \Rightarrow$ car does not move $\Rightarrow$ functions y describing its trajectories do not exist!

differential flatness can be used in conjunction with feedback equivalence $\Rightarrow$ differentially flat control of the car

$$\begin{cases} \dot{x} = v \cos\theta \\ \dot{y} = v \sin\theta \\ \dot{\theta} = \dfrac{v \tan\beta}{L} \\ \dot{\beta} = \omega \end{cases}$$

Steering with sinusoids

$\dot{x}_1 = u_1$
$\dot{x}_2 = u_2$
$\dot{x}_3 = x_2 u_1$
$\dot{x}_4 = x_3 u_1$
$\vdots$
$\dot{x}_n = x_{n-1} u_1$

can be steered arbitrarily using sinusoids at integrally related frequencies

Algorithm

time $\{0, \to T = 1\}$

1) steer x_1 and x_2 at their desired values
2) $\forall x_{k+2}, k \geq 1$ steer x_{k+2} at its value using

$u_1 = a \sin 2\pi t$
$u_2 = b \cos 2\pi k t$

where $x_{k+2}(1) - x_{k+2}(0) = \left(\frac{a}{4\pi}\right)^k \frac{b}{k!}$

proof (sketch)

u_1 and u_2 above $\Rightarrow$ $\dot{x}_3$ has a comp. at freq. $2\pi(k-1)$

$\dot{x}_4$ " " " $2\pi(k-2)$

$\dot{x}_{k+2}$ " " " 0

$\Rightarrow$ net motion in x_{k+2}

while all other variable return at their original values after a period. $q_i(1) = q_i(0)$ $i < k+2$

$$\int \sin mx \sin nx \, dx =$$

$$= \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}$$

$m^2 \neq n^2$

$$X_1 = a\int_0^t \sin 2\pi\tau\, d\tau = \frac{a}{2\pi}\Big[\cos 2\pi\tau\Big]_0^t = \frac{a}{2\pi}\left(\cos 2\pi t - 1\right)$$

$$X_2 = b\int_0^t \cos 2\pi kt = -\frac{b}{2\pi k}\sin 2\pi k t\Big]_0^t = -\frac{b}{2\pi k}\sin 2\pi k t$$

$$\dot{x}_3 = -\frac{ab}{2\pi k}\sin 2\pi k t \sin 2\pi t = -\frac{ab}{4\pi k}\Big[\cos\big(2\pi(k-1)t\big) - \cos\big(2\pi(k+1)t\big)$$

$$X_3 = -\frac{ab}{2\pi k}\int_0^t \sin 2\pi k\tau \sin 2\pi\tau\, d\tau =$$

$$= -\frac{ab}{2\pi k}\left[\frac{\sin[(k-1)2\pi\tau]}{2(k-1)} - \frac{\sin[(k+1)2\pi\tau]}{2(k+1)}\right]_0^t$$

$\Uparrow$

component at freq $\pi(k-1)$

Feedback stabilization (cont)

- If I have a nonlinear system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \qquad m < n \qquad x \in M$$

whose linearization is controllable $\Rightarrow$ locally I can construct a feedback stabilizer using linear system.

$$\begin{cases} A = \left.\dfrac{\partial f}{\partial x}\right|_{x_0} & \text{s.t. } f(x_0) = 0 \\ B = [b_1 \dots b_m] & \text{where } b_i = g_i(x_0) \end{cases}$$

$$\text{rank}\left[B \; AB \; A^2B \; \dots A^k B\right] = n \Rightarrow \exists \; u_i = K_i x \text{ s.t.}$$

$\dot{x} = f(x) + \sum g_i(x) K_i x$ is locally asymptotically stable

$$Re\left[eig\left(A + BK\right)\right] < 0$$

- also stabilizability of the linearization would suffice

If the system is driftless linearization cannot be controllable (when the system is underactuated)

$\rightarrow$ general condition for feedback stabilization (of which driftless systems are an important particular case)

<u>thm</u> (Brockett necessary condition)

Consider the system $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \quad m<n, \; f(x_0)=0$

Assume $\exists$ continuous feedback law $u_i = k_i(x)$ $k_i(x_0)=0$

which make x_0 a (locally) asymptotically stable

equilibrium for the closed loop system. Then the

image of every neighb. of $(x_0, 0)$ under the map

$$f + \sum_i g_i(x) u_i : \mathbb{R}^n \times U \to \mathbb{R}^n$$

$$(x, u) \mapsto f(x) + \sum g_i(x) u_i$$

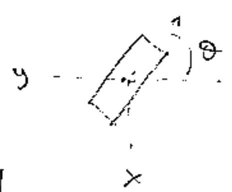
must be onto some open set containing x_0

<u>meaning</u>: locally $\mathbb{R}^{n+m}$ ($x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ can be assumed without loss of generality)

must "cover" $\mathbb{R}^n$

$\Rightarrow$ this is never possible for driftless systems with nonholonomic constraints

<u>example</u> rolling wheel



$$\begin{bmatrix} x \\ y \\ \theta \\ v \\ \omega \end{bmatrix} \to \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega$$

for $|\theta| < \frac{\pi}{2}$ vectors like $\begin{bmatrix} 0 \\ \delta \\ \delta \end{bmatrix}$ $\delta \neq 0$ are never possible

$\Rightarrow$ cannot cover open set in M around equil.

ex chained form of 3 states

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = x_2 u_1 \end{cases}$$

Brockett necessary cond. is not satisfied

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ u_1 \\ u_2 \end{pmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2$$ does not contain vectors like $\begin{bmatrix} 0 \\ 0 \\ d \end{bmatrix}$ $d \neq 0$

meaning: if you are in $x = \begin{bmatrix} 0 \\ 0 \\ d \end{bmatrix}$ and apply any feedb. $u_1 = k_1(x)$
$u_2 = k_2(x)$

then you are "stuck" you do not converge.

$\Rightarrow \nexists$ continuous time-invariant feedback that locally asymptotically stabilizes the system to $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Alternative feedback designs

1) time-varying feedback

2) discontinuous feedb.

ex: chained form: discont. feedback

- when $x_2(0) \neq 0$ apply the feedback $\begin{cases} u_1 = -\dfrac{x_1}{x_2^2} \\ u_2 = -x_2^2 \end{cases}$

$\Rightarrow$ closed loop $\begin{cases} \dot{x}_1 = -\dfrac{x_1}{x_2^2} \\ \dot{x}_2 = -x_2^2 \\ \dot{x}_3 = -x_1 x_2 \end{cases}$

$x_2(t) \xrightarrow{t \to \infty} 0$

$x_2^2(t) \geq 0 \ \forall t \Rightarrow x_1(t) \xrightarrow{t \to \infty} 0$

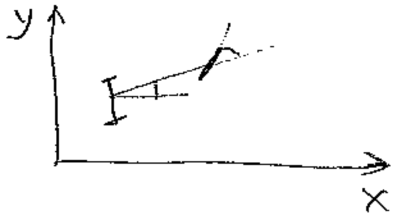
however $x_3 \rightarrow ??$

Jacobian linearization along a trajectory (37)

Kinematic car has non-controllable linearization

however we can fix a trajectory and linearize around it

this linearization will be controllable



- trajectory x axis
- fix $v = const = 1$

$\Rightarrow$ equil. point is $y_0 = 0$, $\theta_0 = 0$, $\beta_0 = 0$; x along traj. has no equil. $\dot{x}_0 = v = const \neq 0$

$\Rightarrow$ we drop x

system with v const has a drift

$$\frac{d}{dt}\begin{bmatrix} y \\ \theta \\ \beta \end{bmatrix} = \begin{bmatrix} \sin\theta \\ \frac{\tan\beta}{L} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega$$

$$\bar{A} = \begin{bmatrix} 0 & -\cos\theta|_{\theta_0} & 0 \\ 0 & 0 & \frac{1}{L\cos^2\beta}\Big|_{\beta_0} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & \frac{1}{L} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \dot{z} = \bar{A}z + \bar{b}\omega$$

$$\bar{A}b = \begin{bmatrix} 0 \\ \frac{1}{L} \\ 0 \end{bmatrix} \quad \bar{A}^2 b = \begin{bmatrix} -\frac{1}{L} \\ 0 \\ 0 \end{bmatrix} \Rightarrow \text{rank } \mathcal{R} = 3 \text{ linearly controllable}$$

$\not\Rightarrow$ STLC

if $v < 0 \Rightarrow \bar{A}$ changes sign

$\Rightarrow$ if $Re[eig(\bar{A})] \geq 0$ for $v > 0$ $\rightarrow$ open loop critically stable

$\Rightarrow$ $Re[eig(\bar{A})] \leq 0$ for $v < 0$ $\rightarrow$ open loop unstable (or critically stable)

Other obstruction to stabilizability of nonlinear systems:
lack of global attractivity on manifolds with nontrivial topology (ex: compact manifolds without boundaries)

given $\dot{x} = f(x)$, $x \in M$ and an asymptotically stable equil $x_0 \in M$, then its region of attraction, $\mathcal{D}$, must be a contractible set.

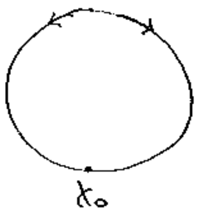
A set $\mathcal{D}$ is contractible if it can be deformed continuously to a point. In term of topology: if $\exists$ a continuous map $h : [0,1] \times \mathcal{D} \to \mathcal{D}$ s.t. $\begin{cases} h(0,x) = x \\ h(1,x) = x_0 \end{cases}$

in this case $\mathcal{D}$ and $\{x_0\}$ are said homotopy equivalent

ex $\mathbb{R}^n$ and $\{x_0\}$ are homotopy equival.
$\to$ the domain of attraction of $\{x_0\}$ can be $\mathbb{R}^n$

ex S^n non $\bar{e}$ homotopy equiv. a $\{x_0 \in S^n\}$

ex: S^1



ex $\dot{x} = Ax u_1 + Bx u_2$ $\quad A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ +1 & 0 & 0 \end{bmatrix}$ $\boxed{f39}$

$x \in S^2$ (see previous exercise) $\quad S^2 = \{x \in \mathbb{R}^3 \mid \|x\|^2 = 1\}$

task: obtain global asymptotic stabilization at a certain $x_0 \in S^2$

question: $\exists$ $u_1 = k_1(x)$, $u_2 = k_2(x)$ s.t.

$\dot{x} = Ax k_1(x) + Bx k_2(x)$ has x_0 as a global attractor on S^2?

consider $\mathbb{R}^3$-induced norm on the sphere

$$V(x_0, x) = \|x - x_0\|^2 = \langle x - x_0, x - x_0 \rangle = (x - x_0)^T (x - x_0)$$

$$= \langle x, x \rangle - \langle x_0, x \rangle - \langle x, x_0 \rangle - \langle x_0, x_0 \rangle$$

$$= \|x\|^2 + \|x_0\|^2 - 2\langle x_0, x \rangle = 2 - 2\langle x_0, x \rangle \quad \text{since } \|x\|^2 = 1,\ \|x_0\|^2 = 1$$

task is then find $u_1 = k_1(x)$ $u_2 = k_2(x)$ s.t. $\|x - x_0\|^2 \to 0$ in closed loop.

$$\dot{V}(x_0, x) = -2\langle x_0, \dot{x} \rangle = -2\langle x_0, Ax u_1 + Bx u_2 \rangle$$

$$= -2\langle x_0, Ax \rangle u_1 - 2\langle x_0, Bx \rangle u_2$$

choosing $\begin{cases} u_1 = \langle x_0, Ax \rangle \\ u_2 = \langle x_0, Bx \rangle \end{cases}$

then $\dot{V}(x_0, x) = -2\langle x_0, Ax \rangle^2 - 2\langle x_0, Bx \rangle^2 \leq 0$

$\Rightarrow$ V is s.t. $V(x_0, x) > 0$ positive def. (it is a distance)

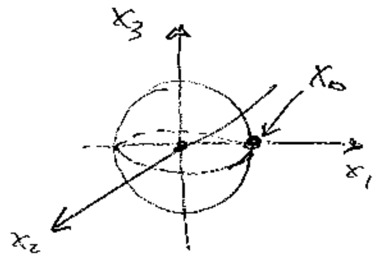
$\dot{V}(x_0, x) \leq 0$

$\Rightarrow$ V is a Lyapunov function for the closed loop syst

$\Rightarrow$ x_0 is at least locally asymptotically stabilized

f40

Q: is the stabilization global?



consider for example $x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

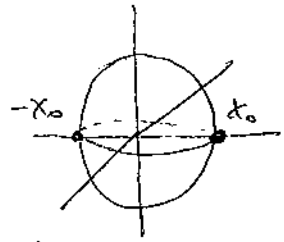
need to find locus where $\dot{V} = 0$ i.e. level sets of the Lyap. funct

$\dot{V} = 0 \iff \begin{cases} \langle x_0, Ax \rangle = 0 \\ \langle x_0, Bx \rangle = 0 \end{cases}$

$x_0^T A x = [1\ 0\ 0] \begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix} = x_2$

$x_0^T B x = [1\ 0\ 0] \begin{bmatrix} -x_3 \\ 0 \\ x_1 \end{bmatrix} = -x_3$

$\iff$ $x_2 = 0$
$x_3 = 0$



this is true also in the antipodal point $-x_0$

$\Rightarrow$ not globally stabilized

This obstruction is topological (S^2 not homotopy equival to $\mathbb{R}^2$)

" " " " to $\{x_0\}$)

$\Rightarrow$ x_0 is not C^1 globally stabilizable.

LIE GROUP

<u>def</u> A <u>Lie group</u> G is a smooth (or analytic) manifold with the algebraic structure of a group i.e. with

- multiplication operation $G \times G \to G$, $(g, h) \mapsto g \cdot h$
- neutral elem. $I_d \in G$ s.t. $g \cdot I_d = g \quad \forall g \in G$
- inverse elem. $\forall g \ \exists g^{-1} \in G$ s.t. $g \cdot g^{-1} = I_d$

<u>ex</u>. General linear group $GL_n(\mathbb{R}) = \{ g \in M_n(\mathbb{R}) \text{ s.t. } \det g \neq 0 \}$

- product is matrix product
- differentiable structure inherited by $\mathbb{R}^{n^2}$

$\dim GL_n(\mathbb{R}) = n^2$

$GL_n(\mathbb{R})$ has two connected components

1) $\{ g \in GL_n(\mathbb{R}) \mid \det g > 0 \} = GL_n^+$

2) $\{ g \in GL_n(\mathbb{R}) \mid \det g < 0 \} = GL_n^-$

$\Rightarrow$ we will consider only GL_n^+ $\quad (I \in GL_n^+)$

<u>def</u> A <u>Lie subgroup</u> $G' \subsetneq G$ is a proper subset of G which is also a Lie group.

<u>ex</u> $SL_n(\mathbb{R})$ = <u>special linear group</u> $= \{ g \in GL_n \mid \det g = 1 \}$

$\dim SL_n(\mathbb{R}) = n^2 - 1$

ex $O(n) = \{ g \in GL_n \mid g^{-1} = g^T \}$ rotations

$=$ orthogonal group 2 connected comp.

ex $SO(n) = \{ g \in GL_n \mid g^{-1} = g^T, \det g = +1 \}$

$= O(n) \cap SL_n$

$=$ special orthogonal group

- $\dim SO(n) = \underbrace{n(n-1)}_{2}$
- group of rotation matrices that preserve orientations

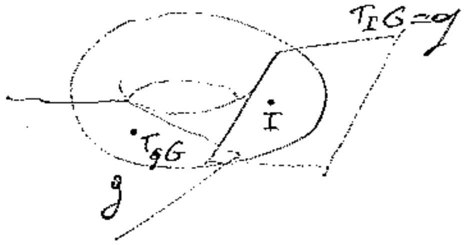
$R \in SO(n)$ $R = [r_{ij}]$ then $\sum_{j=1}^{n} r_{ij}^2 = 1$

$\sum_{i=1}^{n} r_{ij}^2 = 1$

ex $SO(3)$ = group of rigid body rotations in $\mathbb{R}^3$

$R \in SO(3)$ $R = [r_1 \; r_2 \; r_3] \Rightarrow r_i^T r_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

Task I want to study the structure of the tangent bundle TG of G just by looking at the tangent space at the identity $T_I G = g$



To do this I need a way to translate g to I and to express the vector fields at g in terms of the vector fields at I

def Left translation = action of the group on itself

$$L_g : G \longrightarrow G$$
$$h \longmapsto L_g(h) = g \cdot h \qquad \text{(left multiplication)}$$

$\Rightarrow L_g$ diffeomorphism of G $\forall g$. w.r.t. I

$$L_g : G \longrightarrow G$$
$$I \longmapsto L_g(I) = g$$

of inverse $(L_g)^{-1} = L_{(g^{-1})}$

def right translation : same thing from the right

$$R_g : G \longrightarrow G$$
$$h \longmapsto R_g(h) = h \cdot g$$

$\Rightarrow$ for the identity

$$R_g : G \longrightarrow G$$
$$I \longmapsto R_g(I) = g$$

Control system subordinated to a group action

def $\mathcal{F} \in \mathcal{X}(M)$ is said subordinated to a group action if $\exists\, G$ and $\exists\, \Lambda : G \times M \to M$ and $\exists$ a family $\Gamma \in \mathfrak{g}$ s.t. $\Lambda_*(\Gamma)(x) = \mathcal{F}(x) \quad \forall x \in M$

$$\mathcal{F} \triangleq \{ \xi_M \in \mathcal{X}(M) \mid \xi \in \Gamma \}$$

where $\Gamma \in \mathcal{X}_R(G) = \mathfrak{g}$

example invariant system on a matrix Lie group $\Rightarrow$
$\Rightarrow$ bilinear system on a homogeneous space

$$\Gamma = \{ A + \sum B_i u_i \mid u_i \in \mathcal{U} \}, \quad A, B_1, \dots, B_m \in \mathfrak{g}$$

(i.e. $\dot{g} = (A + \sum B_i u_i)\, g$ right invar. syst. on G

$$\Rightarrow \Lambda_*(\Gamma)(x) = \mathcal{F}(x) = \{ Ax + \sum B_i u_i x \} \quad x \in M$$

i.e. $\dot{x} = Ax + \sum B_i x u_i \qquad x \in M$

example syst. on the sphere

$\dot{x} = Ax + Bxu \qquad x \in \mathbb{R}^3 \qquad A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$\mathrm{Lie}\{A, B\} = so(3) \qquad F(x) = Ax + Bxu$

$\Gamma = A + Bu \in so(3) \Rightarrow \dot{R} = (A + Bu) R \qquad R \in SO(3)$

Controllability condition for a control system subordinated to a group action

Thm Let Λ be an action of G (connected) on M

Let Γ = right invariant control system on G

Let $\mathcal{F} = \Lambda_*(\Gamma)$ the induced system on M

We have then:

i) $\mathcal{R}_{\mathcal{F}}(x_0) = \Lambda_{\mathcal{R}_\Gamma}(x_0) = \{\Lambda_g(x_0) \mid g \in \mathcal{R}_\Gamma\} = \mathcal{R}_\Gamma x_0$

ii) Assume action Λ is transitive. If Γ controllable on G then $\mathcal{F}$ controllable on M

iii) $\mathcal{F}$ controllable on $M \iff$ the semigroup $\mathcal{R}_\Gamma$ acts transitively on M

Proof:

i) follows from right invariance

ii) If $\mathcal{R}_\Gamma = G \Rightarrow$ from transitivity $\mathcal{R}_{\mathcal{F}}(x_0) = \mathcal{R}_\Gamma x_0 = G x_0 = M$

iii) "$\Rightarrow$" $\mathcal{F}$ controllable $\Rightarrow$ action of $\mathcal{R}_\Gamma$ must be transitive

"$\Leftarrow$" $\mathcal{R}_{\mathcal{F}} = \mathcal{R}_\Gamma x_0$ with $\mathcal{R}_\Gamma$ transitive $\Rightarrow \mathcal{R}_\Gamma x_0$ is all M //

In general: Γ controllable $\Rightarrow$ $\mathcal{F}$ controllable

$\not\Leftarrow$

Coroll Λ transitive Γ & $\mathcal{F}$ driftless

Γ controllable $\iff$ $\mathcal{F}$ controllable

Action can also be affine ($\Rightarrow$ homogeneous coordinates)

given $\Gamma = \left\{ \bar{A} + \sum_{i=1}^{m} \bar{B}_i u_i \quad u_i \in \mathcal{U} \;\middle|\; \bar{A} = \left[\begin{array}{c|c} A & a \\ \hline 0 & 0 \end{array}\right],\ \bar{B}_i = \left[\begin{array}{c|c} B_i & b_i \\ \hline 0 & 0 \end{array}\right] \quad A, B_i \in gl_n \right\}$

right invariant

$\Rightarrow$ reduced action on $\mathbb{R}^n$

$$\dot{x} = Ax + a + \sum u_i (B_i x + b_i) \qquad x \in \mathbb{R}^n$$

particular cases:

1) linear system $\quad a = B_1 = \dots = B_m = 0$

$$\Rightarrow \dot{x} = Ax + \sum b_i u_i$$

2) bilinear system on $\mathbb{R}^n \quad a = b_1 = \dots = b_m = 0$

$$\dot{x} = Ax + \sum B_i u_i x$$

3) chained form $\quad A = a = 0 \quad m = 2$

$$\dot{x} = \bar{B}_1 u_1 \begin{bmatrix} x \\ 1 \end{bmatrix} + \bar{B}_2 u_2 \begin{bmatrix} x \\ 1 \end{bmatrix} = (B_1 x + b_1) u_1 + b_2 u_2$$

with $\bar{B}_1 = \left[\begin{array}{c|c} B_1 & b_1 \\ \hline 0 & 0 \end{array}\right] \qquad \bar{B}_2 = \left[\begin{array}{c|c} 0 & b_2 \\ \hline 0 & 0 \end{array}\right]$

$$B_1 = \begin{bmatrix} 0 & \cdots & & 0 \\ 0 & \cdots & & 0 \\ 0 & 1 & & \\ & & \ddots & \\ & & 1 & 0 \end{bmatrix} \qquad b_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad b_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow \dot{x} = \begin{bmatrix} 1 \\ 0 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_2$$

STRUCTURE THEORY OF LIE ALGEBRAS

<u>def</u> A Lie algebra $\mathfrak{g}$ is <u>Abelian</u> if $\forall A, B \in \mathfrak{g}$ $[A, B] = 0$

i.e. $[\mathfrak{g}, \mathfrak{g}] = 0$

<u>ex</u> $\mathfrak{g} = \mathbb{R}^n$ is abelian

<u>def</u> A subalgebra $\mathfrak{s} \subseteq \mathfrak{g}$ is an <u>ideal</u> if $\forall A \in \mathfrak{s}, B \in \mathfrak{g}$

$[A, B] \in \mathfrak{s}$ i.e. $[\mathfrak{s}, \mathfrak{g}] \subseteq \mathfrak{s}$

<u>ex</u> $$se(3) = \operatorname{span}\left\{ \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline & & & \end{array}\right], \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline & & & \end{array}\right], \left[\begin{array}{c|c} 0 & \begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \\ \hline & \end{array}\right], \left[\begin{array}{c|c} 0 & \begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix} \\ \hline & \end{array}\right], \left[\begin{array}{c|c} 0 & \begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix} \\ \hline & \end{array}\right] \right\}$$

$$= so(3) \circledS \mathbb{R}^3$$

$\mathbb{R}^3$ is an ideal of $se(3)$ $\quad [so(3), \mathbb{R}^3] \in \mathbb{R}^3$

It is an abelian ideal

<u>def</u> Call $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}], \dots, \mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}] \dots$

$\mathfrak{g}$ is <u>solvable</u> if the derived series (upper series)

$\mathfrak{g}, \mathfrak{g}^{(1)}, \mathfrak{g}^{(2)}, \dots, \mathfrak{g}^{(k)} \dots$ terminates at 0 i.e.

$\exists k$ s.t. $\mathfrak{g}^{(k)} = 0$

- catena di sottosp. vett. $\mathfrak{g} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \dots \supseteq \mathfrak{g}^{(k)}$

ex abelian algebra is always solvable ($[\mathfrak{g},\mathfrak{g}]=0$) $\boxed{f69}$

ex $se(3) = se(3)^{(1)} = \dots = se(3)^{(k)}$ $\forall k$ $\Rightarrow$ $se(3)$ is not solvable

ex $se(2)^{(1)} = \mathbb{R}^2$, $se(2)^{(2)} = 0$ $\Rightarrow$ $se(2)$ is solvable

ex the set of all upper triangular matrices is a solvable Lie algebra

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

thm (Lie thm)

$\mathfrak{g}$ subalgebra of $\mathfrak{gl}_n$.

$\mathfrak{g}$ solvable $\Leftrightarrow$ all elements of $\mathfrak{g}$ can be simultaneously triangularized

. Lie thm : $\mathfrak{g}$ solvable $\Leftrightarrow$ $\exists$ a basis composed of all upper triangular matrices.

- call $\mathfrak{g}_{(1)} = [\mathfrak{g}, \mathfrak{g}]$ $\mathfrak{g}_{(2)} = [\mathfrak{g}, \mathfrak{g}_{(1)}]$ --- $\mathfrak{g}_{(n+1)} = [\mathfrak{g}, \mathfrak{g}_{(n)}]$

→ descending central series (lower series)

$\mathfrak{g}_{(i)}$ ideal on $\mathfrak{g}_{(i-1)}$ $\mathfrak{g}_{(i)} \subseteq \mathfrak{g}_{(i-1)}$

$\mathfrak{g} \supseteq \mathfrak{g}_{(1)} \supseteq \mathfrak{g}_{(2)} \supseteq \dots \supseteq \mathfrak{g}_{(n)}$

- <u>def</u> $\mathfrak{g}$ is <u>nilpotent</u> if the sequence $\mathfrak{g}_{(1)} \dots \mathfrak{g}_{(n)}$ terminates at zero.

- $\mathfrak{g}_{(n)} \supseteq \mathfrak{g}^{(n)}$ → nilpotency $\Rightarrow$ solvability
$\not\Leftarrow$

- <u>ex</u> <u>Heisenberg algebra</u> = algebra of strictly upper triangular 3x3 matrices

$$h(3) = \text{span}\left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

is nilpotent $\quad \dot{X} = \frac{d}{dt}\begin{bmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{bmatrix} = X\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} u_1 \right)$

- <u>ex</u> Lie algebra of the chain form $\begin{bmatrix} B_1 & b_1 \\ 0 - 0 & 0 \end{bmatrix} \dots \begin{bmatrix} 0 & b_n \\ 0 - 0 & 0 \end{bmatrix}$

is nilpotent - $\left\{ \begin{bmatrix} 0 & b_2 \\ 0 & 0 \end{bmatrix} \dots \begin{bmatrix} 0 & b_n \\ 0 & 0 \end{bmatrix} \right\}$ = abelian ideal

- <u>ex</u> algebra of all strictly upper triangular matrices

- <u>ex</u> $\lambda I + \begin{bmatrix} 0 & * & \cdots & * \\ & \ddots & & \vdots \\ & & & * \\ 0 & & & 0 \end{bmatrix}$ algebra of strictly upper triangular matrices + matrices proportional to the identity

f71

ex $se(2)$

$$se(2)_{(1)} = se(2)^{(1)} = \mathbb{R}^2$$

$$se(2)_{(2)} = [se(2), \mathbb{R}^2] = \mathbb{R}^2$$

$$\vdots$$

$$se(2)_{(k)} = [se(2), \mathbb{R}^2] = \mathbb{R}^2$$

$\Rightarrow se(2)$ is solvable but not nilpotent def the <u>radical</u> is the maximal solvable ideal of the Lie algebra $\boxed{f72}$

ex $se(3)$ has radical $\mathbb{R}^3$

def a Lie algebra is said <u>semisimple</u> if its radical is 0 _

$\Rightarrow$ $\mathfrak{g}$ contains no abelian ideals (in order for one of the series to stabilize at zero there has to be an abelian ideal inside $\mathfrak{g}$)

def a semisimple Lie algebra is <u>simple</u> if it contains no ideals other than $\mathfrak{g}$ and $\{0\}$

$\Rightarrow$ semisimple Lie algebra can be splitted into simple ideals

· <u>Levi decomposition</u>: every Lie algebra can be splitted into the semidirect sum of its radical r and a semisimple Lie algebra s

$$\mathfrak{g} = s \circledS r$$

ex $se(3) = so(3) \circledS \mathbb{R}^3$

<u>meaning</u> solvable : part that vanish (when taking brackets)

semisimple : part that keeps winding.

f73

example: the set of all $n\times n$ upper triangular matrices is a solvable Lie algebra

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \ddots & & \vdots \\ 0 & & \ddots & \\ & & & a_{nn} \end{bmatrix}$$

• what we use is a representation of an "abstract" algebra $\mathfrak{g}$

def A representation of the algebra $\mathfrak{g}$ on a v.sp. V is a mapping $\rho : \mathfrak{g} \to gl(V)$ s.t. it is a Lie algebra homomorphism (i.e. a linear map that preserves the Lie bracket) i.e.

- $\rho(\alpha\xi + \beta\eta) = \alpha\rho(\xi) + \beta\rho(\eta)$ $\alpha, \beta \in$ field in which $\mathfrak{g}$ is constructed (for us $\mathbb{R}$)
- $\rho([\xi, \eta]) = [\rho(\xi), \rho(\eta)]$ $\xi, \eta \in \mathfrak{g}$

• $\rho : \mathfrak{g} \to gl(V)$ assigns to each $\xi \in \mathfrak{g}$ a linear operator $\rho(\xi)$ which can be described as an $n\times n$ matrix (if $n = \dim V$.)

• the representation ρ is irreducible if V contains no nontrivial subspaces invariant under the action of all $\rho(\xi)$, $\xi \in \mathfrak{g}$.

f74

- <u>Adjoint representation</u> = linear representation of the Lie algebra in itself - $\rho : \mathfrak{g} \to gl(V)$ - here $V = \mathfrak{g}$

$$\rho : \mathfrak{g} \to \mathfrak{g}$$
$$A \mapsto ad_A$$

- Lie bracket : $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$

$$ad_A = [A, \cdot] : \mathfrak{g} \longrightarrow \mathfrak{g}$$

- adjoint repr. preserves Lie bracket $\quad ad_{[A,B]} = ad\,[A, B] \overset{!}{=} [ad_A , ad_B]$

$\Rightarrow$ Lie algebra homomorphism

- given a basis $A_1 \dots A_n$ of $\mathfrak{g}$

$$[A_i, A_j] = \sum_{k=1}^{n} c_{ij}^k A_k \qquad c_{ij}^k = \underline{\text{structure constants}}$$

- adjoint representation is a matrix representation of $\mathfrak{g}$

matrix associated to ad_{A_i} is $(M_i)_{kj} = c_{ij}^k$

- ex $so(3)$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are also the matrices of the adjoint representation

$$A_1 = c_{1j}^k = (M_1)_{kj}$$

etc.

linearity of the adjoint representation

$A_1 \dots A_n$ basis of $\mathfrak{g}$

$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ bilinear op.

$$A_i, A_j \longmapsto [A_i, A_j] = \sum_{k=1}^{n} c_{ij}^k A_k$$

fix $A_i \Rightarrow [A_i, \cdot]$ becomes a linear oper. in the remaining term

$$[A_i, \cdot] = \mathrm{ad}_{A_i} : \mathfrak{g} \to \mathfrak{g} \quad \text{linear}$$
$$A_j \longmapsto \mathrm{ad}_{A_i} A_j$$

identify A_j with the elementary vector $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{-th}$

$$A_j \sim \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} j\text{-th} \Rightarrow \mathrm{ad}_{A_i} A_j \sim \mathrm{ad}_{A_i} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{-th}$$

since ad_{A_i} is a linear map $\mathrm{ad}_{A_i} = M_i$ $n \times n$ matrix

$$\mathrm{ad}_{A_i} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = M_i \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} (m_i)_{11} & \cdots & (m_i)_{1j} & \cdots & (m_i)_{1n} \\ \vdots & & & & \\ (m_i)_{n1} & \cdots & (m_i)_{nj} & \cdots & (m_i)_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{-th} = \begin{bmatrix} (m_i)_{1j} \\ \vdots \\ \vdots \\ (m_i)_{nj} \end{bmatrix}$$

$$\left.\begin{aligned} &\sim (m_i)_{1j} A_1 + \dots + (m_i)_{nj} A_n \\ &= c_{ij}^1 A_1 + \dots + c_{ij}^n A_n \end{aligned}\right] \Rightarrow (m_i)_{kj} = c_{ij}^k$$

$\Rightarrow$ matrices M_i of elements $(m_i)_{kj} = c_{ij}^k = (\mathrm{ad}_{A_i})_{kj}$ of dim $n \times n$ always

<u>ex</u>: $so(3)$ has dim 3 and $\mathrm{ad}_{so(3)}$ also has matrices 3×3

<u>ex</u> $se(3)$ of dim 6, while A_i are 4×4 the ad_{A_i} are 6×6 matrices

ex se(2) $\quad A_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \mathrm{ad}_{A_i}$ has also dim 3×3

$[A_1, A_2] = A_3 \qquad [A_1, A_3] = -A_2 \qquad [A_2, A_3] = 0$

$A_k \sim \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ k-th position $\qquad \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$

compute the matrices of structure constants:

$\mathrm{ad}_{A_i} A_j = \sum C_{ij}^k A_k \qquad \mathrm{ad}_{A_i} \sim (M_i)_{kj}$ $\quad$ k index : row, j index : column

ad_{A_1} : only nonzero structure const. are $[A_1, A_2] = C_{12}^3 A_3 \Rightarrow C_{12}^3 = 1$

$[A_1, A_3] = +C_{13}^2 A_3 \Rightarrow C_{13}^2 = -1$

$\Rightarrow (M_1)_{32} = 1$

$(M_1)_{23} = -1$

$\Rightarrow \mathrm{ad}_{A_1} = (M_1)_{kj} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathrm{ad}_{A_1} = \begin{bmatrix} c_{11}^1 & c_{12}^1 & c_{13}^1 \\ c_{11}^2 & c_{12}^2 & c_{13}^2 \\ c_{11}^3 & c_{12}^3 & c_{13}^3 \end{bmatrix}$

ad_{A_2} : only nonzero structure const. is $[A_2, A_1] = -A_3 = C_{21}^3 A_3 \Rightarrow C_{21}^3 = -1$

$\Rightarrow (M_2)_{31} = -1$

$\Rightarrow \mathrm{ad}_{A_2} = (M_2)_{kj} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ in fact $\mathrm{ad}_{A_2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \sim -A_3$

ad_{A_3} only nonzero structure const. is $[A_3, A_1] = A_2 \Rightarrow C_{31}^2 = 1$

$\Rightarrow (M_3)_{21} = 1$

$\mathrm{ad}_{A_3} = M_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, in fact $\mathrm{ad}_{A_3} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \sim A_2$

Killing form K

$K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ K is a symmetric bilinear form defined :
quadratic form

$$K(A, B) = \operatorname{tr}(ad_A \cdot ad_B)$$

Cartan criteria

cartan 1st criterion $\mathfrak{g}$ solvable $\Leftrightarrow K(x, y) = 0$
$\underline{\forall}\, x \in \mathfrak{g},\ y \in \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$

i.e. $\mathfrak{g}$ solvable $\Leftrightarrow K(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$

cartan 2nd criterion

$\mathfrak{g}$ semisimple $\Leftrightarrow K(\cdot, \cdot)$ is nondegenerate

$K(\cdot, \cdot)$ non degenerate if $K(x, y) = 0 \ \forall y \in \mathfrak{g} \Rightarrow X = 0$

def
$\mathfrak{g}$ is said compact if $K(\cdot, \cdot) < 0$ negative definite

ex $so(3)$ $K_{ij} = K(A_i, A_j) = \operatorname{tr}(ad_{A_i} \cdot ad_{A_j})$

ex: $K_{11} = \operatorname{tr}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right) = \operatorname{tr}\begin{pmatrix} 0 & & \\ & -1 & \\ & & -1 \end{pmatrix} = -2$

$\Rightarrow (K)_{ij} = -2I \Rightarrow$ negative definite $x^T K x < 0 \ \forall x$

ex Killing form in $SE(2)$

$$ad_{A_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad ad_{A_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad ad_{A_3} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$K_{ij} = K(A_i, A_j) = tr(ad_{A_i} \cdot ad_{A_j})$$

$$K_{11} = tr(ad_{A_1}^2) = tr \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = -2$$

$K_{ij} = 0$ otherwise

$$\Rightarrow K = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow eig(K) = \{-2, 0, 0\}$$

$\Rightarrow se(2)$ is not compact.

Since $se(2)^{(1)}$ (derived series) $= \mathbb{R}^2$

$K(se(2), se(2)^{(1)}) = K(se(2), \mathbb{R}^2) \equiv 0$ (the only nonzero elem. in K is $K_{11} = K(A_1, A_1)$, where $A_1 \in so(2)$)

$\Rightarrow se(2)$ is solvable.

Killing form

$$k(A_i, A_m) = tr(ad_{A_i} \cdot ad_{A_m}) = tr\left(\left(C_{ij}^{k}\right)\left(C_{mp}^{q}\right)\right) = \sum_{r,s=1}^{n} C_{ir}^{s} C_{ms}^{r}$$

metric tensor

- K nondegenerate $\Rightarrow$ can be used as a (pseudo) Riemannian metric tensor $\langle \cdot, \cdot \rangle \stackrel{\Delta}{=} -\alpha K(\cdot, \cdot)$ i.e. inner product
- K neg def. (compact group) $\Rightarrow$ $g_{ij} = +\alpha K(A_i, A_j)$ A_i, A_j basis elem.

$$g_{ij} = -\alpha \sum_{r,s=1}^{n} C_{ir}^{s} C_{js}^{r}$$

$$\Rightarrow \langle \cdot, \cdot \rangle = -\alpha K(\cdot, \cdot) = \text{Riemannian metric}$$

ex SO(3): $g_{ij} = -\frac{1}{2} K$ $\Rightarrow$ representative quadratic form is I

$\langle \cdot, \cdot \rangle$
- symm.
- pos-def.
- Ad_g invariant i.e. biinvariant (invar. to both left and right translations)

$$\langle X, Y \rangle = \langle Ad_g X, Ad_g Y \rangle$$

. Lie group: $\langle \cdot, \cdot \rangle$ inner product is defined on $\mathfrak{g}$ and propagated to $T_g G$ by left or right translation

$$\langle X_g, Y_g \rangle\big|_g = R_g \circ \langle X_I, Y_I \rangle\big|_{\mathfrak{g}}$$

ex $se(3)$ does not have a biinvariant Riemannian metric

$\Rightarrow \nexists$ a natural way, coordinate indep. way to measure distances on $SE(3)$.

$X, Y \in se(3) \Rightarrow K(X, Y) = x^T M y$ with $M = \begin{pmatrix} -2I_3 & 0 \\ 0 & 0_{3\times 3} \end{pmatrix}$

x, y 6-vectors w.r.t. basis $\{A_1 \ldots A_6\}$ $K(\cdot,\cdot)$ is only semidefinite neg.

- two ways to get around it
  1) double geodesic structure
     - discart the semidirect action $SO(3) \to \mathbb{R}^3$
     - consider separately $SO(3)$ and $\mathbb{R}^3$
     - quadratic form $M = \begin{bmatrix} \alpha I_3 & 0 \\ 0 & \beta I_3 \end{bmatrix}$
     - curves are geodesics in $SO(3)$ ~~and~~ or $\mathbb{R}^3$ but not both
  2) pseudo-Riemannian structure
     - quadratic form Killing + Klein form $\begin{bmatrix} \alpha I & \beta I \\ \beta I & 0 \end{bmatrix}$
     - biinvariant
     - nondegenerate but non positive def
     - curves with positive or negative energy $\to$ screw motions

Linear Lie groups can be defined as sets of matrices that leave a quadratic form invariant

ex

$y, x \in \mathbb{R}^n \Rightarrow O_n = \{ R \in Gl_n \mid \langle Rx, Ry \rangle = \langle x, y \rangle = -\alpha K(x,y) \}$

K = quadratic form $\quad I_n$ = symm matrix representing the quadratic form

ex Lorentz group L

special relativity $\Rightarrow$ Minkowsky space $\bar{X} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\quad x_0 = c \cdot t$

Lorentz transf. = transf that leave $\bar{X}$ invariant

$\Lambda \in L$ if $(\Lambda \bar{X})^T S (\Lambda \bar{X}) = \bar{X} S \bar{X}$

where $S = \begin{bmatrix} I_3 & 0 \\ 0 & -1 \end{bmatrix} \in GL_4(\mathbb{R})$

$\Rightarrow L = \{ \Lambda \in Gl_4 \mid \Lambda^T S \Lambda = S \} = O(3,1)$ diagonalized S has 3 pos. eigenv. and 1 neg. $\Rightarrow$ noncompact.

$o(3,1) = \{ A \in M_4(\mathbb{R}) \mid SA + A^T S = 0 \}$

ex <u>Symplectic group</u> $Sp_{2n}(\mathbb{R})$

- $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ skew-symm form

- $Sp_{2n}(\mathbb{R}) = \left\{ g \in Gl_{2n}(\mathbb{R}) \mid g^T J g = J \right\} \Rightarrow \det g = 1$

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp_{2n} \Rightarrow \begin{cases} a^T c \text{ and } b^T d \text{ symmetric} \\ a^T d - c^T b = 1 \end{cases}$$

- $sp_{2n}(\mathbb{R}) = \left\{ A \in M_{2n} \mid A^T J + J A = 0 \right\}$

$$A \in sp_{2n}(\mathbb{R}) \Rightarrow A = \begin{bmatrix} a & b \\ c & -d \end{bmatrix} \quad \text{s.t.} \begin{cases} d = -a^T \\ c = c^T \\ b = b^T \end{cases}$$

- $\dim Sp_{2n} = 2n^2 + n$

- not compact

SWITCHING SYSTEMS

the question we want to study is the following:

Given a family of systems in $\mathbb{R}^n$

$$\dot{x} = f_p(x) \qquad p \in \mathcal{P} = \text{index set} = \{1, \dots, m\}$$

a switched (switching) system is obtained taking a function for p:

$\underbrace{\sigma : [0, \infty]}_{\text{time}} \longrightarrow \underbrace{\mathcal{P}}_{\text{index set}}$ piecewise constant and containing a finite number of discontinuities

ex: $\sigma : \begin{cases} 1 & [0, 2) \\ 5 & [2, 7) \\ 2 & [7, 8) \\ \vdots \end{cases}$ $\Rightarrow$ indicator function of which vector field f_p in the family is taken at each istant.

switching times = jumps in σ

$\Rightarrow \dot{x}(t) = f_{\sigma(t)}(x(t))$ time-varying system (even if each f_p is autonom

<u>example</u> n linear systems $A_1, \dots A_m$ = "modes" of the switching system

$$\dot{x} = A_\sigma x = \begin{cases} A_1 x & t \in [0, 2) \\ A_5 x & t \in [2, 7) \\ A_2 x & t \in [7, 8) \\ \vdots \end{cases}$$

Assume switching times "do not accumulate" in time

$\Rightarrow \dot{x} = f_\sigma(x)$ is at least Lipschitz

$\Rightarrow$ solution $\exists$ and is unique.

Assume each of the modes of the switched system is globally asymptotically stable. Can we conclude that the switched system is globally asymptotically stable $\forall$ switching functions $\sigma : [0,\infty) \to \mathcal{P}$?

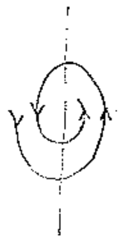
<u>example</u>

$$A_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix}$$

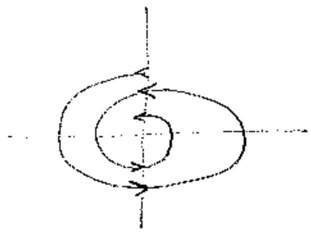
both matrices are Hurwitz.

the trajectories of both systems are ellipses but the "eccentricities" are different.

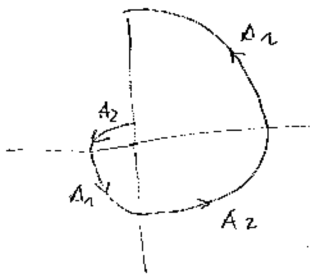
A_1 :



A_2



$\Rightarrow \exists$ a $\sigma : [0,\infty) \to \mathcal{P}$ for which



$\Rightarrow$ system diverges

$\Rightarrow$ stability of each subsystem (mode) is <u>not</u> <u>enough</u> to guarantee stability of the switched system.

Uniform stability

def A function $\beta: [0,\infty) \to [0,\infty)$ is said of class K if α continuous, strictly increasing and $\alpha(0)=0$

def A function $\beta: [0,\infty) \times [0,\infty) \to [0,\infty)$ is said of class KL if $\beta(\cdot,t)$ is of class K for each fixed $t \geq 0$ and $\beta(r,t)$ is decreasing to zero as $t \to \infty$ for each fixed $r \geq 0$

K and KL functions are useful to reformulate the concepts of asymptotic stability

ex: asymptotic stability : $\exists\, \delta > 0$ and $\beta \in KL$ s.t. $\forall$ solutions with $|x(0)| \leq \delta$ one has $|x(t)| \leq \beta(|x(0)|, t)$ $\forall t \geq$

$$\dot{x} = f_p(x) \quad p \in \mathcal{P}$$

def the switched system is uniformly asymptotically stable if $\exists\, \delta > 0$, and $\beta \in KL$ s.t. $\forall$ switching signals $\sigma: [0,\infty) \to \mathcal{P}$ the solutions of $\dot{x} = f_\sigma(x)$ with $|x(0)| < \delta$ satisfy

$$|x(t)| < \beta(|x(0)|, t) \quad \forall t \geq 0$$

def Uniform asymptotic stability (UAS) is global (GUAS) if it holds $\forall$ initial cond. $x(0)$.

For linear systems: UAS $\Leftrightarrow$ GUAS $\Leftrightarrow$ global uniform exponent. stab. (GUES)

Common Lyapunov functions

Lyapunov stability theorem:

thm Given $\dot{x}=f(x)$, assume $\exists$ function $V \in C^1(\mathbb{R})$, $V>0$ (p.d.) s.t. its derivative along the solution of the system satisfies $\dot{V} \leq 0$ $\forall x$. Then the system is stable.
If the derivative satisfy $\dot{V}<0$ $\forall x \neq 0$ ($\dot{V}$ n.d.) then the system is asymptotically stable.

ex linear system $\dot{x}=Ax$ is asymptotically stable if A is Hurwitz matrix i.e. eig(A) have negative real part.
Lyapunov thm corresponds to find $P>0$ (p.d.) s.t. the Lyapunov eq. $A^T P + PA = -Q$ for a given $Q=Q^T>0$ (p.d.)

def Given $\dot{x}=f_p(x)$ $p \in \mathcal{P}$, $V \in C^1(\mathbb{R})$, $V>0$ is a <u>common Lyapunov function</u> for the switching system if $\exists$ $W \in C^0(\mathbb{R})$, $W>0$ (p.d.) s.t. $\mathcal{L}_{f_p} V = \frac{\partial V}{\partial x} f_p(x) \leq -W(x)$ $\forall x$ $\forall p \in \mathcal{P}$

N.B. this condition is stronger than the following

$$\frac{\partial V}{\partial x} f_p(x) < 0 \quad \forall x \quad \forall p \in \mathcal{P}$$

example: $f_p(x) = -px$ $\mathcal{P}=(0,1]$ $\Rightarrow$ one-param. family of switching systems

- Family is Globally asymptotically stable
- ~~common~~ Lyapunov function $V(x)=\frac{x^2}{2}$ is valid $\forall p \in (0,1]$

$$\frac{\partial V}{\partial x} f_p(x) = -px^2 < 0$$

however this is not a common Lyapunov function

$$\frac{\partial V}{\partial x} f_p(x) = -p x^2 \quad \text{when } p \to 0 \text{ as switching mode}$$

this gets "small" $\to 0$ even for x large.

$\Rightarrow$ not a common Lyapunov funct.

this switching system does not admit a common Lyap. funct.: cannot bound $-px^2$ by a negative definite funct. for $0 < p \leq 1$

In fact solving the system for a switching signal $\sigma(t)$

$$x(t) = e^{-\int_0^t \sigma(\tau) d\tau} x(0) \quad \text{does not converge to } 0.$$

- meaning: $\mathcal{P}$ must be taken compact in order to avoid pathological situations $\Rightarrow$ Assume $\mathcal{P}$ compact (e.g. finite set)

Thm If in the switching system $\dot{x} = f_p(x)$ all modes f_p $p \in \mathcal{P}$ share a [radially unbounded] common Lyapunov function then the switching system is GUAS

- radially unbounded function $V: \mathbb{R}^n \to \mathbb{R}$: $V(x) \to \infty$ as $x \to \infty$

Converse Lyapunov thm

thm If switching syst. is GUAS, and $(x,p) \mapsto f_p(x)$ locally Lipschitz in x uniformly over p, then all f_p, $p \in \mathcal{P}$ share a [radially unbounded] common Lyapunov function.

- typical case: convex combinations

Coroll given thm above, the convex combinations

$$\dot{x} = \alpha f_p(x) + (1-\alpha) f_q(x) \quad p, q \in \mathcal{P}, \ \alpha \in [0,1]$$

are globally asympt. stable.

A convex combination $\dot{x} = \alpha f_p(x) + (1-\alpha) f_q(x)$ with f_p and f_q asymptotically stable is not necessarily asymptotically stable.

ex: $f_p(x) = A_1 x$, $f_q(x) = A_2 x$ both linear systems

$$A_1 = \begin{bmatrix} -0.1 & -1 \\ 2 & -0.1 \end{bmatrix} \Rightarrow \text{eig}(A_1) = -0.1 \pm \sqrt{2}$$

$$A_2 = \begin{bmatrix} -0.1 & 2 \\ -1 & -0.1 \end{bmatrix} \Rightarrow \text{eig}(A_2) = -0.1 \pm \sqrt{2}$$

$\alpha = \frac{1}{2}$ convex comb. $\frac{1}{2}A_1 + \frac{1}{2}A_2 = \begin{bmatrix} -0.1 & 0.5 \\ 0.5 & -0.1 \end{bmatrix}$ eig $= \begin{cases} 0.6 \\ 0.4 \end{cases}$

$\Rightarrow$ unstable -

Switched linear system

$$f_p(x) = A_p x \qquad p \in \mathcal{P} \quad \text{(finite) mode set}$$

most common:
Common Lyapunov function : Common Quadratic Lyapunov f.

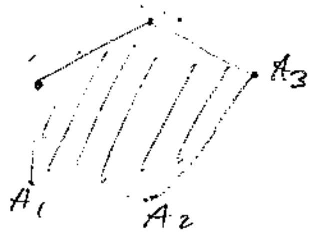
$$V(x) = x^T P x \quad \text{s.t.} \quad A_k^T P + P A_k \leq -Q \quad Q = Q^T > 0$$
$$P = P^T > 0 \qquad \forall k \in \mathcal{P}$$

since $\mathcal{P}$ compact $A_k^T P + P A_k < 0$ is enough $\forall k \in \mathcal{P}$

$\Rightarrow$ CQLF (Common Quadratic Lyapunov function)
can be found by LMI (Linear Matrix Inequalities)

- infeasibility test: a CQLF does not $\exists$ for $\mathcal{P} = \{1, \dots, m\}$
  iff. the eq. $R_0 = \sum_{i=1}^{m} (A_i R_i + R_i A_i^T)$ is satisfied by
  $R_0, R_1, \dots, R_m \geq 0$ (nonnegative) $R_i = R_i^T$

meaning of common Lyapunov funct: use a single Lyap. funct. to "cover" the entire convex polytope whose extremes (vertices) are A_i $i \in \mathcal{P}$.



$\dot{x} = A_\sigma x$ can be anything inside the convex polytope.

Is CQLF enough? No!

<u>example</u> $\mathcal{P} = \{1, 2\}$ $A_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ $A_2 = \begin{bmatrix} -1 & -10 \\ 0.1 & -1 \end{bmatrix}$

<u>fact 1</u> the systems $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$ do not share a CQLF

<u>proof</u>: take $P = \begin{bmatrix} 1 & q \\ q & r \end{bmatrix}$ (without loss of generality)

$$A_1^T P + P A_1 < 0 \iff -A_1^T P - P A_1 = \begin{pmatrix} 2-2q & 2q+1-r \\ 2q+1-r & 2q+2r \end{pmatrix} > 0$$

(in fact $-A_1^T P - P A_1 = -\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & q \\ q & r \end{bmatrix} - \begin{bmatrix} 1 & q \\ q & r \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$)

A necessary cond. for $-A_1^T P - P A_1 > 0$ is that $\det(-A_1^T P - P A_1) > 0$

$$\det(-A_1^T P - P A_1) = (2-2q)(2q+2r) - (2q+1-r)(2q+1-r)$$

$$= \cancel{4q} - 4q^2 + 4r - 4q^2 - \cancel{2q} - \cancel{2qr} - \cancel{2q} - 1 + r + \cancel{2qr} + r - r^2$$

$$= -8q^2 + 6r - r^2 - 1 > 0$$

$$\iff +8q^2 - 6r + r^2 + 1 + 8 - 8 < 0$$

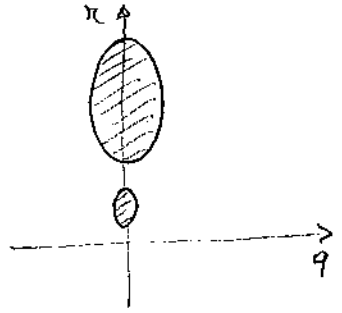
$$8q^2 + (r^2 - 6r + 9) < 8$$

$$q^2 + \frac{(r-3)^2}{8} < 1$$

For the other system:

$$-A_2^T P - P A_2 = \begin{bmatrix} 2 - q/5 & 2q + 10 - r/10 \\ 2q + 10 - r/10 & 20q + 2r \end{bmatrix} > 0 \iff q^2 + \frac{(r-300)^2}{800} < 100$$

in the q-r plane the two inequalities are satisfied by the interiors of the two ellipses



$\Rightarrow$ intersection is empty

$\Rightarrow \nexists$ CQLF

<u>fact 2</u>: the switched system $\dot{x} = A_\sigma x \quad \forall \sigma \in \mathcal{P}$ is GUAS

<u>proof</u> look at the worst case switching

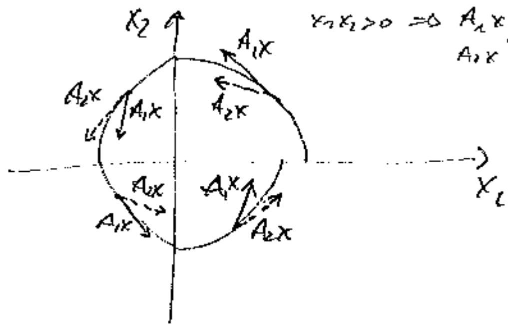
in $\sigma = 1$ $\begin{cases} \dot{x}_1 = -x_1 - x_2 \\ \dot{x}_2 = x_1 - x_2 \end{cases}$

in $\sigma = 2$ $\begin{cases} \dot{x}_1 = -x_1 - 10x_2 \\ \dot{x}_2 = 0.1x_1 - x_2 \end{cases}$

consider $\|x\|^2 = x_1^2 + x_2^2$

$$\frac{d\|x\|^2}{dt} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = \begin{cases} 2x_1(-x_1 - x_2) + 2x_2(x_1 - x_2) & \text{in } \sigma = 1 \\ 2x_1(-x_1 - 10x_2) + 2x_2(0.1x_1 - x_2) & \text{in } \sigma = 2 \end{cases}$$

$$\frac{d\|x\|^2}{dt} = \begin{cases} -2(x_1^2 + x_2^2) \\ -2(x_1^2 + x_2^2) - 19.8\, x_1 x_2 \end{cases}$$



$x_1 x_2 > 0 \Rightarrow A_1 x$ "worst"
$A_2 x$ "better"

worsts case strategy is still such that

$$\frac{d\|x\|^2}{dt} < 0 \text{ always}$$

$\Rightarrow$ switched system is GUAS

Switching systems and structural properties of Lie algebras

Ie: we look at linear switching systems <=> matrices in gl_n

thm If $\{A_p, p \in \mathcal{P}\}$ is a finite set of commuting Hurwitz matrices then the switched system $\dot{x} = A_\sigma x \quad \sigma \in \mathcal{P}$ is GUAS (and GUES)

Proof consider the case of only two systems A_1, A_2 : $\mathcal{P} = \{1, 2\}$.

A_1, A_2 commuting: $[A_1, A_2] = 0 = A_1A_2 - A_2A_1$. Computing exp

$\Rightarrow e^{A_1} e^{A_2} = e^{A_2} e^{A_1}$ or, more generally,

$$e^{A_1 t} e^{A_2 \tau} = e^{A_2 \tau} e^{A_1 t} \qquad \forall t, \tau > 0$$

For an arbitrary switching signal $\sigma = \begin{cases} 1 & \text{for a time } t_1 \\ 2 & \text{'' '' ''} \quad \tau_1 \\ 1 & \text{'' '' ''} \quad t_2 \\ 2 & \text{'' '' ''} \quad \tau_2 \\ \vdots & \end{cases}$

$$\Rightarrow x(t) = \dots e^{A_2 \tau_2} e^{A_1 t_2} e^{A_2 \tau_1} e^{A_1 t_1} x(0)$$

by commutation

$$= \dots e^{A_2 \tau_2} e^{A_2 \tau_1} e^{A_1 t_2} e^{A_1 t_1} x(0)$$

furthermore $\forall [A, B] = 0$ one has $e^A e^B = e^{A+B}$

$$\Rightarrow x(t) = e^{A_2(\tau_1 + \tau_2 + \dots)} e^{A_1(t_1 + t_2 + \dots)} x(0)$$

$$\left(= e^{A_2(\tau_1 + \tau_2 + \dots) + A_1(t_1 + t_2 + \dots)} x(0) \right)$$

when $t \to \infty$ at least one of the two sums $t_1 + t_2 + \dots$ or $\tau_1 + \tau_2 + \dots$ must $\to \infty$. Since both A_1 and A_2 are Hurwitz $e^{A_i t} \xrightarrow{t \to \infty} 0$

$\Rightarrow x(t) \xrightarrow{t \to \infty} 0$

Extension to noncommuting matrices

Thm If $\{A_p \; p \in \mathcal{P}\}$ is a finite set of Hurwitz matrices and the Lie algebra $\mathfrak{g} = \mathrm{Lie}\{A_p \; p \in \mathcal{P}\}$ is solvable then the switched system $\dot{x} = A_\sigma x$ is GUES

proof (sketch) The proof relies on the Lie theorem that affirms that $\mathfrak{g}$ solvable $\Leftrightarrow$ all matrices of $\mathfrak{g}$ can be rendered simultaneously upper triangular by a (possibly complex) linear change of coordinates.

For a finite set of upper triangular matrices one can proceede as follows. Assume $\mathcal{P} = \{1, 2\}$ (no loss of generality)

$$A_1 = \begin{bmatrix} -a_1 & b_1 \\ 0 & -c_1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -a_2 & b_2 \\ 0 & -c_2 \end{bmatrix} \qquad a_i, c_i > 0$$

consider the switched system $\dot{x} = A_\sigma x$. Its second component is $\dot{x}_2 = c_\sigma x_2$ $\Rightarrow$ subsystem decays exponentially to 0

$\Rightarrow$ 1st subsystem $\dot{x}_1 = \underbrace{-a_\sigma x_1}_{\text{exponentially stable}} + \underbrace{b_\sigma x_2}_{\text{exponentially decaying "input"}}$

$\Rightarrow$ also $x_1(t) \xrightarrow{t \to \infty} 0$ exponentially

Consideration can be generalized to arbitrary dimension and arbitrary $\mathcal{P}$.

//

Lie-algebraic interpretation

when you switch system you compose flow. Flow composition must obey to the Baker-Campbell-Hausdorff formula

$$e^{A} e^{B} = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]+[B,[A,B]])+\cdots}$$

$\Rightarrow$ all the higher order bracket count.

If $\mathfrak{g}$ solvable, $A, B \in \mathfrak{g}$ A, B upper triangular

$\Rightarrow$ $[A, B]$ is also upper triangular

es: $\mathfrak{g}$ nilpotent of dim 3

family: $\{A_p, p \in \mathcal{P} = \{1, 2\}\}$ s.t. $[A_1, A_2] = A_3$

B-C-H-formula:

$$e^{A_1} e^{A_2} = e^{A_1 + A_2 + \frac{1}{2}[A_1, A_2]}$$

$\Rightarrow$ can exchange the order of the terms (changing also the signs in $[\cdot,\cdot]$) and regroup

$$[A_3, A_1] = [A_3, A_2] = 0$$

$$\Rightarrow e^{A_1} e^{A_2} = e^{A_1 + A_2} e^{A_3}$$

generalization:

<u>thm</u> If $\{A_p, p \in \mathcal{P}\}$ is a finite set of Hurwitz matrices s.t. the Lie algebra $\mathfrak{g} = \mathrm{Lie}\{A_p \; p \in \mathcal{P}\}$ is a semidirect sum of a compact (semisimple) subalgebra and a solvable ideal then the switched system is GUES

<u>proof</u> $\mathfrak{g} = \underbrace{k}_{\text{compact semisimple}} \circledS \underbrace{r}_{\text{solvable ideal: seen earlier}}$

idea behind the proof: compact part of $\mathfrak{g}$ only "rotates" the solvable ideal (in each $A_p \in$ compact part A_p has eigenvalues of zero real part and/or purely imaginary) $\Rightarrow$ acting on the radical it does not modify the length of vectors $\|x\|$

Alternative charact. of the above thm:

semidirect decomp. $k \circledS r$ with k compact

$\Leftrightarrow K(\mathfrak{g}, \mathfrak{g}) \leq 0$ (Killing form negative semidef.)

<u>ex</u> $A_p = -\lambda_p I + S_p$ where $\lambda_p > 0$, $S_p^T = -S_p$ $\forall p \in \mathcal{P}$

A_p is automatically Hurwitz

$\Rightarrow \mathfrak{g} = k \circledS \mathrm{sp}\{I\}$ where $k = \mathrm{Lie}\{S_p \; p \in \mathcal{P}\}$

$\Rightarrow$ theorem above holds

extension to nonlinear systems : triangular systems

Prop Assume vector field have the following upper triangular form

$$f_p(x) = \begin{bmatrix} f_p^1(x_1, \dots x_n) \\ f_p^2(x_2, \dots, x_n) \\ \vdots \\ f_p^n(x_n) \end{bmatrix} \qquad p \in \mathcal{P}$$

and that their linearizations $\frac{\partial f_p}{\partial x}$ are Hurwitz

then the switched system $\dot{x} = f_\sigma(x)$ $\sigma \in \mathcal{P}$ is locally GUES

· Control theoretic interpretation:

(*) $\dot{x} = A_p x \quad p \in \mathcal{P}$ corresponds to the control system
$\mathcal{P} = \{1, \dots, m\}$

(**) $\dot{x} = \sum_{i=1}^{m} A_i u_i x$ bilinear control system with constrained controls : $u_i \leftrightarrow$ switching signal

$$\Rightarrow u_i = \begin{cases} 0 & \text{inactive} \\ 1 & \text{active} \end{cases}$$

$$u_i \cdot u_j = 0 \qquad \forall t$$

$\Rightarrow$ control set is <u>not</u> symmetric w.r.t. origin

$\Rightarrow$ positive controls

(*) GUAS $\Rightarrow$ (**) cannot be controllable (in a noncompact manifold)

(**) controllable (in noncompact) $\Rightarrow$ $\exists$ switching signal that takes the switched system everywhere (also "arbitrarily far" from $x=0$)

$\Rightarrow$ "sort of duality" between controllability and GUAS of the switched system.

QUANTUM CONTROL

Sp: finite dimensional quantum mechanical system

dim = N = n. of levels of energy for the quantum state

$\Rightarrow$ state lives in a N-dimensional complex Hilbert space $\mathcal{H}$

$$\mathcal{H} = \mathbb{C}^N$$

state $\quad \psi \in \mathcal{H}$

- in Dirac notation: $|\psi\rangle$ ("ket", because the inner product is indicated as $\underbrace{\langle \varphi|}_{\text{"bra"}} \underbrace{|\psi\rangle}_{\text{"ket"}}$)
- $|\psi\rangle$ represents the position of a particle/wave function (momentum is not considered)
- $\dim_{\mathbb{C}} \mathcal{H} = N \qquad \dim_{\mathbb{R}} \mathcal{H} = 2N$
- choose orthonormal basis of $\mathcal{H}$.

$$|1\rangle, \dots, |N\rangle \text{ s.t. } \langle i|j\rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

then $|\psi\rangle = \sum_{i=1}^{N} \alpha_i |i\rangle$

where α_i = probability "amplitude" that a particle materializes somewhere, in a certain (i-th) basis element.

probabilistic interpretation $\Rightarrow \langle \psi|\psi\rangle = \|\psi\|^2 = 1$

$\Rightarrow \sum \alpha_i^2 = 1$

$\Rightarrow |\psi\rangle$ lives in the (complex) unit sphere in $\mathbb{C}^N$

$\Rightarrow |\psi\rangle \in S^{2N-1}_{\mathbb{C}} \subsetneq \mathbb{C}^N$

- complex sphere $S^N_{\mathbb{C}}$ is a homogeneous space of the Lie group of complex rotation $\to$ Unitary group

- Unitary group:

$$U(N) = \{ U \in GL_n(\mathbb{C}) \text{ s.t. } UU^\dagger = U^\dagger U = I \}$$

where $U^\dagger$ = Hermitian (transpose) of U
= conjugate transpose of U

unitary group: $U^{-1} = U^\dagger$

- Lie algebra of the unitary group

$$\mathfrak{u}(N) = \{ A \in \mathfrak{gl}_n(\mathbb{C}) \mid A^\dagger = -A \}$$ skew-Hermitian matrices

- (real) dimension of $U(N) = N^2$

- $U(N)$ is closed, compact

- phase ambiguity: $U \in U(N) \Rightarrow |\det(U)| = 1$

$\Rightarrow$ also $U' = e^{i\varphi} U \in U(N)$ $\varphi \in \mathbb{R}$

$\Rightarrow$ the entire one-param. subgroup of "phase factors"
(i.e. $U \in U(N) \to e^{i\varphi} U = U e^{i\varphi}$) is in $U(N)$

this is $U(1) \sim S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$

these global phase factors are not interesting

$\Rightarrow$ factor out the global phase factor

$\Rightarrow$ special unitary group

- **Special Unitary group** $SU(N)$ subgroup of $U(N)$

$$SU(N) = \{ U \in GL_n(\mathbb{C}) \mid UU^\dagger = U^\dagger U = I \text{ and } \det U = 1 \}$$

$$= U(N) \cap SL_N(\mathbb{C})$$

(in $\mathbb{R}$: $SO(N) = O(N) \cap SL_N(\mathbb{R})$ identifies the component of $GL_N(\mathbb{R})$ having det > 0. GL_n^- $\cdot I$ GL_n^+

Here rather than 2 disconnected components there is an entire 1-param. subgroup of global phases)

$\dim SU(N) = N^2 - 1$ (real dimension)

- **special unitary Lie algebra**

$$su(N) = \{ A \in M_N(\mathbb{C}) \mid A^\dagger = -A, \ tr(A) = 0 \}$$

skew-Hermit, traceless matrices.

$\dim su(N) = N^2 - 1$

$su(N)$: semisimple, compact Lie algebra

semisimple $\uparrow$ Killing form nondegenerate

compact $\uparrow$ Killing form neg. def.

· *ex* $su(2) = \{ A \in M_2(\mathbb{C}) \mid A^\dagger = -A,\ tr(A) = 0 \}$

$\dim su(2) = 3$

basis: take <u>Pauli matrices</u> (Hermitian 2×2 matrices)

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \text{traceless}$$

consider the skew-Hermitian matrices

$$A_i = -\frac{i}{2}\sigma_i$$

$$A_1 = -\frac{i}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \qquad A_2 = -\frac{i}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad A_3 = -\frac{i}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

commutation relations: $[A_1, A_2] = A_3$

$[A_2, A_3] = A_1$

$[A_3, A_1] = A_2$

$\Rightarrow su(2) = \mathrm{span}\{A_1, A_2, A_3\}$

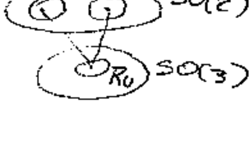
· structure constant are all real $\Rightarrow$ Lie algebra over $\mathbb{R}$ (as field)

· commutation relations are the same as $so(3)$

$\Rightarrow$ adjoint representation of $su(2)$ and $so(3)$ (made with matrices of structure constants) must be the same.

$\Rightarrow$ Lie algebra isomorphism between $so(3)$ (real rotations) and $su(2)$ (complex rotations)

(not between the groups: $SU(2)$ is a 2-to-1 cover of $SO(3)$)



. Schrödinger equation

for $|\psi\rangle \in S_c^{2N-1} \subset \mathbb{C}^N$ we have

- postulate: $|\psi\rangle$ obeys to a "time-dependent" Schrödinger eq.

$$\begin{cases} i\hbar \dfrac{d|\psi\rangle}{dt} = H|\psi\rangle \\ |\psi(0)\rangle = |\psi_0\rangle \end{cases}$$

H = Hamiltonian

. linear, time-independent, Hermitian $H = H^+$

- assume: $\hbar = 1$ (Plank const.)

- multiply by $-i$ ⟹

$$\boxed{|\dot{\psi}\rangle = -iH|\psi\rangle \ , \ |\psi(0)\rangle = |\psi_0\rangle}$$

$A = -iH$ is skew-Hermitian $A^T = -A$

⟹ $A \in \mathfrak{u}(N)$ unitary Lie algebra

- $tr(A)$ is a global phase factor $A = \underbrace{\hat{A}}_{traceless} + \frac{i\varphi}{N} I = \hat{A} + \frac{tr(A)}{N} I$

⟹ $|\psi(t)\rangle = e^{i\varphi t} e^{\hat{A}t} |\psi(0)\rangle$ because $[I, \hat{A}] = 0$

⟹ exponential $e^{At} = e^{\hat{A}t + \frac{i\varphi}{N}It} = e^{\hat{A}t} e^{i\varphi t}$ factorizes

(from the Campbell-Baker-Hausdorff f.)

⟹ can get rid of the global phase factor

⟹ from now on $A \in \mathfrak{su}(N)$ i.e. $tr(A) = 0$

special unitary Lie algebra

• <u>Unitary propagator</u>

- $|\psi\rangle \in S_{\mathbb{C}}^{2N-1}$
- $SU(N)$ acts transitively on $S_{\mathbb{C}}^{2N-1}$: $\forall\ |\psi_1\rangle, |\psi_2\rangle \in S_{\mathbb{C}}^{2N-1}$

  $\exists\ U \in SU(N)$ s.t. $|\psi_2\rangle = U|\psi_1\rangle$

  $\Rightarrow S^{2N-1}$ homogeneous space of $SU(N)$

• $\Rightarrow$ we can lift the Schrödinger eq. to $SU(N)$

$$(*)\quad \begin{cases} \dfrac{dU}{dt} = A\,U(t) \\ U(0) = I \end{cases} \qquad U \in SU(N)$$

$\Rightarrow$ the solution $|\psi(t)\rangle = e^{At}|\psi(0)\rangle$ can be equivalently written as $|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$ where $U(t)$ solves $(*)$

$U(t) \triangleq$ <u>unitary propagator</u>

$\triangleq$ time-evolution operator of the Schrod. eq.

- stationary Schrodinger equation : eigenvalues eq. for H

$$A|\psi\rangle = -iH|\psi\rangle = -iE_k|\psi\rangle$$

$\Rightarrow -iE_k$ eigenvalues of iH (purely imaginary) (E_k real eig. of H)

$\Rightarrow$ can be used to diagonalize A (change of basis)

$\Rightarrow$ assuming to fix a basis $|1\rangle, \dots, |N\rangle$ of eigenvectors (called eigenfunctions) one can always write A as

$$A = -i \begin{bmatrix} E_1 & & 0 \\ & \ddots & \\ 0 & & E_N \end{bmatrix} \quad \text{diagonal.} \quad \in su(N)$$

E_i = energy levels of the system.

$A \in su(N) \Rightarrow tr(A) = 0 \Leftrightarrow E_1 + \dots + E_N = 0$

- On the basis of eigenvectors evolution is trivial:

$$|\psi(t)\rangle = e^{At}|\psi(0)\rangle$$

If $|\psi\rangle = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_N \end{bmatrix}$ $\quad \psi_k(t) = e^{-iE_k t}\psi_k(0)$ $\Rightarrow$ each eigenstate acquire only a phase factor. (relative phase factor this time)

$\Rightarrow$ eigenstructure does not change.

example $N=2$ "two-level" system

quantum bit = qubit

→ used in quantum computations

- computational basis $|0\rangle, |1\rangle$

$\Rightarrow |\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ s.t. $\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \in S^1$ i.e. $|\alpha|^2+|\beta|^2=1$

- quantum computer works by applying unitary gates

i.e. unitary transformations in $SU(2)$

$$|\psi(0)\rangle \longrightarrow U(t)|\psi(0)\rangle = |\psi(t)\rangle$$

- diagonal $A = -i\gamma\underbrace{\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}}_{\text{Pauli matrix } \sigma_3} = -i\gamma\sigma_3 = -i\begin{bmatrix}E_1 & 0\\ 0 & E_2\end{bmatrix}$ i.e. $E_1+E_2=0$
i.e. $E_2=-E_1$

$$|\psi(t)\rangle = e^{At}|\psi(0)\rangle \Rightarrow \begin{bmatrix}\alpha(t)\\ \beta(t)\end{bmatrix} = e^{-i\gamma\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}t}\begin{bmatrix}\alpha(0)\\ \beta(0)\end{bmatrix} = \begin{bmatrix}e^{-i\gamma t} & 0\\ 0 & e^{i\gamma t}\end{bmatrix}\begin{bmatrix}\alpha(0)\\ \beta(0)\end{bmatrix}$$

- if $\begin{bmatrix}\alpha(0)\\ \beta(0)\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix} \Rightarrow \begin{bmatrix}\alpha(t)\\ \beta(t)\end{bmatrix} = \begin{bmatrix}e^{-i\gamma t}\\ 0\end{bmatrix}$ global phase factor

- if $\begin{bmatrix}\alpha(0)\\ \beta(0)\end{bmatrix} = \begin{bmatrix}\alpha_0\\ \beta_0\end{bmatrix}$ $\alpha_0, \beta_0 \neq 0 \Rightarrow \begin{bmatrix}\alpha(t)\\ \beta(t)\end{bmatrix} = \begin{bmatrix}e^{-i\gamma t}\alpha_0\\ e^{i\gamma t}\beta_0\end{bmatrix}$ relative phase factor (cannot neglect)

• Cartan subalgebra

• consider $A = -i \begin{bmatrix} E_1 & & 0 \\ & \ddots & \\ 0 & & E_N \end{bmatrix} \in su(N)$ i.e. s.t. $E_1 + \dots + E_N = 0$

• $\exists$ a $N-1$ dimensional subspace of $su(N)$ of diagonal matrices. Denote $A_1, \dots, A_{N-1}$ a basis of these diag. matrices

$h = \text{span}\{A_1, \dots, A_{N-1}\} \subsetneq su(N)$ is called the

<u>Cartan subalgebra</u> of $su(N)$

since $[A_i, A_j] = 0$ (because they are diagonal)

h is an abelian subalgebra in $su(N)$

h = largest abelian subalgebra in $su(N)$

Driven quantum mechanical system

- quantum mechanical system interacting with an external field: $|\mathring{\psi}\rangle = A|\psi\rangle$ becomes time-varying
  $\Rightarrow |\mathring{\psi}\rangle = A(t)|\psi\rangle$ where $A(t) = A(t,u)$

- semiclassical approximation

  coupling with an external field is represented an an additive perturbation term depending linearly on a "tunable" parameter $\rightarrow$ control

  $\rightarrow$ driven Schrödinger eq.

  $|\mathring{\psi}\rangle = (A + Bu)|\psi\rangle$ or $|\mathring{\psi}\rangle = \left(A + \sum_{i=1}^{m} B_i u_i\right)|\psi\rangle$ $\quad |\psi\rangle \in S^{2N-1}$

  $A, B_1, \dots, B_m \in su(N)$

  $\left\{\begin{array}{ll} A = -iH_0 & H_0 = \text{free Hamilton} \\ B = -iH_1 & H_1 = \text{control Hamilt} \end{array}\right.$

- lifted to the group $SU(N)$

$$\begin{cases} \dfrac{dU}{dt} = \left(A + \sum B_i u_i\right) U & U \in SU(N) \\ U(0) = I \end{cases}$$

right invariant since the control fields are in the inertial frame (lab. frame)

- example: * laser field for molecular systems
  * magnetic fields for nuclear/electron spin
  * photons for cold atoms in a cavity
  * electric fields for "quantum dots"

example dipole approximation:

In the basis in which $A = -i \begin{bmatrix} E_1 & & 0 \\ & \ddots & \\ 0 & & E_n \end{bmatrix}$ is diagonal

assume E_i ordered $E_1 \leq E_2 \leq \ldots \leq E_n$

then the control field couple only nearest energy levels

$$B = \begin{bmatrix} 0 & b_{12} & & 0 \\ b_{12}^* & \ddots & \ddots & \\ & \ddots & \ddots & b_{N-1,N} \\ 0 & & b_{N-1,N}^* & 0 \end{bmatrix}$$

graph(B) = "allowed transitions" between energy levels.

Lemma If A diagonal, a necessary cond for controllab. of

$\Gamma = A + Bu \in su(N)$ is that Graph(B) connected

proof: If Graph(B) not connected $\Rightarrow \exists$ permutation P s.t.

$$P^{-1} B P = \left[\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right]$$

since A diagonal, both $P^{-1}BP$ and $P^{-1}AP$ have invariant subspaces (the diagonal blocks) and therefore controllab. cannot hold //

Controllability of driven Schrodinger eq.

properties to use

- $su(N)$ compact, semisimple
- $\Lambda : SU(N) \times S_{\mathbb{C}}^{2N-1} \to S_{\mathbb{C}}^{2N-1}$ is transitive

- <u>Thm</u>: $\Gamma = A + Bu \in su(N)$ is controllable (for long-enough times) in $SU(N) \iff \mathrm{Lie}(A,B) = su(N)$

- <u>thm</u> Γ controllable on $SU(N)$, Λ-action transitive on $S_{\mathbb{C}}^{2N-1}$
  $\Rightarrow \mathcal{F} = \Lambda_*(\Gamma) = (A+Bu)|\psi\rangle$ is controllable on $S_{\mathbb{C}}^{2N-1}$

- alternative conditions

<u>def</u> A is <u>regular</u> (or nondegenerate) if $E_i \neq E_j \ \forall i,j = 1,\dots,N, i\neq j$
(meaning : all energy levels [eigenvalues of A] are distinct

<u>def</u> A is <u>strongly regular</u> (or with no degenerate transitions) if $E_i - E_j \neq E_k - E_l \ \forall i,j,k,l \in \{1,\dots,N\}$
$i\neq j$, $k \neq l$, $(i,j)\neq(k,l)$

(meaning: all transition frequencies [resonances] are distinct).

thm A strongly regular, Graph(B) connected $\Rightarrow$ $\Gamma = A + Bu \in su(N)$ controllable almost always (i.e. for almost all pairs of $A, B \in su(N)$)

proof Compute the ad-brackets:

$$[A, B] = ad_A B = ad_A \sum_{ij} b_{ij} = \sum (E_i - E_j) b_{ij}$$

$$\vdots$$

$$ad_A^k B = \sum_{ij} (E_i - E_j)^k b_{ij}.$$

If $b_{ij} \neq 0$ $\forall i \neq j$ then the ad-commutators are enough to achieve the LARC

If e.g. $b_{i,i+1} \neq 0$ (dipole approximation) then some more work must be done; however explicit sets of LARC-achieving commutators can be obtained explicitely. //

This is a particular case of the following general thm:

thm For $\mathfrak{g}$ semisimple the set of pairs $A, B \in \mathfrak{g}$ for which $Lie(A, B) = \mathfrak{g}$ is open and dense

meaning: in semisimple Lie algebras new brackets (in new directions) are always produced, and almost always they become involutive only when the entire Lie algebra is spanned.

$\Rightarrow$ control systems on semisimple, compact Lie algebras are almost always controllable.

example: 2-level system.

H_0 = free Hamiltonian $= \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} = \gamma \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \gamma \sigma_3$

H_1 = control Hamiltonian: coupling between the two energy levels

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_1$

u = amplitude of the coupling → control param.

$\Rightarrow A = -i H_0 = -i \gamma \sigma_3$

$B = -i H_1 = -i \sigma_1$

$\Rightarrow \Gamma = A + Bu \in su(2)$

$\mathrm{Lie}\{A, B\} = su(2)$ ⇒ syst has LARC

$\left. \begin{array}{l} su(2) \text{ compact} \\ \text{semisimple} \end{array} \right\}$ ⇒ LARC ≡ controll (for times long enough)

Density operator

$|\psi\rangle \in S^{2N-1}$ s.t. $\langle\psi|\psi\rangle = 1$ (inner product)

. in an orthonormal basis : $|1\rangle, \dots, |N\rangle$

$$|\psi\rangle = \sum_{j=1}^{N} c_j |j\rangle \quad \text{s.t.} \quad \sum_{j=1}^{N} |c_j|^2 = 1$$

(total prob. of finding the state in one of its eigenstates)

def density operator of $|\psi\rangle$ is the outer-product $|\psi\rangle\langle\psi| \triangleq \rho$

$$|\cdot\rangle\langle\cdot| = \text{outer product} = \begin{bmatrix} \\ \\ \end{bmatrix}\begin{bmatrix} & & \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$\Rightarrow$ rank-1 matrix $N \times N$ matrix

$$|\psi\rangle\langle\psi| \sim \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix}\begin{bmatrix} c_1^* & \cdots & c_N^* \end{bmatrix} = \begin{bmatrix} c_1 c_1^* & c_1 c_2^* & \cdots & c_1 c_N^* \\ c_2 c_1^* & & & \\ \vdots & & & \\ c_N c_1^* & \cdots & & c_N c_N^* \end{bmatrix}$$

$$\Rightarrow \rho = |\psi\rangle\langle\psi| = \sum_{jk=1}^{N} c_j c_k^* |j\rangle\langle k| \quad \text{where } |j\rangle \sim \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{-th point}$$

$$\Rightarrow |j\rangle\langle k| \underset{j \to}{\sim} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{c} \scriptstyle k\text{-th} \\ \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots \end{bmatrix} \end{array}$$

statistical ensemble

Assume to have a collection of quantum states $|\psi_1\rangle, \dots, |\psi_K\rangle$.

the density operator is the quantum mechanical equivalent of density of probability and is used to describe the statistical ensemble of states (of identical systems)

$$\rho = \sum_K p_K |\psi_K\rangle\langle\psi_K| \qquad \begin{array}{l} 0 \le p_K \le 1 \quad \sum_K p_K = 1 \\ p_K = \text{probab. to find } |\psi_K\rangle \text{ in the ensemble} \end{array}$$

- summation is up to N because we have at most N indep. quantum

⇒ by regrouping in terms of independent $|\psi_K\rangle$

$$\rho = \sum_{K=1}^{N} p_K |\psi_K\rangle\langle\psi_K| \qquad 0 \le p_K \le 1 \qquad \sum p_K = 1$$

p_K = probab. of the population in the state $|\psi_K\rangle$

- when all members of the collection are in the same state $|\hat{\psi}\rangle$

⇒ $\rho = 1 \cdot |\tilde{\psi}\rangle\langle\tilde{\psi}|$ → rank-1 matrix

→ pure state

- mixed state when at least 2 $p_K \ne 0$ ⇒ not possible to describe the ensemble by means of $|\psi\rangle$

⇒ density operator is more general than wavefunction properties of ρ

1) $tr(\rho) = 1$

<u>proof</u>: $tr(|\psi_k\rangle\langle\psi_k|) = 1 = tr\begin{bmatrix} c_1c_1^* & & & \\ c_2c_1^* & c_2c_2^* & & \\ \vdots & & \ddots & \\ \vdots & & & c_Nc_N^* \end{bmatrix} = \sum c_ic_i^* = 1$

because $\langle\psi_k|\psi_k\rangle = 1$

$\Rightarrow tr\left(\sum_{k=1}^{N} p_k |\psi_k\rangle\langle\psi_k|\right) = \sum_{k=1}^{N} p_k\, tr(|\psi_k\rangle\langle\psi_k|) = \sum_{k=1}^{N} p_k = 1$ //

2) $\rho = \rho^\dagger$ Hermitian

<u>proof</u>: obvious from $\begin{bmatrix} c_1c_1^* & c_1c_2^* & \cdots & c_1c_N^* \\ c_2c_1^* & \ddots & & \\ \vdots & & \ddots & \\ c_Nc_1^* & & & \ddots \end{bmatrix}$

3) $\rho \geq 0$ positive semidefinite

<u>proof</u>: diagonalize ρ $\to$ spectral decomposition of ρ

$\Rightarrow \rho = \sum_{k=1}^{N} \lambda_k |k\rangle\langle k|$

λ_k = eigenvalues of ρ $\equiv$ probabilities in the new basis

$\Rightarrow \lambda_k \geq 0 \qquad \sum_{k=1}^{N} \lambda_k = 1$

$\Rightarrow \rho \geq 0$ //

4) $tr(\rho^2) \leq 1$ follows from previous $0 \leq \lambda_k \leq 1 \quad \sum_{k=1}^{N} \lambda_k = 1$

- <u>pure state</u>: $tr(\rho^2) = 1 \qquad \lambda_1 = 1 \quad \lambda_2 = \ldots = \lambda_N = 0$

$\Rightarrow$ rank-1 matrices

$\rho^2 = \rho\cdot\rho = |\psi\rangle\underbrace{\langle\psi|\psi\rangle}_{=1}\langle\psi| = \rho$ with $tr(\rho) = 1$

• mixed state: $tr(\rho^2) < 1$

$\Rightarrow$ at least 2 eigenvalues $\neq 0$

$rank(\rho) \geq 2$

$tr(\rho^2)$ = degree of mixing

• <u>time evolution</u> for the density operator

$|\psi(t)\rangle = U(t)|\psi(0)\rangle$

$\langle\psi(t)| = (|\psi(t)\rangle)^+ \Rightarrow \langle\psi(t)| = \langle\psi(0)|U^+(t)$

$\Rightarrow$ for a density operator $\rho = \sum_{k=1}^{N} p_k |\psi_k\rangle\langle\psi_k|$ one gets

$$\boxed{\rho(t) = U(t)\rho(0)U^+(t)}$$

proof: for a pure state: $\rho(t) = |\psi(t)\rangle\langle\psi(t)| = U(t)|\psi(0)\rangle\langle\psi(0)|U^+(t)$

$= U(t)\rho(0)U^+(t)$

for a mixed state: proof is analogous //

<u>Liouville equation</u>

from $\begin{cases} \frac{dU(t)}{dt} = -iH(t)U(t) \\ U(0) = I \end{cases}$ for a generic time-varying Hamiltonian $H(t)$

and $UU^+ = I \Rightarrow \frac{dU}{dt}U^+ + U\frac{dU^+}{dt} = 0$

$$\frac{dU^+}{dt} = -U^+\frac{dU}{dt}U^+ = -U^+(-iH(t))\cancel{U}\cancel{U^+} = U^+(iH)$$

one gets $\frac{d\rho}{dt} = \frac{d}{dt}\left(U(t)\,\rho(0)\,U^+(t)\right) =$

$$= \frac{dU(t)}{dt}\rho(0)\,U^+ + U\rho(0)\frac{dU^+}{dt} =$$

$$= -iH\underbrace{U\rho(0)U^+}_{\rho(t)} + \underbrace{U\rho(0)U^+}_{\rho(t)}(iH)$$

$$= -iH\rho(t) + \rho(t)(iH) = -i[H, \rho]$$

$\Rightarrow$ $\boxed{\frac{d\rho}{dt} = -i[H, \rho]}$ <u>Liouville - von Neumann equation</u>

- matrix ODE ρ, H Hermitian
  $-iH \in su(N)$
- [skew Herm, Herm] = Hermit

- ODE is <u>isospectral</u> $\mathrm{eig}(\rho(t)) = \mathrm{eig}(\rho(0))$

$\Rightarrow$ $\lambda_1 \dots \lambda_N$ eigenvalues of ρ are a <u>complete set of invariant</u>

for the Liouville - von Neumann eq.

, alternative set of invariant: <u>symmetric functions</u> = coeff. of the characteristic polynomial $\det(sI - \rho) = 0$

$\Rightarrow$ cone of density operators

$$\mathcal{M} = \{ \rho = \rho^+ \geq 0,\ tr(\rho^2) \leq 1,\ tr(\rho) = 1 \}$$

foliates into leaves determined by $\{\lambda_1, \dots, \lambda_N\}$

the leaves $\{\lambda_1, \dots, \lambda_N\}$ are all homogeneous spaces of $U(N)$

$$\{\lambda_1, \dots, \lambda_N\} = U(N) \Big/ U(j_1) \times \dots \times U(j_e)$$

where $j_1, \dots, j_e$ = multiplicities of the eigenvalues
$j_1 + \dots + j_e = N$

$\to$ flag manifolds