Reflection symmetries for multiqubit density operators

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For multiqubit density operators in a suitable tensorial basis, we show that a number of nonunitary operations used in the detection and synthesis of entanglement are classifiable as reflection symmetries, i.e., orientation changing rotations. While one-qubit reflections correspond to antiunitary symmetries, as is known for example from the partial transposition criterion, reflections on the joint density of two or more qubits are not accounted for by the Wigner Theorem and are well-posed only for sufficiently mixed states. One example of such nonlocal reflections is the unconditional NOT operation on a multiparty density, i.e., an operation yelding another density and such that the sum of the two is the identity operator. This nonphysical operation is admissible only for sufficiently mixed states.

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The Wigner Theorem affirms that unitary and antiunitary operations exhaust all possible symmetric transformations applicable to the wavefunction of a quantum mechanical system. The unitary transformations are physically associated with forward-in-time evolution and the antiunitary ones with backward-in-time evolution, see for example [1]. The characteristic feature of this last class is the presence of a conjugation operation on a wavefunction or a transposition operation on a density operator. It is known [2] that the geometric interpretation of the time reversing operation for a density operator in a two-dimensional Hilbert space (aka "qubit") is a reflection operation, i.e. an orientation changing rotation in $O^-(3) = O(3) \setminus SO(3)$ of the corresponding Bloch vector. A closely related operation variously known as spin flip [3], (unconditional) NOT operation, or universal inverter [4], changes the sign of the entire Bloch vector and hence corresponds geometrically to inversion in the origin (otherwise known as the parity operation, or the "main involution"). While for a single isolated system these operations are indistinguishable from equivalent orientation preserving operations since $O(3)$ and $SO(3)$ both act transitively on the Bloch sphere, for multiqubit systems they correspond to partially antiunitary transformations such as partial transposition and they can be used to detect bipartite entanglement [5]. This highlights the intrinsically "discrete" nature of such tests and their invariance under LOCC (Local Operation and Classical Communication).

In this paper we introduce a more general class of involutory "symmetry" operations, and argue that these are likewise useful (and used) to deal with multiparty nonseparability of density operators. These operations are most easily described in terms of the Stokes tensor [6, 7] (or, equivalently, the coherence vector [8] or the "real density matrix" [9]). This is simply a parametrization of the real linear space of multiqubit density operators by the expectation values of all possible tensor products of the Pauli operators. As is well-known, unitary operations on the usual density operators induce orientation-preserving rotations of the Stokes tensor. We now distinguish the following two types of nonunitary but norm-preserving operations on a multiqubit Stokes tensor:

(i) local reflections applied simultaneously to two or more qubits;

(ii) "nonlocal" reflections, i.e. reflections applied at the *joint density* of two or more qubits.

The two cases are *qualitatively different*: while (i) is equivalent, up to local unitary operations, to multiqubit partial transposition, (ii) is a genuinely new operation and does not correspond to any local operation on two or more qubits. In particular, the total reflection of all components of the Stokes tensor other than the expectation value of the identity does *not* correspond to time-reversal (i.e., to the total transpose of the density matrix) but rather to a multiparty NOT operation. Reflections on more than one qubit are nonunitary operations that do not necessarily yield valid (positive semidefinite) density operators. However, it can be shown that any mixed state with eigenvalues "small enough" (for n qubits, this set turns out to contain only operators of rank 2^{n-1} or higher with eigenvalues in the interval $[0,\, 1/(2^{n-1})]$) is still a density operator when it is totally reflected. In other words, total reflection is a nonunitary involution which preserves such sufficiently mixed sets of density operators. On this set, total reflection behaves like a antiunitary operation in the sense that it preserves the Hermitian structure, the trace and the (Hilbert-Schmidt) inner product. This tells us that when it comes to 3 or more mixed qubits there are more symmetries to be exploited than those of Wigner theorem. For 3 qubits, the set of density operators admitting a total reflection includes for example the Unextendible Product Basis (UPB) states used in Ref. [10] to generate a bound entangled density operator with all positive partial transpositions (PPT). The "complement" operation that turns a separable density into the bound entangled UPB state is in fact a total reflection of the type (ii) above. The various entanglement measures (the concurrence, the negativity and the tangle among them) that rely on the use of spin-flip operations are instead examples of application of multiple one-qubit reflections of the type (i). In between local and total reflections lies a class of "nonlocal yet partial" reflections which also belong to the class (ii) above. These maps resemble very closely those used in the so-called reduction criterion [11, 12].

Beside their unifying mathematical (group theoretic) character, we see reflections as a new tool to "probe" the state space of the set of multiparticle density operators, looking for its nonseparable regions, by means of operations analogous to partial transposition, but inequivalent to it. Hopefully this will eventually lead to a better understanding of bound entanglement in multipartite systems.

ONE QUBIT: TIME-REVERSING AND TRANSPOSITION

Throughout this work we use the notation of [6, 13]. For a single (pure or mixed) qubit this means expressing the density operator ρ in terms of an affine three-vector $[\varrho^0\ \vec{\varrho}^T]^T$ whose linear part $\vec{\varrho} = [\varrho^1\ \varrho^2\ \varrho^3]^T$ is the Bloch vector, $\rho = \varrho^j \lambda_j$ where $\lambda_j = \frac{1}{\sqrt{2}}\sigma_j$, $j = 1, 2, 3$, are the rescaled Pauli matrices, $\lambda_0 = \frac{1}{\sqrt{2}}\mathbb{1}_2$, $\varrho^j = \operatorname{tr}(\rho\lambda_j)$, $j = 1, 2, 3$, and $\varrho^0 = \operatorname{tr}(\rho\lambda_0) = 1/\sqrt{2}$ is a constant. Any antiunitary transformation can be written as a product of a unitary matrix and a conjugation K.

Given a pure state $|\psi\rangle = \sum_{j=1}^2 c_j |j\rangle$, call $|\tilde{\psi}\rangle$ the wavefunction obtained by reversing time by means of K alone: $|\tilde{\psi}\rangle = K|\psi\rangle = \sum_{j=1}^2 c_j^* |j\rangle$. Hence $|\psi\rangle\langle\psi| = \sum_{j,l=1}^2 c_j c_l^* |j\rangle\langle l|$ and $|\tilde{\psi}\rangle\langle\tilde{\psi}| = \sum_{j,l=1}^2 c_j^* c_l |j\rangle\langle l| = (|\psi\rangle\langle\psi|)^T$. Generalizing to mixed states, the effect of K on ρ is transposition, $\rho^T = K\rho K^\dagger = \varrho^0\lambda_0 + \varrho^1\lambda_1 - \varrho^2\lambda_2 + \varrho^3\lambda_3$, i.e., simply a change of sign of the λ_2 component on the Bloch vector, $[\varrho^1\ -\varrho^2\ \varrho^3]^T$, and corresponds (up to a unitary operation) to the so-called spin flip operation [3, 14]. As it is well-known, the group $O(3)$ contains two connected components, the one which preserves the orientation of a frame (i.e. $SO(3)$ which contains the identity matrix $\mathbb{1}_3$) and the one which changes sign to the orientation ($O^-(3)$, $-\mathbb{1}_3$ belongs to it), see Fig. 1. A

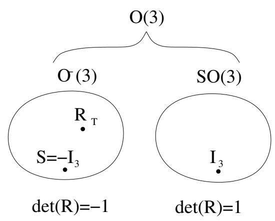

FIG. 1: Structure of the group $O(3)$.

reflection is a rotation which is not preserving the orientation. Our reflections belong to $O^-(3)$ and the particular reflection used in the transpose is $R_T = \operatorname{diag}(1, -1, 1)$. In particular, any $R \in O^-(3)$ is obtained by multiplication of a *spatial inversion* matrix $S = -\mathbb{1}_3$ with a rotation in $SO(3)$. On the Bloch sphere, the spatial inversion corresponds exactly to the spin flip operation. It acts as a reflection around the great plane through $\vec{\varrho} = 0$ and orthogonal to $\vec{\varrho}$ and to $-\vec{\varrho} = S(\vec{\varrho})$. For density matrices: $\rho^S = UK\rho K^\dagger U^\dagger = \varrho^0\lambda_0 - \varrho^1\lambda_1 - \varrho^2\lambda_2 - \varrho^3\lambda_3$ for some $U \in SU(2)$. Notice that

$$\operatorname{eig}(\rho) = \operatorname{eig}(\rho^T) = \operatorname{eig}(\rho^S) = \left\{ \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \pm \|\vec{\varrho}\| \right) \right\} \tag{1}$$

where $\|\vec{\varrho}\| = \sqrt{(\varrho^1)^2 + (\varrho^2)^2 + (\varrho^3)^2}$ is the standard Euclidean inner product in $\mathbb{R}^3$, giving the purity of the density operator. From (1), $\operatorname{eig}(\rho)$ are insensible to the true sign of the terms in $\vec{\varrho}$, meaning that reflections, just like rotations in $SO(3)$, correspond to length preserving actions on the Bloch sphere (just like unitary or antiunitary operations are isospectral for a density operator). For pure states, a difference between S and R_T is that S maps a ket state to an orthogonal one, while R_T does not.

For a single qubit the notion of reflection admits a further interpretation in terms of "co-completely positive" (coCP) maps. From the Størmer-Woronowicz theorem, any positive 2×2 map Φ is decomposable as

$$\Phi = k\Phi_1 + (1-k)\Phi \circ T, \qquad 0 \leqslant k \leqslant 1 \tag{2}$$

where Φ_1, Φ_2 are completely positive (CP) maps and T is transposition. $\Phi \circ T$ is called a *coCP map*. For the Bloch vector, the CP maps form a semigroup in $GL^+(3, \mathbb{R})$Ⓢ$\mathbb{R}^3$, where $GL^+(3, \mathbb{R}) = \{g \in GL(3, \mathbb{R}^3)$ s. t. $\det(g) > 0\}$, [15, 16]. The unital part of the CP maps lives in a component, $GL^+(3, \mathbb{R})$, of the group of actions, while the coCP maps live in the other one, $GL^-(3, \mathbb{R}) = \{g \in GL(3, \mathbb{R}^3)$ s. t. $\det(g) < 0\}$, and are obtained by applying an extra reflection to $GL^+(3, \mathbb{R})$. Restricting to symmetries (intended as trace- and norm-preserving maps) one gets the rotations and reflections introduced above.

TWO QUBITS: PARTIAL TIME REVERSE AND PARTIAL TRANSPOSE; MULTIPLE LOCAL REFLECTIONS AND TOTAL REFLECTIONS

If we have two qubits, then a complete basis for the density ρ is given by $\Lambda_{jk} = \lambda_j \otimes \lambda_k$, $j, k \in \{0, 1, 2, 3\}$. The basis is orthonormal: $\mathrm{tr}\,(\Lambda_{jk}\Lambda_{lm}) = \delta_{jl}\delta_{km}$ for all $j, k, l, m \in \{0, 1, 2, 3\}$, and defines a real two-tensor ϱ^{jk} which gives an equivalent representation of ρ: $\rho = \varrho^{jk}\Lambda_{jk}$. Seen as a 16-vector, ϱ^{jk} is affine: $\varrho^{00} = \mathrm{tr}\,(\rho\Lambda_{00}) = \frac{1}{2}$. The ϱ^{jk} parameterization belongs to a subset of the 15-dimensional sphere in $\mathbb{R}^{16}$ endowed with the Euclidean inner product one gets from the following:

$$\mathrm{tr}\left(\rho^2\right) = \mathrm{tr}\left(\left(\varrho^{jk}\Lambda_{jk}\right)^2\right) = \sum_{j,k=0}^{3}\left(\varrho^{jk}\right)^2 \qquad (3)$$

A two-qubit density ρ is said separable if $\rho = \sum_{r=1}^{s} w^r \rho_{A,r}\otimes\rho_{B,r}$ for real $w^r \geqslant 0$ such that $\sum_{r=0}^{s} w^r = 1$ and $\rho_{A,r}$, $\rho_{B,r}$ are qubit densities. A necessary and sufficient condition for separability of a two-qubit density is provided by the PPT criterion of Peres [5] and Horodeckis [17]. It is shown in [18, 19] that the PPT criterion corresponds to checking the feasibility of the "partial time reverse" operation: changing the time arrow of one of the subsystems alone, for example by means of the mixed unitary/antiunitary operator $K \otimes 1\!\!1_2$ (or $1\!\!1_2 \otimes K$), leads to a new Hermitian matrix which is the partial transpose of the 2-qubit density ρ: $\rho^{T_A} = (K \otimes 1\!\!1_2)\,\rho\,(K \otimes 1\!\!1_2)^\dagger$ (or $\rho^{T_B} = (1\!\!1_2 \otimes K)\,\rho\,(1\!\!1_2 \otimes K)^\dagger$) which is still a well-defined density operator if and only if ρ is separable. In the tensorial parameterization ϱ^{jk}, the formulation of the partial transposition is very intuitive and relies on the observation that λ_2 is the only imaginary observable of the set chosen to describe the Bloch vector.

Proposition 1 *In terms of ϱ^{jk}, the partial transpose operations ρ^{T_A} and ρ^{T_B} become the sign changes on the corresponding index "2":*

$$\begin{aligned} \rho^{T_A} &= \varrho^{0k}\Lambda_{0k} + \varrho^{1k}\Lambda_{1k} - \varrho^{2k}\Lambda_{2k} + \varrho^{3k}\Lambda_{3k} \qquad (4a)\\ \rho^{T_B} &= \varrho^{j0}\Lambda_{j0} + \varrho^{j1}\Lambda_{j1} - \varrho^{j2}\Lambda_{j2} + \varrho^{j3}\Lambda_{j3} \qquad (4b)\end{aligned}$$

The verification is just a straightforward calculation, see Table I below. Concerning ρ^{T_A} for example, the Λ_{2k} are the only basis elements that change sign when applying the partial transposition. Notice that instead for the "total" transpose ρ^T $(= (\rho^{T_A})^{T_B})$ we have

$$\begin{aligned} \Lambda_{jk}^T &= \Lambda_{jk} \qquad \text{if } j,k \neq 2 \text{ or } j=k=2 \qquad (5a)\\ \Lambda_{jk}^T &= -\Lambda_{jk} \qquad \text{if } j=2 \text{ or } k=2,\ j\neq k, \qquad (5b)\end{aligned}$$

that is Λ_{22} behaves differently under partial or total transposition.

The PPT separability test of Peres-Horodecki relies essentially on the decomposability property (2): any 1-qubit positive but not CP map when applied to a 2-qubit density returns a density operator if and only if the density is separable. Restricting from positive maps to symmetric operations means restricting to local reflections. In fact, the map (4a) can be thought of as the linear transformation $\bar{R}_T \otimes 1\!\!1_4$, where $\bar{R}_T$ is the following affine orientation-changing three-dimensional rotation: $\bar{R}_T = \mathrm{diag}\,(1, R_T) = \mathrm{diag}\,(1, 1, -1, 1)$. Any matrix $R \in O^-(3)$ can be used in place of R_T: if $\bar{R} = \mathrm{diag}\,(1, R)$ then local operations from the same connected component of $O(3)$ are such that

$$\mathrm{eig}\left(\left(\bar{R}\otimes 1\!\!1_4\right)(\rho)\right) = \mathrm{eig}\left(\left(\bar{R}_T\otimes 1\!\!1_4\right)(\rho)\right) \qquad (6)$$

where the notation must be intended as follows: $\bar{R}\otimes 1\!\!1_4$ acts on the 16-vector ϱ^{jk} and the resulting 16-vector sums with respect to the basis elements Λ_{jk}, i.e., $(\bar{R}\otimes 1\!\!1_4)(\rho) = \left(\bar{R}\otimes 1\!\!1_4\right)_{jk}^{lm}\varrho^{jk}\Big)\Lambda_{lm}$. In order for ρ to be separable, the matrices obtained by ρ via all affine rotations in $O^-(3)$ must be density operators. Hence we can reformulate the necessary and sufficient condition for separability of two qubits as follows:

Theorem 1 *ρ is separable if and only if $\left(\bar{R}\otimes 1\!\!1_4\right)(\rho)$ is a density matrix for any $R \in O^-(3)$.*

From (6) we have that all maps $\bar{R}\otimes 1\!\!1_4$ are positive but not completely positive. For example, if $\bar{S}_4 = \mathrm{diag}(1, S)$, the map $\bar{S}_4 \otimes 1\!\!1_4$ is a particularly simple such map. We have that $\left(\bar{S}_4\otimes 1\!\!1_4\right)(\rho) = 2\varrho^{0k}\Lambda_{0k} - \rho$ i.e., the sign is changed in all ϱ^{jk} except in the elements that form the reduced density of the second qubit (i.e., the ϱ^{0k}s).

The total time-reversing operation on ρ (yielding ρ^T) corresponds to $\bar{R}_{T,16} = \bar{R}_T \otimes \bar{R}_T = \mathrm{diag}(1, R_{T,15})$ with $R_{T,15} = \mathrm{diag}(1,-1,1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1) \in SO(15)$, where the minus signs correspond to the 6 basis elements obeying (5b). Hence for two qubits total time-reversing is an orientation-preserving operation, since reflections have to have negative determinants. Up to local operations $\bar{R}_T\otimes\bar{R}_T$ is equivalent to the "double local reflection" (or double spin-flip) map $\bar{S}_4\otimes\bar{S}_4$. The difference between $\bar{S}_4\otimes 1\!\!1_4$ and $\bar{S}_4\otimes\bar{S}_4$ is easily explained looking at Fig. 2. While $\bar{S}_4\otimes 1\!\!1_4$ leaves unchanged the reduced density of the second qubit, see (a), under the action of $\bar{S}_4\otimes\bar{S}_4$, shown in (b), the correlation part remains unchanged because its sign is flipped twice.

All local maps in $O(3)$ mentioned so for are orientation-preserving when acting on 2 qubits, although they are orientation-changing when acting on a single, isolated qubit: $\det(\bar{R}_T\otimes 1\!\!1_4) = \det(\bar{R}_T\otimes\bar{R}_T) = \det(\bar{S}_4\otimes 1\!\!1_4) = \det(\bar{S}_4\otimes\bar{S}_4) = 1$. The recovery of "parity" whenever an orientation-changing map is applied to 2 or more qubits is essentially due to the affine structure of the Hilbert

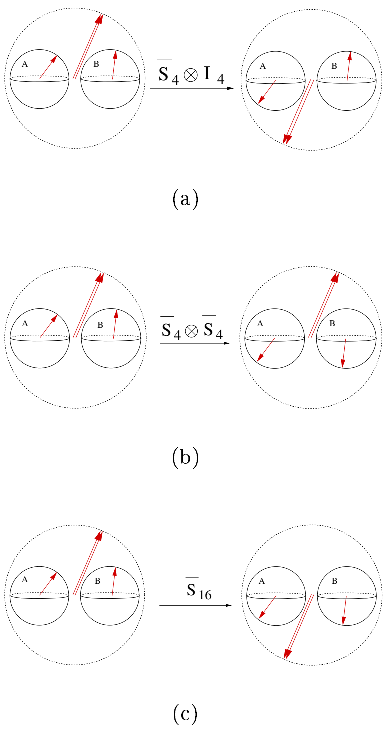

FIG. 2: Reflections on a 2-qubit density. The 2 vectors contained in the smaller spheres correspond to the Bloch vectors ϱ^{j0} and ϱ^{0k} of the 2 reduced densities, the third vector (double arrow) to the 2-body correlation part of the Stokes tensor ϱ^{jk}, $j,k \neq 0$. (a): $\bar{S}_4 \otimes \mathbb{1}_4$ = single qubit reflection (PPT test); (b): $\bar{S}_4 \otimes \bar{S}_4$ = double local reflection (total transpose, up to LOCC); (c): $\bar{S}_{16}$ = nonlocal (total) reflection.

space of a qubit (resulting in a *affine Bloch vector*), itself a consequence of the trace-preserving condition:

$$\begin{bmatrix} 1 & \\ & O^-(3) \end{bmatrix} \otimes \begin{bmatrix} 1 & \\ & SO(3) \end{bmatrix} \in \begin{bmatrix} 1 & \\ & SO(15) \end{bmatrix},$$

$$\begin{bmatrix} 1 & \\ & O^-(3) \end{bmatrix} \otimes \begin{bmatrix} 1 & \\ & O^-(3) \end{bmatrix} \in \begin{bmatrix} 1 & \\ & SO(15) \end{bmatrix}.$$

Hence, a spontaneous question arises, whether there exist any orientation-changing symmetric operation on 2 qubits. One such maps is for instance the 2-qubit total reflection is $\bar{S}_{16} = \mathrm{diag}(1, -\mathbb{1}_{15})$, $-\mathbb{1}_{15} \in O(15) \setminus SO(15)$. Its action is depicted in (c) of Fig. 2 and corresponds to changing sign to the entire tensor ϱ^{jk} except for the trace part ϱ^{00}, thus the name total reflection. This nonlocal operation is genuinely new and inequivalent to any composition of local symmetric operations. The difference is reconducible to the observable Λ_{22}, which changes sign twice under the action of $\bar{R}_{T,16}$ and only once under $\bar{S}_{16}$.

The changes in sign on the Stokes tensor of all the discrete symmetric operations mentioned in this Section are summarized in Table I.

From Table I, observe that $\bar{S}_4 \otimes \bar{S}_4$ has the same sign balance as $\bar{R}_{T,16}$, thus confirming that a total reflection on a 2-qubit joint density is inequivalent to such operations. In terms of density matrices, the operation $\bar{S}_4 \otimes \bar{S}_4$ corresponds to

$$\rho' = \sigma_2 \otimes \sigma_2 \rho^* \sigma_2 \otimes \sigma_2 = 4\Lambda_{22}\rho^*\Lambda_{22}$$

and is of frequent use, for example to compute the concurrence $C(\rho) = \max\{0, \nu_1 - \nu_2 - \nu_3 - \nu_4\}$ where $\nu_j = \mathrm{eig}(\rho\rho')$ [20] or the entanglement measure $\mathrm{tr}\,(\rho\rho')$ which, following [14] also corresponds to the Lorentzian metrix $(\varrho^{00})^2 - \sum_{j=1}^3 \left((\varrho^{0j})^2 + (\varrho^{j0})^2\right) + \sum_{j,k=1}^3 (\varrho^{jk})^2$.

REFLECTIONS ON 3 OR MORE QUBITS

The situation is similar for more than two qubits, since conjugation actions still correspond to real "one sided" rotations for the Stokes tensor and real rotations all split into orientation preserving and orientation changing connected components. For example, for three qubits $\rho = \varrho^{jkl}\Lambda_{jkl}$ we have the following operands:

(ia) the 2-qubit partial transposition is $\bar{R}_{T,16} \otimes \mathbb{1}_4$ (up to a permutation of the three indexes jkl);

(ib) the total transposition is given by a 64 dimensional diagonal rotation matrix with 28 minus signs, $\bar{R}_{T,64} = \mathrm{diag}(1, R_{T,63})$, $R_{T,63} \in SO(63)$. $\bar{R}_{T,64}$ has the effect of changing sign to all and only the terms ϱ^{jkl} with an odd number of indexes "2";

(iia) the 2-qubit reflection is $\bar{S}_{16} \otimes \mathbb{1}_4$ (up to a index permutation);

(iib) the total reflection is given by $\bar{S}_{64} = \mathrm{diag}\,(1, -\mathbb{1}_{63})$, $-\mathbb{1}_{63} \in O(63) \setminus SO(63)$.

The action o f$\bar{S}_{64}$ on ϱ^{jkl} is to change the sign of the entire tensor except for the constant component $\varrho^{000} = \frac{1}{2\sqrt{2}}$:

$$\bar{S}_{64}(\rho) = 2\varrho^{000}\Lambda_{000} - \rho = \frac{1}{4}\mathbb{1}_8 - \rho. \tag{7}$$

This is a nonlocal operation and admits no splitting for any density. Similarly, the action of $\bar{S}_{16} \otimes \mathbb{1}_4$ on ϱ^{jkl}

TABLE I: Action (sign changes) of the rotations $\bar{R}_T$ and of the reflections involving $\bar{S}_4$ and $\bar{S}_{16}$ on the components of the 2-qubit Stokes tensor ϱ^{jk}.

ϱ^{jk}	$\bar{R}_T \otimes \mathbb{1}_4$	$\mathbb{1}_4 \otimes \bar{R}_T$	$\bar{R}_T \otimes \bar{R}_T$	$\bar{S}_4 \otimes \mathbb{1}_4$	$\mathbb{1}_4 \otimes \bar{S}_4$	$\bar{S}_4 \otimes \bar{S}_4$	$\bar{S}_{16}$
ϱ^{00}	+	+	+	+	+	+	+
ϱ^{01}	+	+	+	+	−	−	−
ϱ^{02}	+	−	−	+	−	−	−
ϱ^{03}	+	+	+	+	−	−	−
ϱ^{10}	+	+	+	−	+	−	−
ϱ^{11}	+	+	+	−	−	+	−
ϱ^{12}	+	−	−	−	−	+	−
ϱ^{13}	+	+	+	−	−	+	−
ϱ^{20}	−	+	−	−	+	−	−
ϱ^{21}	−	+	−	−	−	+	−
ϱ^{22}	−	−	+	−	−	+	−
ϱ^{23}	−	+	−	−	−	+	−
ϱ^{30}	+	+	+	−	+	−	−
ϱ^{31}	+	+	+	−	−	+	−
ϱ^{32}	+	−	−	−	−	+	−
ϱ^{33}	+	+	+	−	−	+	−
# sign changes	4	4	6	12	12	6	15

is to change sign to the entire tensor except for the Bloch vector of the 1-qubit reduced density ϱ^{00l}. Items (ia) and (ib) are fundamentally different from (iia) and (iib). Item (ia) produces no action on separable states but gives a Hermitian matrix with negative eigenvalues when the density has bipartite entanglement through the cut. Obviously (ib) only reverses the time arrow on any 3-qubit density. The two reflections instead may map even separable densities to Hermitian matrices with negative eigenvalues, as can be seen looking at the pieces of the UPB state. If $\bar{S}_{64}$ is applied to the (separable) density $\rho_{\rm UPB-sep} = \frac{1}{4}\sum_{j=1}^{4}|\psi_j\rangle\langle\psi_j|$ with $|\psi_j\rangle = |01+\rangle, |1+0\rangle, |+01\rangle, |---\rangle$ and $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, one gets the bound entangled state $\rho_{\rm UPB}$ used in [10]. So in this case a separable state is reflected into an entangled state. However, no one of the 4 components $|\psi_j\rangle\langle\psi_j|$ taken alone (each is obviously separable) is a density when reflected.

In the same fashion as (iia) and (iib), for n qubits, one can define a *m-qubit ($m < n$) nonlocal reflection* $\bar{S}_m \otimes \mathbb{1}_{n-m}$. Assuming the reflection acts on the first m qubits of an n-qubit density operator ρ, then we have

$$\bar{S}_m \otimes \mathbb{1}_{n-m}(\rho) = 2(\varrho^{0\ldots 0 j_{m+1}\ldots j_n}\Lambda_{0\ldots 0 j_{m+1}\ldots j_n}) - \rho \quad (8)$$

For $m = n$ the reflection becomes total:

$$\bar{S}_n(\rho) = 2\varrho^{0\ldots 0}\Lambda_{0\ldots 0} - \rho = \frac{1}{2^{n-1}}\mathbb{1}_{2^n} - \rho.$$

The operation of reflection leaves invariant the norm of the n-qubit tensor $\varrho^{j_1\cdots j_n}$ (i.e., $\operatorname{tr}(\rho^2)$). However, it needs not be isospectral, neither it needs to preserve a positive spectrum, as we saw above. Hence it is a "generically" ill-defined operation on the set of density operators of composite systems, call it $\mathcal{D}_n$. We then have the following.

Proposition 2 *In $\mathcal{D}_n$, the linear map $\bar{S}_{4^n}$*

(i) *preserves trace and Hermiticity;*

(ii) *preserves the inner product;*

(iii) *is neither unitary nor antiunitary;*

(iv) *is not $\mathcal{D}_n$-invariant.*

Property (i) is straightforward to check. Property (ii) can be verified as follows. If $\rho_1 = \varrho_1^{j_1\cdots j_n}\Lambda_{j_1\ldots j_n}$ and $\rho_2 = \varrho_2^{j_1\cdots j_n}\Lambda_{j_1\ldots j_n}$ are two 2-qubit densities in $\mathcal{D}_n$, then from (3) the inner product is $\operatorname{tr}(\rho_1\rho_2) = \sum_{j_1,\ldots,j_n=0}^{3}\varrho_1^{j_1\cdots j_n}\varrho_2^{j_1\cdots j_n}$. For the two reflected densities $\tilde{\rho}_1$ and $\tilde{\rho}_2$, then: $\operatorname{tr}(\tilde{\rho}_1\tilde{\rho}_2) = \varrho_1^{0\ldots 0}\varrho_2^{0\ldots 0} + \sum_{j_1,\ldots,j_n=0,\,(j_1\ldots j_n)\neq(0ldots0)}^{3}(-\varrho_1^{j_1\cdots j_n})(-\varrho_2^{j_1\cdots j_n}) = \operatorname{tr}(\rho_1\rho_2)$.

Properties (i) and (ii) together say that $\bar{S}_{4^n}$ is neither a contraction nor a dilation map.

Notice that $\bar{S}_{4^n}$ does not commute with the operation of convex sum. For example for the 3-qubit UPB state, this means that although *a posteriori* (i.e., after the cancellations that occur in the sum) it holds that

$$\bar{S}_{64}\left(\frac{1}{4}\sum_{j=1}^{4}|\psi_j\rangle\langle\psi_j|\right)=\frac{1}{4}\sum_{j=1}^{4}\bar{S}_{64}\left(|\psi_j\rangle\langle\psi_j|\right), \qquad (9)$$

"formally" the right hand side of (9) contains something else than just density operators, as $\bar{S}_{64}\left(|\psi_j\rangle\langle\psi_j|\right)$ has negative eigenvalues.

The condition (iv) affirms that $\bar{S}_{4^n}$ is not a completely positive map. Nevertheless, writing $\rho = \varrho^{0\ldots0}\Lambda_{0\ldots0}+\xi$ (with ξ the associated homogeneous tensor), it is possible to specify a simple spectral condition that guarantees that the totally reflected matrix $\tilde{\rho}=\bar{S}_{4^n}(\rho)=\varrho^{0\ldots0}\Lambda_{0\ldots0}-\xi$ is still a density.

Theorem 2 *Consider $\rho\in\mathcal{D}_n$. A sufficient condition for $\tilde{\rho}$ to be a density (i.e., $\tilde{\rho}\in\mathcal{D}_n$) is that* $\mathrm{eig}(\rho)\in[0,\,\frac{1}{2^{n-1}}]$.

Proof. The proof is based on the following observation. Given 2 $m\times m$ Hermitian matrices A and B, one has (see [21] Theorem 4.3.1) that if $\mu_j(A)$, $\mu_j(B)$ and $\mu_j(A+B)$ are eigenvalues arranged in increasing order then

$$\mu_j(A)+\mu_1(B)\leqslant\mu_j(A+B)\leqslant\mu_j(A)+\mu_m(B).$$

But if $B=k\mathbb{1}$, $k\in\mathbb{R}$, this boils down to

$$\mu_j(A+B)=\mu_j(A)+k. \qquad (10)$$

From the affine structure of the n-qubit density, one gets that for the random state $\mathrm{eig}\left(\varrho^{0\ldots0}\Lambda_{0\ldots0}\right)=\left\{\frac{1}{2^n}\right\}$ of multiplicity 2^n. Using (10), the statement is obvious: if $\mathrm{eig}(\rho)\in[0,\,\frac{1}{2^{n-1}}]$ then $\mathrm{eig}(\xi)\in[-\frac{1}{2^n},\,\frac{1}{2^n}]$ and $\mathrm{eig}(-\xi)\in[-\frac{1}{2^n},\,\frac{1}{2^n}]$. Therefore $\mathrm{eig}(\tilde{\rho})\in[0,\,\frac{1}{2^{n-1}}]$. □

Hence the linear map $\bar{S}_{4^n}$ is well-defined (positive) in the subset of densities with eigenvalues in the interval $[0,\,\frac{1}{2^{n-1}}]$.

Corollary 1 *A necessary but not sufficient condition for Theorem 2 to hold is that* $\mathrm{tr}\left(\rho^2\right)\leqslant\frac{1}{2^{n-1}}$.

Proof. If $\mu_{\xi_1},\ldots,\mu_{\xi_{2^n}}$ are the eigenvalues of ξ, when Theorem 2 holds it must be $r^2=\mu_{\xi_1}^2+\ldots+\mu_{\xi_{2^n}}^2\leqslant\frac{1}{2^n}$. Hence $r\leqslant\frac{1}{2^{n/2}}$ and $\mathrm{tr}\left(\rho^2\right)=(\varrho^{0\ldots0})^2+r^2\leqslant\frac{1}{2^{n-1}}$. □

Corollary 2 *A necessary but not sufficient condition for $\tilde{\rho}$ to be a density is that* $\mathrm{rank}(\rho)\geqslant 2^{n-1}$.

In fact, only when ρ is a mixture of at least 2^{n-1} pure states one can achieve $\mathrm{eig}(\rho)\in[0,\,\frac{1}{2^{n-1}}]$.

Of course the 3-qubit UPB states satisfy the conditions of Theorem 2 and Corollary 2. They do so in the "limiting" case, $\mathrm{eig}(\rho_{\mathrm{UPB-sep}})=\left\{0,\,\frac{1}{4}\right\}$, each with multiplicity 4, because of the 4 equal weights (which, by the way, make the reflection be isospectral in this case). If the weights are chosen different, then more than 4 summands are needed in the convex combination in order to satisfy Theorem 2.

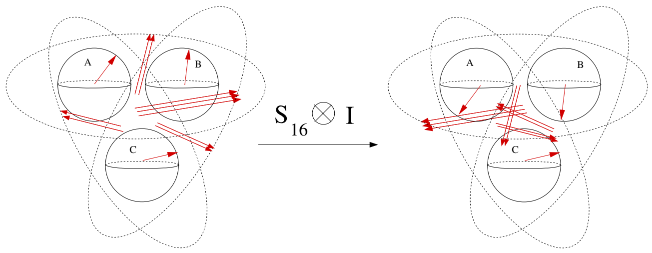

FIG. 3: Action of a two-qubit reflection on 3 qubits $\bar{S}_{16}\otimes\mathbb{1}_4(\rho)$. One-body ($\varrho^{j00}$, ϱ^{0k0} and ϱ^{00l}), two-body (ϱ^{jk0}, ϱ^{j0l} and ϱ^{0kl}) and three-body (ϱ^{jkl}) correlation terms are indicated by single, double and triple arrows. All signs are changed except for ϱ^{00l}.

On a 3-qubit density, the action of $\bar{S}_{16}\otimes\mathbb{1}_4$ is depicted in Fig. 3. Essentially the entire Stokes tensor changes sign, except for the reduced density $\mathrm{tr}_{AB}(\rho)$. Its action resembles very closely the reduction criterion of [11, 12]. This test makes also an implicit use of nonlocal reflections, but it is formulated based on a positive map, hence it is well-posed on the entire $\mathcal{D}_n$. For a 3-qubit density it affirms that a necessary condition for separability is that $\mathbb{1}_2\otimes\mathbb{1}_2\otimes\mathrm{tr}_{AB}(\rho)-\rho\geqslant 0$ (and likewise for the other indexes). Since $\mathrm{tr}\left(\mathbb{1}_2\otimes\mathbb{1}_2\otimes\mathrm{tr}_{AB}(\rho)\right)=4$, one difference between our partial reflection and the reduction criterion is that the latter is not a trace preserving map. Thus it is not a symmetry in the sense used in the paper. The reduction criterion allows to attain a positive map which can be used to test separability. In our case it is not clear whether $\bar{S}_{16}\otimes\mathbb{1}_4$ is positive even when restricted to separable states with eigenvalues in $[0,\,1/4]$, in which a total reflection is always well-posed.

Concerning a total reflection, all pairs ρ and $\tilde{\rho}=\bar{S}_{4^n}(\rho)$ satisfying Theorem 2 are complementary in the sense that their mixture is the random state:

$$\frac{1}{2}\left(\rho+\tilde{\rho}\right)=\frac{1}{2^n}\mathbb{1}_{2^n}. \qquad (11)$$

Eq. (11) implies that $\bar{S}_{2^n}$ corresponds to a multiparty NOT operation. In fact, also in the single qubit case, the NOT operation corresponds to a change of sign to the homogeneous part (i.e., the Bloch vector) but it is not modifying the sign of the trace part and hence a qubit and its reflection obey to (11). Such operation is used for example to map a density operator belonging to a subset of the Hilbert space $\mathcal{D}_n$ to its complement in $\mathcal{D}_n$, for example in the UPB construction mentioned above [10, 22].

CONCLUSION

Reflections are a natural discrete class of transformations for the tensorial parametrization chosen. Their meaning and relation to LOCC is very neat and calls for natural generalizations to nonlocal operations in the way explained above. All the nonlocal reflections, in fact, originate from the nonconnectedness of the group of rotations acting on the Stokes tensor parametrization. In terms of density matrices, this meaning is not as sharp. As a matter of fact, operations reconducible to reflections appear in the PPT test and in the various measures of entanglement relying upon "spin-flip" operations (like concurrence, negativity and tangle) for what concerns (multiple) 1-qubit reflections. Also nonlocal reflections are used: for example a total reflection corresponds to what is normally referred to as "taking the complement of a density", used for example in the construction of Unextendible Product Basis states [10]. Likewise, the reduction critetion makes use of a positive map closely related to our nonlocal reflections.

For the purposes of further understanding the structure of composite quantum systems, we find it useful to have a unifying perspective on these nonunitary yet symmetric (in the sense of Wigner Theorem) transformations.

It is worth noticing that reflections can be defined in the same terms also for SLOOC (stochastic LOCC) [7, 23, 24]. For the Stokes tensor, in fact, this class of operations relaxes the group of admissible local transformations from affine rotations to proper orthochronous local Lorentz transformations $SO(1,3)$. The reflected action in $O^-(3)$ then corresponds to choosing the other connected component of $O(1,3)$ with the same time direction as $SO(1,3)$ (i.e., with positive "time-like" metric element). Also nonlocal reflections fit in with the group structure of nonlocal filtering operations. For example, total reflections belong to $O(1,4^n-1)\setminus SO(1,4^n-1)$.

Notice that, instead, the idea of using reflections does not extend in a straightforward manner to qutrids (nor to higher dimension quantum systems), as in this case the admissible parameters live on a rather complicated subset of the seven dimensional sphere [25, 26] for which the rotation representing transposition is always admissible but spatial inversion may not be. However, the various UPB constructions on 3×3 systems of [10] are no doubt well-defined reflections.

Finally notice that the idea of restricting the set of density operators in order to have a larger set of symmetries is "dual" to the idea of using group actions that are contractions [27, 28].

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