Feedback control of NMR systems: a control-theoretic perspective

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A quote

From R. Ernst, G. Bodenhausen and A. Wokaun "Principles of NMR in one and two dimensions" Claredon Press 1987, p.3

"spectroscopy stands in close analogy to techniques which measure the *transfer function* of an electronic device. (...) It is well known that the transfer function completely characterizes a *linear time-independent system*. Many of the concepts of spectroscopy stems from the consideration of linearly or approximately linear systems for which a simple and elegant mathematical treatement is possible. (...) It has been known for many years that the free induction decay is equivalent to the *impulse response* for linear systems."

Overview of these lectures

1. today:
   - overview and (biased) motivations
   - formulation of the feedback problems for a single spin 1/2 ensemble

2. tomorrow:
   - system theory for bilinear control systems
     - structure
     - controllability
     - feedback stabilization

3. & 4. next week:
   - feedback stabilization for 2 or more spin 1/2
   - disturbance rejection: suppression of unwanted (weak) coupling terms
   - state estimation and density matrix reconstruction

Why feedback on NMR?

- ideal candidate for quantum engineering

- NMR captures most of the features of Quantum Information Processing
  - state space is composed of tensor products of "qubits"
    - exponential growth of the degrees of freedom available
  - correlations are induced by natural infinitesimal couplings
    - nonlocal operations: "more than just tensor products"
    - quantum nature of spin systems

- sufficiently small interaction with environment and long relaxation times

Why feedback may be successfully tried?

- precise knowledge of the dynamics
- extensive expertise on state manipulation
- reliable control methods (RF pulses)
- sufficiently slow evolution $\implies$ possible to pair the system with a computer in real time
- natural "continuous time" evolution
- bulk spin measurement are classical
  - $\implies$ no discontinuity ("collapse") in the state variables (major obstacle to feedback in any other quantum mechanical setting)
  - $\implies$ deterministic feedback is possible
  - $\implies$ feedback is unitary

Why one would like to have it?

- feedback *simplifies the state manipulation problem*

- many tasks, the same controller

- feedback is *intrinsically robust* to model error, disturbances, noise, experimental imperfections

- allows to preserve the state from (unitary) perturbations (unwanted couplings)

- controlled system is still deterministic

- in *any* domain of engineering, whenever you have the chance to apply feedback methods (rather than just open-loop methods) you better take it!

Why nobody has tried it yet (in NMR)?

- lack of real-time equipment

- pulse sequences based on frequency-domain techniques (both hard and shaped pulses, piecewise linear systems $\longrightarrow$ piecewise "transfer function", spectral analysis) are completely dominant over exclusively time-domain techniques

- need to resort to nonlinear system analysis and synthesis (actually bilinear system theory is enough)

- before QIP: lack of interest to state manipulation problems (focus on molecular structure and identification of reaction dynamics)

- after QIP: direct link between "piecewise linear methods" given by pulse sequences and discrete gate formalism of QIP.

Pros: potential results

- *same* device to solve the state transfer problem for *all* superposition states, not just for few well-studied states

- possibility of *suppressing unwanted couplings*

- for "open-loop" control: a tool to design amplitude-modulated controls for
  1. complicated Hamiltonians
  2. decoupling desired subspaces

- reconstruct the density operator (*state tomography*) in a single shot?

Cons: drawbacks/challenges

- building real-time equipment

- invent a completely time-domain formalism for NMR

- forget about the "wires and gates" formalism of QIP

- focus on nonlinear methods for continuous-time state transfer

- cope with very low signal-to-noise ratio

- find tasks not (simply) solvable by current state-of-art RF pulse techniques

- even if successfull, such a control system will *never* be a "quantum computer", rather it will be a *computer controlled quantum system*

Feedback Control of Nonlinear Systems

- Nonlinear system theory
  - deals with systems of ODEs that do not satisfy the superposition principle
  - almost *never* uses frequency methods: lack of a superposition principle implies that the energy that enters at a certain frequency gets "scattered" at other frequencies at the output $\Longrightarrow$ spectral analysis is not useful as in linear systems theory $\Longrightarrow$ *time-domain* metods.

- model of a system

$$
\begin{aligned}
\dot{x} &= f(x, u) \\
y &= g(x)
\end{aligned}
$$

where
  - $f(\,\cdot\,,\,\cdot\,)$ = state equation (system of ODEs), $f \in C^{\infty}(R^n)$
  - $g(\,\cdot\,)$ = output equation, $g \in C^{\infty}(R^p)$

Feedback Control of Nonlinear Systems

$$
\begin{aligned}
\dot{x} &= f(x, u) \\
y &= g(x)
\end{aligned}
$$

- x = *state vector* = all "internal" variables of interest needed to describe the dynamical evolution of the system

- u = *control input*

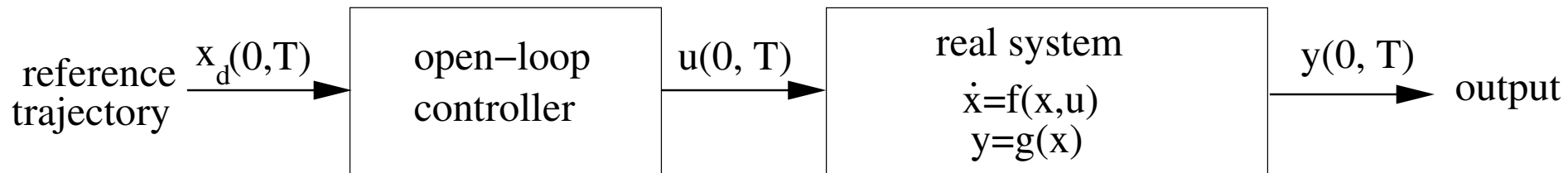- y = *output* = measured quantities
  It can be:
  - $y = x$ (entire state is measurable)
  - y part of the state x
  - memoryless function of x

  y can be
  - measured only at the end of an experiment
  - available on line

Control via pulse sequences

- when are pulse sequences used in modern control theory?
  - system identification (in NMR: process tomography i.e., finding the Hamiltonian)
  - open-loop control methods
- open-loop control

reference trajectory $\xrightarrow{x_d(0,T)}$ [open–loop controller] $\xrightarrow{u(0,\,T)}$ [real system $\dot{x}=f(x,u)$ $y=g(x)$] $\xrightarrow{y(0,\,T)}$ output

- in my knowledge: NMR community has developed more sophisticated and extensive set of tools to do open-loop control than any other engineering community

Open-Loop Control

- use knowledge of the model of the process $f(\,\cdot\,,\,\cdot\,)$, $g(\,\cdot\,)$ and of the initial condition of the state $x(0)$ to design a control function $u : [0,\,T] \rightarrow \mathcal{U}$ such that $\dot{x} = f(x, u(0,\,T))$ is steered from the initial condition $x(0) = x_o$ to a desired terminal condition $x(T) = x_f$.

- $\mathcal{U}$ = functional space of the controls. Examples
  1. $\mathcal{U}$ = { finite amplitude impulses }
  2. $\mathcal{U}$ = { piecewise constant functions }
  3. $\mathcal{U}$ = { smooth functions }

- example: optimal control methods

- prerequisite
  - knowledge of the model
  - knowledge of the initial condition
  - controllability= possibility of arbitrarily manipulate the state by means of u only

Drawbacks of Open-Loop Control

- Accuracy of the model must be very high

- Trust in the model is absolute, there is no way to "correct" the state "on the run".

- Makes no use of the output equation $y = g(x)$ and of the measurements

- It is intrinsecally nonrobust to
  - model error
  - noise and disturbances
  - imprecise initial condition

- computation of open-loop trajectories: "NP hard" search problem in $\mathcal{U}$-space

- need to recompute a control any time $x_o,, x_f$ are changed

- when $y = g(x)$ is available on-line, there is the opportunity to do feedback

Feedback Control

- general idea: use measurement information to correct the state in real-time to
    - impose a desired behavior to the state
    - compensate for modeling errors, disturbancies and noise
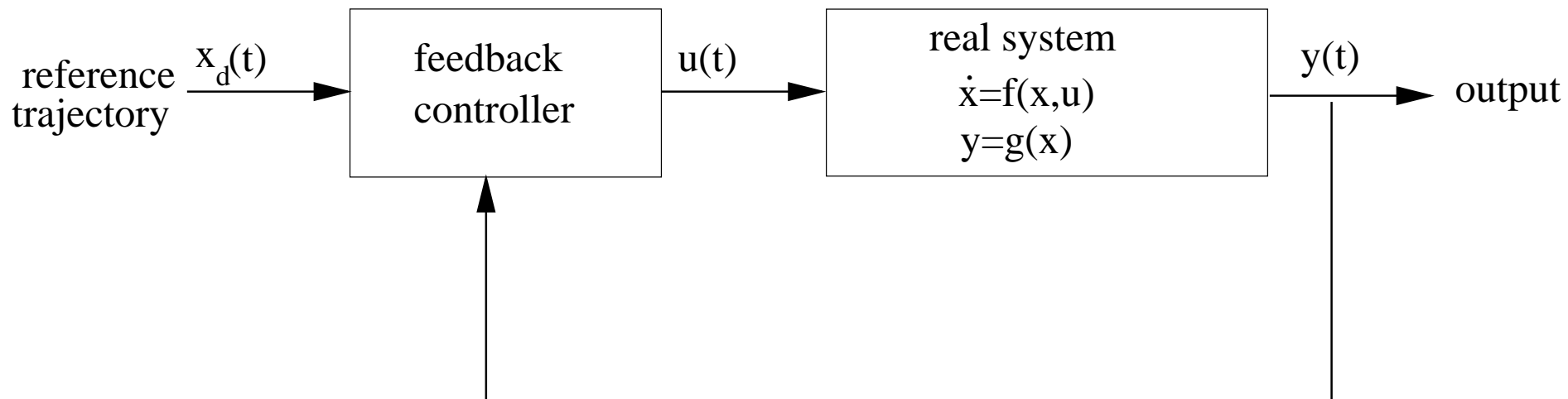
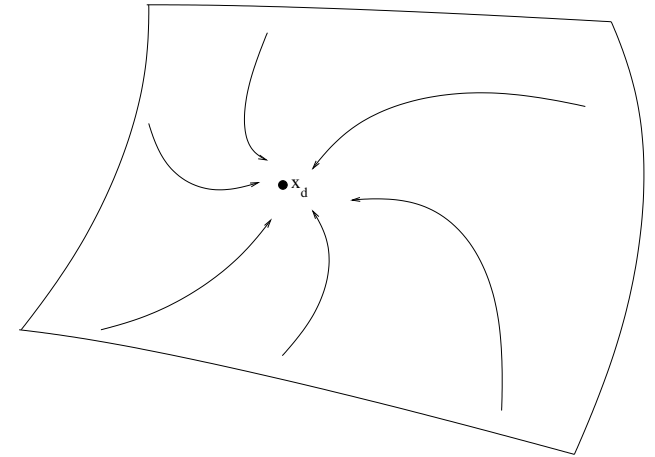- how to build the block "feedback controller"?

Feedback Control

- aim is to have a closed-loop system which is
  - stable
  - converging asymptotically to the target x_d
- asymptotic stability can be
  1. local: holds for all initial conditions in a neighborhood of x_d

  2. global: holds for all initial conditions

- in both cases: no need to know the initial conditions

State Feedback

Case 1 Assume the entire state is measured $y = x \implies$ *state feedback*

- feedback synthesis consists in finding a law $\quad u = k(x, x_d) \quad$ such

that the *closed-loop system*

$$\dot{x} = f\left(x, k(x, x_d)\right)$$

is asymptotically converging to the desired reference trajectory

$$\lim_{t \to \infty} \left(x(t) - x_d(t)\right) = 0$$

- the reference state $x_d(t)$ can be
  1. an equilibrium point $x_d(t) = x_d(0) = \text{const}$
  2. a trajectory defined by e.g. $\dot{x}_d = f_d(x) \implies$ *trajectory tracking*
- to find $u = k(x, x_d)$ there are several methods
- here: *Lyapunov functions*

State Feedback

Lyapunov sufficient condition for stabilization

Theorem If $\exists$ a real-valued function $V = V(x, x_d)$ which is positive definite,

$$V(x, x_d) > 0 \quad \forall\, x, x_d, \qquad V(x, x_d) = 0 \iff x = x_d$$

and such that its total derivative along the trajectories of the closed-loop system is negative definite

$$\dot{V}(x, x_d) = \frac{\partial V}{\partial x} f\left(x, k(x, x_d)\right) + \frac{\partial V}{\partial x_d} f_d\left(x_d\right) < 0,$$

then the closed-loop system is asymptotically tracking the desired reference trajectory $x_d(t)$

- this is a sufficient condition only, but for the class of systems we are interested in it will yield a constructive method to find $k(x, x_d)$.

State Feedback

- Often times one achieves only that $\dot{V}(x, x_d) \leqslant 0$ negative semidefinite.

- this alone guarantees only stability (meaning x and x_d "do not diverge" asymptotically) but not convergence of x to x_d. We need to use *LaSalle invariance principle*

Theorem Assume $V(x, x_d) > 0$ and $\dot{V}(x, x_d) \leqslant 0$. Consider the set $\mathcal{N} = \{x \text{ s. t. } \dot{V}(x, x_d) = 0\}$. If the set of x obeying to $\dot{x} = f(x, k(x, x_d))$ and confined to $\mathcal{N}$ contains only $x = x_d$, then the closed-loop system is asymptotically tracking the reference trajectory x_d.

$$V(\text{x}, \text{x}_d)) = \text{const}$$

$$\dot{\text{x}} = f(\text{x}, k(\text{x}, \text{x}_d))$$
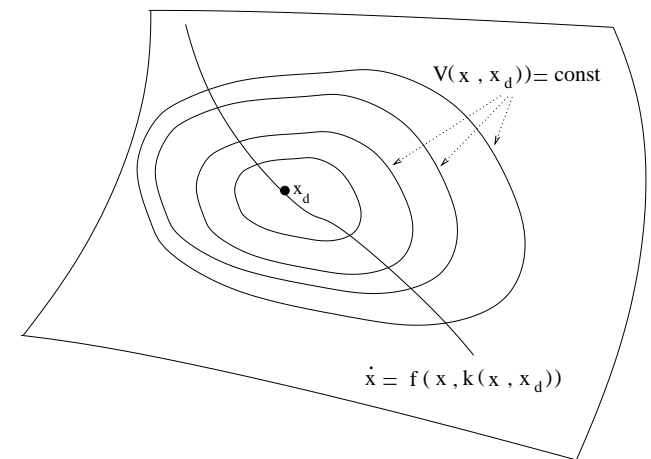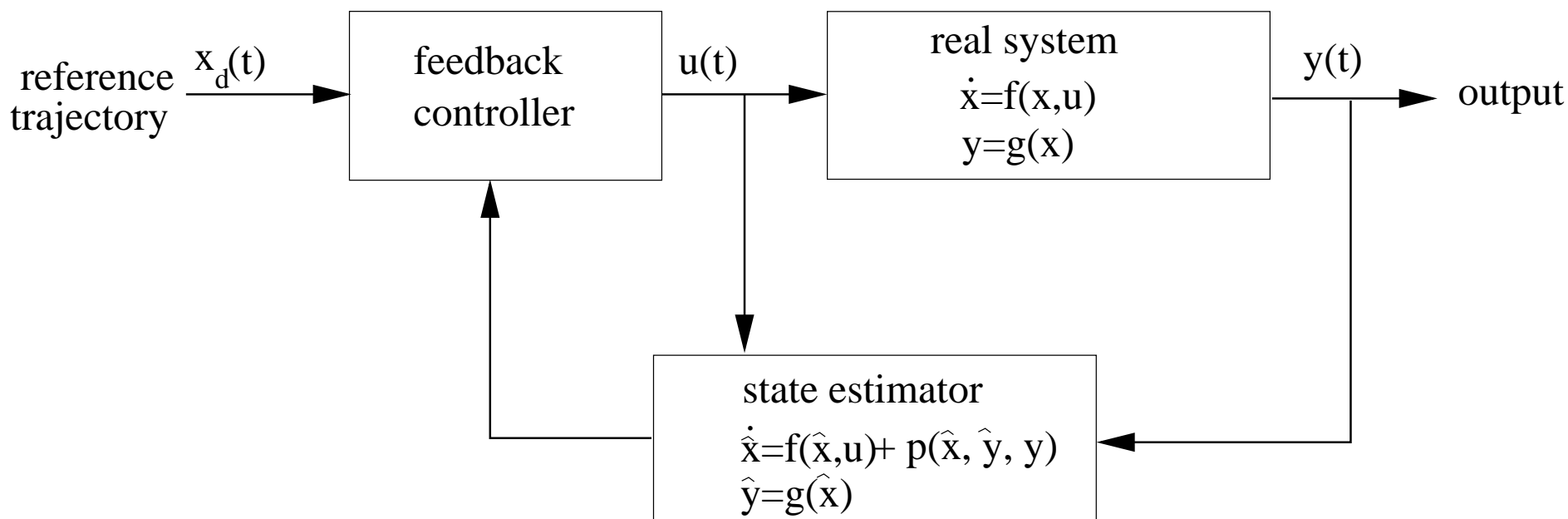
Why no frequency methods in nonlinear feedback?

- in open-loop methods:
  - $u(t)$ is piecewise-constant ("hard pulses")
  - $u(t)$ can be given a shaped time-varying amplitude ("soft pulses")
- in both cases: frequency content of the pulse can be imposed to belong to a certain window $\longrightarrow$ minimal overlap with unwanted frequency regions
- in feedback: amplitude modulation is chosen by the algorithms and changes widely
  - with the initial condition
  - during the evolution
- $\Longrightarrow$ periodicity changes during transient
- $\Longrightarrow$ analysis of resulting spectra during transient are often hopeless...
- steady state is reached only asymptotically...

Output Feedback

Case 2 Assume measures of the entire state are not available $y \neq x$

- Sometimes it is still possible to "reconstruct" on-line the missing state variables $\longrightarrow$ *state estimation problem*
- in control-theoretic language this is the *observability problem* and is "dual" to the controllability problem.
- To set up a state estimator: use a replica (model) of the system
- $\hat{x}(t)$ = estimated state at time t

reference trajectory $\xrightarrow{\ x_d(t)\ }$ feedback controller $\xrightarrow{\ u(t)\ }$ real system $\dot{x}=f(x,u)$ $y=g(x)$ $\xrightarrow{\ y(t)\ }$ output

state estimator $\dot{\hat{x}}=f(\hat{x},u)+ p(\hat{x}, \hat{y}, y)$ $\hat{y}=g(\hat{x})$

State Estimation

- Dynamics of the state estimator

$$\dot{\hat{x}} = f(\hat{x}, u) + \boxed{p(\hat{x}, \hat{y}, y)}$$

$$\hat{y} = g(\hat{x})$$

- need to find $p = p(\hat{x}, \hat{y}, y)$ such that
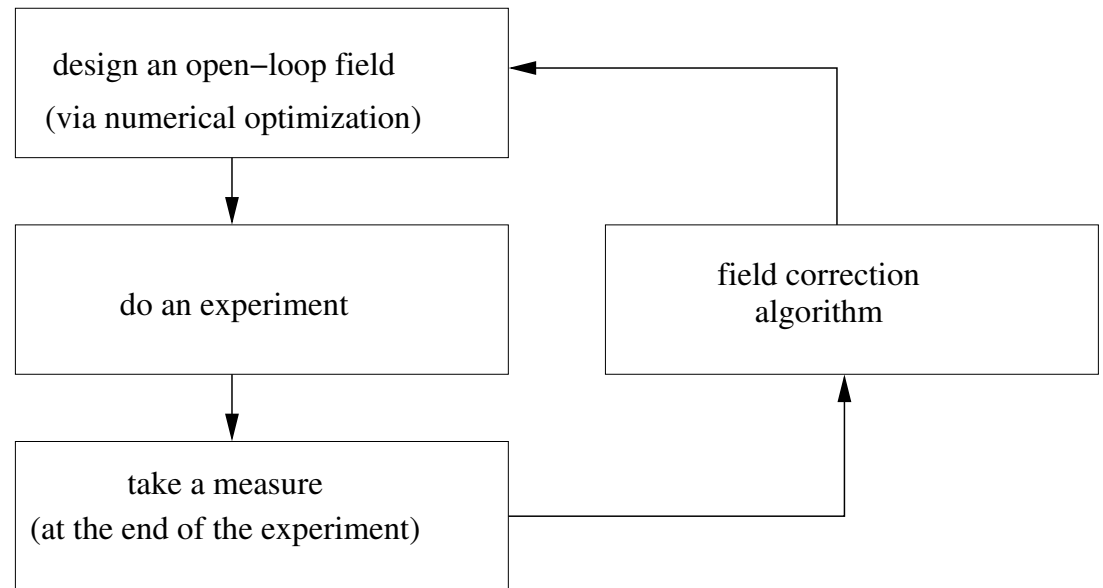
$$\lim_{t \to \infty} (\hat{x}(t) - x(t)) = 0$$

  i.e., estimated state converges to the true one

- The state estimator algorithm play also the role of filter with respect to measurement noise

- for linear systems: *Kalman filter*

- for nonlinear systems: $\nexists$ a general constructive theory

- Prerequisites:
  - knowledge of the model ("Hamiltonian")
  - state "observability"
  - no need to known the initial condition

Feedback in a quantum context

1. _Iterative algorithms_ (H. Rabitz)

```
┌─────────────────────────────┐
│  design an open−loop field  │ ◄──────────────┐
│  (via numerical optimization)│                │
└─────────────────────────────┘                │
              │                                 │
              ▼                                 │
┌─────────────────────────────┐    ┌──────────────────────┐
│      do an experiment       │    │   field correction   │
│                             │    │      algorithm       │
└─────────────────────────────┘    └──────────────────────┘
              │                                 ▲
              ▼                                 │
┌─────────────────────────────┐                │
│       take a measure        │ ───────────────┘
│  (at the end of the experiment)│
└─────────────────────────────┘
```

- this is not _real-time_ $\longrightarrow$ _batch feedback_
- variant: adaptive/genetic/learing algorithms
- useful in quantum chemistry (selective bond dissociation/creation, parameter identification in the Hamiltonian)
- throughput rate of the experiments: thousands per second

Feedback in a quantum context

2. _Feedback control of the stochastic master equation_ (J. Milburn, H. Wiseman, A, Doherty, H. Mabuchi)

    - quantum system is weakly coupled with a monitored bath (e.g. homodyne detection)
    - weak measure is used to correct the stochastic evolution $\longrightarrow$ wavepacket reduction
    - measurement process is nonunitary $\Longrightarrow$ stochastic master equation
    - this is real-time
    - less powerful (and more difficult) than what can be achieved by completely noninvasive measurements
        - possible equilibria are decided by the measurement process (by its Lindbladian) and cannot be altered by feedback
    - more theoretically challenging (quantum measurement)
    - also more practically challenging??

Feedback control of a single spin 1/2 ensemble

- model: controlled Bloch equation

- problem formulation

- two solutions:
  1. orbital stabilization
  2. orbit tracking

- feedback control of nutation

- application: spin squeezing (H. Mabuchi *et al.*)

Model for a single spin $1/2$ ensemble

- State= *density operator* ρ:
  - 2×2 complex matrix
  - positive: $\quad\quad \rho \geq 0$
  - Hermitian: $\quad\quad \rho = \rho^{\dagger}$
  - unit trace: $\quad\quad \text{tr}(\rho) = 1$
  - pure state: $\quad\quad \text{tr}(\rho^2) = 1$
  - mixed state: $\quad\quad \text{tr}(\rho^2) < 1$

Model for a single spin $1/2$ ensemble

- parametrization of ρ:

$$\rho = \varrho^0 \lambda_0 + \varrho^1 \lambda_1 + \varrho^2 \lambda_2 + \varrho^3 \lambda_3$$

- $\lambda_k = \frac{1}{\sqrt{2}} \, \sigma_k$ = Pauli matrices

$$\lambda_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \lambda_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \lambda_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- $\implies$ angular momentum operators for a spin $1/2$

- ϱ^k = *expectation values* of ρ

$$\varrho^k = \mathrm{tr}(\rho \lambda_k), \qquad k = 1,\, 2,\, 3$$

- *Bloch vector* : $\boldsymbol{\varrho} = \begin{bmatrix} \varrho^1 \\ \varrho^2 \\ \varrho^3 \end{bmatrix} \in \mathbb{R}^3$

Model for a single spin $1/2$ ensemble

- $\varrho \in$ Bloch ball
  - complete mixing $\quad \|\varrho\| = 0$
  - pure state $\quad \|\varrho\| = \frac{1}{\sqrt{2}}$

- in NMR: *deviation density operator*

$$(1 - \epsilon)I/2 + \epsilon\rho$$

  - $\longrightarrow$ pseudopure states: only $\epsilon \rho$ matters for the evolution
  - $\epsilon \simeq 10^{-5} \longrightarrow$ very mixed states: $\|\varrho\| \simeq 10^{-5}$
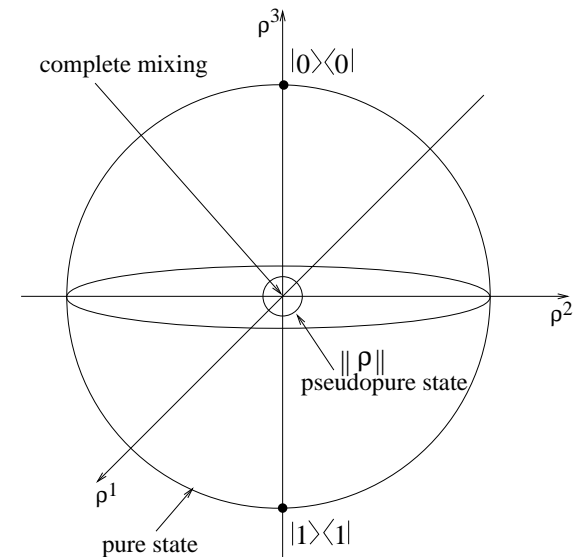
Liouville equation

- differential equation for ρ: Liouville-von Neumann equation

$$\dot{\rho} = -i[H, \rho]$$

- Hamiltonian $\quad H = h^1\lambda_1 + h^2\lambda_2 + h^3\lambda_3$
- in the Bloch vector parametrization:

$$\dot{\varrho} = -i\mathrm{ad}_H\varrho \qquad \Longrightarrow \qquad \textit{Bloch equation}$$
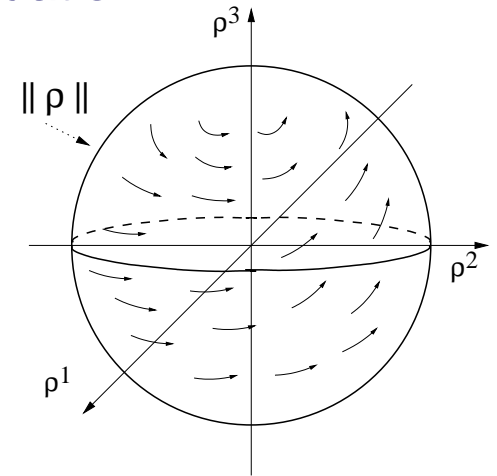
- Bloch equation = "adjoint representation"

$$-i\mathrm{ad}_H = h^1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + h^2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + h^3 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- time-varying evolution $\quad \varrho(t) = \exp\left(\int_0^t -i\mathrm{ad}_{H(\tau)}d\tau\right)\varrho(0)$ is on the sphere
  $\|\varrho\|^2 = \mathrm{const}$
- $\Longrightarrow$ isospectral evolution

Controlled Bloch equation

- Hamiltonian is composed of
  1. a free part (the drift)
  2. a forcing part (the control term)

1. *free part*: due to a strong static magnetic field B_o aligned with the λ_3 axis
   - $B_o \longrightarrow$ induces the deviation $\epsilon\,\rho$
   - $B_o \longrightarrow$ spin ensemble precesses around λ_3
   - in the lab frame
$$H_{f,\ell} = -\gamma B_o \lambda_3$$

   - γ = gyromagnetic ratio

2. *control Hamiltonian*: rotating r.f. field with frequency $\omega_{\mathrm{rf}} \simeq \omega_o = \gamma B_o$
   - in the lab frame

$$H_{\mathrm{rf},\ell} = -\gamma B_1 \left(\cos(\omega_{\mathrm{rf}} t + \phi)\lambda_1 + \sin(\omega_{\mathrm{rf}} t + \phi)\lambda_2 \right)$$

- controllable parameters
  1. amplitude B_1
  2. frequency ω_{rf}
  3. phase ϕ

Bloch equation in the rotating frame

- ω_0 (Larmor frequency) $\simeq$ tens to hundreds of MHz

- $\omega_1 = \gamma B_1 \simeq$ tens to hundreds of kHz

- to get rid of the higher frequency in the Hamiltonian: *rotating frame* = coordinate system rotating around λ_3 at the frequency $\omega_{\mathrm{rf}} \simeq \omega_o$

$$\boldsymbol{\varrho} = e^{-i\omega_{\mathrm{rf}}t\,\mathrm{ad}_{\lambda_3}}\boldsymbol{\varrho}_\ell$$

in the ODE: variation of constants formula

$$\dot{\boldsymbol{\varrho}}_\ell = -i\omega_{\mathrm{rf}}\mathrm{ad}_{\lambda_3}\boldsymbol{\varrho}_\ell, \qquad\qquad \boldsymbol{\varrho}_\ell(0) = \boldsymbol{\varrho}(t)$$
$$\dot{\boldsymbol{\varrho}} = -ie^{-it\omega_{\mathrm{rf}}\mathrm{ad}_{\lambda_3}}\mathrm{ad}_{H_{\mathrm{rf}}}e^{it\omega_{\mathrm{rf}}\mathrm{ad}_{\lambda_3}}\boldsymbol{\varrho}, \qquad \boldsymbol{\varrho}(0) = \boldsymbol{\varrho}_\ell(0)$$

$+$ approximation (Bloch-Siegert shift)

Bloch equation in the rotating frame

- Hamiltonian in the rotating frame $H = H_f + H_c$

$$
\begin{aligned}
H_f &= -(\omega_o - \omega_{\text{rf}})\lambda_3 \\
H_c &= -\omega_1 \left(\cos\phi\,\lambda_1 + \sin\phi\,\lambda_2 \right)
\end{aligned}
$$

- fixing $\phi \implies$ axis at which the control acts is fixed
- call $h^3 = -(\omega_o - \omega_{\text{rf}})$
- assume $\phi = 0$

$$
H = H_f + H_c = h^3\lambda_3 + u\,\lambda_1
$$

  where $u = -\omega_1$ = real valued control, $u \in C^\infty(\mathbb{R})$

- precession h^3 of the order of the kHz $\implies$ much slower than ω_o
- when $\omega_{\text{rf}} = \omega_o \implies$ driftless Hamiltonian $\longrightarrow$ nutation around the λ_1 axis

Bloch equation in the rotating frame

- for density matrix ρ

$$\dot{\rho} = -i \left[h^3 \lambda_3 + u\, \lambda_1 \, , \, \rho \right]$$

  - matrix system, 2×2
  - Hamiltonian: 2×2 matrices
  - solution: conjugation action on a matrix

$$\rho(t) = \exp\left(-i \int_0^t H(\tau)d\tau \right) \rho(0) \, \exp\left(i \int_0^t H(\tau)d\tau \right)$$

- in terms of the Bloch vector:

$$\dot{\boldsymbol{\varrho}} = -i \left(h^3 \mathrm{ad}_{\lambda_3} + u\, \mathrm{ad}_{\lambda_1} \right) \boldsymbol{\varrho}$$

  - bilinear control system living on $\mathbb{S}^2 \subset \mathbb{R}^3$
  - Hamiltonian is 3×3
  - solution: linear action on a vector $\boldsymbol{\varrho}(t) = \exp\left(-i \int_0^t \mathrm{ad}_{H(\tau)} d\tau \right) \boldsymbol{\varrho}(0)$

Formulation of the feedback problem: assumptions

- known Hamiltonian H

- measurement
  - "classical" measurement (no wavefunction collapsing)
  - "continuous" measurement
  - full state is measured and available on-line

$$
\begin{aligned}
\varrho^1 &= \operatorname{tr}(\rho\lambda_1) \\
\varrho^2 &= \operatorname{tr}(\rho\lambda_2) \\
\varrho^3 &= \pm\sqrt{\|\varrho\| - (\varrho^1)^2 - (\varrho^2)^2}
\end{aligned}
$$

- control
  - fully deterministic control problem
  - a single control input
  - unitary
  - real-time feedback

Problem formulation: scheme

| reference trajectory $\rho_d(t)$ | → | feedback law $u = k(\rho_d, \rho^1, \rho^2 \rho^3)$ | → | real system $\dfrac{d\rho}{dt} = -i\,[\,H_f + uH_c\,,\;\rho\,]$ |

'state estimation'
$$\rho^3 = \pm\sqrt{\|\rho\|^2 - (\rho^1)^2 - (\rho^2)^2}$$

measurements
$$\rho_1 = \mathrm{tr}(\rho\lambda_1)$$
$$\rho_2 = \mathrm{tr}(\rho\lambda_2)$$
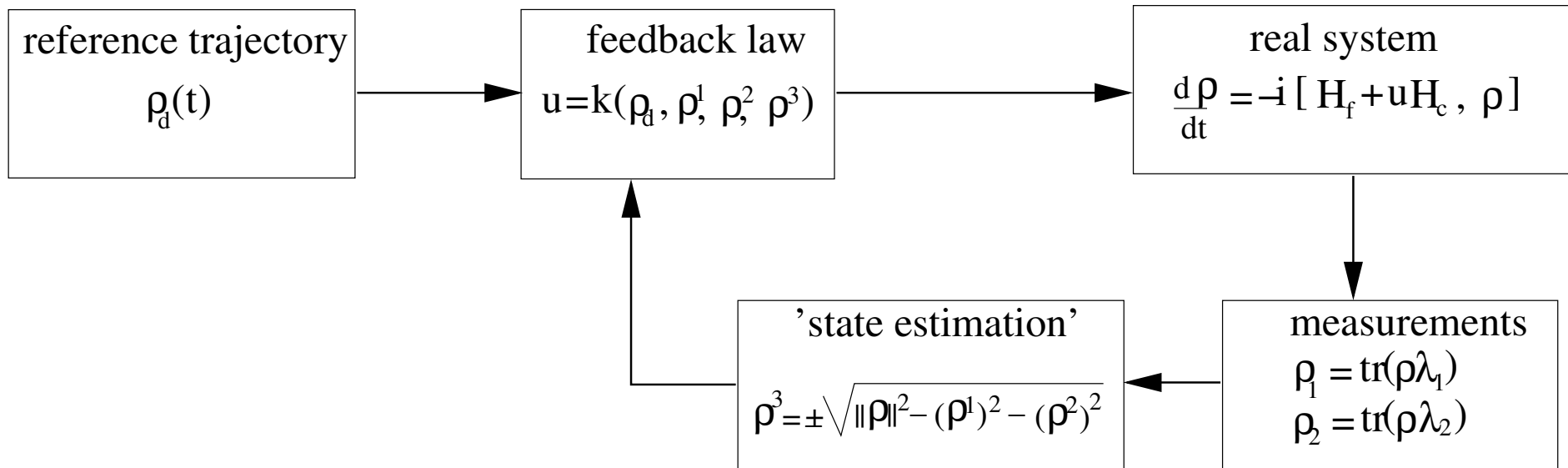
Problem formulation

- given "reference density" $\rho_d(t)$
  - $\|\varrho_d\| = \|\varrho\|$
- "reference evolution" given by $H_{f_d} = h_d^3 \lambda_3$

$$\dot{\rho}_d = -i\left[H_{f_d}, \rho_d\right] \qquad \Longrightarrow \qquad \dot{\boldsymbol{\varrho}}_d = -i\mathrm{ad}_{H_{f_d}} \boldsymbol{\varrho}_d$$

1. periodic orbit $\varrho_d^3(t) = \varrho_d^3(0)$ for $\varrho_d^3 \neq$ north/south poles
2. equilibrium point for $\varrho_d^3 =$ north/south poles
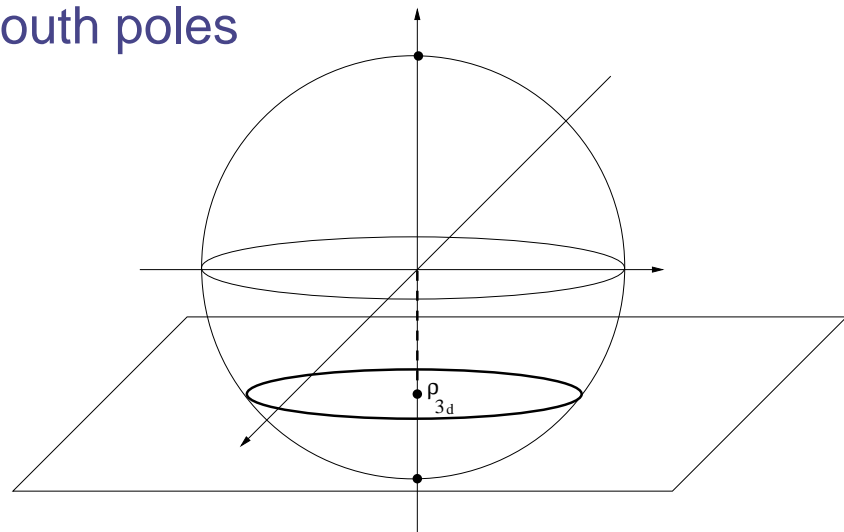
Problem formulation

- *Feedback problem 1:*   orbital stabilization

$$\begin{cases} \quad \varrho^3 & \xrightarrow[\quad]{t \to \infty} \varrho_d^3 = \text{const} \\ \varrho^1, \ \varrho^2 & \text{any} \end{cases}$$

   ○ partial state stabilization

- *Feedback problem 2:*   orbit tracking

$$\boldsymbol{\varrho} \xrightarrow{t \to \infty} \boldsymbol{\varrho}_d \quad \text{where} \quad \dot{\boldsymbol{\varrho}}_d = -i \text{ad}_{H_{f_d}} \boldsymbol{\varrho}_d$$

   ○ full state stabilization

Bloch equation: orbital stabilization

- open-loop system

$$\begin{cases} \dot{\varrho}^1 & = -\varrho^2 \\ \dot{\varrho}^2 & = \varrho^1 - u\varrho^3 \\ \dot{\varrho}^3 & = u\varrho^2 \end{cases}$$

- take candidate Lyapunov function $\longrightarrow$ distance in $\mathbb{R}$

$$V = \frac{1}{2}\left(\varrho_d^3 - \varrho^3\right)^2 > 0$$

- derive: $\dot{V} = -\left(\varrho_d^3 - \varrho^3\right)\dot{\varrho}^3 = -\left(\varrho_d^3 - \varrho^3\right)\varrho^2\, u$

- choosing the *state feedback law*

$$u = \left(\varrho_d^3 - \varrho^3\right)\varrho^2$$

$$\implies \dot{V} = -(\varrho^2)^2\left(\varrho_d^3 - \varrho^3\right)^2 \leqslant 0$$

Bloch equation: orbital stabilization

- close-loop system

$$\begin{cases} \dot{\varrho}^1 & = -\varrho^2 \\ \dot{\varrho}^2 & = \varrho^1 - \varrho^2 \varrho^3 (\varrho_d^3 - \varrho^3) \\ \dot{\varrho}^3 & = (\varrho^2)^2 (\varrho_d^3 - \varrho^3) \end{cases}$$

- use LaSalle invariance principle

- convergence is asymptotic everywhere except at north/south poles of $\mathbb{S}^2$

- nonlinear system

- only ϱ^3 is stabilized

- how to do this in practice? Amplitude B_1 is moduled in real time

$$B_1 = -\frac{u}{\gamma} = -\frac{\varrho^2 (\varrho_d^3 - \varrho^3)}{\gamma}$$

Bloch equation: orbital stabilization

- on the Bloch sphere
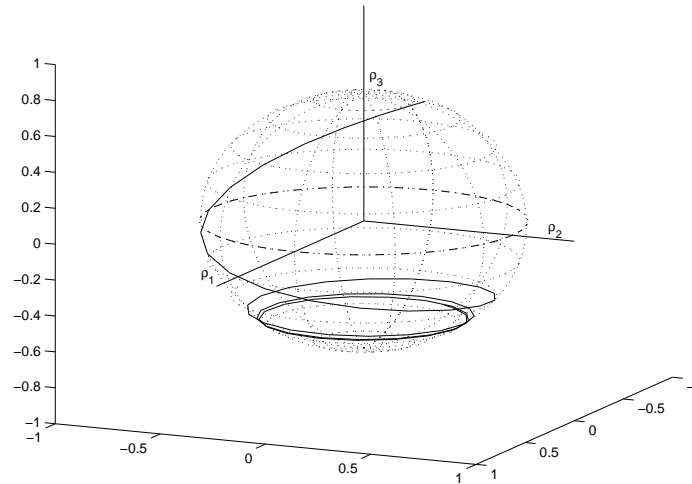
- the 3 components of ϱ
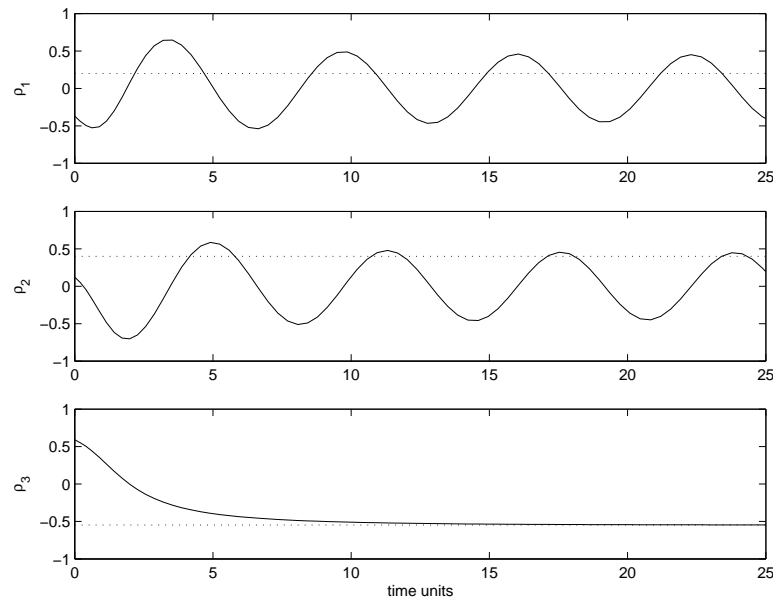
Bloch equation: orbit tracking

- want $\varrho \xrightarrow{t \to \infty} \varrho_d$ where $\dot{\varrho}_d = -i \operatorname{ad}_{H_{f_d}} \varrho_d$

- again, control Lyapunov function based on distance in $\mathbb{S}^2$

$$V = \|\varrho\|^2 - \langle\langle \varrho_d, \varrho \rangle\rangle > 0 \qquad (\text{or } V = \|\varrho_d - \varrho\|^2)$$

- derive

$$
\begin{aligned}
\dot{V} &= -\langle\langle \dot{\varrho}_d, \varrho \rangle\rangle - \langle\langle \varrho_d, \dot{\varrho} \rangle\rangle \\
&= -\langle\langle -i \operatorname{ad}_{H_{f_d}} \varrho_d, \varrho \rangle\rangle - \langle\langle \varrho_d, -i \operatorname{ad}_{H_f} \varrho \rangle\rangle - \langle\langle \varrho_d, -i \operatorname{ad}_{H_c} \varrho \rangle\rangle u
\end{aligned}
$$

- if $H_{f_d} = H_f \Longrightarrow$ desired precession is the true one

$$
\begin{aligned}
\dot{V} &= \langle\langle \varrho_d, -i \operatorname{ad}_{H_f} \varrho \rangle\rangle - \langle\langle \varrho_d, -i \operatorname{ad}_{H_f} \varrho \rangle\rangle - \langle\langle \varrho_d, -i \operatorname{ad}_{H_c} \varrho \rangle\rangle u \\
&= -\langle\langle \varrho_d, -i \operatorname{ad}_{H_c} \varrho \rangle\rangle u
\end{aligned}
$$

- $\dot{V}$ homogeneous in u

Bloch equation: orbit tracking

- *state feedback law*

$$u = \langle\!\langle \boldsymbol{\varrho}_d, \; -i\mathrm{ad}_{H_c}\boldsymbol{\varrho} \rangle\!\rangle$$

$$\implies \dot{V} = -\langle\!\langle \boldsymbol{\varrho}_d, \; -i\mathrm{ad}_{H_c}\boldsymbol{\varrho} \rangle\!\rangle^2 \leqslant 0$$

- LaSalle invariance principle

- convergence is in $\; \mathbb{S}^2 \; - \; \{\pm\boldsymbol{\varrho}_d(0)\} \; - \; \{\boldsymbol{\varrho}(0), \, \boldsymbol{\varrho}_d(0) \; \mathrm{s.t.} \; \varrho^3(0) = \varrho_d^3(0) = 0\}$

- i.e., $\qquad\qquad\qquad \mathbb{S}^2 - \{$ antipodal point$\}$ - $\{$ great horizontal circle $\}$

- $\implies$ stabilization is "almost global"

- if $H_{f_d} \neq H_f$, the system is not asymptotically stable: $\dot{V} \leqslant 0$

- $\implies$ around the λ_3 axis the closed-loop system can catch up a phase difference but not a precession frequency different from its own (at least with this controller)

Bloch equation: orbit tracking

- on the Bloch sphere
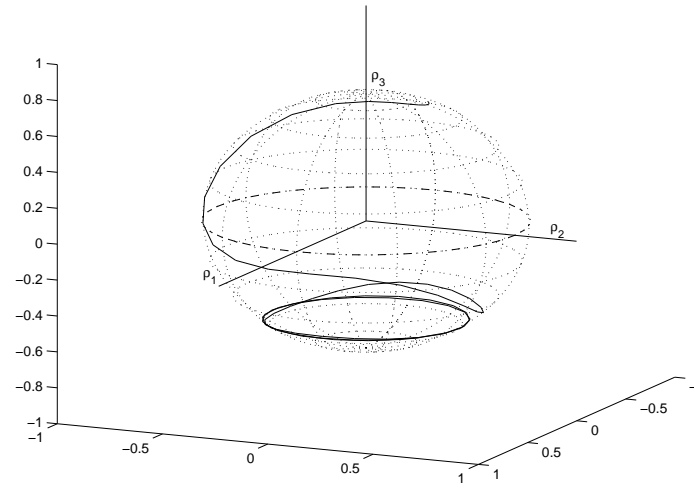
- the 3 components of ϱ
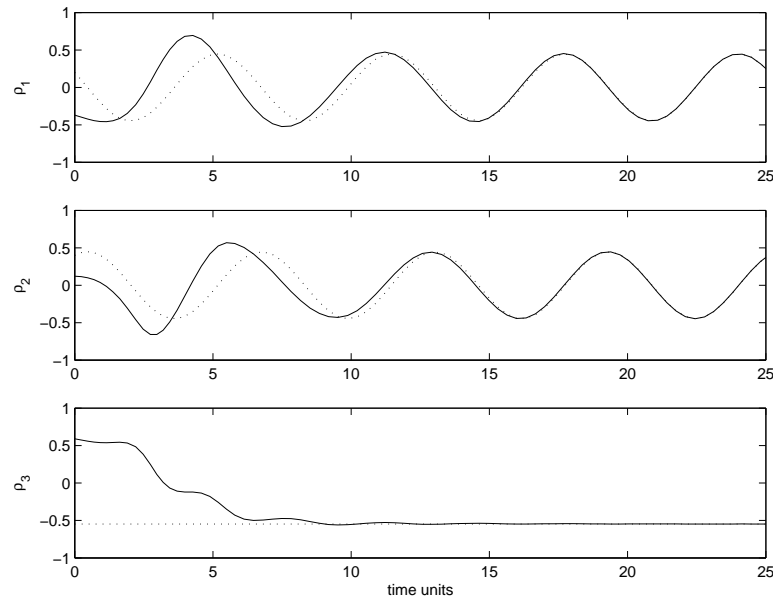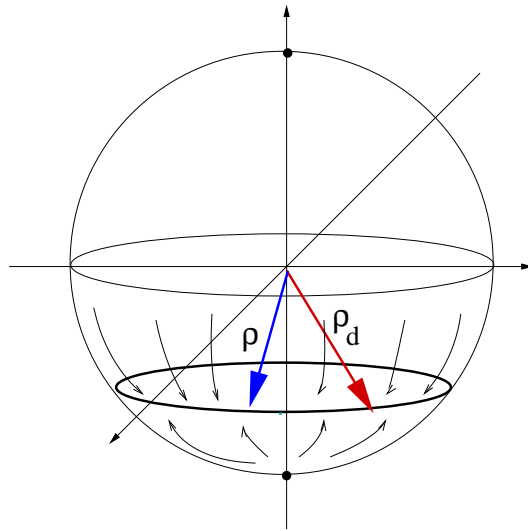
Bloch equation: orbit tracking

- antipodal point is unstable: a small perturbation makes ϱ converge
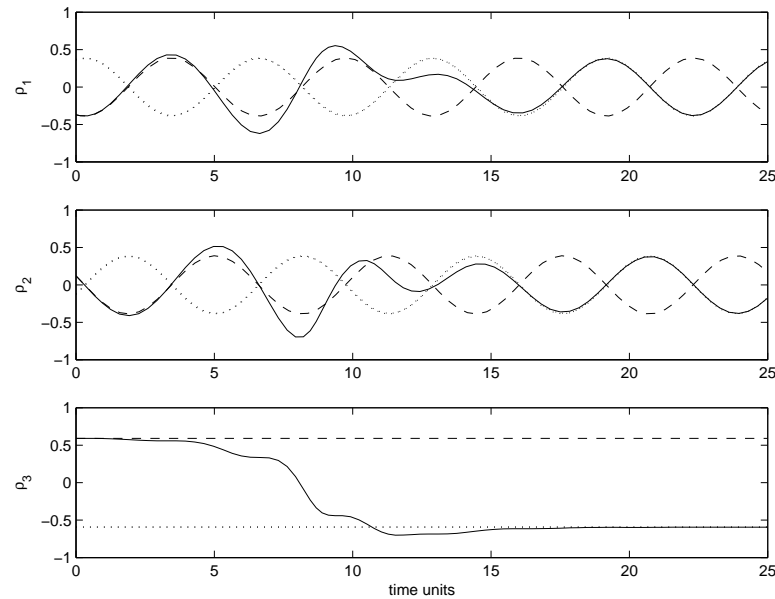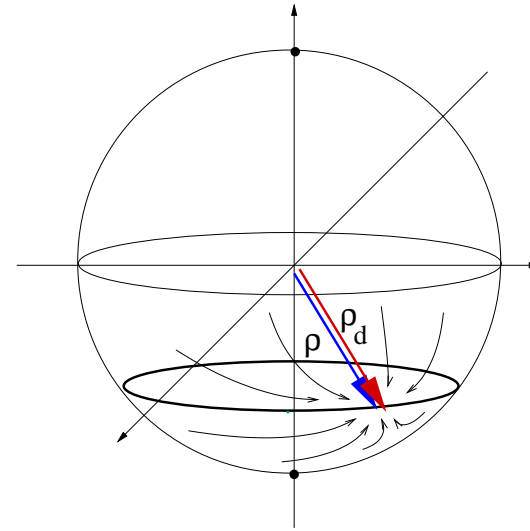

- $\implies$ almost global asymptotic stability
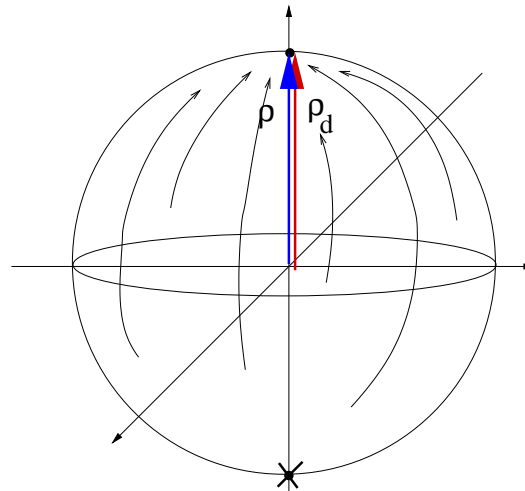
Bloch equation: difference between the two schemes

orbital stability

orbit tracking

for north and south poles: same effect

Bloch equation: inital condition

- convergence holds for (almost) all initial conditions:

  Experiment apply a random pulse sequence then activate feedback: closed-loop system should converge to the desired ϱ_d

- $\longrightarrow$ protection against unknown (unitary) disturbances

Why cannot have global convergence?

There is a *topological obstruction*: a compact manifold (like a sphere) cannot have

- a C^∞ always nonvanishing vector field ("hairy ball" theorem)
- a C^∞ function must vanish in at least as many points as the Euler characteristic of the manifold
- for a sphere $\mathbb{S}^2$: Euler characteristic is 2 $\implies$ antipodal point

Feedback control of nutation

- when $\omega_o = \omega_{\mathrm{rf}}$ system in the rotating frame is driftless $\longrightarrow$ nutation motion

$$\dot{\boldsymbol{\varrho}} = -i\, u \operatorname{ad}_{\lambda_1} \boldsymbol{\varrho} \qquad \text{i.e.} \qquad \begin{cases} \dot{\varrho}^1 &= 0 \\ \dot{\varrho}^2 &= -u\, \varrho^3 \\ \dot{\varrho}^3 &= u\, \varrho^2 \end{cases}$$

- stabilization on the $(y,\, z)$-plane $\Longrightarrow \mathbb{S}^1$-circle

- assume $\boldsymbol{\varrho}$ is in the $(y,\, z)$-plane

- measure ϱ^2

- assume target is north pole

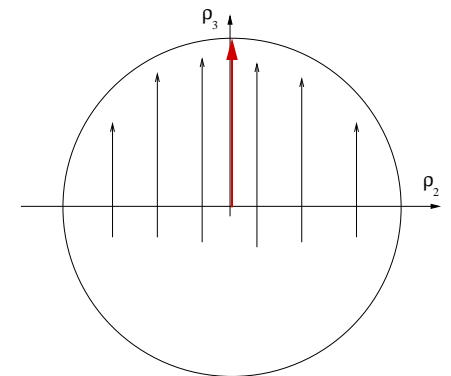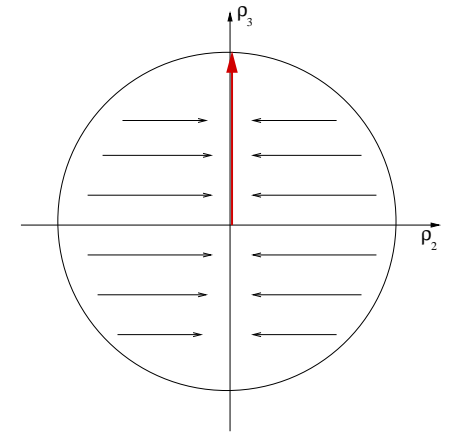- feedback law $u = \varrho^2$
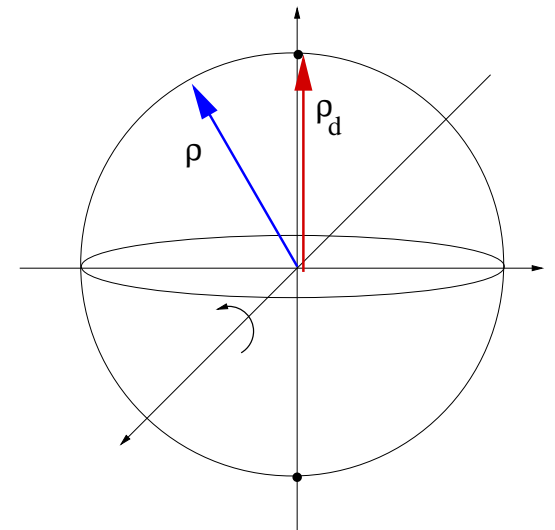
Feedback control of nutation

- closed-loop system

$$
\begin{cases}
\dot{\varrho}^1 & = 0 \\
\dot{\varrho}^2 & = -\varrho^2 \, \varrho^3 \\
\dot{\varrho}^3 & = (\varrho^2)^2
\end{cases}
$$

- $\varrho^3 > 0 \implies$

  - $\dot{\varrho}^2 \leqslant 0 \implies$ converges to $\varrho^2 \to 0$

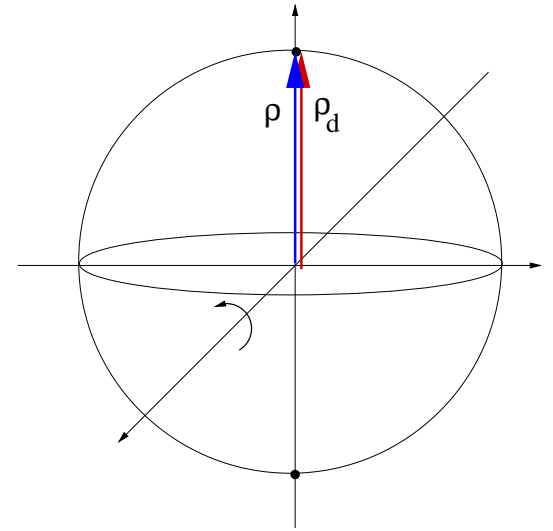 - $\dot{\varrho}^3 \geqslant 0 \implies \varrho^3$ grows

  - $\dot{\varrho}^1 = 0 \implies$ no motion along ϱ^1

Feedback control of nutation: spin squeezing

special case

- start with initial condition on the north pole

- $\implies$ system is at steady state

- $\implies \varrho^2 \simeq 0$

- feedback $u = \varrho^2 \simeq 0$

- $\implies$ no "appreciable" unitary evolution

- however something happens to
  the second order moments...

Feedback control of nutation: spin squeezing

REPORTS

Real-Time Quantum Feedback Control of Atomic Spin-Squeezing

JM Geremia,* John K. Stockton, Hideo Mabuchi

Real-time feedback performed during a quantum nondemolition measurement of atomic spin-angular momentum allowed us to influence the quantum statistics of the measurement outcome. We showed that it is possible to harness measurement backaction as a form of actuation in quantum control, and thus we describe a valuable tool for quantum information science. Our feedback-mediated procedure generates spin-squeezing, for which the reduction in quantum uncertainty and resulting atomic entanglement are not conditioned on the measurement outcome.

Quantum systems evolve deterministically when no one is looking. Free from observation, knowledge of a quantum state at one point in time is in principle sufficient to predict its entire evolution. However, when a measurement is performed, quantum mechanics postulates that the observer will obtain a random postmeasurement outcome. Conveniently, measurement can produce states that are difficult to obtain by other means, such as Hamiltonian evolution, and thus provides a powerful tool for quantum state preparation. But standard quantum mechanics does not predict the outcomes of individual experiments, only their likelihood. Measurement-based state preparation is hindered by nondeterminism, and desirable (for example,

entangled) quantum states often correspond to highly unlikely measurement outcomes. Here we demonstrate that quantum indeterminism can be reduced by suitable intrameasurement feedback, engineered to steer the outcome of an otherwise random quantum process toward a deterministic outcome.

The quantum system in our experiment was provided by a cloud of N atoms each with intrinsic angular momentum, $\hbar f$, because of a combination of nuclear spin, valence electron spin, and orbital angular momentum. The atoms were initially polarized such that their individual moments were oriented along a common longitudinal direction, which we chose to be the x axis. The resulting atomic state displayed a net magnetization, F, along x with magnitude $|\mathbf{F}| = \hbar\sqrt{F(F+1)}$, where $F = Nf$. The cartesian components of F are associated with noncommuting quantum op-

erators, $\hat{F}_x$, $\hat{F}_y$, and $\hat{F}_z$, that obey the Heisenberg uncertainty relation

$$\Delta \hat{F}_y \Delta \hat{F}_z \geq \frac{1}{2}|\langle \hat{F}_x \rangle| \qquad (1)$$

This inequality has the interpretation that an ensemble of measurements (for similarly prepared atomic samples) performed on either $\hat{F}_y$ or $\hat{F}_z$ will yield a distribution of random shot-to-shot outcomes. For a large magnetization, the $\hat{F}_z$ (for example) measurement distribution is essentially Gaussian with mean $\langle \hat{F}_z \rangle$ and variance $\Delta \hat{F}_z^2 \equiv \langle \hat{F}_z^2 \rangle - \langle \hat{F}_z \rangle^2$. The fully polarized atomic state has $\langle \hat{F}_x \rangle = F$ and $\Delta \hat{F}_y = \Delta \hat{F}_z = \sqrt{F/2}$, and is referred to as a coherent spin state (Fig. 1A).

It is possible to reduce the measurement variance in one of the transverse components below the coherent state value of $F/2$ at the expense of increased uncertainty in the orthogonal component, provided that Eq. 1 remains satisfied (Fig. 1A). Polarized states with this property are referred to as spin-squeezed states (1) and have received much attention for their potential to improve the sensitivity of spin-resonance measurements, including magnetometry (2, 3) and atomic clocks (4, 5). Spin-squeezing below the coherent state level is also of fundamental interest in quantum information science for achieving many-particle entanglement (6).

Although several different mechanisms have been explored for the preparation of squeezed atomic states (7, 8), interest has focused on using quantum nondemolition (QND) measurements (9–11), in which the atomic system interacts coherently with an off-resonant optical probe. As a result, the z component of the atomic magnetization, F_z,

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Feedback control of nutation: spin squeezing

- fully polarized atomic cloud

- ensemble feedback is used to squeeze the state

- $\varrho^3 = \operatorname{tr}(\rho\lambda_3) = \langle\lambda_3\rangle$

- variance $(\Delta\lambda_j)^2 = \langle\lambda_j^2\rangle - \langle\lambda_j\rangle^2$

- Heisenberg uncertainty relation:

$$\Delta\lambda_1\Delta\lambda_2 \geqslant \frac{1}{2}|\langle\lambda_3\rangle|$$

- in a fully polarized ensemble:

$$\langle\lambda_3\rangle = F = Nf$$

  - N = n. of atoms
  - f = angular momentum of the single atom

Feedback control of nutation: spin squeezing
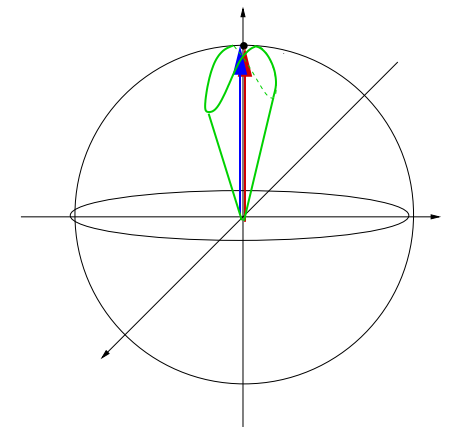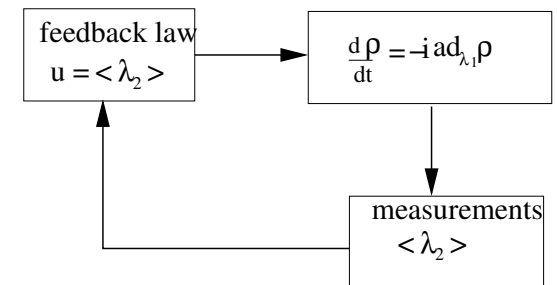
- starting from a coherent state
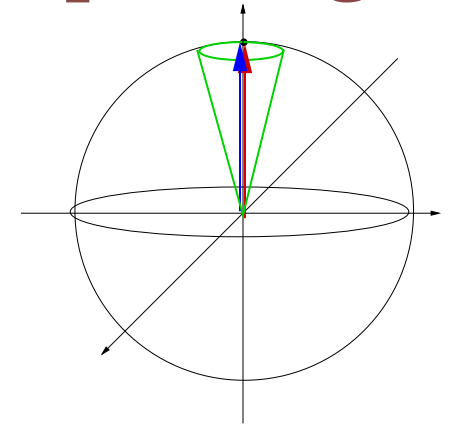
$$\langle \lambda_3 \rangle = F$$

$$\Delta\lambda_1 = \Delta\lambda_2 = \sqrt{\frac{F}{2}}$$

- and using the real-time feedback

- one gets to the sqeezed state
  - uncertainty decreased along λ_2
  - uncertainty increased along λ_1

  $$\Delta\lambda_1 > \Delta\lambda_2$$

feedback law
$u = <\lambda_2>$

$\frac{d\rho}{dt} = -i\, ad_{\lambda_1}\rho$

measurements
$<\lambda_2>$

Feedback control of nutation: spin squeezing

- in NMR systems: only a fraction od the spins is polarized: the bound given by the Heisenberg relation is very far away

- what is the same scheme doing?
  - nothing?
  - reducing dephasing??
  - increasing polarization??

- dephasing:
  - relaxation mechanism that "shrinks" the Bloch ball
  - nonunitary operator (points to the random state)